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A CONDITIONAL LAW OF SUPER ABSORBING BARRIER BROWNIAN MOTION

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Abstract. We give the characterization of a conditional law of super absorbing barrier Brownian motion in terms of solution to the initial value problem of a parabolic differential equation.

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1. Introduction

The method of differential equations plays an important role in the study of superprocesses. Iscoe (1988) characterized some charging probabilities of a super Brownian motion in terms of some partial differential equations with singular boundary conditions. We refer the reader to Dynkin (1994) for systematic study of the interplay of superprocesses and partial differential equations. In this note we show that some conditional laws of superprocesses can also be characterized in terms of solutions to the initial value problem of some differential equations. For concreteness, we shall concentrate to the special case of a super absorbing barrier Brownian motion.

Let $H = (0, \infty)$ and let $\mathbb{R}^+ = [0, \infty)$. Let $M(H)$ be the space of finite Borel measures on H endowed with the weak convergence topology. For a domain $D \subset \mathbb{R}^d$ let $C^2(D)^+$ the space of bounded, positive, twice continuously differentiable functions on D. Let $C_0^2(D)^+$ denote the subset of $C^2(D)^+$ consisting of functions that vanish at the boundary of D. By a critical continuous *super absorbing barrier Brownian motion* we mean a diffusion process $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ with state space $M(H)$ such that

$$\mathbf{Q}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \qquad t \geq 0, f \in C_0^2(H)^+, \tag{1.1}$$

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where $\mu(f) = \int f \mathrm{d}\mu$ and $V_t f(x)$ denotes the mild solution to

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) - \frac{1}{2} u(t,x)^2 & t \geq 0,\ x > 0, \\ u(t,0^+) = 0, u(0,x) = f(x), & t \geq 0,\ x > 0. \end{cases} \tag{1.2}$$

We define the process $\{R_t : t \geq 0\}$ by

$$R_t = \sup\left\{r > 0 : X_s((0,r]) > 0 \text{ for some } 0 \leq s \leq t\right\}, \qquad t \geq 0. \tag{1.3}$$

The main results of this note are the following two theorems.

Theorem 1.1. *Take $a > 0$ and $0 < t \leq \tau$. Suppose that supp$(\mu) \subset [0,a]$. Then for $f \in C_0^2(H)$ we have*

$$\mathbf{Q}_\mu \left\{ \mathrm{e}^{-X_t(f)}; R_\tau \leq a \right\} = \exp\left\{-\mu(V_t^a[f + V_{\tau-t}^a 0])\right\}, \tag{1.4}$$

where $V_t^a f(x)$ is the unique solution to

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) - \frac{1}{2} u(t,x)^2 & t \geq 0,\ 0 < x < a, \\ u(0,x) = f(x), & 0 < x < a, \\ u(t,0^+) = 0,\ u(t,a^-) = \infty, & t > 0, \\ u(t,x) \to u(x)\ (t \to \infty), & 0 < x < a, \end{cases} \tag{1.5}$$

and $u(x)$ is the unique solution to

$$\begin{cases} u''(x) = u(x)^2, & 0 < x < a, \\ u(0^+) = 0,\ u(a^-) = \infty. \end{cases} \tag{1.6}$$

Theorem 1.2. *Take $a > 0$ and $0 < t \leq \tau$. Suppose that supp$(\mu) \subset [0,a]$. Then for $f \in C_0^2(H)$ we have*

$$\mathbf{Q}_\mu \left\{ \mathrm{e}^{-X_t(f)} | R_\tau \leq a \right\} = \exp\left\{-\mu(V_t^a[f + V_{\tau-t}^a 0] - V_\tau^a 0)\right\}, \tag{1.7}$$

where $V_t^a f(x)$ is defined by (1.5).

2. Proofs of the theorems

In this section, we give the proofs of the two theorems, which are based on discussions of parabolic and elliptic equations with singular boundary conditions. Some parts of our proofs follow the lines set out in Iscoe (1988); see also Li and Shiga (1995).

Lemma 2.1 (Iscoe, 1986)**.** *For $t \geq 0$ and $f, g \in C_0^2(H)^+$,*

$$\mathbf{Q}_\mu \exp\left\{ - X_t(f) - \int_0^t X_s(g) \mathrm{d}s \right\} = \exp\left\{-\mu(U_t(f,g))\right\}, \tag{2.1}$$

where $U_t(f,g) = U_t(f,g;x)$ is the unique solution to

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) - \frac{1}{2} u(t,x)^2 + g(x), & t \geq 0,\ x > 0, \\ u(t,0^+) = 0, u(0,x) = f(x), & t > 0,\ x > 0. \end{cases} \tag{2.2}$$

Lemma 2.2. *Fix $a < b$ and $g \in C^2([a,b])^+$. If $u(t,x)$ satisfies*

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}u(t,x)^2 + g(x), \qquad t \geq 0,\ a \leq x \leq b, \tag{2.3}$$

then it satisfies the integral equation

$$\begin{aligned} u(t,x) = \int_a^b u(0,y)g_t^{ab}(x,y)\mathrm{d}y - \int_0^t \mathrm{d}s \int_a^b \left[\frac{1}{2}u(s,y)^2 - g(y)\right] g_{t-s}^{ab}(x,y)\mathrm{d}y \\ + \frac{1}{2}\int_0^t u(s,a)\frac{\partial}{\partial y}g_{t-s}^{ab}(x,a)\mathrm{d}s - \frac{1}{2}\int_0^t u(s,b)\frac{\partial}{\partial y}g_{t-s}^{ab}(x,b)\mathrm{d}s, \end{aligned} \tag{2.4}$$

where $g_t^{ab}(x,y)$ is the transition density of an absorbing barrier Brownian motion in (a,b). Conversely if $u(\cdot,\cdot) \in C([0,\infty)\times[a,b])^+$ satisfies (2.4), then it belongs $C^2([0,\infty)\times[a,b])^+$ and satisfies (2.3).

Proof. Suppose that $u(\cdot,\cdot)$ satisfies (2.3). Let $(B(t), \mathbf{P}_x)$ be a standard one-dimensional Brownian motion, and let $\tau = \inf\{t > 0 : B(t) = a \text{ or } b\}$. By (2.3) and Itô's formula, for $0 \leq s \leq t$ we have

$$u(t - s\wedge\tau, B(s\wedge\tau)) - u(t,x) = \int_0^{s\wedge\tau} \left[\frac{1}{2}u(t-r,B(r))^2 - g(B(r))\right]\mathrm{d}r + \text{mart.}$$

It follows that

$$u(t,x) = \mathbf{P}_x u(t - t\wedge\tau, B(t\wedge\tau)) - \mathbf{P}_x \int_0^t 1_{\{r\leq\tau\}} \left[\frac{1}{2}u(t-r,B(r))^2 - g(B(r))\right]\mathrm{d}r.$$

Then (2.4) holds as we notice the well-known facts

$$\begin{aligned} \mathbf{P}_x\{B(\tau) = a \ \text{ and } \ \tau \in \mathrm{d}s\} &= \frac{1}{2}\frac{\partial}{\partial y}g_s^{ab}(x,a)\mathrm{d}s, \\ \mathbf{P}_x\{B(\tau) = b \ \text{ and } \ \tau \in \mathrm{d}s\} &= -\frac{1}{2}\frac{\partial}{\partial y}g_s^{ab}(x,b)\mathrm{d}s. \end{aligned}$$

Clearly (2.3) follows from (2.4) by differentiating under the integral. $\square$

Lemma 2.3. *Fix $a > 0$. Let $\{f_n\} \subset C_0^2(I\!R^+)^+$ be an increasing sequence such that $f_n \uparrow \infty$ for $x \geq a$ and $f_n(x) \uparrow f(x)$ for $x \in [0, a)$, where $f \in C_0^2([0, b])^+$ for each $0 < b < a$. Let $\{g_n\} \subset C_0^2(I\!R^+)^+$ be an increasing sequence such that $g_n \uparrow \infty$ for $x \geq a$ and $g_n(x) = 0$ for $x \in [0, a)$. Suppose $u_n(t, x)$ is the solution to (2.2) with (f, g) being replaced by (f_n, g_n). Then $u_n(t, x) \uparrow u(t, x)$ uniformly on $[0, l] \times [0, b]$ for each $l > 0$ and $0 < b < a$, where $u(t, x)$ is a solution to*

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) - \frac{1}{2}u(t,x)^2 \qquad t \geq 0,\ 0 < x < a, \\ u(0,x) = f(x), \qquad 0 < x < a, \\ u(t,0^+) = 0,\ u(t,a^-) = \infty, \qquad t > 0. \end{cases} \tag{2.5}$$

Proof. By Lemma 2.1 we have

$$\mathbf{Q}_{\delta_x} \exp\bigg\{ - X_t(f_n) - \int_0^t X_s(g_n)\mathrm{d}s\bigg\} = \exp\{-u_n(t,x)\}. \tag{2.6}$$

Then $u_n(t, x)$ increases to a limit $u(t, x)$ as $n \to \infty$ and

$$\mathbf{Q}_{\delta_x}\{\mathrm{e}^{-X_t(f)}; R_t \leq a\} = \exp\{-u(t,x)\}. \tag{2.7}$$

It is well-known that there is a continuous orthogonal martingale measure $M(\mathrm{d}x, \mathrm{d}s)$ with covariation measure $X_s(\mathrm{d}x)\mathrm{d}s$ such that $\mathbf{Q}_{\delta_x}$-almost surely

$$X_t(h) = P_t h(x) + \int_H \int_0^t P_{t-s}h(x)M(\mathrm{d}x,\mathrm{d}s)$$

for $t \geq 0$ and $h \in B(H)$. On the other hand, note that

$$\mathbf{Q}_{\delta_x}\bigg\{\bigg[\int_H \int_0^t P_{t-s}h(x)M(\mathrm{d}x,\mathrm{d}s)\bigg]^2\bigg\} = \int_0^t \mathrm{d}s \int_H [P_{t-s}h(x)]^2\, P_s(x,\mathrm{d}x),$$

which goes to zero as $t \to 0$. For $h = 1_{(a,a+1)}$, we have $\lim_{t\downarrow 0} P_t h(a) = 1/2$. Then $X_t(a, a+1) \to 1/2$ in probability under $\mathbf{Q}_{\delta_a}$ as $t \to 0$. It follows that, $\mathbf{Q}_{\delta_a}$-almost surely, $\int_0^t X_s(a, a+1)\mathrm{d}s > 0$ for any $t > 0$. By (2.6) we have $u_n(t, a) \uparrow \infty$ as $n \uparrow \infty$. But $u(t, a^-) \geq u_n(t, a)$, so $u(t, a^-) = \infty$.

Applying Lemma 2.2 to $u_n(t, x)$ we have

$$\begin{aligned} u_n(t,x) = &\int_0^b f_n(y)g_t^{0b}(x,y)\mathrm{d}y - \frac{1}{2}\int_0^t \mathrm{d}s \int_0^b u_n(s,y)^2 g_{t-s}^{0b}(x,y)\mathrm{d}y \\ &- \frac{1}{2}\int_0^t u_n(s,b)\frac{\partial}{\partial y}g_{t-s}^{0b}(x,b)\mathrm{d}s \end{aligned}$$

for $0 \leq x \leq b < a$. Letting $n \to \infty$ and using the monotone convergence theorem,

$$\begin{aligned} u(t,x) = &\int_0^b f(y)g_t^{0b}(x,y)\mathrm{d}y - \frac{1}{2}\int_0^t \mathrm{d}s \int_0^b u(s,y)^2 g_{t-s}^{0b}(x,y)\mathrm{d}y \\ &- \frac{1}{2}\int_0^t u(s,b)\frac{\partial}{\partial y}g_{t-s}^{0b}(x,b)\mathrm{d}s. \end{aligned} \tag{2.8}$$

Then $u(t, x)$ satisfies (2.5) by Lemma 2.2. □

Lemma 2.4. *Let $u(t,x)$ be the solution to (2.5) provided by Lemma 2.3. Then we have $u(t,x) \uparrow u(x)$ as $t \to \infty$, where $u(x)$ is the unique solution to*

$$\begin{cases} u''(x) = u(x)^2, & 0 < x < a, \\ u(0) = 0, \quad u(a^-) = \infty, \\ u(x) = \infty, & x \geq a. \end{cases} \tag{2.9}$$

Proof. We first assume $f \equiv 0$. By (2.6) we have that $u(t,x)$ is increasing in $t \geq 0$, so the limit $u(x) = \lim_{t\uparrow\infty} u(t,x)$ exists. By (2.8) we obtain

$$u(x) = -\frac{1}{2}\int_0^\infty \mathrm{d}s \int_0^b u(y)^2 g_s^{0b}(x,y)\mathrm{d}y - \frac{u(b)}{2}\int_0^\infty \frac{\partial}{\partial y} g_s^{0b}(x,b)\mathrm{d}s$$

for $0 \leq x \leq b < a$. Then it is easy to check that $u(x)$ satisfies (2.9).

Suppose that $u(x)$ and $v(x)$ are two solutions to (2.9). For $0 < \theta < 1$, set $v_\theta(x) = \theta^2 v(\theta x)$. We claim that

$$w_\theta(x) := u(x) - v_\theta(x) \geq 0, \qquad t > 0, 0 \leq x < a. \tag{2.10}$$

If this not true, say

$$w_\theta(x_0) = u(x_0) - v_\theta(x_0) = \max_{0 \leq x < a} w_\theta(x) < 0.$$

Then we have

$$0 \leq w_\theta''(x_0) = u''(x_0) - v_\theta''(x_0) = u(x_0)^2 - v_\theta(x_0)^2,$$

yielding a contradiction. Letting $\theta \to 1$ in (2.10) we get $u(x) \geq v(x)$. By the same argument we also have $v(x) \geq u(x)$, proving the uniqueness of the solution to (2.9).

Now let $u(t,x)$ be given by Lemma 2.3 with non-degenerate $f \geq 0$. It is well-known that $X_t(f) \to 0$ almost surely as $t \to \infty$. Then by (2.7)

$$\begin{aligned} \lim_{t\to\infty} \mathrm{e}^{-u(t,x)} &= \lim_{t\to\infty} \mathbf{Q}_{\delta_x}\left\{\mathrm{e}^{-X_t(f)}; R_t \leq a\right\} \\ &= \lim_{t\to\infty} \mathbf{Q}_{\delta_x}\left\{R_t \leq a\right\} = \mathrm{e}^{-u(x)}, \end{aligned} \tag{2.11}$$

as desired. $\square$

Lemma 2.5. *The singular boundary value problem (1.5) has a unique solution $u(t,x)$.*

Proof. That (1.5) does have such a solution follows by Lemmas 2.3 and 2.4. Suppose $v(t,x)$ is another solution to (1.5). Then we have $\lim_{t\uparrow\infty} v(t,x) = u(x)$, where $u(x)$ is the unique solution to (1.6). For $0<\theta<1$, set $v_\theta(t,x) = \theta^2 v(\theta^2 t, \theta x)$. Then $v_\theta(t,x)$ satisfies the first equation in (1.5). We declare that

$$w_\theta(t,x) := u(t,x) - v_\theta(t,x) \geq 0, \qquad t \geq 0, 0 \leq x < a. \tag{2.12}$$

If this is not true, since

$$\lim_{t\uparrow\infty} w_\theta(t,x) = u(x) - \theta^2 u(\theta x) \geq 0, \qquad 0 \leq x < b,$$

it follows that $w_\theta(t,x)$ has a minimum point (t_0,x_0). Then $w(t_0,x_0)<0$ and

$$\frac{\partial}{\partial t} w_\theta(t_0,x_0) = 0, \qquad \frac{\partial^2}{\partial x^2} w_\theta(t_0,x_0) \geq 0. \tag{2.13}$$

By the definition of $w_\theta(t,x)$ we have

$$\frac{\partial^2}{\partial x^2} w_\theta(t_0,x_0) = u(t_0,x_0)^2 - v_\theta(t_0,x_0)^2.$$

Then (2.12) and (2.13) are in contradiction. Letting $\theta \to 1$ in (2.12) we get $u(t,x) \geq v(t,x)$. The desired uniqueness then follows as in the proof of Lemma 2.4. $\square$

Proof of Theorems 1.1 and 1.2. Choose the sequence $\{g_n\}$ as in Lemma 2.3. Using the Markov property and Lemma 2.1 and 2.3, we have

$$\begin{aligned}
&\mathbf{Q}_\mu\left\{\mathrm{e}^{-X_t(f)}; R_\tau \leq a\right\}\\
&= \lim_{n\uparrow\infty} \mathbf{Q}_\mu \exp\left\{-X_t(f) - \int_0^\tau X_s(g_n)\mathrm{d}s\right\}\\
&= \lim_{n\uparrow\infty} \mathbf{Q}_\mu \exp\left\{-X_t(f + U_{\tau-t}(0,g_n)) - \int_0^t X_s(g_n)\mathrm{d}s\right\}\\
&= \lim_{n\uparrow\infty} \exp\left\{-\int_H U_t(f + U_{\tau-t}(0,g_n), g_n; x)\mu(\mathrm{d}x)\right\},
\end{aligned}$$

where $U_t(0,g_n)$ and $U_t(f+U_{\tau-t}(0,g_n),g_n)$ are defined by (2.2). Now Lemmas 2.3 and 2.4 yield (1.4). In particular, letting $t=\tau$ and $f\equiv 0$ in (1.4) we have

$$\mathbf{Q}_\mu\{R_\tau \leq a\} = \exp\{-\mu(V_\tau^a 0)\}.$$

Then (1.7) follows. $\square$

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