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SOME CENTRAL LIMIT THEOREMS FOR SUPER BROWNIAN MOTION

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Dedicated to Professor Wang Zikun on the occasion of his seventieth birthday!

Abstract. We prove a central limit theorem for the critical super Brownian motion, which leads to a Gaussian random field. In the transient case the limiting field is the same as that obtained by Dawson (1977). In the recurrent case it is a spatially uniform field. We also give a central limit theorem for the weighted occupation time of the super Brownian motion with underlying dimension number $d \leq 3$, completing the results of Iscoe (1986).

Key words: super Brownian motion, weighted occupation time, central limit theorem

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1. Introduction

Limit theorems constitute an important part of the branching process theory. It is always interesting to find conditions under which a non-degenerate limit law exists. Since Galton-Watson processes are unstable, people have derived limit theorems for them through devices such as modifying factors, conditioning, immigration, etc. A unified treatment of the limit theory of Galton-Watson processes is given in Athreya and Ney (1972). Some of the above mentioned techniques have also been used in the measure-valued setting to get limit theorems for Dawson-Watanabe superprocesses. See e.g. Evans and Perkins (1990) and Krone (1995) for some limit theorems of the conditioned superprocesses. Indeed, the superprocess provides a richer source for limit theorems. A well-known result of Dawson (1977) is that, if the underlying motion is a transient

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX symmetric stable process, the critical continuous superprocess started with the Lebesgue measure converges to a non-trivial steady state. It was also shown in Dawson (1977) that that the steady random measure has an interesting spatial central limit theorem which leads to a Gaussian random field. Some central limit theorems for the weighted occupation time of the super stable process were proved in Iscoe (1986). Clearly, these results have no counterparts in Galton-Watson processes. In the recurrent underlying motion case, the superprocess exhibits local extinction and there is no central limit theorem on the lines of Dawson (1977) and Iscoe (1986).

In this paper we prove a central limit theorem for the super Brownian motion which covers both the transient and recurrent underlying motions. We show that the renormalized super Brownian motion converges to a limiting Gaussian random field if it is properly started. In the transient case, the covariance kernel of the Gaussian field is given by the potential kernel of the underlying motion as in Dawson (1977). In the recurrent case, the Gaussian field is spatially uniform. We also give a central limit theorem for the weighted occupation time of the super Brownian motion with underlying dimension number $d \leq 3$, completing the results of Iscoe (1986).

2. Limit theorem for the super Brownian motion

Let $(P_t)_{t\geq 0}$ denote the semigroup of a standard Brownian motion in $\mathbb{R}^d$. We fix a strictly positive, twice continuously differentiable function ρ on $\mathbb{R}^d$ with $\rho(x) = \mathrm{e}^{-|x|}$ for $|x| > 1$, where $|\cdot|$ denotes the Euclidean norm. Let $C_\rho(\mathbb{R}^d)$ be the space of all continuous functions on $\mathbb{R}^d$ bounded by $\mathrm{const}\cdot\rho$. Dually, $M_\rho(\mathbb{R}^d)$ denotes the space of Borel measures μ on $\mathbb{R}^d$ such that $\mu(f) := \int f \mathrm{d}\mu < \infty$ for all $f \in C_\rho(\mathbb{R}^d)$. Suppose that $M_\rho(\mathbb{R}^d)$ is endowed with the topology defined by the convention: $\mu_k \to \mu$ if and only if $\mu_k(f) \to \mu(f)$ for every $f \in C_\rho(\mathbb{R}^d)$. The superscripts "+" are used to denote subsets of positive members of function spaces; e.g. $C_\rho(\mathbb{R}^d)^+$. Let $(V_t)_{t\geq 0}$ be the semigroup of nonlinear operators on $C_\rho(\mathbb{R}^d)^+$ defined by the evolution equation

$$V_t f(x) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} [V_s f(y)]^2 P_{t-s}(x, \mathrm{d}y) = P_t f(x). \tag{2.1}$$

By a critical continuous *super Brownian motion* we mean a diffusion process $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ with state space $M_\rho(\mathbb{R}^d)$ such that

$$\mathbf{Q}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \qquad t \geq 0, f \in C_\rho(\mathbb{R}^d)^+. \tag{2.2}$$

Let $S^{d-1} = \{x : x \in \mathbb{R}^d, |x| = 1\}$ be the unit sphere and λ_{d-1} be the surface area on S^{d-1} for $d \geq 2$. We introduce the following condition:

[A] Let $\alpha > 0$ and let $h = h(x)$ be a locally bounded positive Borel function on $\mathbb{R}^d$. When $d = 1$, assume that

$$\lim_{x\to\infty} |x|^{-2\alpha} h(x) = a, \qquad \lim_{x\to-\infty} |x|^{-2\alpha} h(x) = b,$$

where $a \geq 0$ and $b \geq 0$ are constants with $a+b>0$. When $d \geq 2$, assume that

$$\lim_{r\to\infty} r^{-2\alpha} h(rx) = \gamma(x)$$

uniformly on $x \in S^{d-1}$, where γ is a continuous function on S^{d-1} with $\lambda_{d-1}(\gamma) > 0$.

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$ whose all partial derivatives are also rapidly decreasing, and let $\mathcal{S}'(\mathbb{R}^d)$ be the dual space of $\mathcal{S}(\mathbb{R}^d)$. The central limit theorem consists in finding the constants $a_d(t) > 0$ such that

$$\tilde{X}_t(f) = a_d(t)^{-1}\left[X_t(f) - \lambda(hP_t f)\right], \qquad f \in \mathcal{S}'(\mathbb{R}^d), \tag{2.3}$$

converges in distribution as $t \to \infty$. For a measure $\mu \in M_\rho(\mathbb{R}^d)$ and a non-negative Borel function $h = h(x)$ on $\mathbb{R}^d$, let μ_h denote the measure such that $\mu_h(\mathrm{d}x) = h(x)\mu(\mathrm{d}x)$. Let λ be the Lebesgue measure on $\mathbb{R}^d$.

Theorem 2.1. *Suppose that [A] is satisfied with $\alpha > 1/2$ for $d = 1$ and $\alpha > 0$ for $d \geq 2$. For $t > 0$ let*

$$a_1(t)^2 = t^{\alpha+1/2},\ a_2(t)^2 = t^\alpha \log t \ \ \textit{and} \ \ a_d(t)^2 = t^\alpha \ \ \textit{for } d \geq 3.$$

Suppose that $X_0 = \lambda_h$. Then the distribution of $\tilde{X}_t$ converges as $t \to \infty$ to that of a centered Gaussian variable $\tilde{X}$ in $\mathcal{S}'(\mathbb{R}^d)$. If we set

$$\begin{aligned}
\sigma_1 &= \frac{\Gamma(\alpha+1/2)}{2\pi}(a+b)\int_0^1 (2-u)^\alpha u^{-1/2}\mathrm{d}u,\\
\sigma_2 &= \frac{2^\alpha \Gamma(\alpha+1)}{8\pi^2}\lambda_1(\gamma),\\
\sigma_d &= \frac{2^\alpha \Gamma(\alpha+d/2)}{2\pi^{d/2}}\lambda_{d-1}(\gamma) \ \ \textit{for } d \geq 3,
\end{aligned}$$

then for $d \leq 2$ the covariance of Z is given by $\mathbf{Cov}\,(Z(f), Z(g)) = \sigma_d \lambda(f)\lambda(g)$*, and for $d \geq 3$ it is given by* $\mathbf{Cov}\,(Z(f), Z(g)) = \sigma_d \lambda(fGg)$*, where G denotes the potential operator of the Brownian motion.*

The following result was proved in Li-Shiga (1995): Suppose that h satisfies [A]. Then

$$P_t h(x) \leq \mathrm{const}\cdot(1+|x|^{2\alpha})t^\alpha, \tag{2.4}$$

and as $t \to \infty$,

$$P_t h(x) = \frac{2^\alpha \Gamma(\alpha+d/2)}{2\pi^{d/2}}\lambda_{d-1}(\gamma)t^\alpha + o(t^\alpha). \tag{2.5}$$

(We understand that $\lambda_0(\gamma) = a+b$.)

Lemma 2.1. *Let $f \in \mathcal{S}(\mathbb{R}^d)^+$ and let*

$$A_d(t,f) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} [P_s f(x)]^2 P_{t-s} h(x) \mathrm{d}x.$$

Then as $t \to \infty$,

$$\begin{aligned} A_d(t,f) =& \sigma_d \lambda(f)^2 a_d(t)^2 + o(a_d(t)^2) \quad \text{for } d \leq 2 \\ =& \sigma_d \lambda(fGf) a_d(t)^2 + o(a_d(t)^2) \quad \text{for } d \geq 3. \end{aligned}$$

Proof. Note that the Brownian transition density $g_t(x,y) = g_t(x-y)$ satisfies the following relation:

$$g_s(x,y) g_t(x,z) = g_{st/(s+t)}\left(x, (ty+sz)/(s+t)\right) g_{s+t}(y,z). \tag{2.6}$$

Using this one finds that

$$A_d(t,f) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^{2d}} P_{t-s/2} h((y+z)/2) g_{2s}(y,z) f(y) f(z) \mathrm{d}y \mathrm{d}z.$$

Changing the variables and using dominated convergence we have

$$\begin{aligned} & \lim_{t\to\infty} \frac{1}{t^{\alpha+1/2}} A_1(t,f) \\ =& \lim_{t\to\infty} \frac{\sqrt{t}}{t^\alpha} \int_0^1 \mathrm{d}r \int_{\mathbb{R}^2} P_{t(1-r/2)} h\left(\frac{y+z}{2}\right) g_{2rt}(y,z) f(y) f(z) \mathrm{d}y \mathrm{d}z \\ =& \lim_{t\to\infty} \frac{1}{2\sqrt{\pi}} \int_0^1 \mathrm{d}r \int_{\mathbb{R}^2} \frac{1}{t^\alpha \sqrt{r}} P_{t(1-r/2)} h\left(\frac{y+z}{2}\right) \exp\left\{ -\frac{|y-z|^2}{4rt} \right\} f(y) f(z) \mathrm{d}y \mathrm{d}z \\ =& \frac{2^\alpha \Gamma(\alpha+1/2)}{4\pi} (a+b) \lambda(f)^2 \int_0^1 r^{-1/2} (1+r/2)^\alpha \mathrm{d}r, \end{aligned}$$

where we have used (2.4) and (2.5) in the last equality. Similarly, setting $s = t^{1-r}$ we have

$$\begin{aligned} & \lim_{t\to\infty} \frac{1}{t^\alpha \log t} A_2(t,f) \\ =& \lim_{t\to\infty} \frac{1}{t^\alpha \log t} \int_1^t \mathrm{d}s \int_{\mathbb{R}^4} P_{t-s/2} h\left(\frac{y+z}{2}\right) g_{2s}(y,z) f(y) f(z) \mathrm{d}y \mathrm{d}z \\ =& \lim_{t\to\infty} \frac{1}{4\pi} \int_0^1 \mathrm{d}r \int_{\mathbb{R}^4} t^{-\alpha} P_{t[1-1/2t^r]} h\left(\frac{y+z}{2}\right) \exp\left\{ -\frac{|y-z|^2}{4t^{1-r}} \right\} f(y) f(z) \mathrm{d}y \mathrm{d}z \\ =& \frac{2^\alpha \Gamma(\alpha+1)}{8\pi^2} \lambda_1(\gamma) \lambda(f)^2. \end{aligned}$$

Note that $\|P_s f\| \le (1 \wedge s^{-d/2}) \cdot \text{const}$. By (2.4),

$$t^{-\alpha}\int_{\mathbb{R}^d}[P_s f(x)]^2 P_{t-s}h(x)\mathrm{d}x \le t^{-\alpha}\int_{\mathbb{R}^d} f(x)P_t h(x)\mathrm{d}x \cdot (1\wedge s^{-d/2})\cdot \text{const}$$
$$\le \int_{\mathbb{R}^d} f(x)(1+|x|^{2\alpha})\mathrm{d}x \cdot (1\wedge s^{-d/2})\cdot \text{const}.$$

Since $[P_s f]^2$ is rapidly decreasing, using the dominated convergence two times we have

$$\begin{aligned}&\lim_{t\to\infty} t^{-\alpha}A_d(t,f)\\ =&\int_0^\infty \mathrm{d}s\int_{\mathbb{R}^d}[P_s f(x)]^2\lim_{t\to\infty}t^{-\alpha}P_{t-s}h(x)\mathrm{d}x\\ =&\frac{2^\alpha\Gamma(\alpha+d/2)}{2\pi^{-d/2}}\lambda_{d-1}(\gamma)\int_0^\infty \mathrm{d}s\int_{\mathbb{R}^d}[P_s f(x)]^2\mathrm{d}x\end{aligned}$$

for $d\ge 3$. The lemma is proved. $\square$

Proof of Theorems 2.1. For $f \in \mathcal{S}(\mathbb{R}^d)^+$ let $f_t = a_d(t)^{-1}f$. From (2.1) and (2.2) it follows that

$$\mathbf{Q}_{\lambda_h}\exp\left\{-\tilde{X}_t(f)\right\} = \exp\left\{\frac{1}{2}\int_0^t \mathrm{d}s\int_{\mathbb{R}^d}[V_s f_t(x)]^2 P_{t-s}h(x)\mathrm{d}x\right\}, \tag{2.7}$$

where

$$[V_s f_t]^2 = [P_s f_t]^2 - P_s f_t\int_0^s P_{s-r}[(V_r f_t)^2]\mathrm{d}r + \frac{1}{4}\left(\int_0^s P_{s-r}[(V_r f_t)^2]\mathrm{d}r\right)^2. \tag{2.8}$$

By (2.1) we have $V_r f_t \le P_r f_t$, and hence

$$\int_0^s P_{s-r}[(V_r f_t)^2]\mathrm{d}r \le \int_0^s P_{s-r}[(P_r f_t)^2]\mathrm{d}r \le a_d(t)^{-1}P_s f_t\int_0^s \|P_r f\|\mathrm{d}r. \tag{2.9}$$

Under the hypotheses of the theorem we have

$$a_d(t)^{-1}\int_0^s \|P_r f\|\mathrm{d}r \le a_d(t)^{-1}\int_0^t (1\wedge r^{-d/2})\mathrm{d}r \to 0$$

as $t\to\infty$. Combining (2.7) – (2.9) and Lemma 2.1 gives

$$\lim_{t\to\infty}\mathbf{Q}_{\lambda_h}\exp\left\{-\tilde{X}_t(f)\right\} = \exp\left\{\frac{1}{2}C_d(f)^2\right\},$$

where $C_d(f)^2 = \sigma_d\lambda(f)^2$ for $d\le 2$ and $= \sigma_d\lambda(fGf)$ for $d\ge 3$. Then the assertions follow as the limit theorems in Iscoe (1986). $\square$

3. Limit theorem for the weighted occupation time

In this section we give a central limit theorem for the weighted occupation time process of the super Brownian motion with underlying dimension number $d \leq 3$. This completes the results of Iscoe (1986). The weighted occupation time process $\{Y_t : t \geq 0\}$ is defined as

$$Y_t(f) = \int_0^t X_s(f)\mathrm{d}s, \qquad f \in C_\rho(\mathbb{R}^d). \tag{3.1}$$

By Iscoe (1986), the distribution of $Y_t(f)$ is determined by

$$\mathbf{Q}_\mu \exp\left\{-Y_t(f)\right\} = \exp\left\{-\mu(U_t f\right\}, \tag{3.2}$$

where $U_t f$ is the solution to

$$U_t f(x) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} [U_s f(y)]^2 P_{t-s}(x, \mathrm{d}y) = \int_0^t P_s f(x)\mathrm{d}s. \tag{3.3}$$

For $d \leq 3$ and $t > 0$ let $b_d(t) > 0$ be defined by $b_d(t)^2 = t^{\alpha+3-d/2}$ and let

$$\tilde{Y}_t(f) = b_d(t)^{-1}\int_0^t \left[X_s(f) - \lambda(hP_s f)\right]\mathrm{d}s, \qquad f \in \mathcal{S}'(\mathbb{R}^d). \tag{3.4}$$

Theorem 3.1. *Suppose that [A] is satisfied with $\alpha > 1/2$ for $d = 1$, $\alpha > 0$ for $d = 2$, and $\alpha > 1/2$ for $d = 3$. Let $X_0 = \lambda_h$. Then the distribution of $\tilde{Y}_t$ converges as $t \to \infty$ to that of a centered Gaussian variable $\tilde{Y}$ in $\mathcal{S}'(\mathbb{R}^d)$. Let*

$$\kappa_d = \frac{2^\alpha \Gamma(\alpha + d/2)\lambda_{d-1}(\gamma)}{2^{1+d/2}\pi^d}\int_0^1 s^{2-d/2}\mathrm{d}s \int_0^1 \mathrm{d}r \int_0^1 \left(1 - s + \frac{ru}{r+u}\right)^\alpha \frac{\mathrm{d}u}{(r+u)^{d/2}}.$$

Then the covariance of Z is given by $\mathbf{Cov}\,(Z(f), Z(g)) = \kappa_d\lambda(f)\lambda(g)$.

Lemma 3.1. *Let $f \in \mathcal{S}(\mathbb{R}^d)^+$ and let*

$$B_d(t,f) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^d}\left[\int_0^s P_r f(x)\mathrm{d}s\right]^2 P_{t-s}h(x)\mathrm{d}x.$$

If $d \leq 3$, then as $t \to \infty$,

$$B_d(t,f) = \kappa_d\lambda(f)^2 b_d(t)^2 + o(b_d(t)^2).$$

Proof. Using the relation (2.6) one checks that

$$\begin{aligned} B_d(t,f) &= \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^s \mathrm{d}u \int_{\mathbb{R}^d} P_{t-s}h(x)P_r f(x)P_u f(x)\mathrm{d}x \\ &= \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r \int_0^s \mathrm{d}u \int_{\mathbb{R}^{2d}} P_{w(t,s,r,u)}h\left(\frac{uy+rz}{r+u}\right) \\ &\qquad g_{r+u}(y,z)f(y)f(z)\mathrm{d}y\mathrm{d}z, \end{aligned}$$

where $w(t,s,r,u) = t - s + ru(r+u)^{-1}$. Changing the variables we have

$$
\begin{aligned}
B_d(t,f) =& t^3 \int_0^1 s^2 \mathrm{d}s \int_0^1 \mathrm{d}r \int_0^1 \mathrm{d}u \int_{\mathbb{R}^{2d}} P_{tw(1,s,r,u)} h(\frac{uy+rz}{r+u}) \\
& g_{ts(r+u)}(y,z) f(y) f(z) \mathrm{d}y\mathrm{d}z \\
=& \frac{t^3}{(2\pi t)^{d/2}} \int_0^1 s^{2-d/2} \mathrm{d}s \int_0^1 \mathrm{d}r \int_0^1 \frac{\mathrm{d}u}{(r+u)^{d/2}} \int_{\mathbb{R}^{2d}} P_{tw(1,s,r,u)} h\Big(\frac{uy+rz}{r+u}\Big) \\
& f(y) f(z) \exp\Big\{ - \frac{|y-z|^2}{2ts(r+u)} \Big\} \mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

For s, r and u from $(0,1)$ we have $1-s < 1-s+ru(r+u)^{-1} < 1$. Then for $d \leq 3$ we have

$$
0 < \int_0^1 s^{2-d/2} \mathrm{d}s \int_0^1 \mathrm{d}r \int_0^1 \Big(1 - s + \frac{ru}{r+u}\Big)^{\alpha} \frac{\mathrm{d}u}{(r+u)^{d/2}} < \infty.
$$

By dominated convergence we get the result. □

Proof of Theorems 3.1. For $f \in \mathcal{S}(\mathbb{R}^d)^+$ let $f_t = b_d(t)^{-1} f$. Observe that by (3.3) we have

$$
\begin{aligned}
\int_0^s P_{s-r}[(U_r f_t)^2] \mathrm{d}r \leq & \int_0^s P_{s-r} \Big[\Big(\int_0^r P_u f_t \mathrm{d}u \Big)^2 \Big] \mathrm{d}r \\
\leq & \int_0^s P_{s-r} \Big[\int_0^r P_{r-u} f_t \mathrm{d}u \Big] \mathrm{d}r \int_0^t \| P_{r-u} f_t \| \mathrm{d}u \\
\leq & \mathrm{const} \cdot \int_0^s \mathrm{d}r \int_0^r P_{s-u} f_t \mathrm{d}u \cdot b_d(t)^{-1} \int_0^t 1 \wedge u^{d/2} \mathrm{d}u \\
\leq & \mathrm{const} \cdot \int_0^s P_u f_t \mathrm{d}u \cdot t b_d(t)^{-1} \int_0^t 1 \wedge u^{d/2} \mathrm{d}u.
\end{aligned}
$$

Under the hypotheses of the theorem we have

$$
t b_d(t)^{-1} \int_0^t (1 \wedge u^{-d/2}) \mathrm{d}u \to 0
$$

as $t \to \infty$. Then the assertions follow by similar arguments as Theorem 2.1. □

There are similar central limit theorems for the weighted occupation time process with underlying dimension number $d \geq 4$. The modifying factors should be $b_4(t)^2 = t^{\alpha+1} \log t$ and $b_d(t)^2 = t^{\alpha+1}$ for $d \geq 5$. Indeed, as in the proof of Lemma 3.1 one checks that

$$
\begin{aligned}
B_d(t,f) =& t \int_0^1 \mathrm{d}s \int_0^{st} \mathrm{d}r \int_0^{st} \mathrm{d}u \int_{\mathbb{R}^{2d}} P_{w(t,st,r,u)} h(\frac{uy+rz}{r+u}) \\
& g_{r+u}(y,z) f(y) f(z) \mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

Set $g(z) = f(z)(1+|z|^{2\alpha})$. By (2.4) we have immediately

$$t^{-\alpha}\int_{\mathbb{R}^{2d}} P_{w(t,st,r,u)}h\left(\frac{uy+rz}{r+u}\right)g_{r+u}(y,z)f(y)f(z)\mathrm{d}y\mathrm{d}z \le \text{const}\cdot\lambda(fP_{r+u}g)$$

For $d \ge 5$ we have

$$\int_0^\infty \mathrm{d}r\int_0^\infty \lambda(fP_{r+u}g)\mathrm{d}u = \lambda(GfGg) < \infty.$$

Then by dominated convergence, as $t \to \infty$,

$$B_d(t,f) = \frac{2^\alpha \Gamma(\alpha+d/2)\lambda_{d-1}(\gamma)}{2\pi^{d/2}(1+\alpha)}\lambda(fGf)^2 b_d(t) + o(b_d(t)^2).$$

The calculations for $d = 4$ seems more involved and is left to the interested reader. Unfortunately, it seems that the above calculations do not lead to the results of Iscoe (1986), who proved central limit theorems for $d \ge 3$ and $\alpha = 0$.

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