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FLUCTUATION LIMITS OF MEASURE-VALUED IMMIGRATION PROCESSES WITH SMALL BRANCHING

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Abstract

The fluctuation limit of a measure-valued immigration process with small branching rate is considered, which gives the solution to a Langevin type equation driven by a distribution-valued process with independent increments.

1. Measure-valued immigration process. Let E be a locally compact metric space. Let $C_0(E)$ denote the set of continuous functions on E that vanish at infinity. Suppose that $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$ is a Markov process with Feller transition semigroup $(P_t)_{t\geq 0}$, that is, P_t maps $C_0(E)$ to itself and is strongly continuous in $t \geq 0$. Let A denote the strong generator of $(P_t)_{t\geq 0}$ with domain $D(A) \subset C_0(E)$. We choose a strictly positive reference function $\rho \in D(A)$ satisfying $A\rho \in C_\rho(E)$, where $C_\rho(E)$ denotes the set of functions $f \in C_0(E)$ satisfying $|f| \leq \mathrm{const} \cdot \rho$. Then there is some $\beta \geq 0$ such that ρ is a β-excessive function for the semigroup $(P_t)_{t\geq 0}$. Let $D_\rho(A)$ be the set of functions $f \in D(A) \cap C_\rho(E)$ with $Af \in C_\rho(E)$. The subsets of non-negative elements of the function spaces are indicated by the superscript '$^+$', e.g. $C_\rho(E)^+$. For example, if ξ is a Brownian motion on $\mathbb{R}^d$, we may set $\rho(x) = \mathrm{e}^{-|x|}$ for $|x| \geq 1$, where $|\cdot|$ denotes the Euclidean norm.

Let $M_\rho(E)$ denote the space of σ-finite measures μ on $(E, \mathcal{B}(E))$ such that $\mu(f) := \int_E f \mathrm{d}\mu < \infty$ for all $f \in C_\rho(E)$. The topology of $M_\rho(E)$ is defined by the convention:

$\mu_k \to \mu$ if and only if $\mu_k(f) \to \mu(f)$ for all $f \in C_\rho(E)$. Let ϕ be a continuous function on $E \times [0,\infty)$ given by

$$\phi(x,z) = c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)m(x,\mathrm{d}u), \quad x \in E, z \geq 0, \tag{1.1}$$

where $c \geq 0$ is a bounded, continuous function on E and $(u \wedge u^2)m(x,\mathrm{d}u)$ is a bounded kernel from E to $(0,\infty)$. We consider a *Dawson-Watanabe superprocess* X with parameters (ξ,ϕ); see e.g. Dawson (1993). X is an $M_\rho(E)$-valued Markov process with transition semigroup $(Q_t)_{t\geq 0}$ determined by

$$\int_{M_\rho(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu,\mathrm{d}\nu) = \exp\left\{-\mu(V_t f)\right\}, \quad f \in C_\rho(E)^+, \tag{1.2}$$

where $V_t f$ denotes the unique positive solution of the evolution equation

$$V_t f(x) + \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) P_{t-s}(x,\mathrm{d}y) = P_t f(x), \quad t \geq 0, x \in E. \tag{1.3}$$

Let $\mathcal{K}_\rho(P)$ be the set of entrance laws $\kappa = (\kappa_t)_{t>0}$ for the underlying semigroup $(P_t)_{t\geq 0}$ that satisfy $\int_0^1 \kappa_s(\rho)\mathrm{d}s < \infty$. For any $\kappa \in \mathcal{K}_\rho(P)$ we set

$$S_t(\kappa,f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) \kappa_{t-s}(\mathrm{d}y), \quad t > 0, f \in C_\rho(E)^+. \tag{1.4}$$

Theorem 1.1. *The formula*

$$\int_{M_\rho(E)} \mathrm{e}^{-\nu(f)} Q_t^\kappa(\mu,\mathrm{d}\nu) = \exp\left\{ -\mu(V_t f) - \int_0^t S_r(\kappa,f)\mathrm{d}r \right\}, \quad f \in C_\rho(E)^+, \tag{1.5}$$

defines a transition semigroup $(Q_t^\kappa)_{t\geq 0}$ on $M_\rho(E)$.

Proof. Since ρ is a β-excessive function for ξ, we can define a Borel right semigroup $(T_t)_{t\geq 0}$ on E by $T_t f(x) = \mathrm{e}^{-\beta t}\rho(x)^{-1}P_t(\rho f)(x)$; see Sharpe (1988). Let $\psi(x,z) = \rho(x)^{-1}\phi(x,\rho(x)z) - \beta z$ and let $n(x,\mathrm{d}u)$ be the image of $m(x,\mathrm{d}u)$ under the mapping $u \mapsto \rho(x)u$. It is easy to check that $(u \wedge u^2)\rho(x)^{-1}n(x,\mathrm{d}u)$ is a bounded kernel from E to $(0,\infty)$ and ψ has the representation

$$\psi(x,z) = -\beta z + c(x)\rho(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)\rho(x)^{-1}n(x,\mathrm{d}u).$$

Let $U_t f$ be the solution to

$$U_t f(x) + \int_0^t \mathrm{d}s \int_E \psi(y, U_s f(y)) T_{t-s}(x,\mathrm{d}y) = T_t f(x), \quad t \geq 0, x \in E. \tag{1.6}$$

Then $(U_t)_{t\geq 0}$ is the cumulant semigroup of a Dawson-Watanabe superprocess with state space $M(E)$, the totality of finite Borel measures on E. Define the finite measures $(\eta_t)_{t>0}$ on E by $\eta_t(f) = \mathrm{e}^{-\beta t}\kappa_t(\rho f)$. It is easy to check that $(\eta_t)_{t>0}$ is an entrance law for $(T_t)_{t\geq 0}$. Let

$$S_t^T(\eta, f) = \eta_t(f) - \int_0^t \mathrm{d}s \int_E \psi(y, U_s f(y))\eta_{t-s}(\mathrm{d}y).$$

By Theorem 3.2 of Li (1996), there is a Markov process with state space $M(E)$ and semigroup $(R_t^\eta)_{t\geq 0}$ given by

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} R_t^\eta(\mu, \mathrm{d}\nu) = \exp\bigg\{ - \mu(U_t f) - \int_0^t S_r^T(\eta, f)\mathrm{d}r \bigg\}.$$

Let $(G_t^\eta)_{t\geq 0}$ be the image of $(R_t^\eta)_{t\geq 0}$ under the mapping $\mu(\mathrm{d}x) \mapsto \rho(x)^{-1}\mu(\mathrm{d}x)$ from $M(E)$ to $M_\rho(E)$. Then we have

$$\int_{M_\rho(E)} \mathrm{e}^{-\nu(f)} G_t^\eta(\mu, \mathrm{d}\nu) = \exp\bigg\{ - \mu(\rho U_t(\rho^{-1} f)) - \int_0^t S_r^T(\eta, \rho^{-1} f)\mathrm{d}r \bigg\}. \tag{1.7}$$

From (1.6) it follows that

$$\begin{aligned} \rho(x)U_t(\rho^{-1}f)(x) + \int_0^t \mathrm{d}s \int_E [\phi(y, \rho(y)U_s(\rho^{-1}f)(y)) \\ - \beta\rho(x)U_s(\rho^{-1}f)(y)]\mathrm{e}^{-\beta(t-s)}P_{t-s}(x, \mathrm{d}y) = \mathrm{e}^{-\beta t}P_t f(x), \end{aligned}$$

or equivalently,

$$\rho(x)U_t(\rho^{-1}f)(x) + \int_0^t \mathrm{d}s \int_E \phi(y, \rho(y)U_s(\rho^{-1}f)(y))P_{t-s}(x, \mathrm{d}y) = P_t f(x).$$

Therefore, we have $\rho U_t(\rho^{-1}f) = V_t f$. By (1.3) and (1.6) one checks that

$$S_t^T(\eta, f) = \lim_{r\downarrow 0} \eta_r(U_{t-r}\rho^{-1}f) = \lim_{r\downarrow 0} \eta_r(\rho^{-1}V_{t-r}f) = \lim_{r\downarrow 0} \mathrm{e}^{-\beta r}\kappa_r(V_{t-r}f) = S_t(\kappa, f).$$

Returning to (1.7) we have $G_t^\eta = Q_t^\kappa$. $\square$

In the sequel, a Markov process Y with semigroup $(Q_t^\kappa)_{t\geq 0}$ will be called a *measure-valued immigration process* with parameters (A, ϕ, κ). We can show that the immigration process has a right continuous realization if $\kappa_t = \nu P_t$ for some $\nu \in M_\rho(E)$. However, the right continuity of the process in the general case still remains open.

2. Fluctuation around the excessive measure. Let (A, ϕ) be given as the above. Suppose that $\gamma \in M_\rho(E)$ is a purely excessive measure for the semigroup $(P_t)_{t\geq 0}$. Then there is $\kappa \in \mathcal{K}_\rho(P)$ such that $\gamma = \int_0^\infty \kappa_s \mathrm{d}s$; see Dynkin (1980). Let $\{Y_t : t \geq 0\}$ be an immigration process with parameters (A, ϕ, κ). Let $S_\rho^\gamma(E)$ denote the set of all signed-measures μ on E such that $\mu + \gamma \in M_\rho(E)$. We define the process $\{Z_t : t \geq 0\}$ in $S_\rho^\gamma(E)$ by $Z_t = Y_t - \gamma$. Then we have a.s.

$$\begin{aligned}
&\mathbf{E}[\exp\{-Z_{r+t}(f)\}|Z_s : 0 \leq s \leq r] \\
&= \exp\bigg\{ - Z_r(V_t f) + \gamma(f - V_t f) - \int_0^t S_u(\kappa, f)\mathrm{d}u\bigg\},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^t S_u(\kappa, f)\mathrm{d}u &= \int_0^t \kappa_u(f)\mathrm{d}u - \int_0^t \mathrm{d}u \int_0^u \kappa_{u-s}(\phi(V_s f))\mathrm{d}s \\
&= \int_0^t \kappa_u(f)\mathrm{d}u - \int_0^t \mathrm{d}s \int_0^{t-s} \kappa_u(\phi(V_s f))\mathrm{d}u \\
&= \gamma(f - P_t f) - \int_0^t \gamma(\phi(V_s f) - P_{t-s}\phi(V_s f))\mathrm{d}s \\
&= \gamma(f - V_t f) - \int_0^t \gamma(\phi(V_s f))\mathrm{d}s.
\end{aligned}$$

That is, $\{Z_t : t \geq 0\}$ is a Markov process with transition semigroup $(T_t^\kappa)_{t\geq 0}$ given by

$$\int_{S_\rho(E)} \mathrm{e}^{-\nu(f)} T_t^\kappa(\mu, \mathrm{d}\nu) = \exp\bigg\{ - \mu(V_t f) + \int_0^t \gamma(\phi(V_s f))\mathrm{d}s\bigg\}. \tag{2.1}$$

3. A small branching fluctuation limit theorem. For concreteness, we assume that $E = \mathbb{R}^d$ and A is a differential operator. For any $\theta > 0$ let $\phi_\theta(x, z) = \phi(x, \theta z)$ and let b_θ be a continuous function on $\mathbb{R}^d$ which is bounded, positive and bounded away from zero. Assume that $b_\theta \downarrow 0$ boundedly as $\theta \downarrow 0$. Let $(P_t^\theta)_{t\geq 0}$ denote the transition semigroup defined by

$$P_t^\theta f(x) = \mathbf{P}_x \exp\bigg\{ - \int_0^t b_\theta(\xi_s)\mathrm{d}s\bigg\} f(\xi_t). \tag{3.1}$$

Suppose that $\gamma \in M_\rho(\mathbb{R}^d)$ is an excessive measure for the semigroup $(P_t)_{t\geq 0}$. Then it is a purely excessive measure for $(P_t^\theta)_{t\geq 0}$, and hence $\gamma = \int_0^\infty \kappa_s^\theta \mathrm{d}s$ for some $\kappa^\theta \in \mathcal{K}_\rho(P^\theta)$. Suppose that $\{Y_t^\theta : t \geq 0\}$ is an immigration process with parameters $(A - b_\theta, \phi_\theta, \kappa^\theta)$ and $Y_0^\theta = \gamma$. We define the fluctuation process $\{Z_t^\theta : t \geq 0\}$ by

$$Z_t^\theta = \theta^{-1}\big[Y_t^\theta - \gamma\big], \quad t \geq 0. \tag{3.2}$$

Our aim is to obtain the limiting distribution of the process $\{Z_t^\theta : t \geq 0\}$ as $\theta \to 0$ and to show that the limit is a generalized Ornstein-Uhlenbeck process. Observe that

$\phi_\theta(x,z) \downarrow 0$ as $\theta \downarrow 0$, that is, under the rescaling (3.2) the branching rate goes to zero. This sort of fluctuation limit was studied by Gorostiza (1996) for a special type of (non-immigration) superprocesses.

From the discussions in the last section we know that $\{Z_t^\theta : t \geq 0\}$ is a Markov process with $Z_0^\theta = 0$ and with semigroup $(R_t^\theta)_{t\geq 0}$ determined by

$$\int_{S_\rho^\gamma(\mathbb{R}^d)} \mathrm{e}^{-\nu(f)} R_t^\theta(\mu, \mathrm{d}\nu) = \exp\bigg\{ -\mu(\theta V_t^\theta(f/\theta)) + \int_0^t \gamma(\phi(\theta V_s^\theta(f/\theta)))\mathrm{d}s \bigg\}, \qquad (3.3)$$

where $(V_t^\theta)_{t\geq 0}$ is defined by

$$V_t^\theta f(x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \phi_\theta(y, V_s^\theta f(y)) P_{t-s}^\theta(x, \mathrm{d}y) = P_t^\theta f(x). \qquad (3.4)$$

Lemma 3.1. *If $f_\theta \to f \in C(\mathbb{R}^d)^+$ boundedly as $\theta \downarrow 0$, then $\theta V_t^\theta(f_\theta/\theta) \to P_t f$ boundedly as $\theta \downarrow 0$.*

Proof. By (3.4) we have $V_t^\theta f(x) \leq P_t f(x)$ and hence $\theta V_t^\theta(f_\theta/\theta)(x) \leq P_t f(x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. On the other hand, (3.4) is equivalent to

$$V_t^\theta f(x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} [\phi_\theta(y, V_s^\theta f(y)) - b_\theta(y) V_s^\theta f(y)] P_{t-s}(x, \mathrm{d}y) = P_t f(x).$$

Then we have

$$\begin{aligned} \theta V_t^\theta(f_\theta/\theta)(x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \theta[\phi(y, V_s^\theta(f_\theta/\theta)(y)) \\ - b_\theta(y) V_s^\theta(f_\theta/\theta)(y)] P_{t-s}(x, \mathrm{d}y) = P_t f_\theta(x). \end{aligned}$$

Since the second term on the left hand side goes to zero as $\theta \downarrow 0$, we have $\theta V_t^\theta(f_\theta/\theta) \to P_t f$ boundedly as $\theta \downarrow 0$. $\square$

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of infinitely differentiable, rapidly decreasing functions all of whose derivatives are also rapidly decreasing. Let $\mathcal{S}'(\mathbb{R}^d)$ denote the dual space of $\mathcal{S}(\mathbb{R}^d)$. Then we have the following fluctuation limit theorem.

Theorem 3.2. *The finite-dimensional distributions of $\{Z_t^\theta : t \geq 0\}$ converges as $\theta \downarrow 0$ to those of the $\mathcal{S}'(\mathbb{R}^d)$-valued Markov process $\{Z_t : t \geq 0\}$ with $Z_0 = 0$ and with semigroup $(R_t^\gamma)_{t\geq 0}$ determined by*

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{i\nu(f)} R_t^\gamma(\mu, \mathrm{d}\nu) = \exp\bigg\{ i\mu(P_t f) + \int_0^t \gamma(\phi(-iP_s f))\mathrm{d}s \bigg\}, \quad f \in \mathcal{S}(\mathbb{R}^d), \qquad (3.5)$$

where $\phi(-iP_s f)$ is given by (1.1) with z replaced by $-iP_s f(x)$.

Proof. For $0 \leq t_1 < \cdots < t_n$ and $f_1, \cdots, f_n \in \mathcal{S}(\mathbb{R}^d)$ set

$$h_j^{(\theta)} = f_j + V_{t_{j+1}-t_j}^{(\theta)}(f_{j+1} + \cdots + V_{t_n - t_{n-1}}^{(\theta)} f_n),$$

where $V_t^{(\theta)} f(x) = \theta V_t^\theta (f/\theta)(x)$. Using (3.3) inductively we get

$$\mathbf{E}\exp\bigg\{-\sum_{j=1}^n Z_{t_j}^\theta(f_j)\bigg\} = \exp\bigg\{\sum_{j=1}^n \int_0^{t_j - t_{j-1}} \gamma(\phi(V_s^{(\theta)} h_j^{(\theta)}))\mathrm{d}s\bigg\}. \tag{3.6}$$

By Lemma 3.1 it can be proved inductively that

$$h_j^{(\theta)} \to h_j := f_j + P_{t_{j+1}-t_j}(f_{j+1} + \cdots + P_{t_n - t_{n-1}} f_n)$$

boundedly as $\theta \to \infty$. Returning to (3.6) we get

$$\lim_{\theta\to\infty} \mathbf{E}\exp\bigg\{-\sum_{j=1}^n Z_{t_j}^\theta(f_j)\bigg\} = \exp\bigg\{\sum_{j=2}^n \int_0^{t_j - t_{j-1}} \gamma(\phi(P_s h_j))\mathrm{d}s\bigg\}.$$

As in Iscoe (1986), it follows that the finite-dimensional distributions of $\{Z_t^\theta : t \geq 0\}$ converge to those of the $\mathcal{S}'(\mathbb{R}^d)$-valued Markov process $\{Z_t : t \geq 0\}$ with $Z_0 = 0$ and with transition semigroup $(R_t^\gamma)_{t\geq 0}$. $\square$

For the special branching mechanism $\phi(x,z) \equiv c(x)z^2$, Li (1998a) proved that the family $\{Z_t^\theta : t \geq 0\}$ is tight in the space $C([0,\infty), \mathcal{S}'(\mathbb{R}^d))$ and hence the fluctuation limit $\{Z_t : t \geq 0\}$ has a continuous realization. For the one-dimensional immigration process with general branching mechanism, a fluctuation limit theorem was given in Li (1998b) where the tightness follows from the convergence of the corresponding transition semigroups. However, the tightness problem for $\{Z_t^\theta : t \geq 0\}$ in the general setting is still unsolved.

By Bojdecki and Gorostiza (1991; p. 1139), if $\{Z_t : t \geq 0\}$ is a cadlag Markov process with transition semigroup given by (3.5), then it solves the Langevin type equation

$$\mathrm{d}Z_t = A^* Z_t \mathrm{d}t + \mathrm{d}W_t,$$

where A^* is the adjoint of A and $\{W_t : t \geq 0\}$ is an $\mathcal{S}'(\mathbb{R}^d)$-valued martingale with independent increments given by

$$\mathbf{E}\exp\{i[W_t(f) - W_r(f)]\} = \exp\{(t-r)\gamma(\phi(-if))\}, \quad t \geq r \geq 0,$$

where $\phi(-if)$ is given by (1.1) with z replaced by $-if(x)$.

4. General Ornstein-Uhlenbeck process. Now we fix A and ϕ as the above. Let φ be a function on $\mathbb{R}^d \times \mathbb{R}$ with the representation

$$\varphi(x,z) = -c(x)z^2 + \int_{-\infty}^{\infty} (\mathrm{e}^{izu} - 1 - izu) m(x, \mathrm{d}u), \quad x \in \mathbb{R}^d, z \in \mathbb{R},$$

where $c \in C(\mathbb{R}^d)^+$ and $(|u| \wedge |u|^2)m(x, \mathrm{d}u)$ is a bounded kernel from $\mathbb{R}^d$ to $(-\infty, \infty)$. It is more natural to consider the Langevin equation driven by an $\mathcal{S}'(\mathbb{R}^d)$-valued martingale $\{L_t : t \geq 0\}$ which has independent increments determined by

$$\mathbf{E} \exp \{i[L_t(f) - L_r(f)]\} = \exp \{(t-r)\gamma(\varphi(f))\}, \quad t \geq r \geq 0. \tag{4.1}$$

Let $\psi_1(x, z) = \phi(x, z)$ and let

$$\psi_2(x, z) = \int_{-\infty}^{0} (\mathrm{e}^{zu} - 1 - zu)m(x, \mathrm{d}u), \quad x \in \mathbb{R}^d, z \geq 0.$$

Suppose that we have two independent cadlag Markov processes $\{Z_t^1 : t \geq 0\}$ and $\{Z_t^2 : t \geq 0\}$ whose transition semigroups are given by Theorem 3.2 with parameters (A, ψ_1, γ) and (A, ψ_2, γ), respectively. Let $Z_t = Z_t^1 - Z_t^2$ for $t \geq 0$. It is easy to check that $\{Z_t : t \geq 0\}$ is a *generalized Ornstein-Uhlenbeck process* which by definition is the solution to the Langevin equation

$$\mathrm{d}Z_t = A^* Z_t \mathrm{d}t + \mathrm{d}L_t,$$

where $\{L_t : t \geq 0\}$ is an $\mathcal{S}'(\mathbb{R}^d)$-valued martingale with independent increments determined by (4.1). This gives an interpretation for the generalized Langevin equation in terms of the measure-valued immigration process.

5. Fluctuation of a stationary process. Suppose that $\gamma \in M_\rho(E)$ is a purely excessive measure for $(P_t)_{t\geq 0}$ represented as $\gamma = \int_0^\infty \kappa_s \mathrm{d}s$ for $\kappa \in \mathcal{K}_\rho(P)$. One may check that the semigroup $(Q_t^\kappa)_{t\geq 0}$ defined by (1.5) has a stationary distribution Q_∞^κ with Laplace functional

$$\int_{M_\rho(E)} \mathrm{e}^{-\nu(f)} Q_\infty^\kappa(\mathrm{d}\nu) = \exp\bigg\{ -\gamma(f) + \int_0^\infty \gamma(\phi(V_s f))\mathrm{d}s \bigg\}, \quad f \in C_\rho(E)^+. \tag{5.1}$$

Moreover, if $\mu(P_t\rho) \to 0$ as $t \to \infty$, then $Q_t^\kappa(\mu, \cdot) \to Q_\infty^\kappa$ as $t \to \infty$.

Let $(R_t^\gamma)_{t\geq 0}$ be given by (3.5). If $\mu(P_t\rho) \to 0$ as $t \to \infty$, then $R_t^\gamma(\mu, \cdot) \to R_\infty^\gamma$ as $t \to \infty$, where R_∞^γ is a stationary distribution of $(R_t^\gamma)_{t\geq 0}$ determined by

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{i\nu(f)} R_\infty^\gamma(\mathrm{d}\nu) = \exp\bigg\{ \int_0^\infty \gamma(\phi(-iP_s f))\mathrm{d}s \bigg\}, \quad f \in \mathcal{S}(\mathbb{R}^d). \tag{5.2}$$

If we start from a stationary immigration process with one-dimensional distribution Q_∞^κ given by (5.1) and take the fluctuation limit as in section 3, we get a stationary $\mathcal{S}'(\mathbb{R}^d)$-valued Markov process with semigroup $(R_t^\gamma)_{t\geq 0}$ and one-dimensional distribution R_∞^γ. That is, the small branching limit and the long time limit are interchangeable to some extent.

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