Published in: *Advances in Applied Probability* **30** (1998), 657–675.

IMMIGRATION PROCESSES ASSOCIATED WITH BRANCHING PARTICLE SYSTEMS

Zeng-Hu LI[1] *Beijing Normal University*

Abstract. The immigration processes associated with a given branching particle system are formulated by skew convolution semigroups. It is shown that every skew convolution semigroup corresponds uniquely to a locally integrable entrance law for the branching particle system. The immigration particle system may be constructed using a Poisson random measure based on a Markovian measure determined by the entrance law. In the special case where the underlying process is a minimal Brownian motion in a bounded domain, a general representation is given for locally integrable entrance laws for the branching particle system. The convergence of immigration particle systems to measure-valued immigration processes is also studied.

Key words: branching particle system; immigration; Dawson-Watanabe superprocess; skew convolution semigroup; entrance law; minimal Brownian motion; convergence

AMS 1991 Subject Classifications: Primary 60J80; Secondary 60G57.

1. Introduction

The Dawson-Watanabe superprocess (measure-valued branching process) and the branching particle system are two mathematical models from biology and physics. Typical examples of the models are biological populations in isolated regions, families of neutrons in nuclear reactions and so on. Some immigration processes associated with the models have been introduced and studied involving different motivations; see e.g. Bojdecki and Gorostiza (1986), Dawson and Ivanoff (1978), Gorostiza (1988), Gorostiza and Lopez-Mimbela (1990), Ivanoff (1981), Li (1992), Li and Shiga (1995), Shiga (1990). From the view point of applications, those immigration processes are clearly of great importance. For instance, a typical unadulterated branching process started with a finite initial state goes either extinction or explosion at large times, which is not desired for the transformation process of particles in a nuclear reactor, but the situation can be changed

*Postal address: Department of Mathematics, Beijing Normal University, Beijing 100875, P. R. China. e-mail: lizh@bnu.edu.cn

[1]Research supported by the National Natural Science Foundation of China (Grant No.19361060).

if we consider the process with immigration. A type of *Markovian* immigration may be formulated by using skew convolution semigroups. Each skew convolution semigroup determines uniquely the transition semigroup of an immigration process. It was proved in Li (1996a) that a skew convolution semigroup associated with the Dawson-Watanabe superprocess corresponds uniquely to an infinitely divisible probability entrance law for the process. The infinitely divisible probability entrance laws were characterized and some path properties of the corresponding immigration processes were studied in Li (1996b).

In this paper, we study immigration processes associated with branching particle systems and their convergence to the measure-valued immigration processes. As in Li (1996ab) we concentrate our discussion to the time homogeneous situation. We shall see that a skew convolution semigroup associated with the branching particle system can also be characterized in terms of an infinitely divisible probability entrance law. The major difference between the two models is that, started with any deterministic state, the superprocess is infinitely divisible and the branching particle system, which can only be started with an integer-valued measure, is not. This causes some difficulties for the study of the immigration particle system – a minimal probability entrance law for the superprocess is always infinitely divisible, but that for the branching particle system is usually not. Indeed, the characterization of all infinitely divisible probability entrance laws for a general branching particle system still remains an open problem although we shall give some partial results.

In section 2 we introduce the notion of branching particle system considered in this paper. Their associated immigration particle systems are formulated using skew convolution semigroups in section 3. We show that every skew convolution semigroup corresponds uniquely to a locally integrable entrance law for the branching particle system. In section 4, we give a general representation for entrance laws of the branching particle system where the underlying process is a minimal Brownian motion in a bounded domain. Some examples of immigration particle systems are discussed in section 5. In section 6, we show that a general immigration process associated with the Dawson-Watanabe superprocess may arise as the high density limit in finite-dimensional distributions of a sequence of immigration particle systems. The convergence of immigration particle systems in path spaces is discussed in section 7.

2. Branching particle systems

Suppose that E is a Lusin topological space, i.e., a homeomorphism of a Borel subset of a compact metric space, with the Borel σ-algebra $\mathcal{B}(E)$. Denote by $B(E)$ the set of bounded $\mathcal{B}(E)$-measurable functions on E, and $C(E)$ the subspace of $B(E)$ comprising continuous functions. The subsets of positive members of the function spaces are denoted by the superscript "+"; e.g., $B(E)^+$, $C(E)^+$. Let $M(E)$ be the totality of finite measures on $(E, \mathcal{B}(E))$, and $N(E)$ the subspace of finite integer-valued measures. Topologize $M(E)$ and $N(E)$ by the weak convergence topology, so they also become Lusin spaces. Put $M(E)^\circ = M(E) \setminus \{0\}$ and $N(E)^\circ = N(E) \setminus \{0\}$, where 0 denotes the null measure on E. The unit mass concentrated at a point $x \in E$ is denoted by δ_x. For $f \in B(E)$ and $\mu \in M(E)$, write $\mu(f)$ for $\int_E f \mathrm{d}\mu$.

Let $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\sigma)$ be a Borel right Markov process in $N(E)$ with transition semigroup $(Q_t)_{t\geq 0}$. We call X a *branching particle system* provided

$$Q_t(\sigma_1 + \sigma_2, \cdot) = Q_t(\sigma_1, \cdot) * Q_t(\sigma_2, \cdot), \qquad t \geq 0,\ \sigma_1, \sigma_2 \in N(E), \tag{2.1}$$

where "$*$" denotes the convolution operation. For $f \in B(E)^+$, let

$$U_t f(x) = -\log \int_{N(E)} \mathrm{e}^{-\nu(f)} Q_t(\delta_x, \mathrm{d}\nu), \qquad t \geq 0, x \in E. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\int_{N(E)} \mathrm{e}^{-\nu(f)} Q_t(\sigma, \mathrm{d}\nu) = \exp\{-\sigma(U_t f)\}, \qquad t \geq 0, \sigma \in N(E). \tag{2.3}$$

We always assume that the branching particle system satisfies

(2A) for every $l \geq 0$ and $f \in B(E)^+$, the function $U_t f(x)$ of (t, x) restricted to $[0, l] \times E$ belongs to $B([0, l] \times E)^+$.

For $\mu \in M(E)$, let $\mathbf{Q}_{(\mu)}$ denote the conditional law of $\{X_t : t \geq 0\}$ given "X_0 is Poisson random measure with intensity μ". Then

$$\mathbf{Q}_{(\mu)} \exp\{-X_t(f)\} = \exp\{-\mu(J_t f)\}, \tag{2.4}$$

where

$$J_t f(x) = 1 - \exp\{-U_t f(x)\}. \tag{2.5}$$

Let us describe a special form of the branching particle system. Let ξ be a Borel right process in E with conservative transition semigroup $(P_t)_{t\geq 0}$. Let $\gamma(\cdot) \in B(E)^+$ and $g(\cdot, \cdot) \in B(E \times [0, 1])$. Suppose that for each fixed $x \in E$, $g(x, \cdot)$ coincides on $[0, 1]$ with a probability generating function and that $g'_z(\cdot, 1^-) \in B(E)^+$. Set $\rho(r, t) = \exp\left\{-\int_r^t \gamma(\xi_s)\mathrm{d}s\right\}$. A branching particle system X is called a (ξ, γ, g)-*particle system* if its transition probabilities are determined by (2.3) with $u_t(x) = U_t f(x)$ being the unique positive solution to the evolution equation

$$\mathrm{e}^{-u_t(x)} = \mathbf{P}_x \rho(0, t) \mathrm{e}^{-f(\xi_t)} + \mathbf{P}_x \left\{ \int_0^t \rho(0, s) g(\xi_s, \exp\{-u_{t-s}(\xi_s)\}) \gamma(\xi_s) \mathrm{d}s \right\}. \tag{2.6}$$

The heuristic meaning of the (ξ, γ, g)-particle system is clear from the equation (2.6): The particles in E move randomly according to the laws given by the transition probabilities of ξ. For a particle which is alive at time r and follows the path $\{\xi_s : s \geq r\}$, the conditional probability of survival during the time interval $[r, t]$ is $\rho(r, t)$. When the particle dies at a point $x \in E$, it gives birth to a random number of offspring according to the generating function $g(x, \cdot)$ and the offspring then move and propagate in E in the same fashion as their parents. It is assumed that the migrations, the life times and the branchings of the particles are independent of each other. See e.g. Dynkin (1991) for a vigorous construction of the (ξ, γ, g)-particle system. Note that (2.6) is equivalent to the equation

$$\begin{aligned}\exp\{-u_t(x)\} =& \mathbf{P}_x \exp\{-f(\xi_t)\} - \int_0^t \mathbf{P}_x \left[\gamma(\xi_{t-s}) \exp\{-u_s(\xi_{t-s})\}\right] \mathrm{d}s \\ &+ \int_0^t \mathbf{P}_x \left[\gamma(\xi_{t-s}) g(\xi_{t-s}, \exp\{-u_s(\xi_{t-s})\})\right] \mathrm{d}s.\end{aligned}$$

We shall simply write the above equation as

$$\mathrm{e}^{-u_t} = P_t \mathrm{e}^{-f} - \int_0^t P_{t-s} \left[\gamma \left(\mathrm{e}^{-u_s} - g(\mathrm{e}^{-u_s})\right)\right] \mathrm{d}s. \tag{2.7}$$

We sometimes need to consider the first moments of the (ξ, γ, g)-particle system. Let $\beta(x) = \gamma(x)[1 - g'_z(x, 1)]$ and let $(P_t^\beta)_{t \geq 0}$ be defined by

$$P_t^\beta f(x) = \mathbf{P}_x f(\xi_t) \exp\left\{ - \int_0^t \beta(\xi_s) \mathrm{d}s \right\}. \tag{2.8}$$

Using (2.3) and (2.7) one may check that

$$\int_{N(E)} \nu(f) Q_t(\sigma, \mathrm{d}\nu) = \sigma(P_t^\beta f). \tag{2.9}$$

By Jensen's inequality we have

$$J_t f(x) \leq U_t f(x) \leq P_t^\beta f(x), \qquad x \in E. \tag{2.10}$$

Therefore (2A) is satisfied for the (ξ, γ, g)-particle system. Note also that if we let

$$\varphi(x, z) = \gamma(x)[g(x, 1 - z) - (1 - z)], \qquad x \in E, 0 \leq z \leq 1, \tag{2.11}$$

then by (2.7),

$$J_t f(x) + \int_0^t \mathrm{d}s \int_E \varphi(y, J_s f(y)) P_{t-s}(x, \mathrm{d}y) = P_t \left(1 - \mathrm{e}^{-f}\right)(x). \tag{2.12}$$

The transition semigroup of the (ξ, γ, g)-particle system can also be determined by (2.3), (2.5) and (2.12). Clearly, this characterization of the system applies even for a non-conservative underlying semigroup $(P_t)_{t \geq 0}$.

The Dawson-Watanabe superprocess is a continuous state analogue of the (ξ, γ, g)-particle system. Let ξ be as the above and let $\phi(x,z)$ be a function having the representation

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x, \mathrm{d}u), \qquad x \in E, z \geq 0, \tag{2.13}$$

where $b \in B(E)$, $c \in B(E)^+$ and $(u \wedge u^2) m(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$. For $f \in B(E)^+$, let $V_t f(x)$ be the unique solution in $B(E)^+$ to the evolution equation

$$V_t f(x) + \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) P_{t-s}(x, \mathrm{d}y) = P_t f(x), \qquad t \geq 0, x \in E. \tag{2.14}$$

A Markov process $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ in $M(E)$ is called a *Dawson-Watanabe superprocess* provided

$$\mathbf{Q}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \qquad t \geq 0, \mu \in M(E), f \in B(E)^+. \tag{2.15}$$

See e.g. Dawson (1992, 1993) for systematic discussions on superprocesses and branching particle systems.

3. Skew convolution semigroups and immigration

Suppose that X is a branching particle system with transition semigroup $(Q_t)_{t\geq 0}$. Let $(N_t)_{t\geq 0}$ be a family of probability measures on $N(E)$. We call $(N_t)_{t\geq 0}$ a *skew convolution semigroup* associated with X or $(Q_t)_{t\geq 0}$ provided

$$N_{r+t} = (N_r Q_t) * N_t, \qquad r, t \geq 0. \tag{3.1}$$

The relation (3.1) is necessary and sufficient to ensure that

$$Q_t^N(\sigma, \cdot) := Q_t(\sigma, \cdot) * N_t, \qquad t \geq 0, \ \sigma \in N(E), \tag{3.2}$$

defines a transition semigroup $(Q_t^N)_{t\geq 0}$ on $N(E)$. (Indeed, (3.1) is equivalent to the Chapman-Kolmogorov equation satisfied by $(Q_t^N)_{t\geq 0}$.) Let $Y = \{Y_t : t \in T\}$ be a Markov process in $N(E)$, where T is an interval in $\mathbb{R}$. We call Y an *immigration process* or *immigration particle system* associated with X if it has $(Q_t^N)_{t\geq 0}$ as its transition semigroup. The intuitive meaning of the immigration process is clear from (3.2), that is, $Q_t(\sigma, \cdot)$ is the distribution of descendants of the people distributed as $\sigma \in N(E)$ at time zero and N_t is the distribution of descendants of the people immigrating to E during the time interval $(0, t]$.

It is known that a metric r can be introduced into E so that (E, r) becomes a compact metric space while the Borel σ-algebra induced by r coincides with $\mathcal{B}(E)$. Let $\mathcal{D}(E)^+$ be a countable dense subset of the space of strictly positive, continuous functions on (E, r) containing all rational-valued constant functions.

Lemma 3.1. *Suppose that $\{H_n : n = 1, 2, \cdots\}$ is a sequence of finite measures on $N(E)^\circ$. If*

$$\lim_{n\to\infty} \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_n(\mathrm{d}\nu) = R(f), \qquad f \in \mathcal{D}(E)^+, \tag{3.3}$$

and $\lim_{\mathrm{rat}.\theta\downarrow 0} R(\theta) = 0$, *then there is a unique finite measure H on $N(E)^\circ$ such that*

$$\int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H(\mathrm{d}\nu) = R(f), \qquad f \in \mathcal{D}(E)^+. \tag{3.4}$$

Proof. Let $N(E_r)^\circ$ denote the set $N(E)^\circ$ endowed with the topology of weak convergence on (E, r). Then $N(E_r)^\circ$ is a locally compact separable space. Let $N(E_r)^\circ \cup \{\partial\}$ be its one point compactification. Define $\mathrm{e}^{-\partial(f)} = 0$ for all $f \in \mathcal{D}(E)^+$. We may regard $\{H_n : n = 1, 2, \cdots\}$ as a bounded sequence of finite measures on $N(E_r)^\circ \cup \{\partial\}$. Let $H(\mathrm{d}\nu)$ be any limit point of this sequence. Then we have

$$\int_{N(E_r)^\circ \cup \{\partial\}} \left(1 - \mathrm{e}^{-\nu(f)}\right) H(\mathrm{d}\nu) = R(f), \qquad f \in \mathcal{D}(E)^+.$$

Taking $f = 1/n$ and letting $n \to \infty$ we see that $H(\mathrm{d}\nu)$ is in fact supported by $N(E_r)^\circ$, and hence the assertion follows. $\square$

Let $(Q_t^\circ)_{t\geq 0}$ denote the restriction of $(Q_t)_{t\geq 0}$ to the subspace $N(E)^\circ$. The following theorem characterizes completely the immigration structures associated with the branching particle system.

Theorem 3.1. *Suppose $(N_t)_{t\geq 0}$ is a family of probability measures on $N(E)$. Then $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with $(Q_t)_{t\geq 0}$ if and only if there is a locally integrable entrance law $(H_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$ such that*

$$\int_{N(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \exp\left\{ - \int_0^t \mathrm{d}s \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu) \right\} \tag{3.5}$$

for all $f \in B(E)^+$.

Proof. If $(N_t)_{t\geq 0}$ is given by (3.5), then clearly (3.1) holds. Conversely, suppose $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with $(Q_t)_{t\geq 0}$. Define

$$J_t(f) = -\log \int_{N(E)^\circ} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu), \qquad t \geq 0, f \in B(E)^+.$$

Then the relation (3.1) is equivalent to

$$J_{r+t}(f) = J_t(f) + J_r(U_t f), \qquad r, t \geq 0, f \in B(E)^+. \tag{3.6}$$

Consequently $J_t(f)$ is a non-decreasing function of $t \geq 0$. By a similar argument as in Li (1996a) one finds that $J_t(f) = \int_0^t I_s(f)\mathrm{d}s$ for $I_s(f) \geq 0$ given by

$$I_s(f) = \lim_{r \downarrow 0} \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s^{(r)}(\mathrm{d}\nu), \qquad 0 \leq s \notin N, f \in \mathcal{D}(E)^+, \tag{3.7}$$

where N is a Lebesgue null subset of $[0, \infty)$, and

$$H_s^{(r)}(\mathrm{d}\nu) = r^{-1} \int_{N(E)^\circ} N_r(\mathrm{d}\mu) Q_s(\mu, \mathrm{d}\nu).$$

Since clearly $J_t(f) \to 0$ as $f \to 0$, by Lemma 3.1, enlarging the Lebesgue null set N if it is necessary, we may assume $I_s(f)$ has the representation

$$I_s(f) = \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu), \qquad 0 \leq s \notin N, f \in \mathcal{D}(E)^+,$$

where $H_s(\mathrm{d}\nu)$ is a finite measure on $N(E)^\circ$. Consequently,

$$J_t(f) = \int_0^t \mathrm{d}s \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu), \qquad t \geq 0, f \in B(E)^+. \tag{3.8}$$

Now (3.1) and (2.3) yield that for all $r, t \geq 0$, $f \in B(E)^+$,

$$\int_0^r \mathrm{d}s \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_{t+s}(\mathrm{d}\nu) = \int_0^r \mathrm{d}s \int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(U_t f)}\right) H_s(\mathrm{d}\nu).$$

By Fubini's theorem, there are Lebesgue null subsets N' and N'_s of $[0, \infty)$ such that

$$H_{t+s} = H_s Q_t^\circ, \qquad 0 \leq s \notin N', 0 \leq t \notin N'_s.$$

Choose a sequence $0 < s_n \notin N'$ with $s_n \downarrow 0$, and define

$$H_t = H_{s_n} Q_{t-s_n}^\circ, \qquad t \geq s_n.$$

Under this modification, $(H_t)_{t>0}$ becomes an entrance law for $(Q_t^\circ)_{t\geq 0}$, and (3.8) remains unchanged. $\square$

An immediate consequence of Theorem 3.1 is the following construction of the immigration particle system, which gives a probabilistic interpretation of the equation (3.5) and explains the role of the entrance law $(H_t)_{t>0}$ in the phenomenon. Let $W_0(N(E))$ be the space of all right continuous paths $\{w : t > 0\}$ from $(0, \infty)$ to $N(E)$ with the natural σ-algebra. By the theory of Markov processes, there exists a σ-finite measure $\mathbf{Q}_H$ on $W_0(N(E))$ under which the coordinate process $\{w : t > 0\}$ is a Markov process with one dimensional distributions $(H_t)_{t>0}$ and semigroup $(Q_t^\circ)_{t\geq 0}$. Suppose $N^H(\mathrm{d}s, \mathrm{d}w)$ is a Poisson random measure on $[0,\infty) \times W_0(N(E))$ with intensity $\mathrm{d}s \times \mathbf{Q}_H(\mathrm{d}w)$. Define the process $\{Y_t^H : t \geq 0\}$ by

$$Y_t^H = \int_{[0,t]} \int_{W_0(M(E))} w_{t-s} N^H(\mathrm{d}s, \mathrm{d}w), \qquad t \geq 0, \tag{3.9}$$

where $w_0 = 0$ by convention. It is easy to check that $\{Y_t^H : t \geq 0\}$ is a Markov process in $N(E)$ with transition semigroup $(Q_t^H)_{t\geq 0}$ such that

$$\begin{aligned} &\int_{N(E)} \mathrm{e}^{-\nu(f)} Q_t^H(\sigma, \mathrm{d}\nu) \\ &= \exp\bigg\{ - \sigma(U_t f) - \int_0^t \mathrm{d}s \int_{N(E)^\circ} \Big(1 - \mathrm{e}^{-\nu(f)}\Big) H_s(\mathrm{d}\nu)\bigg\}. \end{aligned} \tag{3.10}$$

That is, $\{Y_t^H : t \geq 0\}$ is an immigration particle system corresponds to the skew convolution semigroup given by (3.5). Some special cases of the immigration particle system will be discussed latter.

A probability measure Q on $N(E)$ is said to be *infinitely divisible on* $N(E)$ if for each integer $n \geq 1$ there exists a probability measure Q_n on $N(E)$ such that $Q = Q_n * \cdots * Q_n$ ($n-1$ times). It is well-known that Q is an infinitely divisible probability on $N(E)$ if and only if

$$\int_{N(E)} \mathrm{e}^{-\nu(f)} Q(\mathrm{d}\nu) = \exp\bigg\{ - \int_{N(E)^\circ} \Big(1 - \mathrm{e}^{-\nu(f)}\Big) H(\mathrm{d}\nu)\bigg\}, \qquad f \in B(E)^+, \tag{3.11}$$

for a finite measure H on $N(E)^\circ$; see e.g. Kallenberg (1975; p44) for a related result. We write $Q = I(0, H)$ if Q and H are related by (3.11). The next theorem follows immediately from (3.11) and the branching property (2.3).

Theorem 3.2. *If $(H_t)_{t>0}$ is a finite entrance law for $(Q_t^\circ)_{t\geq 0}$, then $(K_t)_{t>0} := I(0, H_t)_{t>0}$ is an infinitely divisible probability entrance law for $(Q_t)_{t\geq 0}$. Conversely, if $(K_t)_{t>0}$ is an infinitely divisible probability entrance law for $(Q_t)_{t\geq 0}$, then $(K_t)_{t>0} = I(0, H_t)_{t>0}$ for a finite entrance law $(H_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$.*

We may give some particular entrance laws for the (ξ, γ, g)-particle system. Let $\mathcal{K}^1(Q)$ be the set of probability entrance laws $(K_t)_{t>0}$ for $(Q_t)_{t\geq 0}$ that satisfy

$$\int_0^1 \mathrm{d}s \int_{N(E)^\circ} \nu(E) K_s(\mathrm{d}\nu) < \infty. \tag{3.12}$$

Let $\mathcal{K}(Q^\circ)$ be the set of finite entrance laws $(K_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$ that satisfy (3.12). We use the subscript "m" to denote subsets of minimal elements, e.g., $\mathcal{K}_m^1(Q)$, $\mathcal{K}_m(Q^\circ)$, etc. Denote by $\mathcal{K}(P)$ the set of entrance laws $(\kappa_t)_{t>0}$ for the underlying semigroup $(P_t)_{t\geq 0}$ that satisfy $\int_0^1 \kappa_s(E)\mathrm{d}s < \infty$. Then we have

Theorem 3.3. *To each $(\kappa_t)_{t>0} \in \mathcal{K}(P)$ there corresponds an infinitely divisible $(K_t)_{t>0} \in \mathcal{K}^1(Q)$ which is given by*

$$\int_{N(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\left\{-R_t(\kappa, f)\right\}, \qquad t > 0, f \in B(E)^+, \tag{3.13}$$

where

$$R_t(\kappa, f) = \kappa_t\left(1 - \mathrm{e}^{-f}\right) - \int_0^t \kappa_{t-s}\left(\varphi(J_s f)\right) \mathrm{d}s. \tag{3.14}$$

Proof. By (2.4) and (3.14) we have

$$\mathbf{Q}_{(\kappa_r)} \exp\{-X_{t-r}(f)\} = \exp\left\{-\kappa_r(J_{t-r}f)\right\} \to \exp\left\{-R_t(\kappa, f)\right\} \tag{3.15}$$

uniformly on $f \in B(E)^+$ as $r \downarrow 0$. It follows that (3.13) indeed defines a family of infinitely divisible probability measures $(K_t)_{t>0}$ on $N(E)$. From (3.15) and (2.5) we have that

$$R_s(\kappa, U_t f) = \lim_{r\downarrow 0} \kappa_r(J_{s-r}U_t f) = \lim_{r\downarrow 0} \kappa_r(J_{s+t-r}f) = R_{s+t}(\kappa, f) \tag{3.16}$$

for all $s, t > 0$. Then we have $(K_t)_{t>0} \in \mathcal{K}(Q)$. $\square$

4. Characterization of entrance laws

With Theorems 3.1 and 3.2 in hands, one naturally hopes to characterize all infinitely divisible probability entrance laws for a given branching particle system. The analogous problem for a (ξ, ϕ)-superprocess is more tractable and has been solved in Li (1996b); see also Dynkin (1989) and Fitzsimmons (1988) for results of related interest. For the branching particle system, we can only solve this problem in some special cases up to now.

In this section, we suppose that $E = D$ is a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary ∂D and closure $\bar{D}$. Assume that both $g(x,z)$ and $g'_z(x,z)$ can be extended to continuous functions on $\bar{D} \times [0,1]$. Let ξ be a minimal (absorbing barrier) Brownian motion in D with transition density $p_t(x,y)$. It is well-known that $p_t(x,y) = p_t(y,x)$ is continuously differentiable in x and y to the boundary; see e.g. Friedman (1984: p82). We also use ∂ to denote the inward normal derivative operator at ∂D. Let $h(x) = \int_0^t P_s 1(x)\mathrm{d}s$. Then h is a continuous, excessive function of $(P_t)_{t\geq 0}$.

Lemma 4.1. *The function h is continuously differentiable to the boundary and ∂h is continuous, bounded and bounded away from zero on ∂D.*

Proof. Denote the life time of ξ in D by τ_D. Because of the sample path continuity, $\xi(\tau_D)$ lies on ∂D a.s. For $x \in D$ we have

$$\partial p_s(x, \cdot)(y)\mathrm{d}s\sigma(\mathrm{d}y) = 2\mathbf{P}_x\{\tau_D \in \mathrm{d}s, \xi(\tau_D) \in \mathrm{d}y\},$$

where $\sigma(\mathrm{d}y)$ is the volume element on ∂D; see e.g. Hsu (1986). Since D is bounded, integrating the above gives

$$\int_{\partial D} \partial h(y)\sigma(\mathrm{d}y) = 2\int_D \mathbf{P}_x\{\tau_D \leq 1\}\mathrm{d}x < \infty.$$

Therefore, if $D = \{x : a < |x| < b\}$ for some $0 < a < b < \infty$, the conclusion holds by symmetry. The general result can be proved by comparison method. $\square$

Let $N_h(D)$ be the set of integer-valued measures σ on D satisfying $\sigma(h) < \infty$, and $N_h(\bar{D})$ the set of measures μ on $\bar{D}$ such that $\mu_D := \mu|_D \in N_h(D)$ and $\mu_\partial := \mu|_{\partial D} \in M(\partial D)$. Then we have

Theorem 4.1. *For any $\mu \in N_h(\bar{D})$ there corresponds an entrance law $(K_t)_{t>0} \in \mathcal{K}^1(Q)$ given by*

$$\int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\left\{-\mu_D(U_t f) - \mu_\partial(\partial U_t f)\right\}, \qquad f \in C(D)^+. \tag{4.1}$$

If we denote this entrance law by $(K_t^{\mu_D,\mu_\partial})_{t>0}$ to indicate its dependence on μ_D and μ_∂, then $(K_t^{0,\mu_\partial})_{t>0} \in \mathcal{K}^1(Q)$ is infinitely divisible, and $(K_t^{\mu_D,0})_{t>0} \in \mathcal{K}^1(Q)$ is not unless we have $\mu_D = 0$.

Proof. Since $p_t(x,y)$ is continuously differentiable in x to the boundary ∂D, so are $U_t f(x)$ and $J_t f(x)$ by (2.7) and (2.12). It is clear that (4.1) with $\mu_\partial = 0$ defines a probability entrance law $(K_t^{\mu_D,0})_{t>0}$ for $(Q_t)_{t\geq 0}$, which is not infinitely divisible unless $\mu_D = 0$. As for (2.4) one may check that

$$\int_{N(E)^\circ} \nu(f) K_t^{\mu_D,0}(\mathrm{d}\nu) = \mu_D(P_t^\beta f) \leq \mathrm{e}^{\|\beta\|t}\mu_D(P_t f).$$

Since $\mu_D \in N_h(\bar{D})$, we have $(K_t^{\mu_D,0})_{t>0} \in \mathcal{K}^1(Q)$. Next we claim that (4.1) with $\mu_D = 0$ defines an infinitely divisible probability entrance law $(K_t^{0,\mu_\partial})_{t>0}$ for $(Q_t)_{t\geq 0}$. Clearly, both $U_t f(x)$ and $J_t f(x)$ go to zero as x tends to the boundary ∂D. Let n_z be the inward unit norm at $z \in \partial D$ and let $z_k = z + k^{-1}n_z$. Then $z_k \in D$ for large enough k and

$$\partial U_t f(z) = \partial J_t f(z) = \lim_{k\to\infty} k\left(1 - \exp\left\{-U_t f(z_k)\right\}\right). \tag{4.2}$$

From (2.10) it is immediate that $\partial U_t f(z) \leq \mathrm{e}^{\|\beta\|t}\partial P_t f(z)$. By the smoothness of $p_t(x,y)$ we have

$$\partial P_t f(z) = \int_D f(y)\partial p_t(\cdot,y)(z)\mathrm{d}y, \qquad z \in \partial D.$$

Therefore, $\partial P_t f$ is bounded on ∂D and decreases to zero as $f \downarrow 0$. By (4.2) and Lemma 3.1, there is a family of finite measures $(H_t^{\mu_\partial})_{t>0}$ on $N(D)^\circ$ such that

$$\int_{N(D)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t^{\mu_\partial}(\mathrm{d}\nu) = \mu_\partial(\partial U_t f) = \mu_\partial(\partial J_t f). \tag{4.3}$$

From the semigroup property of $(U_t)_{t\geq 0}$ it follows that $\partial U_r U_t f = \partial U_{r+t} f$ for all $r, t > 0$. Then $(H_t^{\mu_\partial})_{t>0}$ form an entrance law for $(Q_t^\circ)_{t\geq 0}$. By Theorem 3.2, (4.1) with $\mu_D = 0$ defines an infinitely divisible probability entrance law $(K_t^{0,\mu_\partial})_{t>0}$ for $(Q_t)_{t\geq 0}$ as claimed. By (4.1) and (2.10) we have

$$\int_{N(D)^\circ} \nu(f) K_t^{0,\mu_\partial}(\mathrm{d}\nu) = \lim_{k\to\infty} \mu_\partial(k\partial U_t(f/k)) \leq \mathrm{e}^{\|\beta\| t} \mu_\partial(\partial P_t f).$$

By Lemma 4.1, ∂h is a bounded function on ∂D, so we have $(K_t^{0,\mu_\partial})_{t>0} \in \mathcal{K}^1(Q)$. Finally, since $K_t^{\mu_D,0} * K_t^{0,\mu_\partial} = K_t^{\mu_D,\mu_\partial}$, by (2.3) one checks that $(K_t^{\mu_D,\mu_\partial})_{t>0} \in \mathcal{K}^1(Q)$ and the theorem is proved. $\square$

Theorem 4.2. *For any $\mu \in N_h(\bar{D})$, the entrance law $(K_t)_{t>0} \in \mathcal{K}^1(Q)$ defined by (4.1) is minimal. Conversely, if $(K_t)_{t>0} \in \mathcal{K}_m^1(Q)$, then there exists some $\mu \in N_h(\bar{D})$ such that we have (4.1).*

Proof. Recall that for any $(K_t)_{t>0} \in \mathcal{K}^1(Q)$ there is a probability measure $\mathbf{Q}_K$ on $W_0(N(D))$ under which the coordinate process $\{w_t : t > 0\}$ is a Markov process with one-dimensional distributions $(K_t)_{t>0}$ and semigroup $(Q_t)_{t\geq 0}$. We first show that any $(K_t)_{t>0} \in \mathcal{K}_m^1(Q)$ has Laplace functional given by (4.1) for some $\mu \in N_h(\bar{D})$. Since $(K_t)_{t>0}$ is minimal, for any $f \in C(D)^+$ we have $\mathbf{Q}_K$-a.s.,

$$\int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \lim_{\mathrm{rat.}r\downarrow 0} \exp\left\{-w_r(U_{t-r}f)\right\}, \tag{4.4}$$

where "rat. $r \downarrow 0$" means "r decreases to zero along the rational". For any $\bar{g} \in C(\bar{D})^+$ let g denote its restriction to D and let

$$\bar{W}_t\bar{g}(x) = h(x)^{-1} U_t(hg)(x) \ \text{ for } x \in D, \ = \partial h(x)^{-1} \partial U_t(hg)(x) \ \text{ for } x \in \partial D.$$

Set $hw_t(\mathrm{d}x) = h(x)w(\mathrm{d}x)$. We regard $\{hw_t : t > 0\}$ as a path in $W_0(M(\bar{D}))$. From (4.4) it follows that, if f/h can be extended to some $\overline{f/h} \in C(\bar{D})^+$, then $\mathbf{Q}_K$-a.s.,

$$\int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \lim_{\mathrm{rat.}r\downarrow 0} \exp\left\{-hw_r(\bar{W}_{t-r}(\overline{f/h}))\right\}. \tag{4.5}$$

By Lemma 4.1, ∂h is bounded and bounded away from zero on ∂D. Then we may define a strongly continuous Feller semigroup $(\bar{T}_t)_{t\geq 0}$ on $\bar{D}$ by

$$\begin{aligned}\bar{T}_t\bar{g}(x) &= h(x)^{-1}\int_D h(y)\bar{g}(y)p_t(x,y)\mathrm{d}y \quad \text{for } x\in D,\\ &= \partial h(x)^{-1}\int_D h(y)\bar{g}(y)\partial p_t(\cdot,y)(x)\mathrm{d}y \quad \text{for } x\in \partial D.\end{aligned}$$

Define the continuous function $\bar{\phi}(\cdot,\cdot)$ on $\{(x,z): x\in D \text{ and } 0\leq z\leq h(x)^{-1}$, or $x\in\partial D$ and $0\leq z<\infty\}$ by

$$\bar{\phi}(x,z) = h(x)^{-1}\varphi(x,h(x)z) \quad \text{for } x\in D, \quad = \varphi_z'(x,0^+)z \quad \text{for } x\in\partial D.$$

For $\bar{g}\in C(\bar{D})^+$ let $\bar{R}_t\bar{g}$ be the solution to

$$\bar{R}_t\bar{g}(x) = \bar{T}_t[(1-\mathrm{e}^{-hg})/h](x) - \int_0^t \bar{T}_{t-s}\bar{\phi}(\bar{R}_s\bar{g})(x)\mathrm{d}s, \qquad t\geq 0, x\in\bar{D}, \tag{4.6}$$

with $(1-\mathrm{e}^{-hg})/h=\bar{g}$ on ∂D defined by continuity. One checks easily that

$$\bar{R}_t\bar{g}(x) = h(x)^{-1}J_t(hg)(x) \quad \text{for } x\in D, \quad = \partial h(x)^{-1}\partial U_t(hg)(x) \quad \text{for } x\in\partial D.$$

(Recall that $\partial U_t(hg)(x) = \partial J_t(hg)(x)$ for $x\in\partial D$.) By (4.6), $\bar{R}_t\bar{g}\in C(\bar{D})^+$ is strongly continuous in $t\geq 0$; see e.g. Pazy (1983). From (2.5) it follows that

$$|U_t(hg)(x) - U_r(hg)(x)| \leq C|J_t(hg)(x) - J_r(hg)(x)|, \qquad 0\leq r\leq t\leq q, x\in D.$$

where $C=C(q,\|hg\|)$ is a positive constant. By continuity we get

$$|\bar{W}_t\bar{g}(x) - \bar{W}_r\bar{g}(x)| \leq C|\bar{R}_t\bar{g}(x) - \bar{R}_r\bar{g}(x)|, \qquad 0\leq r\leq t\leq q, x\in\bar{D}.$$

Then $\bar{W}_t\bar{g}\in C(\bar{D})^+$ is also strongly continuous in $t\geq 0$. Let $\mathcal{D}(\bar{D})^+$ be a countable, dense subset of $C(\bar{D})^+$ consisting strictly positive functions. Take any path $\{w_t: t>0\}$ along which (4.5) holds if $\overline{f/h}\in\mathcal{D}(\bar{D})^+$. Since $\overline{f/h}\in\mathcal{D}(\bar{D})^+$ is bounded away from zero, so is $\bar{W}_t(\overline{f/h})$ for small $t>0$. Then $hw_r(\bar{D})$ is bounded on rat. $r\in(0,\delta]$ for small enough $\delta>0$. Choosing a sequence $r_k\downarrow 0$ such that $hw_{r_k}\to$ some $\eta\in M(\bar{D})$ we see easily that

$$\int_{N(D)} \mathrm{e}^{-\nu(f)}K_t(\mathrm{d}\nu) = \exp\left\{-\eta(\bar{W}_t(\overline{f/h}))\right\}, \qquad \overline{f/h}\in\mathcal{D}(\bar{D})^+.$$

(Now it is clear that $\mathbf{Q}_K$-a.s. $hw_t\to\eta$ as $t\downarrow 0$.) Therefore (4.1) follows with

$$\mu_D(\mathrm{d}x) = h(x)^{-1}\eta(\mathrm{d}x) \quad \text{for } x\in D, \quad \mu_\partial(\mathrm{d}x) = \partial h(x)^{-1}\eta(\mathrm{d}x) \quad \text{for } x\in\partial D,$$

Obviously $\mu_D\in N_h(D)$ and $\mu_\partial\in M(\partial D)$. By a similar argument as in Dynkin (1989) one can show that for any $\mu\in N_h(\bar{D})$ the entrance law defined by (4.1) is minimal. Then the theorem is proved. $\square$

Theorem 4.3. *Let $\gamma \in M(\partial D)$ and let G be a measure on $N_h(\bar{D})$ such that*

$$\int_{N_h(\bar{D})} [\nu(h) + \nu(\partial h)] G(\mathrm{d}\nu) < \infty. \tag{4.7}$$

Then there is an infinitely divisible probability entrance law $(K_t)_{t>0} \in \mathcal{K}^1(Q^\circ)$ whose Laplace functional is given by

$$\begin{aligned} \int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\bigg\{ & -\gamma(\partial U_t f) \\ & - \int_{N_h(\bar{D})} (1 - \exp\{-\nu(U_t f) - \nu(\partial U_t f)\}) \, G(\mathrm{d}\nu)\bigg\}, \qquad f \in C(D)^+. \end{aligned} \tag{4.8}$$

Conversely, every infinitely divisible $(K_t)_{t>0} \in \mathcal{K}^1(Q^\circ)$ has Laplace functional given by (4.8).

Proof. By Theorem 4.1, (4.8) defines an entrance law $(K_t)_{t>0} \in \mathcal{K}^1(Q^\circ)$ which is infinitely divisible. Conversely, we assume that $(K_t)_{t>0} \in \mathcal{K}^1(Q^\circ)$ is infinitely divisible. By Theorem 4.2,

$$\int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \int_{N_h(\bar{D})} \exp\{-\nu_D(U_t f) - \nu_\partial(\partial U_t f)\} \, F_1(\mathrm{d}\nu)$$

for a probability measure F_1 on $N_h(\bar{D})$, or using the notation introduced in the proof of Theorem 4.2,

$$\int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \int_{M(\bar{D})} \exp\Big\{-\nu(\bar{W}_t(\overline{f/h}))\Big\} \, F_2(\mathrm{d}\nu), \qquad \overline{f/h} \in C(\bar{D})^+,$$

for a probability measure F_2 on $M(\bar{D})$. Clearly, both F_1 and F_2 are infinitely divisible, hence

$$\begin{aligned} \int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\bigg\{ & -\eta(\bar{W}_t(\overline{f/h})) \\ & - \int_{M(\bar{D})} \Big(1 - \exp\Big\{-\nu(\bar{W}_t(\overline{f/h}))\Big\}\Big) \, H(\mathrm{d}\nu)\bigg\}, \qquad \overline{f/h} \in C(\bar{D})^+, \end{aligned} \tag{4.9}$$

where $\eta \in M(\bar{D})$ and H is a σ-finite measure on $M(\bar{D})$. Let $M_h(\bar{D})$ denote the space of σ-finite measures ν on $\bar{D}$ such that $\nu(h) < \infty$ and $\nu(\partial D) < \infty$. Then we have immediately

$$\begin{aligned} \int_{N(D)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\bigg\{ & -\gamma(U_t f) - \gamma(\partial U_t f) \\ & - \int_{M_h(\bar{D})} (1 - \exp\{-\nu(U_t f) - \nu(\partial U_t f)\}) \, G(\mathrm{d}\nu)\bigg\}, \qquad f \in C(D)^+, \end{aligned}$$

where $\gamma \in M_h(\bar{D})$ and G is a σ-finite measure on $M_h(\bar{D})$. Since K_t is an infinitely divisible probability measure on $N(D)$, $\gamma(D) = 0$ and G is supported by $N_h(\bar{D})$; see e.g. Kallenberg (1975). $\square$

By Theorems 3.2 and 4.3, $(H_t)_{t>0} \in \mathcal{K}(Q^\circ)$ if and only if we have

$$\begin{aligned} \int_{N(D)} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) &= \gamma(\partial U_t f) \\ &+ \int_{N_h(\bar{D})} (1 - \exp\{-\nu_D(U_t f) - \nu_\partial(\partial U_t f)\})\, G(\mathrm{d}\nu), \qquad f \in C(D)^+, \end{aligned} \tag{4.10}$$

where $\gamma \in M(\partial D)$ and G is a measure on $N_h(\bar{D})$ satisfying (4.7).

Corollary 4.4. *For any $(H_t)_{t>0} \in \mathcal{K}_m(Q^\circ)$, either there are $0 < q < \infty$ and $\mu \in N_h(\bar{D})$ such that*

$$\int_{N(D)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = q\left(1 - \exp\{-\mu_D(U_t f) - \mu_\partial(\partial U_t f)\}\right), \quad f \in C(D)^+,$$

or there are $0 < q < \infty$ and $z \in \partial D$ such that

$$\int_{N(D)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = q \partial U_t f(z), \qquad f \in C(D)^+,$$

5. Examples of immigration particle systems

Let us see some examples of immigration particle system over a minimal Brownian motion in the domain D. Let $\{Y_t^H : t \geq 0\}$ be constructed by (3.9). We shall use the notation introduced in the last section.

Example 5.1. Fix $0 < q < \infty$ and $\sigma \in N_h(D) \setminus \{0\}$, and let $(H_t)_{t>0} \in \mathcal{K}_m(Q^\circ)$ be given by

$$\int_{N(D)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = q\left(1 - \exp\{-\sigma(U_t f)\}\right).$$

By the construction (3.9), immigration times of $\{Y_t^H : t \geq 0\}$ are given by a Poisson random measure on $[0, \infty)$ with intensity $q\mathrm{d}s$. It is easy to see that $w_{0+} = \sigma$ for $\mathbf{Q}_H$-a.a. $w \in W_0(M(D))$. Intuitively, at each immigration time a clique of immigrants land in D according to the point measure σ. Note that $\sigma(D) = \infty$ is possible.

Example 5.2. Fix $0 < q < \infty$ and $\mu \in M(\partial D) \setminus \{0\}$, and let $(H_t)_{t>0} \in \mathcal{K}_m(Q^\circ)$ be given by

$$\int_{N(D)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = q\left(1 - \exp\{-\mu(\partial U_t f)\}\right).$$

As in the last example, the immigration times of $\{Y_t^H : t \geq 0\}$ are given by a Poisson random measure on $[0, \infty)$ with intensity $q\mathrm{d}s$. By the proof of Theorem 4.2,

$hw_{0+}(\mathrm{d}z) = \partial h(z)\mu(\mathrm{d}z)$ for $\mathbf{Q}_H$-a.a. $w \in W_0(M(D))$. Then $w_{0+}(D) = \infty$ for $\mathbf{Q}_H$-a.a. $w \in W_0(M(D))$. Therefore, at each immigration time, a clique with infinite number of immigrants land at ∂D, some of which succeed in escaping from the absorbing boundary and entering the inner space D. Then the immigrants propagate and move in D according to the transition law of the Brownian branching particle system.

Example 5.3. Fix $\gamma \in M(\partial D) \setminus \{0\}$, and let $(H_t)_{t>0} \in \mathcal{K}(Q^\circ)$ be given by

$$\int_{N(D)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = \gamma(\partial U_t f).$$

Since $\gamma(\partial U_t 1) \to \infty$ as $t \downarrow 0$, the Markovian measure $\mathbf{Q}_H$ is infinite, but it is σ-finite. Therefore, the immigration of $\{Y_t^H : t \geq 0\}$ occurs countably infinite times in each non-empty open time interval. As in Li (1996b) one may show that $w_t(h)^{-1} hw_t \to \delta_{z(w)}$ with $z(w) \in \partial D$ for $\mathbf{Q}_H$-a.a. $w \in W_0(M(D))$. That is, the immigrants of $\{Y_t^H : t \geq 0\}$ enter D from the boundary points.

Example 5.4. Suppose that $(H_t)_{t>0} \in \mathcal{K}(Q^\circ)$ is given by (4.10). This is the general form of the entrance law and the corresponding immigration particle system involves a combination of the phenomena described in Examples 5.1 – 5.3.

6. Convergence in finite-dimensional distributions

The problem of characterizing of all skew convolution semigroups associated with a general (ξ, γ, g)-particle system still remains unsolved. In this section we shall see that a general immigration process associated the (ξ, ϕ)-superprocess may arise as the high density limit in finite-dimensional distributions of the immigration particle systems given by Theorems 3.1 and 3.3. In a certain sense, this shows the range of the immigration phenomenon associated with the branching particle system.

Suppose we have a sequence of parameters (ξ, γ_k, g_k), $k = 1, 2, \cdots$. Let $u_t(k, x) \equiv u_t(k, x, f)$ denote the solution to (2.7) with (γ, g) replaced by (γ_k, g_k), and let $(Q_t(k))_{t\geq 0}$ be the transition semigroup of the corresponding (ξ, γ_k, g_k)-particle systems. Fix $(\kappa_t)_{t>0} \in \mathcal{K}(P)$. Define $R_t(k, \kappa, f)$ by (3.14) with (γ, g) replaced by (γ_k, g_k). By Theorems 3.2 and 3.3,

$$\int_{N(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t^{(k)}(\mathrm{d}\nu) = kR_t(k, \kappa, f), \qquad f \in B(E)^+, \tag{6.1}$$

defines an entrance law $H^{(k)} \in \mathcal{K}(Q(k)^\circ)$. The parameters $(\xi, \gamma_k, g_k; H^{(k)})$ determines a sequence of immigration particle systems $\{Y_t(k) : t \geq 0\}$ via Theorems 3.1. Take $\eta \in M(E)$ and assume $Y_0(k)$ is a Poisson random measure on E with intensity $k\eta$. Then $\{Y_t^{(k)} := k^{-1} Y_t(k) : t \geq 0\}$ is a Markov process in $M_k(E) := \{\sigma/k : \sigma \in N(E)\}$. Let $\mathbf{Q}_{(\eta)}^{(k)}$ denote the law of $\{Y_t^{(k)} : t \geq 0\}$, and let

$$\phi_k(x, z) = k\gamma_k[g_k(x, 1 - z/k) - (1 - z/k)], \qquad 0 \leq z \leq k. \tag{6.2}$$

Then we have the following

Theorem 6.1. *Suppose that, for each $l \geq 0$, on the set $E \times [0, l]$ of (x, z), the sequence $\phi_k(x, z)$ defined by (6.2) is uniformly Lipschitz in z and $\phi_k(x, z) \to \phi(x, z)$ uniformly as $k \to \infty$. Then $\phi(x, z)$ has the representation (2.13), and the finite-dimensional distributions of $\{Y_t^{(k)} : t \geq 0\}$ under $\mathbf{Q}_{(\eta)}^{(k)}$ converge to those of a Markov process $\{Y_t : t \geq 0\}$ in $M(E)$ with initial state η and semigroup $(Q_t^\kappa)_{t\geq 0}$ defined by*

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^\kappa(\mu, \mathrm{d}\nu) = \exp\bigg\{ -\mu(V_t f) - \int_0^t S_r(\kappa, f)\mathrm{d}r \bigg\}, \qquad f \in B(E)^+, \tag{6.3}$$

where $(V_t)_{t\geq 0}$ is given by (2.14) and

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y))\kappa_{t-s}(\mathrm{d}y). \tag{6.4}$$

Now we proceed to the proof of the above theorem. Let $u_t^{(k)}(x)$ be the solution to

$$\mathrm{e}^{-u_t^{(k)}} = P_t \mathrm{e}^{-f/k} - \int_0^t P_{t-s}\left[\gamma_k\left(\mathrm{e}^{-u_s^{(k)}} - g_k(\mathrm{e}^{-u_s^{(k)}})\right)\right]\mathrm{d}s, \qquad t \geq 0, \tag{6.5}$$

and let

$$v_t^{(k)}(x) = k[1 - \exp\{-u_t^{(k)}(x)\}]. \tag{6.6}$$

Then we have

$$v_t^{(k)} + \int_0^t P_{t-s}[\phi_k(v_s^{(k)})]\mathrm{d}s = P_t k(1 - \mathrm{e}^{-f/k}). \tag{6.7}$$

Denote by $(Q_t^{(k)})_{t\geq 0}$ the transition semigroup of $\{Y_t^{(k)} : t \geq 0\}$. One may check that

$$\int_{M_k(E)} \mathrm{e}^{-\nu(f)} Q_t^{(k)}(\sigma, \mathrm{d}\nu) = \exp\bigg\{ -\sigma(k u_t^{(k)}) - \int_0^t R_s^{(k)}(\kappa, f)\mathrm{d}s \bigg\}, \tag{6.8}$$

where

$$R_t^{(k)}(\kappa, f) = k\kappa_t(1 - \mathrm{e}^{-f/k}) - \int_0^t \kappa_{t-s}(\phi_k(v_s^{(k)}))\mathrm{d}s. \tag{6.9}$$

Note that the sequence $\{Y_t^{(k)} : t \geq 0\}$ can be characterized by (6.6) – (6.9), which are applicable even when $(P_t)_{t\geq 0}$ is non-conservative. For any $\eta \in M(E)$ we have

$$\mathbf{Q}_{(\eta)}^{(k)} \exp\{ -Y_t^{(k)}(f)\} = \exp\bigg\{ -\eta(v_t^{(k)}) - \int_0^t R_s^{(k)}(\kappa, f)\mathrm{d}s \bigg\}. \tag{6.10}$$

Lemma 6.1. *Assume that the conditions of Theorem 6.1 are fulfilled. Then for any* $t \geq 0$ *and* $f \in B(E)^+$, *the sequence*

$$\mathbf{Q}_{(\eta)}^{(k)}\{Y_t^{(k)}(f)\}, \qquad k = 1, 2, \cdots,$$

is bounded.

Proof. Let $b_k(x) = (\mathrm{d}/\mathrm{d}z)\phi_k(x, 0)$ and define the semigroup of bounded kernels $(P_t^{b_k})_{t\geq 0}$ on E by (2.8) with β placed by b_k. By (6.7), $v_t^{(k)}$ satisfies

$$v_t^{(k)} + \int_0^t P_{t-s}^{b_k}[\phi_{0k}(v_s^{(k)})]\mathrm{d}s = P_t^{b_k} k(1 - \mathrm{e}^{-f/k}), \tag{6.11}$$

where $\phi_{0k}(x, z) = \phi_k(x, z) - b_k(x)z$. By Lemma 2.1 in Li (1996b),

$$\zeta_t^{(k)}(f) = \lim_{r\downarrow 0} \kappa_r P_{t-r}^{b_k}(f) \tag{6.12}$$

defines an entrance law $(\zeta_t^{(k)})_{t>0}$ for $(P_t^{b_k})_{t\geq 0}$, and

$$\mathrm{e}^{-\|b_k\|t}\kappa_t(f) \leq \zeta_t^{(k)}(f) \leq \mathrm{e}^{\|b_k\|t}\kappa_t(f), \qquad t > 0. \tag{6.13}$$

Using (6.9), (6.11) and (6.12) we get

$$R_t^{(k)}(\kappa, f) = k\zeta_t^{(k)}(1 - \mathrm{e}^{-f/k}) - \int_0^t \zeta_{t-s}^{(k)}(\phi_{0k}(v_s^{(k)}))\mathrm{d}s. \tag{6.14}$$

Replacing f with θf in (6.10), differentiating in θ and using (6.11) and (6.14) we get

$$\mathbf{Q}_{(\eta)}^{(k)}\{Y_t^{(k)}(f)\} = \eta(P_t^{b_k} f) + \int_0^t \zeta_s^{(k)}(f)\mathrm{d}s,$$

which is less than

$$\mathrm{e}^{\|b_k\|t}\eta(P_t f) + \int_0^t \mathrm{e}^{\|b_k\|s}\kappa_s(f)\mathrm{d}s.$$

Under the conditions of Theorem 6.1, the sequence $\|b_k\|$, $k = 1, 2, \cdots$, is bounded, and hence the lemma is proved. $\square$

The following result was proved in Dynkin (1991) and Li (1991, 1992).

Lemma 6.2. *Assume that the conditions of Theorem 6.1 are fulfilled. Then $\phi(x,z)$ has the representation (2.13) and $v_t^{(k)}(x,f)$ converges as $k\to\infty$ to $V_tf(x)$ boundedly and uniformly on the set $[0,l]\times E$ of (t,x) for every $l>0$.*

Proof of Theorem 6.1. We first note that (6.3) really defines a Markov semigroup $(Q_t^\kappa)_{t\geq 0}$ on $M(E)$; see Li (1996b). By (6.6) and Lemma 6.2, the sequence $ku_t^{(k)}(x)$ converges boundedly and uniformly on each set $[0,l]\times E$ to $V_tf(x)$. Therefore,

$$R_t^{(k)}(\kappa,f)\to S_t(\kappa,f) \qquad (k\to\infty) \tag{6.15}$$

boundedly and uniformly on each set $[0,l]\times E$ of (t,x), and the convergence of the one dimensional distributions follows. Take $0\leq t_1<\cdots<t_n$ and $f_1,\cdots,f_n\in B(E)^+$. Using the Markov property of $\{Y_t^{(k)}:t\geq 0\}$ we have

$$\begin{aligned}&\mathbf{Q}_{(\eta)}^{(k)}\exp\Big\{-\sum_{i=1}^n Y_{t_i}^{(k)}(f_i)\Big\}\\ =&\mathbf{Q}_{(\eta)}^{(k)}\exp\Big\{-\sum_{i=1}^{n-1} Y_{t_i}^{(k)}(f_i)-Y_{t_{n-1}}^{(k)}(ku_{\delta_n}^{(k)})\Big\}\exp\Big\{\int_0^{\delta_n}R_s^{(k)}(\kappa,f)\mathrm{d}s\Big\},\end{aligned} \tag{6.16}$$

where $\delta_n=t_n-t_{n-1}$. By Lemma 6.1 we have

$$\begin{aligned}&\lim_{k\to\infty}\mathbf{Q}_{(\eta)}^{(k)}\Big|\exp\big\{-Y_{t_{n-1}}^{(k)}(ku_{\delta_n}^{(k)})\big\}-\exp\big\{-Y_{t_{n-1}}^{(k)}(V_{\delta_n}f_n)\big\}\Big|\\ \leq&\lim_{k\to\infty}\mathbf{Q}_{(\eta)}^{(k)}\Big\{Y_{t_{n-1}}^{(k)}(\|ku_{\delta_n}^{(k)}-V_{\delta_n}f_n\|)\Big\}=0.\end{aligned} \tag{6.17}$$

From (6.15), (6.16) and (6.17) it follows that

$$\begin{aligned}&\lim_{k\to\infty}\mathbf{Q}_{(\eta)}^{(k)}\exp\Big\{-\sum_{i=1}^n Y_{t_i}^{(k)}(f_i)\Big\}\\ =&\lim_{k\to\infty}\mathbf{Q}_{(\eta)}^{(k)}\exp\Big\{-\sum_{i=1}^{n-1} Y_{t_i}^{(k)}(f_i)-Y_{t_{n-1}}^{(k)}(V_{\delta_n}f_n)\Big\}\exp\Big\{\int_0^{\delta_n}S_r(\kappa,f)\mathrm{d}r\Big\},\end{aligned}$$

so the desired assertion holds by induction in n. □

One can also construct a sequence of immigration particle systems that converge in finite dimensional distributions to a Markov process with transition semigroup $(Q_t^G)_{t\geq 0}$ given by

$$\begin{aligned}&\int_{M(E)}\mathrm{e}^{-\nu(f)}Q_t^G(\mu,\mathrm{d}\nu)\\ =&\exp\Big\{-\mu(V_tf)-\int_0^t\mathrm{d}r\int_{\mathcal{K}(P)}\left(1-\exp\{-S_r(\eta,f)\}\right)G(\mathrm{d}\eta)\Big\},\end{aligned} \tag{6.18}$$

where $(V_t)_{t\geq 0}$ is given by (2.14) and G is a σ-finite measure on $\mathcal{K}(P)$ satisfying

$$\int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) G(\mathrm{d}\eta) < \infty.$$

A combination of (6.3) and (6.18) gives the general immigration structure associated with a (ξ, ϕ)-superprocess; see Li (1996b).

7. Convergence in cadlag path spaces

In this section we consider the convergence of particle systems in path spaces. We shall operate in the simple situation where ξ be a minimal Brownian motion in a bounded smooth domain D. For $y \in \partial D$,

$$\kappa_t(y, f) = \partial P_t f(y), \qquad t > 0, f \in C(D), \tag{7.1}$$

determines an entrance law $\kappa(y) \in \mathcal{K}(P)$. Clearly we have

$$S_t(\kappa(y), f) = \partial V_t f(y), \qquad t > 0, f \in C(D)^+. \tag{7.2}$$

By Theorem 2.2 of Li (1996b) there is a probability entrance law $(K_t(y, \cdot))_{t>0}$ for the Brownian motion such that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(y, \mathrm{d}\nu) = \exp\left\{-\partial V_t f(y)\right\}, \qquad t > 0, f \in C(D)^+. \tag{7.3}$$

Let $\mathbf{Q}_{\kappa(y)}$ denote the probability measure on $W_0(M(D))$ under which the coordinate process is a Markov process having the same semigroup as super Brownian motion and 1-dimensional distributions $(K_t(y, \cdot))_{t>0}$. Fix $F \in M(\partial D)$ and let

$$\mathbf{Q}^F(\mathrm{d}w) = \int_{\partial D} F(\mathrm{d}y) \mathbf{Q}_{\kappa(y)}(\mathrm{d}w). \tag{7.4}$$

Suppose that $N^F(\mathrm{d}s, \mathrm{d}w)$ is a Poisson random measure on $[0, \infty) \times W_0(M(D))$ with intensity $\mathrm{d}s \times \mathbf{Q}^F(\mathrm{d}w)$. By setting

$$Y_t = \int_{[0,t)} \int_{W_0(M(D))} w_{t-s} \, N^F(\mathrm{d}s, \mathrm{d}w), \qquad t \geq 0,$$

we construct a measure-valued immigration process $\{Y_t : t \geq 0\}$ over D. This process represents a population generated by cliques of immigrants with infinite mass which arrive at points in ∂D at occurring times of $N^F(\mathrm{d}s, \mathrm{d}w)$, and it has no cadlag modifications; see Li (1996b).

The transition semigroup $(Q_t^F)_{t\geq 0}$ of $\{Y_t : t\geq 0\}$ is determined by

$$\begin{aligned}&\int_{M(D)} \mathrm{e}^{-\nu(f)} Q_t^F(\mu, \mathrm{d}\nu)\\ &= \exp\bigg\{ -\mu(V_t f) - \int_0^t \mathrm{d}s \int_{\partial D} (1 - \exp\{-\partial V_s f(y)\})\, F(\mathrm{d}y)\bigg\}.\end{aligned} \tag{7.5}$$

As mentioned in the last section, the process $\{Y_t : t\geq 0\}$ is the limit in finite dimensional distributions of a sequence of renormalized particle systems $\{Y_t^{(k)} : t\geq 0\}$ with $Y_0^{(k)} = 0$ and with transition semigroups given by

$$\begin{aligned}&\int_{M_k(D)} \mathrm{e}^{-\nu(f)} Q_t^{(k)}(\sigma, \mathrm{d}\nu)\\ &= \exp\bigg\{ -\sigma(ku_t^{(k)}) - \int_0^t \mathrm{d}s \int_{\partial D} \Big(1 - \exp\{-\partial v_s^{(k)}(y)\}\Big)\, F(\mathrm{d}y)\bigg\},\end{aligned} \tag{7.6}$$

where $u_t^{(k)}$ and $v_t^{(k)}$ are determined by (6.6) and (6.7), respectively. The processes $\{Y_t^{(k)} : t\geq 0\}$ can be constructed in the same way as $\{Y_t : t\geq 0\}$, and has similar trajectory singularities; see Example 5.2. Naturally, one wishes to show the convergence $\{Y_t^{(k)} : t\geq 0\} \to \{Y_t : t\geq 0\}$ in a path space. Because of the singularities mentioned above, it is impossible to formulate the convergence in the cadlag space $D([0,\infty), M(D))$ with the usual topology. However, we can use a transformation and prove a limit theorem in the space $D([0,\infty), M(\bar{D}))$. Define

$$\begin{aligned}\bar{\psi}(x,z) &= h(x)^{-1}\phi(x, h(x)z) \quad \text{for } x\in D,\\ &= 0 \quad \text{for } x\in\partial D.\end{aligned} \tag{7.7}$$

Define $\bar{\psi}_k$ in the same manner with ϕ replaced by ϕ_k. Assume that $\bar{\psi}(x,z)$ and $\bar{\psi}_k(x,z)$ are continuous functions of (x,z). For $f\in B(\bar{D})^+$ let $\bar{U}_t f$ denote the solution to

$$\bar{U}_t f(x) = \bar{T}_t f(x) - \int_0^t \mathrm{d}s \int_{\bar{D}} \bar{\psi}(y, \bar{U}_s f(y)) \bar{T}_{t-s}(x, \mathrm{d}y), \qquad t\geq 0, x\in\bar{D}, \tag{7.8}$$

where $(\bar{T}_t)_{t\geq 0}$ is the strongly continuous Feller semigroup defined in the proof of Theorem 4.2. Starting from the process $\{Y_t : t\geq 0\}$, let us define the process $\{\bar{Z}_t : t\geq 0\}$ by

$$\bar{Z}_t(\partial D) = 0 \quad\text{and}\quad \bar{Z}_t(\mathrm{d}x) = h(x) Y_t(\mathrm{d}x) \quad \text{for } x\in D. \tag{7.9}$$

It is easy to check that $\{\bar{Z}_t : t\geq 0\}$ is a Markov process in $M(\bar{D})$ with transition semigroup $(R_t^F)_{t\geq 0}$ determined by

$$\begin{aligned}\int_{M(D)} \mathrm{e}^{-\nu(f)} R_t^F(\mu, \mathrm{d}\nu) = \exp\bigg\{ &-\mu(\bar{U}_t f)\\ &- \int_0^t \mathrm{d}s \int_{\partial D} \big(1 - \exp\{-\partial h(y) \bar{U}_s f(y)\}\big)\, F(\mathrm{d}y)\bigg\}.\end{aligned} \tag{7.10}$$

Let $\bar{U}_t^{(k)} f$ be the solution to

$$\begin{aligned}\bar{U}_t^{(k)} f(x) =& \bar{T}_t[(1-\mathrm{e}^{-hf/k})/h](x) \\ & - \int_0^t \mathrm{d}s \int_D \bar{\psi}_k(y, \bar{U}_s^{(k)} f(y)) \bar{T}_{t-s}(x, \mathrm{d}y), \qquad t \geq 0, x \in \bar{D}.\end{aligned} \tag{7.11}$$

We introduce the space

$$M_k^h(\bar{D}) = \{\mu : \mu \in M(\bar{D}) \;\text{ and }\; h(x)^{-1}\mu|_D(\mathrm{d}x) \in M_k(D)\}. \tag{7.12}$$

Let $\{\bar{Z}_t^{(k)} : t \geq 0\}$ be defined by (7.9) with $\{Y_t : t \geq 0\}$ replaced by $\{Y_t^{(k)} : t \geq 0\}$. Let

$$\begin{aligned}\hat{U}_t^{(k)} f(x) &= -kh^{-1} \log \Big[1 - k^{-1} h \bar{U}_t^{(k)} f(x)\Big] \quad \text{for } x \in D, \\ &= \bar{U}_t^{(k)} f(x) \quad \text{for } x \in \partial D.\end{aligned} \tag{7.13}$$

Then $\{\bar{Z}_t^{(k)} : t \geq 0\}$ is a Markov process in $M_k^h(\bar{D})$ having the transition semigroup given by

$$\begin{aligned}\int_{M_k^h(D)} \mathrm{e}^{-\nu(f)} R_t^{(k)}(\sigma, \mathrm{d}\nu) = \exp\bigg\{ & -\sigma(\hat{U}_t^{(k)} f) \\ & - \int_0^t \mathrm{d}s \int_{\partial D} \Big(1 - \exp\{-\partial h(y) \bar{U}_s^{(k)} f(y)\}\Big) F(\mathrm{d}y)\bigg\}.\end{aligned} \tag{7.14}$$

Theorem 7.1. *Assume that $\bar{\psi}_k(x,z)$ and $\bar{\psi}(x,z)$ are uniformly Lipschitz in z on the set $D \times [0,l]$ for each finite $l > 0$. Then the processes $\{\bar{Z}_t^{(k)} : t \geq 0\}$ and $\{\bar{Z}_t : t \geq 0\}$ have modifications in $D([0,\infty), M(\bar{D}))$. If, in addition, $\bar{\psi}_k(x,z) \to \bar{\psi}(x,z)$ uniformly on each set $D \times [0,l]$, then*

$$\{\bar{Z}_t^{(k)} : t \geq 0\} \;\to \{\bar{Z}_t : t \geq 0\} \quad \textit{weakly in } D([0,\infty), M(\bar{D})). \tag{7.15}$$

Proof. Clearly $(\bar{T}_t)_{t\geq 0}$ preserves $C(\bar{D})^{++}$, strictly positive continuous functions, then so do $(\bar{U}_t)_{t\geq 0}$ and $(\bar{U}_t^{(k)})_{t\geq 0}$ by (7.8) and (7.11). On the other hand, since $(\bar{T}_t)_{t\geq 0}$ is strongly continuous on $C(\bar{D})^+$, so are $(\bar{U}_t)_{t\geq 0}$ and $(\bar{U}_t^{(k)})_{t\geq 0}$; see Pazy (1983) and the proof of Theorem 4.2. By (7.13) one may check that $(\hat{U}_t^{(k)})_{t\geq 0}$ preserves $C(\bar{D})^{++}$ and is strongly continuous on $C(\bar{D})^+$. It follows that $\{\bar{Z}_t : t \geq 0\}$ and $\{\bar{Z}_t^{(k)} : t \geq 0\}$ both have strongly continuous Feller transition semigroups, and hence they have modifications in $D([0,\infty), M(\bar{D}))$. By Theorem 2.11 of Ethier-Kurtz (1986; p172), to get (7.15) it suffices to show that as $k \to \infty$,

$$\sup_{\sigma \in M_k^h(D)} \bigg| \int_{M(D)} \mathrm{e}^{-\nu(f)} Q_t^F(\sigma, \mathrm{d}\nu) - \int_{M_k^h(D)} \mathrm{e}^{-\nu(f)} Q_t^{(k)}(\sigma, \mathrm{d}\nu) \bigg| \;\to\; 0$$

for every $f \in C(\bar{D})^{++}$, which follows by a careful application of Lemma 6.2. $\square$

Corollary 7.2. *Assume the conditions in Theorem 7.1. If $f/h \in C(\bar{D})^+$ vanishes at ∂D, then the processes $\{Y_t^{(k)}(f) : t \geq 0\}$ and $\{Y_t(f) : t \geq 0\}$ have modifications in $D([0,\infty), I\!R)$ and*

$$\{Y_t^{(k)}(f) : t \geq 0\} \ \to \{Y_t(f) : t \geq 0\} \ \ \textit{weakly in } D([0,\infty), I\!R). \tag{7.16}$$

Proof. Clearly, $Y_t^{(k)}(f) = Z_t^{(k)}(f/h)$ and $Y_t(f) = Z_t(f/h)$, so (7.16) follows from (7.15). □

Acknowledgment. It is my pleasure to thank a referee for his/her comments which helped me to improve the presentation of the paper.

References

Bojdecki, T. and Gorostiza, L.G. (1986), *Langevin equation for $\mathcal{S}'$-valued Gaussian processes and fluctuation limits of infinite particle system*, Probab. Theory Related Fields **73**, 227-244.

Dawson, D.A. (1992), *Infinitely Divisible Random Measures and Superprocesses*, In: Proceedings of 1990 Workshop on Stochastic Analysis and Related Topics, Silivri, Turkey.

Dawson, D.A. (1993), *Measure-valued Markov Processes*, Lecture Notes Math. **1541**, Springer-Verlag, Berlin.

Dawson, D.A. and Ivanoff, D. (1978), *Branching diffusions and random measures*, In: Advances in Probability and Related Topics **5**, 61-103, A. Joffe and P. Ney ed.

Dynkin, E.B. (1989), *Three classes of infinite dimensional diffusion processes*, J. Funct. Anal. **86**, 75-110.

Dynkin, E.B. (1991), *Branching particle systems and superprocesses*, Ann. Probab. **19**, 1157-1194.

Ethier, S.N. and Kurtz, T.G. (1986), *Markov Processes: Characterization and Convergence*, John Wiley & Sons, New York.

Fitzsimmons, P.J. (1988), *Construction and regularity of measure-valued Markov branching processes*, Israel J. Math. **64**, 337-361.

Friedman, A. (1984), *Partial Differential Equations of Parabolic Type*, Englewood Cliffs, NJ, Prentice Hall.

Gorostiza, L.G. (1988), *Limit theorems for supercritical branching random fields with immigration*, Adv. Appl. Math. **9**, 56-86.

Gorostiza, L.G. and Lopez-Mimbela, J.A. (1990), *The multitype measure branching process*, Adv. Appl. Probab. **22**, 49-67.

Hsu, P. (1986), *Brownian exit distribution of a ball*, Seminar on Stochastic Process. **1985**, 108-116.

Ivanoff, D. (1981), *The branching diffusion with immigration*, J. Appl. Probab. **17**, 1-15.

Kallenberg, O. (1975), *Random measures*, Academic Press, New York.

Li, Z.H. (1991), *Integral representations of continuous functions*, Chinese Sci. Bull. **36**, 979-983.

Li, Z.H. (1992), *Measure-valued branching processes with immigration*, Stochastic Process. Appl. **43**, 249-264.

Li, Z.H. (1996a), *Convolution semigroups associated with measure-valued branching processes*, Chinese Sci. Bull. **41**, 276-280.

Li, Z.H. (1996b), *Immigration structures associated with Dawson-Watanabe superprocesses*, Stochastic Process. Appl. **62**, 73-86.

Li, Z.H. and Shiga, T. (1995), *Measure-valued branching diffusions: immigrations, excursions and limit theorems*, J. Math. Kyoto Univ. **35** (1995), 233-274.

Pazy, A. (1983), *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag.

Shiga, T. (1990), *A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes*, J. Math. Kyoto Univ. **30**, 245-279.

Department of Mathematics
Beijing Normal University
Beijing 100875, P. R. China
e-mail: lizh@ bnu.edu.cn