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Entrance laws for Dawson-Watanabe superprocesses with nonlocal branching*

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Abstract. We prove a 1-1 correspondence between minimal probability entrance laws for the superprocess and entrance laws for its underlying process. From this we deduce that an infinitely divisible probability entrance law for the superprocess is uniquely determined by an infinitely divisible probability measure on the space of the underlying entrance laws. Under an additional condition, a characterization is given for all entrance laws for the superprocess, generalizing the results of Dynkin (1989). An application to immigration processes is also discussed.

Key words: superprocess; entrance law; 1-1 correspondence

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1. Introduction

Let E be a Lusin topological space with the Borel σ-algebra denoted by $\mathcal{B}(E)$. Let $B(E)$ denote the set of all bounded $\mathcal{B}(E)$-measurable functions on E and $B(E)^+$ the subspace of $B(E)$ comprising of non-negative elements. Denote by $M(E)$ the space of finite Borel measures on E equipped with the topology of weak convergence. For $f \in B(E)$ and $\mu \in M(E)$, write $\mu(f)$ for $\int_E f \mathrm{d}\mu$. We fix the underlying process $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$, which is a Borel right process in E, with transition semigroup $(P_t)_{t\geq 0}$. Let ϕ be the local branching mechanism given by

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x, \mathrm{d}u), \qquad x \in E, z \geq 0, \tag{1.1}$$

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where $b \in B(E)$, $c \in B(E)^+$ and $[u \wedge u^2]m(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$. Let $M(E)^\circ = M(E) \setminus \{0\}$, where 0 denotes the null measure. The non-local branching mechanism φ is given by

$$\varphi(x, f) = d(x, f) + \int_{M(E)^\circ} (1 - \mathrm{e}^{-\nu(f)}) n(x, \mathrm{d}\nu), \qquad x \in E, f \in B(E)^+, \tag{1.2}$$

where $d(x, \mathrm{d}y)$ is a bounded kernel from E to $(0, \infty)$ and $\nu(E)n(x, \mathrm{d}\nu)$ is a bounded kernel from E to $M(E)^\circ$. It is known that for each $f \in B(E)^+$ there exists a unique solution $V_t f \in B(E)^+$ to the evolution equation

$$V_t f(x) = P_t f(x) - \int_0^t \mathrm{d}s \int_E \left[\phi(y, V_s f(y)) - \varphi(y, V_s f)\right] P_{t-s}(x, \mathrm{d}y), \tag{1.3}$$

and there is a Markov semigroup $(Q_t)_{t \geq 0}$ on $M(E)$ such that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\left\{-\mu(V_t f)\right\}, \qquad t \geq 0, \mu \in M(E). \tag{1.4}$$

See e.g. Dynkin (1993). A Markov process $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ in $M(E)$ is called a (ξ, ϕ, φ)-*superprocess* (Dawson-Watanabe) if it has transition semigroup $(Q_t)_{t \geq 0}$. The (ξ, ϕ, φ)-superprocess is the continuous state approximation for a branching particle system where the offspring of a dying particle are displaced randomly into the entire space. In the special case $\varphi \equiv 0$, the (ξ, ϕ, φ)-superprocess becomes a (ξ, ϕ)-superprocess; see e.g. Fitzsimmons (1988).

In this paper, we show that minimal probability entrance laws for the (ξ, ϕ, φ)-superprocess X are in 1-1 correspondence with entrance laws for the process ξ. From this correspondence we deduce that an infinitely divisible probability entrance law for X is determined uniquely by an infinitely divisible probability measure on the space of entrance laws for ξ. Under an additional condition, a characterization is given for all entrance laws for the superprocess, generalizing the results of Dynkin (1989). We shall see that the description of entrance laws for the superprocess is important in the study of measure-valued immigration process.

For simplicity we shall only work with superprocesses having state space $M(E)$. There is no much change when $M(E)$ is replaced by the more general space $M_\rho(E) := \{$ Borel measures μ on E satisfying $\mu(\rho) < \infty\}$, where ρ is some bounded, strictly positive, continuous function on E. Indeed, most of our results can be translated into the $M_\rho(E)$ space case using the mapping $\mu(\mathrm{d}x) \mapsto \rho(x)^{-1}\mu(\mathrm{d}x)$.

2. Main results and proofs

Given a semigroup of bounded kernels $(T_t)_{t \geq 0}$ on E, we denote by $\mathcal{K}(T)$ the set of entrance laws $\kappa = (\kappa_t)_{t > 0}$ for $(T_t)_{t \geq 0}$ such that $\int_0^1 \kappa_s(E)\mathrm{d}s < \infty$. For $\kappa \in \mathcal{K}(P)$ we note

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \left[\phi(x, V_s f(y)) - \varphi(x, V_s f)\right] \kappa_{t-s}(\mathrm{d}x) \tag{2.1}$$

where $t > 0$ and $f \in B(E)^+$. Let $\mathcal{K}^1(Q)$ denote the set of probability entrance laws $K = (K_t)_{t>0}$ for $(Q_t)_{t\geq 0}$ satisfying

$$\int_0^1 \mathrm{d}s \int_{M(E)^\circ} \nu(E) K_s(\mathrm{d}\nu) < \infty, \tag{2.2}$$

and let $\mathcal{K}^1_m(Q)$ denote the subset of $\mathcal{K}^1(Q)$ comprising of all minimal (extremal) elements. See e.g. Dynkin (1978) or Sharpe (1988) for the definition of an entrance law.

Theorem 2.1. *There is a 1-1 correspondence between $K \in \mathcal{K}^1_m(Q)$ and $\kappa \in \mathcal{K}(P)$, which is given by*

$$\kappa_t(f) = \lim_{r\downarrow 0} \int_{M(E)} \nu(P_{t-r} f) K_r(\mathrm{d}\nu), \tag{2.3}$$

and

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\left\{-S_t(\kappa, f)\right\}. \tag{2.4}$$

Dynkin (1989) proved this in the case where $\varphi \equiv 0$ and $\phi(x,z) \equiv c(x)z^2$ but ξ was allowed to be non-homogeneous and X was allowed to take values in a space of σ-finite measures. The basic idea of our proof of Theorem 2.1 is the same with that of Dynkin (1989) and based on a lifting and projecting argument. The non-local branching mechanism causes some difficulties, for projecting an entrance law for the (ξ, ϕ, φ)-superprocess does not readily give an entrance law for the underlying process.

To give the proof of Theorem 2.1 we need some preparation. It is well-known that $(V_t)_{t\geq 0}$ has the canonical representation

$$V_t f(x) = \lambda_t(x, f) + \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) L_t(x, d\nu), \tag{2.5}$$

where $\lambda_t(x, \mathrm{d}y) \in M(E)$ and $\nu(E) L_t(x, \mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$. Define the kernels $(\Pi_t)_{t\geq 0}$ on E by

$$\Pi_t f(x) = \lambda_t(x, f) + \int_{M(E)^\circ} \nu(f) L_t(x, d\nu). \tag{2.6}$$

One can check that $(\Pi_t)_{t\geq 0}$ form a semigroup and

$$\int_{M(E)} \nu(f) Q_t(\mu, \mathrm{d}\nu) = \mu(\Pi_t f). \tag{2.7}$$

From the equation (1.3) we get

$$\Pi_t f(x) = P_t f(x) - \int_0^t \mathrm{d}s \int_E \left[b(y)\Pi_s f(y) - D(y, \Pi_s f)\right] P_{t-s}(x, \mathrm{d}y), \tag{2.8}$$

where $D(x,\cdot)$ is the bounded kernel on E defined by

$$D(x,f) = d(x,f) + \int_{M(E)^\circ} \nu(f) n(x,\mathrm{d}\nu).$$

Let $(P^b_t)_{t\geq 0}$ be the semigroup of bounded kernels on E defined by

$$P^b_t f(x) = \mathbf{P}_x f(\xi_t)\exp\left\{-\int_0^t b(\xi_s)\mathrm{d}s\right\}. \tag{2.9}$$

By a standard argument one sees that (2.8) is equivalent to

$$\varPi_t f(x) = P^b_t f(x) + \int_0^t \mathrm{d}s \int_E D(y,\varPi_s f) P^b_{t-s}(x,\mathrm{d}y). \tag{2.10}$$

Set $\bar\varphi = \sup_{x\in E} D(x,E)$. Applying Gronwall's inequality to (2.8) yields

$$\|\varPi_t f\| \leq \|f\| \exp\{(\|b\|+\bar\varphi)t\},$$

where $\|\cdot\|$ denotes the supreme norm. In particular the semigroup $(\varPi_t)_{t\geq 0}$ is locally bounded.

Lemma 2.1. *There is a 1-1 correspondence between $\kappa\in\mathcal{K}(P)$ and $\eta\in\mathcal{K}(\varPi)$, which is given by*

$$\eta_t(f) = \lim_{r\downarrow 0}\kappa_r(\varPi_{t-r}f) \quad \text{and} \quad \kappa_t(f) = \lim_{r\downarrow 0}\eta_r(P_{t-r}f). \tag{2.11}$$

Furthermore, if the two entrance laws κ and η are related by (2.11), then

$$\eta_t(f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \left[b(x)\varPi_s f(x) - D(x,\varPi_s f)\right]\kappa_{t-s}(\mathrm{d}x). \tag{2.12}$$

Proof. Suppose that $\kappa\in\mathcal{K}(P)$. By the equation (2.8) it is immediate that

$$\lim_{r\downarrow 0}\kappa_r(\varPi_{t-r}f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \left[b(x)\varPi_s f(x) - D(x,\varPi_s f)\right]\kappa_{t-s}(\mathrm{d}x).$$

Then the first equation in (2.11) defines an entrance law $\eta\in\mathcal{K}(\varPi)$ and (2.12) holds. The second equation in (2.11) follows by (2.12).

Conversely, suppose $\eta\in\mathcal{K}(\varPi)$. By (2.10) we see that $P^b_t f\leq \varPi_t f$ for $t\geq 0$ and $f\in B(E)^+$. Thus we can define an entrance law $\gamma\in\mathcal{K}(P^b)$ by $\gamma_t(f) = \lim_{r\downarrow 0}\eta_r(P^b_{t-r}f)$. It follows that $\kappa_t(f) = \lim_{r\downarrow 0}\gamma_r(P_{t-r}f)$ defines some $\kappa\in\mathcal{K}(P)$; see e.g. Li (1996b). Since clearly $\eta_r\geq\gamma_r$, for $t>0$ and $f\in B(E)^+$ we have,

$$\begin{aligned}\lim_{r\downarrow 0}|\eta_r(P_{t-r}f) - \gamma_r(P_{t-r}f)| &\leq \lim_{r\downarrow 0}\left|\eta_r(P^b_{t-r}f) - \gamma_r(P^b_{t-r}f)\right| \mathrm{e}^{\|b\|(t-r)}\\ &\leq \lim_{r\downarrow 0}\left|\eta_r(P^b_{t-r}f) - \gamma_t(f)\right| \mathrm{e}^{\|b\|t} = 0.\end{aligned}$$

Then the second relation in (2.11) holds. By (2.8), for $0 < r < t$,

$$\eta_t(f) = \eta_r(P_{t-r}f) - \int_0^{t-r} \mathrm{d}s \int_E \eta_r(\mathrm{d}x) \int_E [b(x)\Pi_s f(y) - D(y, \Pi_s f)]\, P_{t-r-s}(x, \mathrm{d}y).$$

Letting $r \downarrow 0$ in the above equality gives (2.12). $\square$

By (2.5) and (2.6), $V_t f \leq \Pi_t f$ for all $t \geq 0$ and $f \in B(E)^+$. Consequently, if $\eta \in \mathcal{K}(\Pi)$, $\eta_r(V_{t-r}f)$ is an increasing function of $r \in (0, t]$. Put

$$S_t^*(\eta, f) = \lim_{r\downarrow 0} \eta_r(V_{t-r}f), \qquad t > 0, f \in B(E)^+. \tag{2.13}$$

It is easy to check that, if the two entrance laws $\kappa \in \mathcal{K}(P)$ and $\eta \in \mathcal{K}(\Pi)$ are related by (2.11), then

$$S_t(\kappa, f) = S_t^*(\eta, f), \qquad t > 0, f \in B(E)^+. \tag{2.14}$$

With these in hands, we now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Step 1. For $K \in \mathcal{K}^1(Q)$ it follows from (2.7) that

$$\eta_t(f) = \int_{M(E)} \nu(f) K_t(\mathrm{d}\nu),$$

defines some $\eta \in \mathcal{K}(\Pi)$. Then we can define $\kappa \in \mathcal{K}(P)$ by the second equation in (2.11). Write $\eta = \pi K$ and $\kappa = pK$.

Step 2. Suppose $\kappa \in \mathcal{K}(P)$ and $\eta \in \mathcal{K}(\Pi)$ are related by (2.11). By (2.13) and (2.14) we have

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_{t-r}(\eta_r, \mathrm{d}\nu) = \exp\{-\eta_r(V_{t-r}f)\} \uparrow \exp\{-S_t^*(\eta, f)\} \tag{2.15}$$

as $r \downarrow 0$. It follows that (2.4) really defines a $K \in \mathcal{K}^1(Q)$. Write $K = l\kappa = \lambda\eta$. By (2.1) and (2.4) on may check that

$$\int_{M(E)} \nu(f) K_t(\mathrm{d}\nu) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E [b(x)\Pi_s f(x) - D(x, \Pi_s f)]\, \kappa_{t-s}(\mathrm{d}x).$$

Therefore $\pi K = \eta$ and $pK = \kappa$ by Lemma 2.1.

Step 3. We claim that $\lambda\pi K = K$ for all $K \in \mathcal{K}_m^1(Q)$. Indeed if $K \in \mathcal{K}_m^1(Q)$, then there is a probability measure $\mathbf{Q}_K$ on $M(E)^{(0,\infty)}$ under which the coordinate process $\{w_t : t > 0\}$ is a Markov process with one-dimensional distributions $(K_t)_{t>0}$ and semigroup $(Q_t)_{t\geq 0}$. Since K is minimal, $\mathbf{Q}_K$-a.s.,

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \lim_{\mathrm{rat}.r\downarrow 0} \exp\{-w_r(V_{t-r}f)\}, \tag{2.16}$$

where "rat. $r \downarrow 0$" means "$r \to 0$ decreasingly along the rational". See Dynkin (1978). Clearly $\mathbf{Q}_K$-a.s.,

$$w_r(V_{t-r}f) \leq w_r(\Pi_{t-r}f) = \mathbf{Q}_K\left\{w_t(f)|w_s : 0 < s \leq r\right\}.$$

Then the family of random variables $\{w_r(V_{t-r}f) : 0 < \text{rat. } r \leq t\}$ is uniformly $\mathbf{Q}_K$-integrable. By (2.16) we have

$$-\log \int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \lim_{\text{rat.}r\downarrow 0} \mathbf{Q}_K\left\{w_r(V_{t-r}f)\right\} = S_t^*(\pi K, f),$$

as claimed.

Step 4. Finally we show $\lambda\eta \in \mathcal{K}_m^1(Q)$ for all $\eta \in \mathcal{K}(\Pi)$. Since $\lambda\eta \in \mathcal{K}^1(Q)$, there is a probability measure F on $\mathcal{K}_m^1(Q)$ such that

$$\lambda\eta_t = \int_{\mathcal{K}_m^1(Q)} H_t\, F(\mathrm{d}H).$$

See e.g. Dynkin (1978). Let G be the image of F under mapping $\pi : \mathcal{K}_m^1(Q) \to \mathcal{K}(\Pi)$. By the results proved in steps 2 and 3 it follows that

$$\exp\left\{-S_t^*(\eta, f)\right\} = \int_{\mathcal{K}(\Pi)} \exp\left\{-S_t^*(\gamma, f)\right\} G(\mathrm{d}\gamma), \tag{2.17}$$

and

$$\eta_t = \int_{\mathcal{K}(\Pi)} \gamma_t\, G(\mathrm{d}\gamma). \tag{2.18}$$

Since e^{-u} is a strictly convex function of $u \geq 0$, (2.17) and (2.18) imply that G is the unit mass concentrated at γ; hence F is the unit mass at $\lambda\eta$, yielding $\lambda\eta \in \mathcal{K}_m^1(Q)$. $\square$

Based on Theorem 2.1 we may give a description of the infinitely divisible entrance laws in $\mathcal{K}^1(Q)$ as follows. (See Li (1996b) for the proof of the next theorem in the special case where $\varphi \equiv 0$.)

Theorem 2.2. *An entrance law $K \in \mathcal{K}^1(Q)$ is infinitely divisible if and only if its Laplace functional has the representation*

$$\begin{aligned}&\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) \\ &= \exp\left\{ - S_t(\kappa, f) - \int_{\mathcal{K}(P)} \left(1 - \exp\left\{-S_t(\eta, f)\right\}\right) F(\mathrm{d}\eta)\right\},\end{aligned} \tag{2.19}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying

$$\int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) F(\mathrm{d}\eta) < \infty. \tag{2.20}$$

Let $(Q_t^\circ)_{t\geq 0}$ denote the restriction of $(Q_t)_{t\geq 0}$ to $M(E)^\circ$ and $\mathcal{K}(Q^\circ)$ the set of entrance laws $(H_t)_{t>0}$ for the semigroup $(Q_t^\circ)_{t\geq 0}$ satisfying (2.2). We introduce the condition

$$\begin{aligned}&(V_t)_{t\geq 0} \text{ has the representation (2.5) with}\\ &\lambda_t(x,E)=0 \text{ for all } t>0 \text{ and } x\in E.\end{aligned} \tag{2.21}$$

Note that (2.21) holds if there is some constant $a>0$ such that

$$\int_a^\infty \big[\sup_{x\in E}|\phi(x,z)^{-1}|\big]\mathrm{d}z<\infty; \tag{2.22}$$

see Dawson (1993; pp195-196). It is easy to check that, when (2.21) holds, an entrance law $K\in\mathcal{K}^1(Q)$ is infinitely divisible if and only if

$$\int_{M(E)} \mathrm{e}^{-\nu(f)}K_t(\mathrm{d}\nu)=\exp\bigg\{-\int_{M(E)^\circ}\Big(1-\mathrm{e}^{-\nu(f)}\Big)H_t(\mathrm{d}\nu)\bigg\} \tag{2.23}$$

for some $H\in\mathcal{K}(Q^\circ)$. Then we have

Theorem 2.3. *Assume (2.21) holds. Then $H\in\mathcal{K}(Q^\circ)$ if and only if*

$$\int_{M(E)^\circ}\Big(1-\mathrm{e}^{-\nu(f)}\Big)H_t(\mathrm{d}\nu)=S_t(\kappa,f)+\int_{\mathcal{K}(P)}\left(1-\exp\left\{-S_t(\eta,f)\right\}\right)F(\mathrm{d}\eta) \tag{2.24}$$

where $\kappa\in\mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying (2.20).

Let $\mathcal{K}_m(Q^\circ)$ denote the set of minimal elements of $\mathcal{K}(Q^\circ)$. Let $L\kappa$ denote the entrance law for $(Q_t^\circ)_{t\geq 0}$ defined by

$$\int_{M(E)^\circ}\Big(1-\mathrm{e}^{-\nu(f)}\Big)L\kappa_t(\mathrm{d}\nu)=S_t(\kappa,f). \tag{2.25}$$

Since $S_t(\kappa,f)$ is linear in κ, we conclude from Theorem 2.3 that $\mathcal{K}_m(Q^\circ)$ consists of two parts: (i) $ql\kappa$ for all $0<q<\infty$ and all $\kappa\in\mathcal{K}(P)$, and (ii) $L\kappa$ with κ being extremal elements of $\mathcal{K}(P)$. These generalize the results of Dynkin (1989).

3. Application to immigration processes

Let us give an application of the above results to the study of immigration structures associated with the (ξ,ϕ,φ)-superprocess. Suppose that $(N_t)_{t\geq 0}$ is a family of probability measures on $M(E)$. We call $(N_t)_{t\geq 0}$ a *skew convolution semigroup* associated with the (ξ,ϕ,φ)-superprocess X or its semigroup $(Q_t)_{t\geq 0}$ if

$$N_{r+t}=(N_rQ_t)*N_t,\qquad r,t\geq 0, \tag{3.1}$$

where "$*$" denotes the convolution operation. The relation (3.1) holds if and only if

$$Q_t^N(\mu,\cdot) := Q_t(\mu,\cdot) * N_t, \qquad t\geq 0,\ \mu\in M(E), \tag{3.2}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. If Y is a Markov process in $M(E)$ having transition semigroup $(Q_t^N)_{t\geq 0}$, we call it an *immigration process* associated with X. See Li (1996ab) for the intuitive meaning of the immigration process. By the results of Li (1996a), $(N_t)_{t\geq 0}$ form a skew convolution semigroup if and only if there is an infinitely divisible probability entrance law $(K_t)_{t>0}$ for $(Q_t)_{t\geq 0}$ such that

$$\log\int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \int_0^t \Big[\log\int_{M(E)} \mathrm{e}^{-\nu(f)} K_s(\mathrm{d}\nu)\Big]\mathrm{d}s \tag{3.3}$$

for all $t\geq 0$ and $f\in B(E)^+$.Therefore, the immigration structures associated with the (ξ,ϕ,φ)-superprocess can be characterized by its infinitely divisible probability entrance laws. Combining (2.23) with (3.3), we get the following

Theorem 3.1. *Suppose that (2.21) holds and $(N_t)_{t\geq 0}$ is a family of probability measures on $M(E)$ satisfying*

$$\int_{M(E)} \nu(1) N_t(\mathrm{d}\nu) < \infty \quad \textit{for all } t\geq 0. \tag{3.4}$$

Then $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with the (ξ,ϕ,φ)-superprocess if and only if its Laplace functional has the representation

$$\begin{aligned}\int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \exp\bigg\{ &-\int_0^t S_u(\kappa,f)\mathrm{d}u \\ &-\int_0^t \mathrm{d}u\int_{\mathcal{K}(P)} \left(1-\exp\left\{-S_u(\eta,f)\right\}\right) F(\mathrm{d}\eta)\bigg\}\end{aligned} \tag{3.5}$$

where $\kappa\in\mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying (2.20).

An immediate consequence of this theorem is a construction of the immigration process by a Poisson system of measure-valued paths. Let $W_0(M(E))$ denote the space of all right continuous paths $\{w_t : t>0\}$ from $(0,\infty)$ to $M(E)$. Let $(\mathcal{G}^\circ,\mathcal{G}_t^\circ)$ denote the natural σ-algebras on $W_0(M(E))$. Suppose that $H\in\mathcal{K}(Q^\circ)$ is given by (2.24). By the theory of Markov processes, there is a σ-finite measure $\mathbf{Q}_H$ on $(W_0(M(E)),\mathcal{G}^\circ)$ under which $\{w_t : t>0\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$ and one dimensional distributions $(H_t)_{t>0}$. Suppose that $N(\mathrm{d}s,\mathrm{d}w)$ is a Poisson random measure on $[0,\infty)\times W_0(M(E))$ with intensity $\mathrm{d}s\times\mathbf{Q}_H(\mathrm{d}w)$. Let

$$Y_t = \int_{[0,t]}\int_{W_0(M(E))} w_{t-s}\, N(\mathrm{d}s,\mathrm{d}w), \qquad t\geq 0, \tag{3.6}$$

where $w_0 = 0$ by convention. It is easy to check that $\{Y_t : t \geq 0\}$ is an immigration process corresponding to the skew convolution semigroup $(N_t)_{t\geq 0}$ given by (3.5). The construction (3.6) of the immigration process explains the role of the entrance law $H \in \mathcal{K}(Q^\circ)$ in the phenomenon and gives a probabilistic interpretation of the equation (3.5).

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