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ABSOLUTE CONTINUITY OF MEASURE BRANCHING PROCESSES WITH INTERACTION*

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Abstract. We study the state properties of the measure branching process over $\mathbb{R}$ with mean field interaction constructed by Méléard and Roelly (1992, 1993) and Métivier (1987). It is proved that, under natural hypotheses, the process is absolutely continuous with respect to the Lebesgue measure and the density process has a continuous version which satisfies a stochastic partial differential equation.

Key words: measure branching process; mean field interaction; stochastic partial differential equation

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1. Introduction

Let $M(\mathbb{R}^d)$ denote the totality of finite Borel measures on $\mathbb{R}^d$ endowed with the weak convergence topology. Let $C(\mathbb{R}^d)$ be the space of all bounded continuous functions on $\mathbb{R}^d$ with the supreme norm $\|\cdot\|$, and $C_0(\mathbb{R}^d)$ the set of functions in $C(\mathbb{R}^d)$ vanishing at infinity. For $\mu \in M(\mathbb{R}^d)$ and $f \in C(\mathbb{R}^d)$ write $\mu(f)$ for $\int f \mathrm{d}\mu$. Let $C([0,\infty), M(\mathbb{R}^d))$ be the space of $M(\mathbb{R}^d)$-valued continuous paths $\{w_t : t \geq 0\}$ with the coordinate process denoted by $X_t(w) = w_t$. Let $\{P_t(\mu) : \mu \in M(\mathbb{R}^d)\}$ be Feller semigroups on $C(\mathbb{R}^d)$ with generators $\{A(\mu) : \mu \in M(\mathbb{R}^d)\}$. Assume that $\{A(\mu) : \mu \in M(\mathbb{R}^d)\}$ have domains that all contain a vector space $\mathcal{D}$ which is dense in $C_0(\mathbb{R}^d)$ and independent of μ. Assume that the constant functions belong to $\mathcal{D}$ and $A(\mu)1 = 0$ for each $\mu \in M(\mathbb{R}^d)$. Furthermore, we assume that, for each $f \in \mathcal{D}$,

(A1) there is a constant $K(f) > 0$ such that $\|A(\mu)f\| \leq K(f)\mu(1)$;

(A2) $\mu(A(\mu)f)$ is continuous in $\mu \in M(\mathbb{R}^d)$.

We fix two bounded, continuous functions $c = c(\mu, x) \geq 0$ and $b = b(\mu, x)$ on the product space $M(\mathbb{R}^d) \times \mathbb{R}^d$. It follows from the construction in Méléard and Roelly

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(1992, 1993) and Métivier (1987) that for each $\mu \in M(I\!R^d)$ there is a probability measure $\mathbf{Q}_\mu$ on $C([0,\infty), M(I\!R^d))$ such that, for any $f \in \mathcal{D}$,

$$M_t(f) := X_t(f) - \mu(f) - \int_0^t X_s\big(A(X_s)f + b(X_s)f\big)\mathrm{d}s, \quad t \geq 0, \tag{1.1}$$

is a $\mathbf{Q}_\mu$-martingale starting at zero with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t \int_{I\!R^d} c(X_s, x) f(x)^2 X_s(\mathrm{d}x)\mathrm{d}s, \quad t \geq 0. \tag{1.2}$$

We shall call $\{\mathbf{Q}_\mu : \mu \in M(I\!R^d)\}$ a measure branching process with mean field interaction, or simply an interacting (Dawson-Watanabe) superprocess in keeping with Méléard and Roelly (1992, 1993). The term "mean field interaction" refers to the fact that the migrating and branching of each particle is influence by the entire population; see Dawson (1993) and Perkins (1992) for related models.

If A, b and c are all independent of μ, then $\{\mathbf{Q}_\mu : \mu \in M(I\!R^d)\}$ is uniquely determined by (1.1) and (1.2), which is the classical non-interacting superprocess. In that case, the Laplace functional of the process can be given as

$$\mathbf{Q}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \quad f \in C(I\!R^d)^+, \tag{1.3}$$

where $V_t f$ is the mild solution of the evolution equation

$$\begin{aligned} &\frac{\partial}{\partial t} V_t f(x) = A V_t f(x) - \frac{1}{2} c(x) V_t f(x)^2 + b(x) V_t f(x), \\ &V_0 f(x) = f(x). \end{aligned} \tag{1.4}$$

In general, the uniqueness of solutions to the martingale problem (1.1) and (1.2) is still unknown and the Laplace functional of $\{\mathbf{Q}_\mu : \mu \in M(I\!R^d)\}$ cannot be expressed explicitly. One important property implied by (1.3) is the multiplicative property of the family $\{\mathbf{Q}_\mu : \mu \in M(I\!R^d)\}$, which has lead to many deep results concerning the trajectory structures of the non-interacting superprocess. The loss of this property in the general situation makes the study of the interacting superprocess much more difficult.

The following results were proved in Méléard and Roelly (1993): If the underlying motion is a symmetric stable process with index α $(0 < \alpha \leq 2)$ independent of μ, then for each $t > 0$ the Hausdorff dimension of the Borel support $\mathrm{supp}(X_t)$ is almost surely not less than $d \wedge \alpha$. Only under the additional condition $c(\mu, x) \equiv \mathrm{const}$, it was proved that the Hausdorff dimension of $\mathrm{supp}(X_t) = d \wedge \alpha$ for all $t > 0$ almost surely. Therefore, compared with what we have known about the non-interacting superprocess, the interacting one is much less understood.

It is known that when $d = 1$, the non-interacting superprocess is absolutely continuous with respect to the Lebesgue measure on $I\!R$ for a large class of admissible generators

A; see Konno and Shiga (1988) and Reimers (1989). The same result for the interacting superprocess was conjectured by Méléard and Roelly (1992; p256). In the special case where A is independent of $\mu \in M(I\!R^d)$, the problem has been studied in Zhao (1997), where a measurable density $\{X(t,x) : t > 0, x \in I\!R\}$ was obtained as the limit

$$X(t,x) = \lim_{r \downarrow 0} \int_{I\!R} X_t(\mathrm{d}z) p_r(z,x) \tag{1.5}$$

with $p_r(z,x)$ denoting the transition density of A.

In this paper we establish the absolute continuity of the interacting superprocess in a typical situation where the underlying motion and the branching mechanism are both dependent on $\mu \in M(I\!R^d)$. The result is proved here using a Fubini's theorem of stochastic integrals with respect to a martingale measure induced by (1.1) and (1.2). In the special case where A is independent of $\mu \in M(I\!R^d)$, we show that the density process $\{X(t,x) : t > 0, x \in I\!R\}$ has a continuous version and satisfies a stochastic partial differential equation. One essential step of the proof is some moment estimates for the superprocess. In the non-interacting case, Konno and Shiga (1988) proved those by using the log-Laplace equation, which is not available in our situation. We obtain the estimates from the martingale characterization by an induction argument. This paper is in fact an extended version of the unpublished manuscript Li (1995), where we considered the special case $A = \Delta$ and $b = 0$. A combination of the techniques of Li (1995) and Zhao (1997) has been given by Liang (1996).

2. Absolute continuity of the superprocess

We first give a moment estimate for the interacting superprocess which will be used in the proofs of the main results. Let $B \geq 1$ be a fixed common upper bound for the functions $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$. Then we have

Proposition 2.1. *There is a family of locally bounded functions $F_n = F_n(t)$ on $[0,\infty)$ such that*

$$\mathbf{Q}_\mu \left\{ X_t(1)^n \right\} \leq F_n(t) \sum_{k=1}^{n} \mu(1)^k \tag{2.1}$$

for $\mu \in M(I\!R)$ and $n = 1, 2, \cdots$.

Proof. Since $\{X_t(1) : t \geq 0\}$ is $\mathbf{Q}_\mu$-a.s. continuous, we may take an increasing sequence of optional times $\tau_k \to \infty$ $(k \to \infty)$ such that $\mathbf{Q}_\mu$-a.s. $\sup\{X_t(1) : t \leq \tau_k\} \leq k$. By (1.1), (1.2) and Itô's formula,

$$\begin{aligned} X_{t\wedge\tau_k}(1)^n - \mu(1)^n - n \int_0^{t\wedge\tau_k} & X_s(1)^{n-1} X_s(b(X_s)) \mathrm{d}s \\ - \frac{n(n-1)}{2} \int_0^{t\wedge\tau_k} & X_s(1)^{n-2} X_s(c(X_s)) \mathrm{d}s \end{aligned}$$

is a $\mathbf{Q}_\mu$-martingale. Then we have

$$\begin{aligned}\mathbf{Q}_\mu\{X_{t\wedge\tau_k}(1)^n\} \le& \mu(1)^n + nB\int_0^t \mathbf{Q}_\mu\{X_{s\wedge\tau_k}(1)^n\}\mathrm{d}s \\ &+\frac{n(n-1)}{2}B\int_0^t \mathbf{Q}_\mu\{X_{s\wedge\tau_k}(1)^{n-1}\}\mathrm{d}s < \infty.\end{aligned}$$

Applying Gronwall's inequality gives

$$\mathbf{Q}_\mu\{X_{t\wedge\tau_k}(1)^n\} \le \mathrm{e}^{nBt}\mu(1)^n + \frac{n(n-1)}{2}B\mathrm{e}^{nBt}\int_0^t \mathbf{Q}_\mu\{X_{s\wedge\tau_k}(1)^{n-1}\}\mathrm{d}s.$$

Using this inductively we get

$$\mathbf{Q}_\mu\left\{X_{t\wedge\tau_k}(1)^n\right\} \le F_n(t)\sum_{k=0}^n \mu(1)^k,$$

where the F_n are locally bounded functions on $[0,\infty)$. The desired inequality follows by Fatou's lemma. $\square$

In the sequel of the paper we shall always assume $d=1$. Suppose that there is a continuous function $p_t(\sigma;x,y)$ of $(t,\sigma,x,y)\in(0,\infty)\times I\!R^3$ which is twice continuously differentiable in $\sigma\in I\!R$ such that

$$P_t(\mu)f(x) = \int_{I\!R} p_t(\mu(g);x,y)f(y)\mathrm{d}y, \tag{2.2}$$

where $g\in\mathcal{D}$ is fixed. Set $P_t(\mu)f(x)=0$ for $t<0$ and $p_t(\sigma;x,y)=0$ for $t\le 0$. We introduce the notations

$$\begin{aligned}P_t'(\mu)f(x) &= \int_{I\!R} p_t'(\mu(g);x,y)f(y)\mathrm{d}y,\\ P_t''(\mu)f(x) &= \int_{I\!R} p_t''(\mu(g);x,y)f(y)\mathrm{d}y,\end{aligned}$$

where $p_t'(\sigma;x,y) = (\partial/\partial\sigma)p_t(\sigma;x,y)$ and $p_t''(\sigma;x,y) = (\partial^2/\partial\sigma^2)p_t(\sigma;x,y)$. Assume further that, for each $f\in\mathcal{D}$,

(A3) $P_t(\mu)f$ is strongly continuous in $(t,\mu)\in[0,\infty)\times M(I\!R)$;

(A4) $A(\mu)P_t(\mu)f = P_t(\mu)A(\mu)f$ is strongly continuous in $(t,\mu)\in[0,\infty)\times M(I\!R)$;

(A5) $P_t'(\mu)f$ is strongly continuous in $(t,\mu)\in[0,\infty)\times M(I\!R)$;

(A6) $P_t''(\mu)f$ is strongly continuous in $(t,\mu)\in[0,\infty)\times M(I\!R)$.

Of course, (A4) implies (A2). Since (1.1) is linear in $f\in\mathcal{D}$, one can extend the system $\{M_t(f): f\in\mathcal{D}, t\ge 0\}$ to a continuous orthogonal martingale measure $\{M_t(B): B\in\mathcal{B}(I\!R), t\ge 0\}$ with covariant measure $c(X_s,x)X_s(\mathrm{d}x)\mathrm{d}s$ in the sense of Walsh (1986). See also Méléard and Roelly (1993). Let $M(\mathrm{d}s,\mathrm{d}x)$ denote the stochastic integral with respect to this martingale measure.

Proposition 2.2. *Under the assumptions (A1) – (A6), for any $t > r \geq 0$ and $f \in C(I\!R)$, we have $\mathbf{Q}_\mu$-a.s.,*

$$\begin{aligned} X_t(f) =& X_r(P_{t-r}(X_r)f) + \int_r^t \int_{I\!R} P_{t-s}(X_s)f(x)M(\mathrm{d}s, \mathrm{d}x) \\ &+ \int_r^t X_s(b(X_s)P_{t-s}(X_s)f)\mathrm{d}s + \int_r^t X_s(P'_{t-s}(X_s)f)\mathrm{d}M_s(g) \\ &+ \int_r^t X_s(P'_{t-s}(X_s)f)X_s(A(X_s)g + b(X_s)g)\mathrm{d}s \\ &+ \frac{1}{2}\int_r^t X_s(P''_{t-s}(X_s)f)X_s(c(X_s)g^2)\mathrm{d}s. \end{aligned} \tag{2.3}$$

Proof. We may assume $f \in \mathcal{D}$ in this proof since the extension of (2.3) to a general $f \in C(I\!R)$ is easy. Suppose that $r = t_0 < t_1 < \cdots < t_n = t$ is a partition of $[r, t]$. We have

$$\begin{aligned} X_t(f) =& X_r(P_{t-r}(X_r)f) + \sum_{i=1}^n X_{t_i}(P_{t-t_i}(X_{t_i})f - P_{t-t_{i-1}}(X_{t_i})f) \\ &+ \sum_{i=1}^n \big[X_{t_i}(P_{t-t_{i-1}}(X_{t_i})f) - X_{t_{i-1}}(P_{t-t_{i-1}}(X_{t_{i-1}})f)\big]. \end{aligned} \tag{2.4}$$

Let $\delta_n = \max\{|t_i - t_{i-1}| : i = 1, \cdots, n\}$. We choose the partition $r = t_0 < t_1 < \cdots < t_n = t$ in a way such that $\delta_n \to 0$ as $n \to \infty$. By (A4),

$$\begin{aligned} &\lim_{n\to\infty} \sum_{i=1}^n X_{t_i}(P_{t-t_i}(X_{t_i})f - P_{t-t_{i-1}}(X_{t_i})f) \\ =& -\lim_{n\to\infty} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_{t_i}(A(X_{t_i})P_{t-s}(X_{t_i})f)\mathrm{d}s \\ =& -\int_r^t X_s(A(X_s)P_{t-s}(X_s)f)\mathrm{d}s. \end{aligned} \tag{2.5}$$

Applying (1.1) term by term we get

$$\begin{aligned} &\sum_{i=1}^n \big[X_{t_i}(P_{t-t_{i-1}}(X_{t_i})f) - X_{t_{i-1}}(P_{t-t_{i-1}}(X_{t_i})f)\big] \\ =& \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \int_{I\!R} p_{t-t_{i-1}}(X_{t_i})f(x)M(\mathrm{d}s, \mathrm{d}x) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s(A(X_{t_i})P_{t-t_{i-1}}(X_{t_i})f)\mathrm{d}s \\ &+ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s(b(X_s)P_{t-t_{i-1}}(X_{t_i})f)\mathrm{d}s. \end{aligned}$$

By the hypothesis (A3) and (A4), as $n \to \infty$ the above value converges to

$$\begin{aligned}\int_r^t \int_{I\!R} P_{t-s}(X_s)f(x)M(\mathrm{d}s,\mathrm{d}x) + \int_r^t X_s(A(X_s)P_{t-s}(X_s)f)\mathrm{d}s \\ + \int_r^t X_s(b(X_s)P_{t-s}(X_s)f)\mathrm{d}s.\end{aligned} \tag{2.6}$$

Using (1.1), (1.2) and Itô's formula we have

$$\begin{aligned}&\sum_{i=1}^n \left[X_{t_{i-1}}(P_{t-t_{i-1}}(X_{t_i})f) - X_{t_{i-1}}(P_{t-t_{i-1}}(X_{t_{i-1}})f)\right] \\ =&\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_{t_{i-1}}(P'_{t-t_{i-1}}(X_s)f)\mathrm{d}M_s(g) \\ &+ \sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_{t_{i-1}}(P'_{t-t_{i-1}}(X_s)f)X_s(A(X_s)g + b(X_s)g)\mathrm{d}s \\ &+ \frac{1}{2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_{t_{i-1}}(P''_{t-t_{i-1}}(X_s)f)X_s(c(X_s)g^2)\mathrm{d}s.\end{aligned}$$

By (A5) and (A6), as $n \to \infty$ this converges to

$$\begin{aligned}\int_r^t X_s(P'_{t-s}(X_s)f)\mathrm{d}M_s(g) + \int_r^t X_s(P'_{t-s}(X_s)f)X_s(A(X_s)g + b(X_s)g)\mathrm{d}s \\ + \frac{1}{2}\int_r^t X_s(P''_{t-s}(X_s)f)X_s(c(X_s)g^2)\mathrm{d}s\end{aligned} \tag{2.7}$$

Combining (2.4) – (2.7) in the above we get (2.3). $\square$

Theorem 2.3. *Suppose that (A1) – (A6) hold. If there exist* $0 < \delta, \theta < 1$ *such that*

$$\sup\{p_t(\sigma;x,y) + p'_t(\sigma;x,y)^2 + p''_t(\sigma;x,y) : (\sigma,x,y) \in I\!R^3\} \le t^{-\theta} \tag{2.8}$$

for all $0 < t < \delta$, *then there exists a two-parameter process* $\{X(t,x) : t > 0, x \in I\!R\}$ *such that* $\mathbf{Q}_\mu\{X_t(\mathrm{d}x)$ *has density* $X(t,x)$ *for all* $t > 0\} = 1$ *for each* $\mu \in M(I\!R)$.

Proof. We fix $f \in C(I\!R)$ with compact support. Using Burkholder-Davis-Gundy's inequality and condition (2.8),

$$\begin{aligned}&\int_{I\!R} \mathbf{Q}_\mu\Big\{\int_0^t \int_{I\!R} p_{t-s}(X_s(g);z,x)^2 c(X_s)X_s(\mathrm{d}z)\Big\} f(x)\mathrm{d}x \\ \le&\mathrm{const}\cdot \int_0^t \frac{1}{(t-s)^\theta}\mathbf{Q}_\mu\big\{X_s(P_{t-s}(X_s)f)\big\}\mathrm{d}s < \infty.\end{aligned}$$

By Theorem 2.6 in Walsh (1986), for every $t > 0$ we have, $\mathbf{Q}_\mu$-a.s.,

$$\int_0^t \int_{I\!R} P_{t-s}(X_s) f(x) M(\mathrm{d}s, \mathrm{d}x) = \int_{I\!R} f(x) Z(t,x) \mathrm{d}x, \tag{2.9}$$

where

$$Z(t,x) = \int_0^t \int_{I\!R} p_{t-s}(X_s(g); z, x) M(\mathrm{d}s, \mathrm{d}z), \quad x \in I\!R \setminus N_1(t,f),$$

for some Lebesgue null set $N_1(t,f) \subset I\!R$. In the same way, since

$$\begin{aligned} &\int_{I\!R} \mathbf{Q}_\mu \Big\{ \int_0^t X_s(p'_{t-s}(X_s(g); \cdot, x))^2 X_s(cg^2) \mathrm{d}s \Big\} f(x) \mathrm{d}x \\ \leq &\mathrm{const} \cdot \int_0^t \frac{1}{(t-s)^\theta} \mathbf{Q}_\mu \{ X_s(1)^2 X_s(cg^2) \} \mathrm{d}s \cdot \int_{I\!R} f(x) \mathrm{d}x < \infty \end{aligned}$$

by (2.8) and Proposition 2.1, it follows that, $\mathbf{Q}_\mu$-a.s.,

$$\int_0^t X_s(P'_{t-s}(X_s) f) \mathrm{d}M_s(g) = \int_{I\!R} f(x) U(t,x) \mathrm{d}x, \tag{2.10}$$

where $U(t,x)$ defined by

$$U(t,x) = \int_0^t \mathrm{d}M_s(g) \int_{I\!R} p'_{t-s}(X_s(g); z, x) X_s(\mathrm{d}z), \quad x \in I\!R \setminus N_2(t,f),$$

for another Lebesgue null set $N_2(t,f) \subset I\!R$. Under condition (2.8) the integrals

$$\begin{aligned} Y_1(t,x) &= \int_0^t \mathrm{d}s \int_{I\!R} b(X_s, z) p_{t-s}(X_s(g); z, x) X_s(\mathrm{d}z), \\ Y_2(t,x) &= \int_0^t \mathrm{d}s \int_{I\!R} p'_{t-s}(X_s(g); z, x) X_s(A(X_s)g + b(X_s)g) X_s(\mathrm{d}z), \\ Y_3(t,x) &= \frac{1}{2} \int_0^t \mathrm{d}s \int_{I\!R} p''_{t-s}(X_s(g); z, x) X_s(c(X_s)g^2) X_s(\mathrm{d}z). \end{aligned}$$

are well-defined. It is not difficult to get versions of $Z(t,x)$ and $U(t,x)$ such that (2.9) and (2.10) hold for all f in a countable dense subset of $C(I\!R)$. Using those to define

$$X(t,x) = \int_{I\!R} p_t(\mu; z, x) \mu(\mathrm{d}z) + Z(t,x) + U(t,x) + \sum_{i=1}^3 Y_i(t,x).$$

The assertion follows from (2.9), (2.10) and Fubini's lemma. $\square$

Example 2.4. We remark that (A1) – (A6) and (2.8) are satisfied when $A(\mu) = \Delta/2 + \mu(g)(\mathrm{d}/\mathrm{d}x)$. In this case,

$$p_t(\sigma; x, y) = \frac{1}{\sqrt{2\pi t}} \exp\Big\{ - \frac{(y - x - \sigma t)^2}{2t} \Big\}.$$

It is easy to check that (2.8) is satisfied with $\delta = 1$ and $\theta = 1/2$.

A natural generalization of the situation considered above is as follows. Suppose that $g_1, \cdots, g_k \in \mathcal{D}$ are fixed functions and

$$P_t(\mu)f(x) = \int_{I\!R} p_t(\mu(g_1), \cdots, \mu(g_k); x, y) f(y) \mathrm{d}y, \tag{2.11}$$

where $p_t(\sigma_1, \cdots, \sigma_k; x, y)$ is a continuous function of $(t, \sigma_1, \cdots, \sigma_k; x, y)$ twice continuously differentiable in $(\sigma_1, \cdots, \sigma_k) \in I\!R^k$. Put

$$P_t^j(\mu)f(x) = \int_{I\!R} p_t^j(\mu(g_1), \cdots, \mu(g_k); x, y) f(y) \mathrm{d}y,$$
$$P_t^{jl}(\mu)f(x) = \int_{I\!R} p_t^{jl}(\mu(g_1), \cdots, \mu(g_k); x, y) f(y) \mathrm{d}y,$$

where

$$p_t^j(\sigma_1, \cdots, \sigma_k; x, y) = (\partial/\partial\sigma_j) p_t(\sigma_1, \cdots, \sigma_k; x, y)$$

and

$$p_t^{jl}(\sigma_1, \cdots, \sigma_k; x, y) = (\partial^2/\partial\sigma_j\partial\sigma_l) p_t(\sigma_1, \cdots, \sigma_k; x, y).$$

Assume that, for each $f \in \mathcal{D}$,

(A5′) $P_t^j(\mu)f$ is strongly continuous in $(t, \mu) \in [0, \infty) \times M(I\!R)$;

(A6′) $P_t^{jl}(\mu)f$ is strongly continuous in $(t, \mu) \in [0, \infty) \times M(I\!R)$.

By a similar arguments as the above it can be proved that for any $t > r \geq 0$ and $f \in C(I\!R)$ we have

$$\begin{aligned} X_t(f) =& X_r(P_{t-r}(X_r)f) + \int_r^t \int_{I\!R} P_{t-s}(X_s) f(x) M(\mathrm{d}s, \mathrm{d}x) \\ &+ \int_r^t X_s(b(X_s)P_{t-s}(X_s)f)\mathrm{d}s + \sum_{j=1}^k \int_r^t X_s(P_{t-s}^j(X_s)f)\mathrm{d}M_s(g_j) \\ &+ \sum_{j=1}^k \int_r^t X_s(P_{t-s}^j(X_s)f) X_s(A(X_s)g_j + b(X_s)g_j)\mathrm{d}s \\ &+ \frac{1}{2} \sum_{j,l=1}^k \int_r^t X_s(P_{t-s}^{jl}(X_s)f) X_s(c(X_s)g_j g_l)\mathrm{d}s. \end{aligned}$$

Assume further that there exist $0 < \delta, \theta < 1$ such that

$$\sup\{p_t(\sigma_1, \cdots, \sigma_k; x, y) + p_t^i(\sigma_1, \cdots, \sigma_k; x, y)^2 + p_t^{jl}(\sigma_1, \cdots, \sigma_k; x, y)\} \le t^{-\theta}$$

for all $0 < t < \delta$ and $i, j, l = 1, \cdots, k$, where the suprum is taken over $(\sigma_1, \cdots, \sigma_k; x, y) \in I\!R^{k+2}$. Then there exists a two-parameter process $\{X(t, x) : t > 0, x \in I\!R\}$ such that $\mathbf{Q}_\mu\{X_t(\mathrm{d}x)$ has density $X(t, x)$ for all $t > 0\} = 1$ for each $\mu \in M(I\!R)$.

3. A stochastic partial differential equation

In this section, we consider the special case of the interacting superprocess where the underlying motion is given by a Feller semigroup $(P_t)_{t\ge0}$ independent of $\mu \in M(I\!R)$ and the interaction only appears in the branching coefficients b and c. In this case we can give a sharper estimate for the moments of the superprocess. Let $(T_t)_{\ge0}$ be the semigroup of operators on $C(I\!R)$ defined by $T_t f(x) = \mathrm{e}^{Bt} P_t f(x)$.

Proposition 3.1. *In the situation described as above, define inductively $A_0(t, f) = 1$ and*

$$A_k(t, f) = \int_0^t \|T_{t-s} f\| A_{k-1}(s, T_{t-s} f)\mathrm{d}s.$$

Then we have

$$\mathbf{Q}_\mu \{X_t(f)^n\} \le \frac{(n!)^2}{2^n} \sum_{k=0}^{n-1} B^{k+1} A_k(t, f)\mu(T_t f)^{n-k} \tag{3.1}$$

for all $f \in C(I\!R)^+$, $\mu \in M(I\!R)$ and $n = 1, 2, \cdots$.

Proof. Using the same argument as for Proposition 2.2 one shows that

$$X_t(f) = \mu(T_t f) + \int_0^t \int_{I\!R^d} T_{t-s} f(x) M(\mathrm{d}s, \mathrm{d}x) + \int_0^t X_s(d(X_s)T_{t-s} f)\mathrm{d}s, \tag{3.2}$$

where $d(x, \mu) = b(x, \mu) - B \le 0$. Therefore $\mathbf{Q}_\mu\{X_t(f)\} \le \mu(T_t f)$. Taking a constant $u > 0$ and replacing f by $T_{u-t} f$ in (3.2),

$$X_t(T_{u-t} f) = \mu(T_u f) + \int_0^t \int_{I\!R} T_{u-s} f(x) M(\mathrm{d}s, \mathrm{d}x) + \int_0^t X_s(d(X_s)T_{u-s} f)\mathrm{d}s.$$

By Itô's formula we have

$$\begin{aligned} X_t(T_{u-t} f)^n =& \mu(T_u f)^n + \text{martingale} \\ &+ n \int_0^t X_s(T_{u-s} f)^{n-1} X_s(d(X_s)T_{u-s} f)\mathrm{d}s \\ &+ \frac{n(n-1)}{2} \int_0^t X_s(T_{u-s} f)^{n-2} X_s(c(X_s)[T_{u-s} f]^2)\mathrm{d}s. \end{aligned}$$

Taking the expectation with $t = u$ gives

$$\mathbf{Q}_\mu\{X_t(f)^n\} \leq \mu(T_t f)^n + \frac{n(n-1)}{2} B \int_0^t \|T_{t-s} f\| \mathbf{Q}_\mu\{X_s(T_{t-s}f)^{n-1}\}\mathrm{d}s.$$

Now (3.1) follows by induction in $n = 1, 2, \cdots$. $\square$

Now we assume that $(P_t)_{t\geq 0}$ has continuous density $p_t(x, y)$ and there exist $0 < \delta, \theta < 1$ such that

$$\sup\{p_t(x, y) : (x, y) \in I\!R^2\} \leq t^{-\theta} \tag{3.3}$$

for $0 < t < \delta$. By Theorem 2.3, the interacting superprocess $\{X_t : t > 0\}$ is absolutely continuous with respect to the Lebesgue measure with density $\{X(t, x) : t > 0\}$. Assume further that there exist $\beta, \gamma > 0$ such that for all $0 < r < t < q$ and $x, y \in I\!R$,

$$\int_0^t \mathrm{d}s \int_{I\!R} [p_{t-s}(z, x) - p_{r-s}(z, y)]^2 \, \mathrm{d}z \leq C(q) \cdot \left(|t - r|^\beta + |y - x|^\gamma\right), \tag{3.4}$$

where $C(q) > 0$ is a constant only depending on $q > 0$. Let $C_1(I\!R)^+$ be the set of non-negative, continuous, integrable functions on $I\!R$. Under the above conditions, the density process $\{X(t, \cdot) : t > 0\}$ has a $C_1(I\!R)^+$-valued version which is pointwise continuous and satisfies the expected stochastic partial differential equation. These are summarized in the following

Theorem 3.2. *Assume (3.3) and (3.4) hold. Then there exists a continuous $C_1(I\!R)^+$-valued process $\{X_t(\cdot) : t > 0\}$ such that $\mathbf{Q}_\mu$-a.s. $X_t(\mathrm{d}x) = X_t(x)\mathrm{d}x$ for all $t > 0$ and $x \in I\!R$. Let λ denote the Lebesgue measure on $I\!R$, and let $b(h, x) = b(\lambda_h, x)$ and $c(h, x) = c(\lambda_h, x)$ for $x \in I\!R$ and $h \in C_1(I\!R)^+$, where $\lambda_h \in M(I\!R)$ is defined by $\lambda_h(\mathrm{d}x) = h(x)\mathrm{d}x$. Then the density process $\{X_t(x) : t > 0, x \in I\!R\}$ solves the following stochastic partial differential equation:*

$$\frac{\partial}{\partial t} X_t(x) = \sqrt{c(X_t, x)X_t(x)}\dot{W}_t(x) + A^* X_t(x) + b(X_t, x)X_t(x), \tag{3.5}$$

where A^* is the adjoint operator of A, and $\dot{W}_t(x)$ is a time-space white noise defined on an extension of the original probability space.

More precisely, the equation (3.5) should be understood in the sense of distribution, that is, $\mathbf{Q}_\mu$-a.s.,

$$\begin{aligned}\int_{I\!R} X_t(x)f(x)\mathrm{d}x - \int_{I\!R} f(x)X_0(\mathrm{d}x) &= \int_0^t \int_{I\!R} \sqrt{c(X_s, x)X_s(x)} f(x)\dot{W}_s(x)\mathrm{d}x\mathrm{d}s \\ &\quad + \int_0^t \int_{I\!R} X_s(x)\big[Af(x) + b(X_s, x)f(x)\big]\mathrm{d}x\mathrm{d}s\end{aligned}$$

for all $t \geq 0$ and $f \in C(I\!R)$.

Proof. Step 1. Clearly, the following integral is well-defined:

$$Y_t(x) = \int_0^t \int_{I\!R} b(X_s, z) p_{t-s}(z,x) X_s(\mathrm{d}z)\mathrm{d}s, \quad t > 0, x \in I\!R. \tag{3.6}$$

By Proposition 3.1, for $t > 0$ and $x \in I\!R$,

$$\begin{aligned} \mathbf{Q}_\mu \left\{ Y_t(x)^2 \right\} \leq& Bt \int_0^t \mathbf{Q}_\mu\{X_s(p_{t-s}(\cdot,x))^2\}\mathrm{d}s \\ \leq& Bt \int_0^t \left[B\mathrm{e}^{2Bt}\mu(p_t(\cdot,x))^2 + B^2\mathrm{e}^{Bt}\mu(p_t(\cdot,x))A_1(s, p_{t-s}(\cdot,x)\right]\mathrm{d}s, \end{aligned} \tag{3.7}$$

where

$$A_1(s, p_{t-s}(\cdot,x)) = \int_0^s \|T_{s-r}p_{t-s}(\cdot,x)\|\mathrm{d}r \leq \mathrm{e}^{Bt} \int_0^s \|p_{t-r}(\cdot,x)\|\mathrm{d}r.$$

Step 2. As in the proof of Theorem 2.3, we may choose a version of

$$Z(t,x) = \int_0^t \int_{I\!R} p_{t-s}(z,x) M(\mathrm{d}s, \mathrm{d}z), \quad t > 0, x \in I\!R, \tag{3.8}$$

such that for every $t > 0$ and $f \in C(I\!R)$, $\mathbf{Q}_\mu$-a.s.,

$$\int_{I\!R} f(x)Z(t,x)\mathrm{d}x = \int_0^t \int_{I\!R} P_{t-s}f(x)M(\mathrm{d}s, \mathrm{d}x). \tag{3.9}$$

Using Proposition 3.1, for $t > 0$ and Leb.-a.a. $x \in I\!R$,

$$\begin{aligned} \mathbf{Q}_\mu \left\{ Z(t,x)^2 \right\} =& \mathbf{Q}_\mu \left\{ \int_0^t \int_{I\!R} p_{t-s}(z,x)^2 c(X_s,z) X_s(\mathrm{d}z)\mathrm{d}s \right\} \\ \leq& \frac{B}{\sqrt{2\pi}}\mathrm{e}^{Bt} \int_0^t \frac{\mathrm{d}s}{\sqrt{t-s}} \int_{I\!R} p_{t-s}(z,x)\mu P_s(\mathrm{d}z) \\ \leq& \frac{\sqrt{2t}\, B}{\sqrt{\pi}}\mathrm{e}^{Bt} \int_{I\!R} p_t(z,x)\mu(\mathrm{d}z). \end{aligned} \tag{3.10}$$

Step 3. As in the proof of Theorem 2.3 we see that for every $t > 0$ the random measure $X_t(\mathrm{d}x)$ has density

$$X(t,x) = \int_{I\!R} p_t(z,x)\mu(\mathrm{d}z) + Z_t(x) + Y_t(x), \quad x \in I\!R. \tag{3.11}$$

Combining (3.7), (3.10), (3.11) and the C_r-inequality we have

$$\mathbf{Q}_\mu\{X(t,x)^2\} \leq C_2(t), \quad t > 0, \text{ Leb.-a.a. } x \in I\!R, \tag{3.12}$$

where $C_2 = C_2(t)$ is a locally bounded function on $(0, \infty)$, i.e., bounded on $[\delta, q]$ for any $0 < \delta < q$.

Step 4. Using Theorem 2.6 of Walsh (1986) gives that for $t > r \geq 0$, $\mathbf{Q}_\mu$-a.s.

$$X(t,x) = \int_{I\!R} p_{t-r}(z,x) X_r(\mathrm{d}z) + Z_t^r(x) + Y_t^r(x), \qquad \text{Leb.-a.a. } x \in I\!R, \tag{3.13}$$

where

$$Y_t^r(x) = \int_r^t \int_{I\!R} b(X_s, z) p_{t-s}(z,x) X_s(\mathrm{d}z)\mathrm{d}s, \tag{3.14}$$

and $Z_t^r(x)$ is defined for Leb.-a.a. $x \in I\!R$ by

$$Z_t^r(x) = \int_r^t \int_{I\!R} p_{t-s}(z,x) M(\mathrm{d}s, \mathrm{d}z). \tag{3.15}$$

By Proposition 2.1, we have for $n = 1, 2, \cdots$,

$$\mathbf{Q}_\mu \Big\{ \Big[\int_{I\!R} p_{t-r}(z,x) X_r(\mathrm{d}z) \Big]^n \Big\} \leq \frac{F_n(t)}{[2\pi(t-r)]^{n/2}} \sum_{k=1}^n \mu(1)^k. \tag{3.16}$$

By a similar technique as in (3.7) it follows that

$$\mathbf{Q}_\mu \left\{ Y_t^r(x)^n \right\} \leq D_n(t) \sum_{k=0}^{n-1} H_k(t) \mu(p_t(\cdot, x))^{n-k}, \tag{3.17}$$

where $D_n(t)$ and $H_k(t)$ are locally bounded functions on $(0, \infty)$.

Step 5. Using Burkholder-Davis-Gundy's and Hölder's inequalities to (3.15) we have for $t > r \geq 0$ and Leb.-a.a. $x \in I\!R$,

$$\begin{aligned} &\mathbf{Q}_\mu \{ Z_t^r(x)^{2n} \} \\ &\leq C \mathbf{Q}_\mu \Big\{ \Big[\int_r^t \mathrm{d}s \int_{I\!R} p_{t-s}(z,x)^2 c(X_s, z) X(s,z) \mathrm{d}z \Big]^n \Big\} \\ &\leq C B^n \mathbf{Q}_\mu \Big\{ \int_r^t \mathrm{d}s \int_{I\!R} p_{t-s}(z,x)^2 X(s,z)^n \mathrm{d}z \Big\} \cdot \Big\{ \int_r^t \mathrm{d}s \int_{I\!R} p_{t-s}(z,x)^2 \mathrm{d}z \Big\}^{n-1} \\ &\leq C B^n t^{(n-1)/2} \int_r^t \mathrm{d}s \int_{I\!R} p_{t-s}(z,x)^2 \mathbf{Q}_\mu \{ X(s,z)^n \} \mathrm{d}z, \end{aligned} \tag{3.18}$$

where C is a universal constant. Combining this with the estimates in steps 3 and 4 one shows by induction that for $n = 1, 2, \cdots$

$$\mathbf{Q}_\mu \{ X(t,x)^n \} \leq C_n(t), \quad t > 0, \ \text{Leb.-a.a. } x \in I\!R, \tag{3.19}$$

where the $C_n(t)$ are locally bounded functions on $(0,\infty)$.

Step 6. By a similar technique as in (3.18), for $0<r<t<u$ and Leb.-a.a. $x,y\in I\!\!R$,

$$\begin{aligned}&\mathbf{Q}_0\{|Z_t^r(x)-Z_u^r(y)|^{2n}\}\\ &\leq CB^n\int_r^u \mathrm{d}s\int_{I\!\!R}[p_{t-s}(z,x)-p_{u-s}(z,y)]^2\mathbf{Q}_\mu\{X(s,z)^n\}\mathrm{d}z\\ &\qquad\cdot\Big\{\int_r^u \mathrm{d}s\int_{I\!\!R}[p_{t-s}(z,x)-p_{u-s}(z,y)]^2\mathrm{d}z\Big\}^{n-1},\end{aligned}\tag{3.20}$$

where $p_s(z,y)=0$ for $s\leq 0$ by convention. Taking n large enough in (3.20) we conclude as in Konno and Shiga (1988) or Li and Shiga (1995) that $\{Z_t^r(x):\text{Leb.-a.a. } x\in I\!\!R, t>r\}$ satisfies Kolmogorov's criterion, and hence it has a continuous modification. A glance at (3.13) and (3.14) shows that $\{X(t,x):x\in I\!\!R,t>r\}$ also has a continuous version, which we denote by $\{X_t(x):x\in I\!\!R,t>r\}$. The equation (3.5) then follows by a standard argument. $\square$

We remark that (3.3) and (3.4) are satisfied if $(P_t)_{t\geq 0}$ is the semigroup of a symmetric stable process.

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