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IMMIGRATION STRUCTURES ASSOCIATED WITH DAWSON-WATANABE SUPERPROCESSES

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Abstract

The immigration structure associated with a measure-valued branching process may be described by a skew convolution semigroup. For a special type of measure-valued branching process, the Dawson-Watanabe superprocess, we show that a skew convolution semigroup corresponds uniquely to an infinitely divisible probability measure on the space of entrance laws for the underlying process. An immigration process associated with a Borel right superprocess does not always have a right continuous realization, but it can always be obtained by transformation from a Borel right one in an enlarged state space.

Key words: Dawson-Watanabe superprocess; skew convolution semigroup; entrance law; immigration process; Borel right process

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1. Introduction

Let E be a Lusin topological space, i.e., a homeomorphism of a Borel subset of a compact metric space, with the Borel σ-algebra $\mathcal{B}(E)$. Let $B(E)$ denote the set of all bounded $\mathcal{B}(E)$-measurable functions on E and $B(E)^+$ the subspace of $B(E)$ comprising of non-negative elements. Denote by $M(E)$ the space of finite measures on $(E, \mathcal{B}(E))$ equipped with the topology of weak convergence. For $f \in B(E)$ and $\mu \in M(E)$, write

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$\mu(f)$ for $\int_E f \mathrm{d}\mu$. Suppose that $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$ is a Borel right process in E with semigroup $(P_t)_{t\geq 0}$ and ϕ is a branching mechanism given by

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x, \mathrm{d}u), \qquad x \in E, z \geq 0, \tag{1.1}$$

where $b \in B(E)$, $c \in B(E)^+$ and $[u \wedge u^2] m(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$. From a general construction in Fitzsimmons (1988, 1992) we have that for each $f \in B(E)^+$ the evolution equation

$$V_t f(x) + \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) P_{t-s}(x, \mathrm{d}y) = P_t f(x), \qquad t \geq 0, x \in E, \tag{1.2}$$

has a unique solution $V_t f \in B(E)^+$, and there is a Markov semigroup $(Q_t)_{t\geq 0}$ on $M(E)$ such that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\left\{-\mu(V_t f)\right\}, \qquad t \geq 0, \mu \in M(E). \tag{1.3}$$

Furthermore, $(Q_t)_{t\geq 0}$ is the transition semigroup of a Borel right process $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$. The process X is called a *Dawson-Watanabe superprocess* with parameters (ξ, ϕ), or simply a (ξ, ϕ)-*superprocess*.

The (ξ, ϕ)-superprocess is a special form of the measure-valued branching process (MB-process), which is a kind of measure-valued Markov process with transition semigroup satisfying the *branching property* (1.3) with $V_t f$ defined by

$$V_t f(x) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\delta_x, \mathrm{d}\nu), \qquad t \geq 0, x \in E, \tag{1.4}$$

where δ_x denotes the unit mass at $x \in E$. See e.g. Dawson (1993) and Watanabe (1968). An MB-process X is the mathematical model for the evolution of a population in some region whose growth and decay is subject to the law of chance. If we consider a situation where there are some additional sources of population from which immigration into the region occurs during the evolution, we need to introduce a measure-valued immigration process. Several authors have constructed measure-valued immigration processes under different hypotheses; see e.g. Dynkin (1991), Gorostiza-Lopez-Mimbela (1990), Li (1992) and Shiga (1990).

As observed in Li (1995), a special type of immigration associated with the MB-process may be described by a flow of probability measures that solves an equation with a kind of skew product: Let $(N_t)_{t\geq 0}$ be probability measures on $M(E)$. We call $(N_t)_{t\geq 0}$ a *skew convolution semigroup* associated with X or $(Q_t)_{t\geq 0}$ if

$$N_{r+t} = (N_r Q_t) * N_t, \qquad r, t \geq 0, \tag{1.5}$$

where "$*$" denotes the convolution operation. The relation (1.5) holds if and only if

$$Q_t^N(\mu,\cdot) := Q_t(\mu,\cdot) * N_t, \qquad t \geq 0,\ \mu \in M(E), \tag{1.6}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. In view of (1.6), if Y is a Markov process in $M(E)$ having transition semigroup $(Q_t^N)_{t\geq 0}$, we call it an *immigration process* associated with X. It was proved in Li (1995) that the family of probability measures $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with $(Q_t)_{t\geq 0}$ if and only if there is an infinitely divisible probability entrance law $(K_t)_{t>0}$ for $(Q_t)_{t\geq 0}$ such that

$$\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \int_0^t \left[\log \int_{M(E)} \mathrm{e}^{-\nu(f)} K_s(\mathrm{d}\nu)\right] \mathrm{d}s \tag{1.7}$$

for all $t \geq 0$ and $f \in B(E)^+$. Therefore the immigration structures associated with an MB-process may be characterized by its infinitely divisible probability entrance laws.

The purpose this paper is to describe the set of infinitely divisible probability entrance laws for the (ξ,ϕ)-superprocess and to discuss the regularities of the corresponding immigration processes. By characterizing all those entrance laws we find some immigration processes which have not been studied in the literature. An example given at the end of the paper shows that an immigration process associated with the Borel right (ξ,ϕ)-superprocess may have no right continuous realization. The general theory of Markov processes developed in Sharpe (1988) provides important tools for the study.

In section 2 we prove a 1-1 correspondence between minimal probability entrance laws for the (ξ,ϕ)-superprocess and entrance laws for the underlying process ξ, using an argument of lifting and projecting adapted from Dynkin (1989).

In section 3 we show that each infinitely divisible probability entrance law for the superprocess is determined uniquely by an infinitely divisible probability measure on the space of entrance laws for the underlying process.

In section 4 we study the basic regularities of the immigration processes. If the infinitely divisible probability entrance law for the superprocess can be closed by a probability measure on $M(E)$, it yields a Borel right immigration process.

In section 5 it is shown that a "good" version of the general immigration process may be obtained by transformation from a Borel right one in an enlarged state space.

The problems considered in this paper are similar to those in Li-Li-Wang (1993) and Li-Shiga (1995), although the basic hypotheses and formulations are different. In Li-Li-Wang (1993) we discussed Feller processes and in Li-Shiga (1995) we were only interested in diffusions.

2. Minimal probability entrance laws

Let us consider a change of form of the equation (1.2). Define the semigroup of bounded kernels $(P_t^b)_{t\geq 0}$ on E by

$$P_t^b f(x) = \mathbf{P}_x f(\xi_t) \exp\left\{ -\int_0^t b(\xi_s)\mathrm{d}s \right\}. \tag{2.1}$$

Then (1.2) is equivalent to

$$V_t f(x) + \int_0^t \mathrm{d}s \int_E \phi_0(y, V_s f(y)) P^b_{t-s}(x, \mathrm{d}y) = P^b_t f(x), \tag{2.2}$$

where

$$\phi_0(x, z) = c(x) z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x, \mathrm{d}u). \tag{2.3}$$

Note that the first moments of the (ξ, ϕ)-superprocess are given by

$$\int_{M(E)} \nu(f) Q_t(\mu, \mathrm{d}\nu) = \mu(P^b_t f). \tag{2.4}$$

See e.g. Fitzsimmons (1988).

Given a semigroup of bounded kernels $(T_t)_{t\geq 0}$ on E, we denote by $\mathcal{K}(T)$ the set of entrance laws $\kappa = (\kappa_t)_{t>0}$ for $(T_t)_{t\geq 0}$ that satisfy

$$\int_0^t \kappa_s(E) \mathrm{d}s < \infty \quad \text{for all } t > 0. \tag{2.5}$$

Let $\mathcal{K}^1(Q)$ denote the set of probability entrance laws $K = (K_t)_{t>0}$ for the semigroup $(Q_t)_{t\geq 0}$ such that

$$\int_0^t \mathrm{d}s \int_{M(E)} \nu(E) K_s(\mathrm{d}\nu) < \infty \quad \text{for all } t > 0, \tag{2.6}$$

and let $\mathcal{K}^1_m(Q)$ denote the subset of $\mathcal{K}^1(Q)$ comprising of minimal elements. See e.g. Sharpe (1988) for the definition of an entrance law.

Theorem 2.1. *There is a one-to-one correspondence between $K \in \mathcal{K}^1_m(Q)$ and $\gamma \in \mathcal{K}(P^b)$, which is given by*

$$\gamma_t(f) = \int_{M(E)} \nu(f) K_t(\mathrm{d}\nu), \tag{2.7}$$

and

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\left\{-S^b_t(\gamma, f)\right\}, \tag{2.8}$$

where

$$S^b_t(\gamma, f) = \gamma_t(f) - \int_0^t \mathrm{d}s \int_E \phi_0(y, V_s f(y)) \gamma_{t-s}(\mathrm{d}y). \tag{2.9}$$

We omit the proof of the above theorem, which follows from (1.3) and (2.4) by the same argument as Dynkin (1989). To describe the class $\mathcal{K}^1_m(Q)$ we need to clarify a connection between $\mathcal{K}(P)$ and $\mathcal{K}(P^b)$.

Lemma 2.1. *There is a one-to-one correspondence between $\kappa \in \mathcal{K}(P)$ and $\gamma \in \mathcal{K}(P^b)$, which is given by*

$$\gamma_t = \lim_{r\downarrow 0} \kappa_r P^b_{t-r} \quad \text{and} \quad \kappa_t = \lim_{r\downarrow 0} \gamma_r P_{t-r}. \tag{2.10}$$

Moreover, if the two entrance laws κ and γ are related by (2.10), then we have

$$\mathrm{e}^{-\|b\|t}\kappa_t(f) \le \gamma_t(f) \le \mathrm{e}^{\|b\|t}\kappa_t(f), \qquad t > 0, f \in B(E)^+. \tag{2.11}$$

Proof. The assertions follow from the inequalities

$$\mathrm{e}^{-\|b\|t}P_t f \le P^b_t f \le \mathrm{e}^{\|b\|t}P_t f, \qquad t \ge 0, f \in B(E)^+.$$

We omit the details. □

For $\kappa \in \mathcal{K}(P)$ we note

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y))\kappa_{t-s}(\mathrm{d}y), \qquad t > 0, f \in B(E)^+. \tag{2.12}$$

If the entrance laws $\kappa \in \mathcal{K}(P)$ and $\gamma \in \mathcal{K}(P^b)$ are related by (2.10), then clearly $S_t(\kappa, f) = S^b_t(\gamma, f)$. Combining those with Theorem 2.1 we get the following

Theorem 2.2. *There is a one-to-one correspondence between $K \in \mathcal{K}^1_m(Q)$ and $\kappa \in \mathcal{K}(P)$, which is given by*

$$\kappa_t(f) = \lim_{r\downarrow 0} \int_{M(E)} \nu(P_{t-r}f)K_r(\mathrm{d}\nu), \tag{2.13}$$

and

$$\int_{M(E)} \mathrm{e}^{-\nu(f)}K_t(\mathrm{d}\nu) = \exp\left\{-S_t(\kappa, f)\right\}, \tag{2.14}$$

where $S_t(\kappa, f)$ is defined by (2.12). □

If ξ is conservative, each $\kappa \in \mathcal{K}(P)$ is uniquely determined by a measure $\kappa_0 \in M(E_D)$, where E_D is the entrance space of ξ; see Sharpe (1988). In that case, Theorem 2.2 follows from a result of Fitzsimmons (1988). See also Dynkin (1989) for the analogous results in the case where $\phi(x, z) \equiv c(x)z^2$ but ξ is allowed to be non-homogeneous and X is allowed to take values in a space of σ-finite measures.

3. Infinitely divisible probability entrance laws

The goal of this section is to describe the class of infinitely divisible probability entrance laws for the (ξ, ϕ)-superprocess in terms of its underlying process. The following kind of h-transform will be useful. Set

$$h(x) = \int_0^1 P_s 1(x)\mathrm{d}s, \qquad x \in E. \tag{3.1}$$

Since $h \in B(E)^+$ is a strictly positive excessive function of $(P_t)_{t\geq 0}$, the formula

$$T_t f(x) = h(x)^{-1} \int_E f(y)h(y)P_t(x, \mathrm{d}y) \tag{3.2}$$

defines a Borel right semigroup $(T_t)_{t\geq 0}$ with state space E. See e.g. Sharpe (1988). Let $(T_t^\partial)_{t\geq 0}$ be the conservative extension of $(T_t)_{t\geq 0}$ to $E^\partial := E \cup \{\partial\}$, where ∂ is the cemetery point. Let $(\bar{T}_t^\partial)_{t\geq 0}$ be a Ray extension of $(T_t^\partial)_{t\geq 0}$ to its entrance space $E_D^{\partial,T}$ with the Ray topology. Denote by $(\bar{T}_t)_{t\geq 0}$ the restriction of $(\bar{T}_t^\partial)_{t\geq 0}$ to $E_D^T := E_D^{\partial,T} \setminus \{\partial\}$. Then $(\bar{T}_t^\partial)_{t\geq 0}$ and $(\bar{T}_t)_{t\geq 0}$ are also Borel right semigroups. Define a branching mechanism $\bar{\psi}$ on E_D^T by

$$\bar{\psi}(x, z) = h(x)^{-1}\phi(x, h(x)z) \ \text{ for } x \in E, \quad = 0 \ \text{ for } x \in E_D^T \setminus E.$$

Let $(\bar{U}_t)_{t\geq 0}$ be the cumulant semigroup on $B(E_D^T)^+$ determined by the equation

$$\bar{U}_t \bar{f}(x) = \bar{T}_t \bar{f}(x) - \int_0^t \mathrm{d}s \int_{E_D^T} \bar{\psi}(y, \bar{U}_s \bar{f}(y))\bar{T}_{t-s}(x, \mathrm{d}y), \qquad t \geq 0, x \in E_D^T. \tag{3.3}$$

Note that for $t > 0$ and $x \in E_D^T$, the measure $\bar{T}_t(x, \cdot)$ is supported by E, so $\bar{T}_t \bar{f}(x)$ and $\bar{U}_t \bar{f}(x)$ are independent of the values of $\bar{f}$ on $E_D^T \setminus E$. In the sequel, we may write $\bar{T}_t f(x)$ and $\bar{U}_t f(x)$ instead of $\bar{T}_t \bar{f}(x)$ and $\bar{U}_t \bar{f}(x)$ respectively, where f is the restriction of $\bar{f}$ to E. Note also that the definitions of $\bar{T}_t f$ and $\bar{U}_t f$ can be extended to all non-negative Borel functions f on E by increasing limit.

Lemma 3.1. *For each $\rho \in M(E_D^T)$, the formula*

$$\kappa_t(f) = \rho(\bar{T}_t(h^{-1}f)), \qquad t > 0, f \in B(E)^+, \tag{3.4}$$

defines a $\kappa \in \mathcal{K}(P)$. Conversely, if $\kappa \in \mathcal{K}(P)$, there is a unique $\rho \in M(E_D^T)$ such that κ is given by (3.4). Moreover, if κ and ρ are related by (3.4), then we have

$$S_t(\kappa, f) = \rho(\bar{U}_t(h^{-1}f)), \qquad t > 0, f \in B(E)^+, \tag{3.5}$$

where $S_t(\kappa, f)$ is defined by (2.12).

Proof. For each $\rho \in M(E_D^T)$, (3.4) clearly defines an entrance law $\kappa \in \mathcal{K}(P)$. Conversely, for $\kappa \in \mathcal{K}(P)$ we can define an entrance law $\eta = (\eta_t)_{t>0}$ for the semigroup $(T_t)_{t\geq 0}$ by $\eta_t(f) = \kappa_t(hf)$. Since

$$\lim_{t\downarrow 0} \eta_t(1) = \lim_{t\downarrow 0} \kappa_t(h) = \int_0^1 \kappa_s(1)\mathrm{d}s < \infty,$$

there exists a unique measure $\rho \in M(E_D^T)$ such that $\eta_t = \rho\bar{T}_t$ for all $t > 0$; see Sharpe (1988). Then (3.4) follows.

If (3.4) holds, by (1.2), (2.12) and (3.3) we have

$$S_t(\kappa, f) = \lim_{r\downarrow 0} \kappa_r(V_{t-r}f) = \lim_{r\downarrow 0} \rho\bar{T}_r(\bar{U}_{t-r}(h^{-1}f)) = \rho(\bar{U}_t(h^{-1}f))$$

for all $t > 0$ and $f \in B(E)^+$. $\square$

Theorem 3.1. *The probability entrance law $K \in \mathcal{K}^1(Q)$ is infinitely divisible if and only if its Laplace functional has the representation*

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\bigg\{ - S_t(\kappa, f) - \int_{\mathcal{K}(P)} (1 - \exp\{-S_t(\eta, f)\})\, J(\mathrm{d}\eta)\bigg\}, \quad (3.6)$$

where $\kappa \in \mathcal{K}(P)$ and J is a σ-finite measure on $\mathcal{K}(P)$ satisfying

$$\int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) J(\mathrm{d}\eta) < \infty. \quad (3.7)$$

Proof. It is easy to see that (3.6) defines an infinitely divisible probability entrance law $K \in \mathcal{K}^1(Q)$. Conversely, suppose $K \in \mathcal{K}^1(Q)$ is infinitely divisible. By Lemma 2.1, Theorem 2.2 and a result of Dynkin (1978), K admits the following representation

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \int_{\mathcal{K}(P)} \exp\{-S_t(\eta, f)\}\, F(\mathrm{d}\eta), \quad (3.8)$$

where F is a probability measure on $\mathcal{K}(P)$ satisfying

$$\int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) F(\mathrm{d}\eta) < \infty. \quad (3.9)$$

It follows by Lemma 3.1 that there is a probability measure H on $M(E_D^T)$ such that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \int_{M(E_D^T)} \exp\big\{-\rho(\bar{U}_t(h^{-1}f))\big\}\, H(\mathrm{d}\rho), \quad (3.10)$$

with

$$\int_{M(E_D^T)} \rho(1) H(\mathrm{d}\rho) = \int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) F(\mathrm{d}\eta). \quad (3.11)$$

Since K is infinitely divisible, so is H by (3.10). Thus

$$\int_{M(E_D^T)} \mathrm{e}^{-\rho(f)} H(\mathrm{d}\rho) = \exp\bigg\{ - \gamma(f) - \int_{M(E_D^T)^\circ} \Big(1 - \mathrm{e}^{-\nu(f)}\Big)\, G(\mathrm{d}\nu)\bigg\}, \quad (3.12)$$

where $\gamma \in M(E_D^T)$ and $\nu(1)G(\mathrm{d}\nu)$ is a finite measure on $M(E_D^T)^\circ := M(E_D^T) \setminus \{o\}$. Then the expression (3.6) follows from (3.10) and (3.12), and (3.7) holds by (3.9) and (3.11). $\square$

The next result, which is an immediate consequence of Theorem 3.1, characterizes completely the immigration structures associated with the (ξ, ϕ)-superprocess.

Theorem 3.2. *Suppose that $(N_t)_{t\geq 0}$ is a family of probability measures on $M(E)$ satisfying*

$$\int_{M(E)} \nu(1) N_t(\mathrm{d}\nu) < \infty \quad \text{for all } t \geq 0. \tag{3.13}$$

Then $(N_t)_{t\geq 0}$ is a skew convolution semigroup associated with the (ξ, ϕ)-superprocess if and only if its Laplace functional has the representation

$$\begin{aligned}&\int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu)\\ &= \exp\bigg\{ - \int_0^t \bigg[S_r(\kappa, f) + \int_{\mathcal{K}(P)} (1 - \exp\{-S_r(\eta, f)\})\, J(\mathrm{d}\eta) \bigg] \mathrm{d}r \bigg\},\end{aligned} \tag{3.14}$$

where $\kappa \in \mathcal{K}(P)$ and J is a σ-finite measure on $\mathcal{K}(P)$ satisfying (3.7). $\square$

4. Borel right immigration processes

Note that if $\kappa \in \mathcal{K}(P)$ is given by $\kappa_t = \nu P_t$ for $\nu \in M(E)$, then $S_t(\kappa, f) = \nu(V_t f)$. Let $\gamma \in M(E)$ and let $\nu(E)G(\mathrm{d}\nu)$ be a finite measure on $M(E)^\circ := M(E) \setminus \{o\}$. By Theorem 3.2,

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \exp\bigg\{ - \int_0^t \bigg[\gamma(V_s f) + \int_{M(E)^\circ} \Big(1 - \mathrm{e}^{-\nu(V_s f)}\Big) G(\mathrm{d}\nu) \bigg] \mathrm{d}s \bigg\} \tag{4.1}$$

defines a skew convolution semigroup $(N_t)_{t\geq 0}$ associated with the (ξ, ϕ)-superprocess X. This $(N_t)_{t\geq 0}$ corresponds to an infinitely divisible probability entrance law for X which can be closed by a probability measure on $M(E)$. We shall prove in this section that $(N_t)_{t\geq 0}$ yields a Borel right immigration process.

We shall need to consider two topologies on the space E: the original topology and the Ray topology of ξ. We write E_r for the set E furnished with the Ray topology of ξ. Let $C(E)^+$ be the set of bounded non-negative functions that are continuous in the original topology. The notations $C(E_r)^+$ and $M(E_r)$ are self-explanatory. Let $W_0(M(E))$ denote the space of all paths $\{w_t : t \geq 0\}$ from $[0, \infty)$ to $M(E)$ that are right continuous both in $M(E)$ and in $M(E_r)$. Let $(\mathcal{G}^\circ, \mathcal{G}_t^\circ)$ denote the natural σ-algebras on $W_0(M(E))$. By the results of Fitzsimmons (1988, 1992), for each $\mu \in M(E)$, there is a unique probability measure $\mathbf{Q}_\mu$ on $(W_0(M(E)), \mathcal{G}^\circ)$ such that $\mathbf{Q}_\mu\{w_0 = \mu\} = 1$ and $\{w_t : t \geq 0\}$ under $\mathbf{Q}_\mu$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. Furthermore, the system $(W_0(M(E)), \mathcal{G}, \mathcal{G}_t, w_t, \mathbf{Q}_\mu)$ is a Borel right process, where $(\mathcal{G}, \mathcal{G}_t)$ is the augmentation of $(\mathcal{G}^\circ, \mathcal{G}_{t+}^\circ)$ by the system $\{\mathbf{Q}_\mu : \mu \in M(E)\}$. Define the σ-finite measure $\mathbf{Q}^G$ on $W_0(M(E))$ by

$$\mathbf{Q}^G(\mathrm{d}w) = \int_{M(E)^\circ} G(\mathrm{d}\mu) \mathbf{Q}_\mu(\mathrm{d}w). \tag{4.2}$$

Suppose that $N(\mathrm{d}s, \mathrm{d}w)$ is a Poisson random measure on $[0,\infty) \times W_0(M(E))$ with intensity $\mathrm{d}s \times \mathbf{Q}^G(\mathrm{d}w)$. Let

$$Y_t = \int_{[0,t]} \int_{W_0(M(E))} w_{t-s}\, N(\mathrm{d}s, \mathrm{d}w), \qquad t \geq 0. \tag{4.3}$$

Theorem 4.1. *The process $\{Y_t : t \geq 0\}$ defined by (4.3) is a Markov process with semigroup $(Q_t^G)_{t\geq 0}$ given by*

$$\begin{aligned}&\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^G(\mu, \mathrm{d}\nu) \\ =&\exp\bigg\{ -\mu(V_t f) - \int_0^t \mathrm{d}s \int_{M(E)^\circ} \Big(1 - \mathrm{e}^{-\nu(V_s f)}\Big)\, G(\mathrm{d}\nu)\bigg\}.\end{aligned} \tag{4.4}$$

Furthermore, $\{Y_t : t \geq 0\}$ is a.s. right continuous both in $M(E)$ and in $M(E_r)$.

Proof. The Markov property of $\{Y_t : t \geq 0\}$ follows by a standard argument. For $k = 1, 2, \cdots$, we let

$$W_k = \{w \in W_0(M(E)) : w_0(1) \geq 1/k\},$$

and define

$$Y_t^{(k)} = \int_{[0,t]} \int_{W_k} w_{t-s}\, N(\mathrm{d}s, \mathrm{d}w), \qquad t \geq 0. \tag{4.5}$$

Then $\{Y_t^{(k)} : t \geq 0\}$ is a Markov process in $M(E)$ with semigroup $(Q_t^{(k)})_{t\geq 0}$ given by

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^{(k)}(\mu, \mathrm{d}\nu) = \exp\bigg\{ -\mu(V_t f) - \int_0^t \mathrm{d}s \int_{M_k} \Big(1 - \mathrm{e}^{-\nu(V_s f)}\Big)\, G(\mathrm{d}\nu)\bigg\}, \tag{4.6}$$

where $M_k = \{\mu \in M(E) : \mu(E) \geq 1/k\}$. Observe that for each $l > 0$, the process $\{Y_t^{(k)} : 0 \leq t \leq l\}$ is a.s. a finite sum of right continuous paths, so $\{Y_t^{(k)} : t \geq 0\}$ is a.s. right continuous. Since $Y_t^{(k)} \to Y_t$ increasingly as $k \to \infty$, it follows that $\{Y_t(f) : t \geq 0\}$ is a.s. right lower semi-continuous for each $f \in C(E)^+ \cup C(E_r)^+$. Let $\mathscr{F}_t^\circ = \sigma\{Y_s^{(k)} : 0 \leq s \leq t, k = 1, 2, \cdots\}$ and let $(\mathscr{F}_t)_{t\geq 0}$ be the augmentation of $(\mathscr{F}_{t+}^\circ)_{t\geq 0}$. From (2.2) we have the following inequalities:

$$V_t f \leq P_t^b f \leq \mathrm{e}^{\|b\| t} P_t f, \qquad t \geq 0, f \in B(E)^+. \tag{4.7}$$

By (4.7), for any $\beta > \|b\|$ and $q \geq 0$,

$$\begin{aligned}&\mathbf{E}\left\{\mathrm{e}^{-\beta(r+t)} Y_{r+t}^{(k)}(q) \big| \mathscr{F}_r\right\} \\ =&\mathrm{e}^{-\beta(r+t)}\left[Y_r^{(k)}(P_t^b q) + \int_0^t \mathrm{d}s \int_{M_k} \nu(P_s^b q) G(\mathrm{d}\nu)\right] \\ \leq&\mathrm{e}^{-\beta r}\left[Y_r^{(k)}(q) + \beta^{-1} \int_{M_k} \nu(q) G(\mathrm{d}\nu)\right] - \beta^{-1} \mathrm{e}^{-\beta(r+t)} \int_{M_k} \nu(q) G(\mathrm{d}\nu).\end{aligned}$$

Therefore

$$\mathrm{e}^{-\beta t}\bigg[Y_t^{(k)}(q)+\beta^{-1}\int_{M_k}\nu(q)G(\mathrm{d}\nu)\bigg],\qquad t\geq 0, \tag{4.8}$$

is an a.s. right continuous $(\mathcal{J}_t)$-supermartingale, which converges increasingly as $k\to\infty$ to the $(\mathcal{J}_t)$-adapted process

$$\mathrm{e}^{-\beta t}\bigg[Y_t(q)+\beta^{-1}\int_{M(E)^\circ}\nu(q)G(\mathrm{d}\nu)\bigg],\qquad t\geq 0. \tag{4.9}$$

Thus (4.9) is a.s. right continuous; see Dellacherie-Meyer (1982). For any $f\in C(E)^+\cup C(E_r)^+$, choose a constant q such that $q\geq f(x)$ for all $x\in E$. By the above arguments, both $Y_t(f)$ and $Y_t(q-f)=Y_t(q)-Y_t(f)$ are a.s. right lower semi-continuous, and $Y_t(q)$ is a.s. right continuous. Those clearly yields the a.s. right continuity of $Y_t(f)$ and the a.s. right continuity of $\{Y_t: t\geq 0\}$ follows immediately. $\square$

Theorem 4.2. *For each $\mu\in M(E)$ there is a unique probability measure $\mathbf{Q}_\mu^N$ on $(W_0(M(E)),\mathcal{G}^\circ)$ such that $\mathbf{Q}_\mu^N\{w_0=\mu\}=1$ and $\{w_t: t\geq 0\}$ under $\mathbf{Q}_\mu^N$ is a Markov process having semigroup $(Q_t^N)_{t\geq 0}$ defined by*

$$\begin{aligned}&\int_{M(E)}\mathrm{e}^{-\nu(f)}Q_t^N(\mu,\mathrm{d}\nu)\\ &=\exp\bigg\{-\mu(V_tf)-\int_0^t\bigg[\gamma(V_sf)+\int_{M(E)^\circ}\Big(1-\mathrm{e}^{-\nu(V_sf)}\Big)G(\mathrm{d}\nu)\bigg]\mathrm{d}s\bigg\}.\end{aligned} \tag{4.10}$$

Proof. We first note that the formula

$$\int_{M(E)}\mathrm{e}^{-\nu(f)}Q_t^\gamma(\mu,\mathrm{d}\nu)=\exp\bigg\{-\mu(V_tf)-\int_0^t\gamma(V_sf)\mathrm{d}s\bigg\} \tag{4.11}$$

determines a transition semigroup $(Q_t^\gamma)_{t\geq 0}$ on $M(E)$ and for each $\mu\in M(E)$ there is a unique probability measure $\mathbf{Q}_\mu^\gamma$ on $(W_0(M(E)),\mathcal{G})$ such that $\mathbf{Q}_\mu^\gamma\{w_0=\mu\}=1$ and $\{w_t: t\geq 0\}$ under $\mathbf{Q}_\mu^\gamma$ is a Markov process having semigroup $(Q_t^\gamma)_{t\geq 0}$. Those facts together with some other regularities of measure-valued processes were discussed in Dynkin (1993). Let $\mathbf{Q}_o^G$ denote the distribution on $W_0(M(E))$ of the process $\{Y_t: t\geq 0\}$ defined by (4.3). One may simply define $\mathbf{Q}_\mu^N=\mathbf{Q}_\mu^\gamma*\mathbf{Q}_o^G$. $\square$

Theorem 4.3. *Let* $\{\mathbf{Q}^N_\mu : \mu \in M(E)\}$ *be provided by Theorem 4.2, and let* $(\mathcal{H}, \mathcal{H}_t)$ *be the corresponding augmentation of* $(\mathcal{G}^\circ, \mathcal{G}^\circ_{t+})$. *Then the system* $(W_0(M(E)), \mathcal{H}, \mathcal{H}_t, w_t, \mathbf{Q}^N_\mu)$ *is a Borel right process.*

Proof. Let $\mathcal{R}$ be a countable Ray cone for (the conservative extension of) ξ as constructed in Sharpe (1988). Assume that each $f \in \mathcal{R}$ is bounded away from 0. Let $\bar{E}$ be the corresponding Ray-Knight compactification of $E \cup \{\partial\}$ with the Ray topology. We regard $M(E_r)$ as a topological subspace of $M(\bar{E})$ in the usual way. Since $\bar{E}$ is a compact metric space, $M(\bar{E})$ is locally compact and separable. Let $C_0(M(\bar{E}))$ denote the space of continuous functions on $M(\bar{E})$ vanishing at infinity. Note that each $f \in \mathcal{R}$ admits a unique extension $\bar{f}$ to $\bar{E}$ by continuity; we write $\bar{\mathcal{R}}$ for the set of all those extensions. In view of (4.10) and the totality of $\{\nu \mapsto \mathrm{e}^{-\nu(\bar{f})} : \bar{f} \in \bar{\mathcal{R}}\}$ in $C_0(M(\bar{E}))$, the assertion follows from Theorem 7.4 of Sharpe (1988) once it is shown that $s \mapsto w_s(V_{t-s}f)$ is $\mathbf{Q}^N_\mu$-a.s. right continuous on $[0, t]$ for each $t \geq 0$ and $f \in \mathcal{R}$.

Recall some notations from the proofs of Theorems 4.1 and 4.2. Let $(\mathcal{H}^\gamma, \mathcal{H}^\gamma_t)$ be the augmentation of $(\mathcal{G}^\circ, \mathcal{G}^\circ_{t+})$ by $\{\mathbf{Q}^\gamma_\mu : \mu \in M(E)\}$. Using a similar argument as Fitzsimmons (1988) one can show that $(W_0(M(E)), \mathcal{H}^\gamma, \mathcal{H}^\gamma_t, w_t, \mathbf{Q}^\gamma_\mu)$ is a Borel right process both in $M(E)$ and in $M(E_r)$. It follows that $s \mapsto w_s(V_{t-s}f)$ is $\mathbf{Q}^\gamma_\mu$-a.s. right continuous on $[0, t]$. On the other hand, since $(W_0(M(E)), \mathcal{G}, \mathcal{G}_t, w_t, \mathbf{Q}_\mu)$ is a Borel right process and since for each $t \geq 0$, the process $\{Y^{(k)}_s : 0 \leq s \leq t\}$ is a.s. a finite sum, (4.6) reveals that

$$\exp\bigg\{ - Y^{(k)}_s(V_{t-s}f) - \int_0^{t-s} \mathrm{d}r \int_{M_k} \Big(1 - \mathrm{e}^{-\nu(V_r f)}\Big) G(\mathrm{d}\nu)\bigg\}, \qquad s \in [0, t], \tag{4.12}$$

is an a.s. right continuous $(\mathcal{J}_s)$-martingale. As $k \to \infty$, (4.12) converges decreasingly to the $(\mathcal{J}_s)$-adapted process

$$\exp\bigg\{ - Y_s(V_{t-s}f) - \int_0^{t-s} \mathrm{d}r \int_{M(E)^\circ} \Big(1 - \mathrm{e}^{-\nu(V_r f)}\Big) G(\mathrm{d}\nu)\bigg\}, \qquad s \in [0, t], \tag{4.13}$$

so (4.13), and hence $Y_s(V_{t-s}f)$, is a.s. right continuous. Since $\mathbf{Q}^N_\mu = \mathbf{Q}^\gamma_\mu * \mathbf{Q}^G_o$, the theorem is proved. □

5. General immigration processes

By Theorem 3.2, a general skew convolution semigroup $(N_t)_{t\geq 0}$ associated with the (ξ, ϕ)-superprocess is represented by (3.14). Let $(\bar{U}_t)_{t\geq 0}$ be defined by (3.3) and let (γ, G) be provided by the proof of Theorem 3.1. Denote by $W_0(M(E^T_D))$ the space of all right continuous paths $\{\bar{w}_t : t \geq 0\}$ from $[0, \infty)$ to $M(E^T_D)$. Given $\mu \in M(E)$ we define $h\bar{\mu} \in M(E^T_D)$ by

$$h\bar{\mu}(E^T_D \setminus E) = 0 \ \text{ and } \ h\bar{\mu}(\mathrm{d}x) = h(x)\mu(\mathrm{d}x), \qquad x \in E.$$

Theorems 4.2 and 4.3 guarantee a unique probability measure $\bar{\mathbf{Q}}^N_\mu$ on $W_0(M(E_D^T))$ under which $\{\bar{w}_t : t \geq 0\}$ is an immigration process in $M(E_D^T)$ starting at $h\bar{\mu}$ with the skew convolution semigroup $(\bar{N}_t)_{t\geq 0}$ given by

$$\begin{aligned}&\int_{M(E_D^T)} \mathrm{e}^{-\nu(f)} \bar{N}_t(\mathrm{d}\nu)\\ &= \exp\bigg\{ -\int_0^t \bigg[\gamma(\bar{U}_s f) + \int_{M(E_D^T)^\circ} \Big(1 - \mathrm{e}^{-\nu(\bar{U}_s f)}\Big) G(\mathrm{d}\nu)\bigg]\mathrm{d}s\bigg\}.\end{aligned} \tag{5.1}$$

Define the measure-valued process $\{Y_t : t \geq 0\}$ by

$$Y_t(\mathrm{d}x) = h(x)^{-1}\bar{w}_t(\mathrm{d}x), \qquad t \geq 0, x \in E. \tag{5.2}$$

It is easy to check that $\bar{\mathbf{Q}}^N_\mu\{Y_0 = \mu\} = 1$ and $\{Y_t : t \geq 0\}$ under $\bar{\mathbf{Q}}^N_\mu$ is an immigration process corresponding to the skew convolution semigroup given by (3.14). That is, a general immigration process may be obtained from a Borel right one by the transformation (5.2).

The process $\{Y_t : t \geq 0\}$ constructed by (5.2) is not necessarily a.s. right continuous in the topology of $M(E)$. The trajectory structures of the general immigration process can be worse than those of the (ξ, ϕ)-superprocess. This is illustrated by the following example which describes the immigration to some region from an absorbing boundary.

Example 5.1. We consider the case where E is the positive half line $H := (0, \infty)$. Suppose that ξ is an absorbing barrier Brownian motion in H. The transition semigroup $(P_t)_{t\geq 0}$ of ξ is determined by

$$P_t f(x) = \int_H \left[g_t(x-y) - g_t(x+y)\right] f(y)\mathrm{d}y, \tag{5.3}$$

where

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left\{-x^2/2t\right\}, \qquad t > 0, x \in \mathbb{R}. \tag{5.4}$$

We call the corresponding (ξ, ϕ)-superprocess X a *super absorbing barrier Brownian motion* following the common usage. Let $W_+(M(H))$ denote the space of all right continuous paths $\{w_t : t > 0\}$ from $(0, \infty)$ to $M(H)$. Consider the entrance law $(\kappa_t)_{t>0}$ for ξ given by

$$\kappa_t(f) = \frac{2}{t}\int_H x g_t(x) f(x)\mathrm{d}x. \tag{5.5}$$

By Theorem 2.2, for each $q > 0$,

$$\int_{M(H)} \mathrm{e}^{-\nu(f)} K_t^q(\mathrm{d}\nu) = \exp\left\{-qS_t(\kappa, f)\right\} \tag{5.6}$$

determines a $K^q \in \mathcal{K}_m^1(Q)$. Accordingly, there is a unique probability measure $\mathbf{Q}_{q\kappa}$ on $W_+(M(H))$ under which $\{w_t : t > 0\}$ is a Markov process with one dimensional distributions $(K_t^q)_{t>0}$ and semigroup $(Q_t)_{t\geq 0}$. In the present case we may identify H_D^T with $\mathbb{R}^+ := [0, \infty)$. Let $(\bar{U}_t)_{t\geq 0}$ be defined by (3.3). Then we have

$$h'(0^+)\bar{U}_t f(0) = S_t(\kappa, hf), \qquad t > 0, f \in B(H)^+. \tag{5.7}$$

Using this one can show that $\mathbf{Q}_{q\kappa}$-a.s.

$$w_t(H) \to \infty \ \text{ and } \ h\bar{w}_t \to qh'(0^+)\delta_0 \text{ in } M(\mathbb{R}^+) \text{ as } t \downarrow 0, \tag{5.8}$$

where $h\bar{w}_t(\{0\}) = 0$ and $h\bar{w}_t(\mathrm{d}x) = h(x)w_t(\mathrm{d}x)$ for $t > 0$ and $x \in H$. Suppose $qF(\mathrm{d}q)$ is a non-degenerate finite measure on $(0, \infty)$. Define the σ-finite measure $\mathbf{Q}^F$ on $W_+(M(H))$ by

$$\mathbf{Q}^F(\mathrm{d}w) = \int_0^\infty F(\mathrm{d}q)\mathbf{Q}_{q\kappa}(\mathrm{d}w).$$

Let $N^F(\mathrm{d}s, \mathrm{d}w)$ be a Poisson random measure on $[0, \infty) \times W_+(M(H))$ with intensity $\mathrm{d}s \times \mathbf{Q}^F(\mathrm{d}w)$, and let

$$Y_t^F = \int_{[0,t]} \int_{W_+(M(H))} w_{t-s}\, N^F(\mathrm{d}s, \mathrm{d}w), \qquad t \geq 0, \tag{5.9}$$

where $w_0 = o$ by convention. As for the proof of Theorem 4.1 one may check that $\{Y_t^F : t \geq 0\}$ is an immigration process corresponding to the skew convolution semigroup $(N_t^F)_{t\geq 0}$ given by

$$\int_{M(H)} \mathrm{e}^{-\nu(f)} N_t^F(\mathrm{d}\nu) = \exp\bigg\{ - \int_0^t \mathrm{d}r \int_0^\infty \Big(1 - \mathrm{e}^{-qS_r(\kappa, f)}\Big) F(\mathrm{d}q) \bigg\}.$$

In view of (5.8), the immigration process $\{Y_t^F : t \geq 0\}$ represents a population generated by cliques with infinite masses which arrive at the origin at occurring times of $N^F(\mathrm{d}s, \mathrm{d}w)$. It is easy to check that

$$\mathbf{E}\{Y_t^F(1)\} \leq \int_0^\infty qF(\mathrm{d}q) \int_0^t \mathrm{e}^{\|b\|s}\kappa_s(1)\mathrm{d}s < \infty,$$

so $Y_t^F \in M(H)$ a.s. for every $t \geq 0$. The process $\{Y_t^F : t \geq 0\}$ is certainly not right continuous. It even has no right continuous modification. Otherwise, let $\{Y_t : t \geq 0\}$ be such a modification. Define

$$h\bar{Y}_t(\{0\}) = 0 \ \text{ and } \ h\bar{Y}_t(\mathrm{d}x) = h(x)Y_t(\mathrm{d}x), \qquad t \geq 0, x \in H. \tag{5.10}$$

Then $\{h\bar{Y}_t : t \geq 0\}$ is an a.s. right continuous Markov process in $M(\mathbb{R}^+)$ having semigroup $(\bar{Q}_t^F)_{t\geq 0}$ determined by

$$\begin{aligned}\int_{M(\mathbb{R}^+)} &\mathrm{e}^{-\nu(\bar{f})}\bar{Q}_t^F(\mu, \mathrm{d}\nu) \\ = \exp\bigg\{ -\mu(\bar{U}_t\bar{f}) - &\int_0^t \mathrm{d}s \int_0^\infty \big(1 - \exp\{-qh'(0^+)\bar{U}_s\bar{f}(0)\}\big)\, F(\mathrm{d}q)\bigg\},\end{aligned} \tag{5.11}$$

From (5.10) we have a.s.

$$T_0(h\bar{Y}) := \inf\{t > 0 : h\bar{Y}_t(\{0\}) > 0\} = \infty. \tag{5.12}$$

On the other hand, by a special form of (4.3) one can construct an a.s. right continuous immigration process $\{\bar{Z}_t : t \geq 0\}$ starting at o with semigroup $(\bar{Q}_t^F)_{t\geq 0}$. Since F is non-degenerate, it follows from the construction that a.s.

$$T_0(\bar{Z}) := \inf\{t > 0 : \bar{Z}_t(\{0\}) > 0\} < \infty. \tag{5.13}$$

By Theorem 4.3, $(\bar{Q}_t^F)_{t\geq 0}$ is a Borel right semigroup on $M(\mathbb{R}^+)$, so (5.12) and (5.13) are in contradiction.

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