Published in: *Chinese Science Bulletin* (English Edition) **41** (1996), 276–280.

CONVOLUTION SEMIGROUPS ASSOCIATED WITH MEASURE-VALUED BRANCHING PROCESSES

LI Zeng-hu

(*Department of Mathematics, Beijing Normal University, Beijing 100875, PRC*)

Keywords: Measure-valued branching process, entrance law, convolution semigroup.

Suppose that E is a Lusin topological space. We let $\mathcal{B}(E)$ denote the σ-algebra on E generated by all open sets, which is referred to as the Borel σ-algebra on E. $B(E)$ denotes the set of all bounded $\mathcal{B}(E)$-measurable functions on E and $B(E)^+$ denotes the subspace of $B(E)$ comprising of non-negative elements. Let $M(E)$ be the totality of finite measures on $(E, \mathcal{B}(E))$. Topologize $M(E)$ by the weak convergence topology, so it also becomes a Lusin space. Put $M(E)^\circ = M(E) \setminus \{o\}$, where o denotes the null measure on E. The unit mass concentrated at a point $x \in E$ is denoted by δ_x. For $f \in B(E)$ and $\mu \in M(E)$, write $\mu(f)$ for $\int_E f \mathrm{d}\mu$. Suppose that $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ is a Markov process in the space $M(E)$. For $f \in B(E)^+$ set

$$V_t f(x) = -\log \mathbf{Q}_{\delta_x} \exp\{-X_t(f)\}, \quad t \geq 0, x \in E. \tag{0.1}$$

The process X is called a *measure-valued branching process*, or simply an *MB-process* if for every $t \geq 0$ and $f \in B(E)^+$, the function $V_t f$ belongs to $B(E)^+$ and

$$\mathbf{Q}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \qquad \mu \in M(E). \tag{0.2}$$

Here $\mathbf{Q}_\mu$ denotes the conditional expectation given $X_0 = \mu$.

If X is an MB-process then the family of operators $(V_t)_{t\geq 0}$ form a semigroup, which is called the *cumulant semigroup* of X. The equation (0.2) implies that $(V_t)_{t\geq 0}$ has the following canonical representation

$$V_t f(x) = \lambda_t(x, f) + \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) L_t(x, \mathrm{d}\nu), \qquad f \in B(E)^+, \tag{0.3}$$

where for every $t \geq 0$ and $x \in E$, $\lambda_t(x, \mathrm{d}y) \in M(E)$ and $1 \wedge \nu(E) L_t(x, \mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$.

Let $(Q_t)_{t\geq 0}$ be the transition semigroup of the MB-process X. Suppose that $(N_t)_{t\geq 0}$ is a family of probability measures on $M(E)$. We say that $(N_t)_{t\geq 0}$ is a *(skew) convolution semigroup* associated with X or $(Q_t)_{t\geq 0}$ provided

$$N_{r+t} = (N_r Q_t) * N_t, \qquad r, t \geq 0, \tag{0.4}$$

where '$*$' denotes the convolution operation. Clearly, (0.4) holds if and only if

$$Q_t^N(\mu, \mathrm{d}\nu) := Q_t(\mu, \mathrm{d}\nu) * N_t(\mathrm{d}\nu), \qquad t \geq 0, \tag{0.5}$$

defines a Markov transition semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. The formula (0.5) is similar to the construction of a Lévy transition semigroup from the usual convolution semigroup. It is well-known that a convolution semigroup on the Euclidean space $\mathbb{R}^d$ is uniquely determined by an infinitely divisible probability measure on $\mathbb{R}^d$. In this note, we prove an analogous result for the convolution semigroup associated with a general MB-process, which asserts a one-to-one correspondence between the convolution semigroups and infinitely divisible probability entrance laws for the MB-process. In many cases, the entrance laws for the MB-process are known[1,2], so our one-to-one correspondence describes completely the class of transition semigroups defined by (0.5). In view of (0.5), if $Y = \{Y_t : t \geq 0\}$ is a Markov process having transition semigroup $(Q_t^N)_{t\geq 0}$, we call it an *immigration process* associated with the MB-process X, where $(N_t)_{t\geq 0}$ describes the rate of the immigration[3].

I. Entrance Laws and Convolution Semigroups

A family of σ-finite measures $(K_t)_{t>0}$ on $M(E)$ is called an *entrance law* for the MB-process X or its semigroup $(Q_t)_{t\geq 0}$ if $K_{r+t} = K_r Q_t$ for all $r, t > 0$. It is called a *probability entrance law* if each K_t is a probability measure on $M(E)$, an *infinitely divisible probability entrance law* if, in addition, each K_t is infinitely divisible. An entrance law $(K_t)_{t>0}$ is *minimal* if every entrance law dominated by $(K_t)_{t>0}$ is proportional to it. See e.g. [4].

Theorem 1. *A family of probability measures $(K_t)_{t>0}$ on $M(E)$ is an infinitely divisible probability entrance law for $(Q_t)_{t\geq 0}$ if and only if its Laplace functional has the representation*

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\left\{ -\eta_t(f) - \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu)\right\}, \tag{1.1}$$

where for each $t > 0$, $\eta_t \in M(E)$, and $1 \wedge \nu(E) H_t(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$ such that

$$\eta_{r+t} = \int_E \eta_r(\mathrm{d}x)\lambda_t(x, \cdot), \tag{1.2}$$

$$H_{r+t} = \int_E \eta_r(\mathrm{d}x) L_t(x, \cdot) + \int_{M(E)^\circ} H_r(\mathrm{d}\mu) Q_t(\mu, \cdot), \tag{1.3}$$

for all $r, t > 0$. In particular, all minimal probability entrance laws for $(Q_t)_{t\geq 0}$ are infinitely divisible.

For the MB-process X defined by (0.1) and (0.2) we consider the following conditions:

(1A) for every $l \geq 0$ and $f \in B(E)^+$, the function $V_t f(x)$ of (t, x) restricted to $[0, l] \times E$ belongs to $B([0, l] \times E)^+$;

(1B) for every $f \equiv \text{const.} \geq 0$, it holds that $\lim_{t\downarrow 0} V_t f(x) = f$ for all $x \in E$.

Our next theorem gives the one-to-one correspondence between the convolution semigroups associated with a general MB-process and its infinitely divisible probability entrance laws.

Theorem 2. *Suppose that (1A) and (1B) are satisfied. Then a family of probability measures $(N_t)_{t\geq 0}$ on $M(E)$ is a convolution semigroup associated with $(Q_t)_{t\geq 0}$ if and only if there is an infinitely divisible probability entrance law $(K_t)_{t>0}$ for $(Q_t)_{t\geq 0}$ such that*

$$\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) = \int_0^t \left[\log \int_{M(E)} \mathrm{e}^{-\nu(f)} K_s(\mathrm{d}\nu) \right] \mathrm{d}s, \tag{1.4}$$

for all $t \geq 0$ and $f \in B(E)^+$.

Now we consider a special case. Suppose that ξ is a conservative Borel right process in E with transition semigroup $(P_t)_{t\geq 0}$, and that ϕ is given by

$$\phi(x, z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x, \mathrm{d}u), \qquad x \in E, z \geq 0, \tag{1.5}$$

where $b \in B(E)$, $c \in B(E)^+$, and $u \wedge u^2 m(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$. Then the following evolution equation defines a cumulant semigroup:

$$V_t f(x) + \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) P_{t-s}(x, \mathrm{d}y) = P_t f(x), \qquad t \geq 0, x \in E. \tag{1.6}$$

The corresponding MB-process X is called a (ξ, ϕ)-*superprocess*. Let E_D be the entrance space of ξ and let $(\bar{P}_t)_{t\geq 0}$ be the Ray extension of $(P_t)_{t\geq 0}$ to E_D; see e.g. [4]. For $f \in B(E)^+$ set

$$\bar{V}_t f(x) = \bar{P}_t f(x) - \int_0^t \mathrm{d}s \int_{E_D} \phi(y, V_s f(y)) \bar{P}_{t-s}(x, \mathrm{d}y), \qquad t > 0, x \in E_D. \tag{1.7}$$

Using Theorems 1 and 2 above and Theorem 3.7 of [2] one can show that $(N_t)_{t\geq 0}$ is a convolution semigroup associated with the (ξ, ϕ)-superprocess if and only if its Laplace functional has the representation

$$\begin{aligned}
&\int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu) \\
&= \exp\bigg\{ - \int_0^t \bigg[\gamma(\bar{V}_s f) + \int_{M(E_D)} \big(1 - \exp\{-\nu(\bar{V}_s f)\}\big) G(\mathrm{d}\nu) \bigg] \mathrm{d}s \bigg\},
\end{aligned}$$

where $\gamma \in M(E_D)$, and $1\wedge\nu(E)G(\mathrm{d}\nu)$ is a finite measure on $M(E_D)^\circ$. Although $(Q_t)_{t\geq 0}$ is a Borel right semigroup[2], the transition semigroup $(Q_t^N)_{t\geq 0}$ defined by (0.5) is not a Borel right semigroup on $M(E)$ in general, but it can be extended to a Borel right semigroup on $M(E_D)$.

II. Proofs of the Theorems

Proof of Theorem 1. The formula (1.1) gives the canonical representation for the Laplace functionals of the infinitely divisible probability measures $(K_t)_{t>0}$ on $M(E)$, while (1.2) and (1.3) give an alternative expression for the relation $K_{r+t} = K_r Q_t$. Therefore the former part of the theorem follows.

Now we suppose that $(K_t)_{t>0}$ is a minimal probability entrance law for X. By a well-known result on entrance laws, there is a probability measure $\mathbf{Q}_K$ on the space $M(E)^{(0,\infty)}$ under which the coordinate process $\{w_t : t > 0\}$ is a Markov process having transition semigroup $(Q_t)_{t\geq 0}$ and one-dimensional distributions $(K_t)_{t>0}$. Since $(K_t)_{t>0}$ is minimal, for each $f \in B(E)^+$ we have, $\mathbf{Q}_K$-almost surely,

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} K_t(d\nu) = \lim_{\mathrm{rat}.r\downarrow 0} \exp\{-w_r(V_{t-r}f)\}, \tag{2.1}$$

where "rat. $r \downarrow 0$" means "$r > 0$ tends to 0 decreasingly along rational". See e.g. [4]. It is known that a metric r can be introduced into E so that (E, r) become a compact metric space, while the Borel σ-algebra induced by r coincides with $\mathcal{B}(E)$. Let $D(E)^+$ be a countable dense subset of the space of strictly positive continuous functions on (E, r). Choosing any path $\{w_t : t > 0\}$ in $M(E)^{(0,\infty)}$ along which (2.1) holds for all $f \in D(E)^+$ one sees the infinite divisibility of $(K_t)_{t>0}$. $\square$

Proof of Theorem 2. If $(N_t)_{t\geq 0}$ is given by (1.4), then (0.4) clearly holds. We shall prove the converse. Suppose $(N_t)_{t\geq 0}$ is a convolution semigroup associated with $(Q_t)_{t\geq 0}$. Define

$$J_t(f) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_t(\mathrm{d}\nu), \qquad t \geq 0, f \in B(E)^+. \tag{2.2}$$

Then the relation (0.4) is equivalent to

$$J_{r+t}(f) = J_t(f) + J_r(V_t f), \qquad r, t \geq 0, f \in B(E)^+. \tag{2.3}$$

Consequently for $f \in B(E)^+$, $J_t(f)$ is a non-decreasing function of $t \geq 0$. By (2.2) and (2.3) together with the assumptions (1A) and (1B) one can show that $J_t(f) \to 0$ as $t \to 0$. Let $q = q(l, f) > 0$ be such that $V_t f(x) \leq q$ for all $0 \leq t \leq l$ and $x \in E$. For

$0 \le c_1 < d_1 < \cdots < c_n < d_n \le l$, set $\sigma_n = \sum_{i=1}^n (d_i - c_i)$. By induction in $n = 1, 2, \cdots$, one finds easily

$$\sum_{i=1}^n \left[J_{d_i}(f) - J_{c_i}(f)\right] \le J_{\sigma_n}(q). \tag{2.4}$$

Therefore $J_t(f)$ is absolutely continuous in $t \ge 0$, say, $J_t(f) = \int_0^t I_s(f)\mathrm{d}s$. Let $D(E)^+$ be as defined in the proof of Theorem 1. Then there is a Lebesgue null subset N of $[0, \infty)$ such that

$$I_s(f) = \lim_{r \downarrow 0} r^{-1} \left[J_{s+r}(f) - J_s(f)\right], \qquad 0 \le s \notin N, f \in D(E)^+. \tag{2.5}$$

Using (2.2) – (2.5) we get

$$I_s(f) = \lim_{r \downarrow 0} \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s^{(r)}(\mathrm{d}\nu), \qquad 0 \le s \notin N, f \in D(E)^+, \tag{2.6}$$

where

$$H_s^{(r)}(\mathrm{d}\nu) = r^{-1} \int_{M(E)} N_r(\mathrm{d}\mu) Q_s(\mu, \mathrm{d}\nu).$$

Then, by enlarging the Lebesgue null set N, we can assume $I_s(f)$ has the following representation:

$$I_s(f) = \eta_s(f) + \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu), \qquad 0 \le s \notin N, f \in D(E)^+,$$

where $\eta_s \in M(E)$ and $1 \wedge \nu(E) H_s(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$. Consequently,

$$J_t(f) = \int_0^t \left[\eta_s(f) + \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu)\right] \mathrm{d}s, \qquad t \ge 0, f \in B(E)^+. \tag{2.7}$$

Now the equation (2.3) yields

$$\int_0^r \eta_{t+s}(f)\mathrm{d}s = \int_0^r \mathrm{d}s \int_E \eta_s(\mathrm{d}x) \lambda_t(x, f), \qquad r, t \ge 0, f \in B(E)^+.$$

By Fubini's theorem, there are Lebesgue null subsets N' and N'_s of $[0, \infty)$ such that

$$\eta_{t+s} = \int_E \eta_s(\mathrm{d}x) \lambda_t(x, \cdot), \qquad 0 \le s \notin N', 0 \le t \notin N'_s.$$

Choose a sequence $0 < s_n \notin N'$ with $s_n \downarrow 0$, and define

$$\eta_t = \int_E \eta_{s_n}(\mathrm{d}x) \lambda_{t - s_n}(x, \cdot), \qquad t \ge s_n.$$

Under this modification, $(\eta_t)_{t>0}$ clearly satisfy (1.2), and for each $t \geq 0$ and $f \in B(E)^+$ the value $\int_0^t \eta_s(f)\mathrm{d}s$ remains unchanged. By a similar procedure, one can modify the definition of $(H_t)_{t>0}$ to make it satisfy (1.3) while (2.7) remains valid. Then the desired result follows by Theorem 1. $\square$

References

1. Dynkin, E.B., *Three classes of infinite dimensional diffusion processes*, J. Funct. Anal., **86** (1989), 75-110.
2. Fitzsimmons, P.J., *Construction and regularity of measure-valued Markov branching processes*, Israel J. Math., **64** (1988), 337-361.
3. Li, Z.H., *Measure-valued branching processes with immigration*, Stoch. Proc. Appl., **43** (1992), 249-264.
4. Sharpe, M.J., *General Theory of Markov Processes*, Academic Press, New York, 1988.
5. Watanabe, S., *A limit theorem of branching processes and continuous state branching processes*, J. Math. Kyoto Univ., **8** (1968), 141-167.