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ASYMPTOTIC BEHAVIOR OF THE MEASURE-VALUED BRANCHING PROCESS WITH IMMIGRATION

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Abstract. The measure-valued branching process with immigration is defined as $Y_t = X_t + I_t,\ t \geq 0$, where X_t satisfies the branching property and I_t with $I_0 = 0$ is independent of X_t. This formulation leads to the model of [12,14,15]. We prove a large number law for Y_t. Equilibrium distributions and spatial transformations are also studied.

Key words: branching process, immigration, large number law, equilibrium distribution, transformation

1. Introduction

The measure-valued branching process with immigration (MBI-process) arises as the high density limit of a certain branching particle system with immigration; see Kawazu and Watanabe [12], Konno and Shiga [13], Shiga [19], Dynkin [4,5] and Li [14,15] for construction of the MBI-process and for detailed descriptions of the particle system. Multitype MBI-processes have been considered by Gorostiza and Lopez-Mimbela [9] and Li [16].

Suppose (Y_t) is a measure-valued Markov process. It is natural to call (Y_t) an MBI-process provided

$$(Y_t|Y_0 = \mu) = (X_t|X_0 = \mu) + I_t, \quad t \geq 0, \tag{1.1}$$

(in distribution) where (X_t) is an MB-process, and (I_t) with $I_0 = 0$ (the null measure) a.s. is independent of (X_t). In section 2 of this draft we prove that under certain regularity conditions this formulation will lead to the model proposed by Kawazu and Watanabe [12]. Section 3 concerns the convergence of $a_t Y_t$ as $t \to \infty$, where (a_t) is a suitable family of constants. Equilibrium distributions and spatial transformations are discussed in section 4.

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2. The MBI-process

Let E be a topological Lusin space (that is, a homeomorph of a Borel subset of a compact metric space) with the Borel σ-algebra $\mathcal{B}(E)$. We shall use the notation introduced in Li [14]:

$B(E)^+ = \{$ bounded nonnegative Borel functions on E $\}$,

$M = \{$ finite Borel measures on E $\}$,

$M_0 = \{\pi : \ \pi \in M$ and $\pi(E) = 1\}$.

Suppose that M and M_0 are equipped with the usual weak topology. We call w a *cumulant* and put $w \in \mathcal{W}$ if it is a functional on $B(E)^+$ with representation

$$w(f) = \iint\limits_{R^+ \times M_0} \left(1 - e^{-u\langle \pi, f\rangle}\right) \frac{1+u}{u} G(du, d\pi), \tag{2.1}$$

where $R^+ = [0, \infty)$, G is a finite measure on $R^+ \times M_0$, and the value of the integrand at $u = 0$ is defined as $\langle \pi, f\rangle := \int f d\pi$. It is well known that P is an infinitely divisible probability measure on M if and only if the Laplace functional L_P has the canonical representation $L_P(f) = \exp\{-w(f)\}$, where $w \in \mathcal{W}$.

A Markov process (X_t, P_μ) in the space M is an *MB-process* if it satisfies the *branching property,*

$$P_\mu \exp\langle X_t, -f\rangle = \exp\langle \mu, -w_t\rangle, \tag{2.2}$$

where P_μ denotes the conditional expectation given $X_0 = \mu$, and $W_t : f \mapsto w_t$ is a semigroup of operators on $B(E)^+$, the so-called *cumulant semigroup.*

Let Q_μ denote the conditional law of the MBI-process (Y_t) in (1.1) given $Y_0 = \mu$. Assume (X_t) satisfies (2.2). Then

$$Q_\mu \exp\langle Y_t, -f\rangle = \exp\left\{-\langle \mu, w_t\rangle - j_t(f)\right\}, \tag{2.3}$$

where $j_t(f) = -\log \mathbf{E} \exp\langle I_t, -f\rangle$. To ensure that (Y_t) is Markovian, the (I_t) is not arbitrary. Typically,

$$j_t(f) = \int_0^t i(w_s) ds, \quad t \geq 0, \tag{2.4}$$

for some $i \in \mathcal{W}$. The process (Y_t, Q_μ) defined by (2.3) and (2.4) will be called an *MBI-process with parameters* (W, i). This is the case studied by Li [14,15].

When E is a one point set, Kawazu and Watanabe [12] proved that (2.3) and (2.4) represent the most general form of the MBI-process (Y_t) given by (1.1). The following Theorem 2.1 shows that this generality remains valid in the present case. Following Kawazu and Watanabe [12] and Watanabe [20], we call the cumulant semigroup (W_t) a Ψ-semigroup if E is a compact metric space and (W_t) preserves $C(E)^{++}$, the strictly positive continuous functions on E.

Theorem 2.1. *Suppose $W_t : f \mapsto w_t$ is a Ψ-semigroup and is (weakly) continuous on $C(E)^{++}$. Then (j_t) has the form (2.4) with $i \in \mathcal{W}$.*

Proof. Since any probability measure P is uniquely determined by the Laplace functional L_P restricted to $C(E)^{++}$, it is sufficient to prove (2.4) for all $f \in C(E)^{++}$. We shall follow the lead of [12]. For $f \in C(E)^{++}$ and $(u, \pi) \in [0, \infty] \times M_0$ define

$$\xi(u, \pi; f) = \begin{cases} \left(1 - e^{-u\langle \pi, f\rangle}\right) \frac{1+u}{u}, & 0 < u < \infty, \\ \langle \pi, f\rangle, & u = 0, \\ 1, & u = \infty. \end{cases} \tag{2.5}$$

Then $\xi(u, \pi; f)$ is jointly continuous in (u, π) for fixed f. It is easy to see that $\exp(-j_t)$ has the form

$$\exp\{-j_t(f)\} = 1 - \iint_{[0,\infty]\times M_0} \xi(u, \pi; f) G_t(du, d\pi), \tag{2.6}$$

where G_t is actually carried by $(0, \infty) \times M_0$. The Chapman-Kolmogorov equation yields

$$j_{t+s}(f) = j_t(f) + j_s(w_t), \quad t, s \geq 0. \tag{2.7}$$

Thus $j_t(f)$ is increasing in $t \geq 0$ for all f, so the limit

$$i_t(f) := \lim_{s\to 0^+} \frac{j_{t+s}(f) - j_t(f)}{s} \equiv \lim_{s\to 0^+} s^{-1} j_s(w_t) \tag{2.8}$$

exists for almost all $t \geq 0$. Consequently,

$$i_0(f) = \lim_{s\to 0^+} s^{-1} j_s(f) \tag{2.9}$$

exists for f in a dense subset D of $C(E)^{++}$, and by (2.5) and (2.6)

$$\sup_{0<s<\delta} s^{-1} G_s([0, \infty] \times M_0) < \infty$$

for some $\delta > 0$. Since $[0, \infty] \times M_0$ is a compact metric space, this implies $\{s^{-1}G_s : 0 < s < \delta\}$ is relatively compact under the weak convergence. Suppose $s_n \to 0$ and $s_n^{-1} G_{s_n} \to G$ as $n \to \infty$. Then

$$\lim_{n\to\infty} s_n^{-1} j_{s_n}(f) = i(f), \quad f \in C(E)^{++},$$

where

$$i(f) = \iint_{[0,\infty]\times M_0} \xi(u, \pi; f) G(du, d\pi). \tag{2.10}$$

By a standard argument one gets the existence of (2.9), and hence (2.8), for all $f \in C(E)^{++}$, $t \geq 0$ and $i_t(f) = i(w_t)$. Then (2.4) follows. Letting $f \to 0^+$ in (2.3) gives $G(\{\infty\} \times M_0) = 0$. $\square$

3. The (ξ, ϕ, i)-superprocess

3.1. Usually, the cumulant semigroup of the MBI-process is given by an evolution equation. Let $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, \xi_t, \mathit{\Pi}_x)$ be a Borel right Markov process[18] in space E with semigroup $(\mathit{\Pi}_t)$ and ϕ a "branching mechanism" represented by

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty \left(e^{-zu} - 1 + zu\right) m(x,du), \quad x \in E, z \geq 0, \tag{3.1}$$

where $c \geq 0$ and b are $\mathcal{B}(E)$-measurable functions and m is a kernel from E to $\mathcal{B}((0,\infty))$ such that $\int u \wedge u^2 m(\cdot, du) \in B(E)^+$. The MBI-process Y defined by (2.3) and (2.4) is called a (ξ, ϕ, i)-*superprocess* if the associated cumulant semigroup is uniquely determined by

$$w_t + \int_0^t \mathit{\Pi}_{t-s}\phi(w_s)ds = \mathit{\Pi}_t f, \quad t \geq 0. \tag{3.2}$$

Several authors [4,5,13,etc] have studied the (ξ, ϕ, i)-superprocess in the special case $i(f) = \langle \lambda, f \rangle$ for some $\lambda \in M$.

In this section we study the limiting behavior of the MBI-process. A typical case is where

$$Q_\mu \exp\langle Y_t, -f \rangle = \exp\left\{ -\langle \mu, w_t \rangle - \int_0^t \langle \lambda, w_s \rangle^\theta ds \right\}, \tag{3.3}$$

where $\lambda \in M,\ 0 < \theta \leq 1$ and w_t satisfies

$$w_t + \int_0^t \mathit{\Pi}_{t-s}(w_s)^{1+\beta} ds = \mathit{\Pi}_t f, \quad t \geq 0, \tag{3.4}$$

with $0 < \beta \leq 1$. For $f(x) \equiv \gamma > 0$, (3.2) has the solution

$$w_t = \gamma(1 + \beta\gamma^\beta t)^{-1/\beta}.$$

Thus

$$Q_\mu \exp\{-\gamma Y_t(E)\} = \exp\left\{ \frac{-\gamma\mu(E)}{(1+\beta\gamma^\beta t)^{1/\beta}} - \int_0^t \frac{\gamma^\theta \lambda(E)^\theta ds}{(1+\beta\gamma^\beta s)^{\theta/\beta}} \right\}. \tag{3.5}$$

It is clear from (3.5) that $Y_t(E) \to 0$ $(t \to \infty)$ in probability if $\lambda(E) = 0$. When $\lambda(E) > 0$ and $\beta < \theta$, we have

$$\int_0^\infty \frac{\gamma^\theta \lambda(E)^\theta ds}{(1+\beta\gamma^\beta s)^{\theta/\beta}} = \text{const.}\gamma^{\theta-\beta} < \infty,$$

and

$$Q_\mu \exp\langle Y_t, -f \rangle \ \to\ \exp\left\{ -\int_0^\infty \langle \lambda, w_s \rangle^\theta ds \right\} \qquad (t \to \infty) \tag{3.6}$$

uniformly on the set $\{0 \le f \le \gamma\}$. Therefore $Y_t \to Y_\infty$ set-wise in distribution, where Y_∞ is an M-valued random measure with Laplace functional given by the r.h.s. of (3.6). When $\lambda(E) > 0$ and $\beta \ge \theta$, (3.5) shows that $Y_t(E) \to \infty$ (explodes) in probability as $t \to \infty$, so it can be vacuous to discuss the convergence of (Y_t). In the next paragraph we shall study the convergence of the process $(a_t Y_t)$ for a suitably chosen family (a_t) of constants.

3.2. We now assume ξ is the Brownian motion in R^d. The name *super Brownian motion* is used at this time for the (ξ, ϕ, i)-superprocess. Let λ denote the Lebesgue measure on R^d and let

$$M_p(R^d) := \{\ \sigma\text{-finite Borel measures } \mu \text{ on } R^d \text{ such that } \int (1+|x|^p)^{-1}\mu(dx) < \infty\ \}$$

for $p > d$. Suppose w_t is determined by

$$w_t + \int_0^t \Pi_{t-s}(w_s)^2 ds = \Pi_t f, \quad t \ge 0, \tag{3.7}$$

where (Π_t) denotes the semigroup of ξ. Then

$$Q_\mu \exp\langle Y_t, -f\rangle = \exp\left\{-\langle \mu, w_t\rangle - \int_0^t \langle \lambda, w_s\rangle ds\right\} \tag{3.8}$$

defines a super Brownian motion (Y_t) in space $M_p(R^d)$ (cf. [13,14]). Since the "immigration measure" λ is nonzero, more and more "people" immigrate to the space E as time goes on. The following theorem gives a large number law for (Y_t) and completes the observation of paragraph 3.1.

Theorem 3.1. *For any bounded Borel set $B \subset R^d$ and finite measure $\mu \in M_p(R^d)$,*

$$t^{-1} Y_t(B) \ \to\ \lambda(B) \quad (t \to \infty) \tag{3.9}$$

in probability w.r.t. Q_μ.

Proof. It is, obviously, sufficient to prove the result for $\mu = 0$. Our method of the proof relies on the estimates of the moments of Y_t. For fixed $f \in B_p(R^d)^+$, the members of $B(R^d)^+$ upper bounded by const.$(1+|x|^p)^{-1}$, we define

$$\varphi_t^{1*}(x) = \varphi_t(x) = \Pi_t f(x), \qquad u_t * v_t = \int_0^t \Pi_{t-s} u_s v_s ds,$$

$$\varphi_t^{n*} = \sum_{k=1}^{n-1} \varphi_t^{k*} * \varphi_t^{(n-k)*}, \qquad \Phi_n(t) = \int_0^t \langle \lambda, \varphi_s^{n*}\rangle ds.$$

Put

$$M_n(t) = Q_0\langle Y_t, f\rangle^n,$$
$$C_n(t) = Q_0[\langle Y_t, f\rangle - M_1(t)]^n, \quad n = 1, 2, \cdots .$$

Routine computations give

$$M_n(t) = \sum_{k=1}^{n} \binom{n-1}{k-1} k! \Phi_k(t) M_{n-k}(t),$$

and

$$\begin{aligned} C_n(t) =& \sum_{i=0}^{n} \binom{n}{i} (-1)^i M_{n-i}(t) M_1(t)^i \\ =& n!\Phi_n(t) + \sum \text{const.} \Phi_2^{k_2}(t) \cdots \Phi_{n-2}^{k_{n-2}}(t). \end{aligned} \tag{3.10}$$

Here the last summation is taken for all possible $\{k_2, \cdots, k_{n-2}\}$ satisfying $2k_2 + \cdots + (n-2)k_{n-2} = n$, for instance,

$$C_2(t) = 2\Phi_2(t),\ C_3(t) = 6\Phi_3(t),$$
$$C_4(t) = 24\Phi_4(t) + 12\Phi_2^2(t),$$
$$C_5(t) = 120\Phi_5(t) + 120\Phi_2(t)\Phi_3(t).$$

Let $p_t(x-y) \equiv p_t(x,y)$ denote the Brownian transition density. We have

$$\begin{aligned} C_2(t) &= 2\int_0^t ds \big\langle \lambda, \int_0^s \Pi_{s-u}(\Pi_u f)^2 du \big\rangle \\ &= 2\int_0^t ds \int_0^s du \int [\Pi_u f(x)]^2 dx \\ &= 2\int_0^t ds \int_0^s du \int dx \int f(y) p_u(y-x) dy \Pi_u f(x) \\ &= 2\int_0^t ds \int_0^s du \int f(y) dy \int p_u(y-x) \Pi_u f(x) dx \\ &= 2\int_0^t ds \int_0^s du \int f(y) dy \int p_{2u}(y-z) f(z) dz \\ &= \int_0^t ds \int_0^{2s} du \int dy \int dz f(y) f(z) p_u(y-z). \end{aligned}$$

If f is supported boundedly, then

$$\int dy \int dz f(y) f(z) p_u(y-z) < 1 \wedge u^{-d/2} \cdot \text{const.}$$

Now we use Chebyshev's inequality to obtain

$$Q_0 \left\{ \left| t^{-1}\langle Y_t, f\rangle - \langle \lambda, f\rangle \right| > \varepsilon \right\} \\ \leq \varepsilon^{-2} t^{-2} C_2(t) \;\to\; 0 \;\; (t \to \infty)$$

for every $\varepsilon > 0$, as desired. $\square$

It is interesting and, undoubtedly, possible to extend the above theorem to some more general cases. We shall leave the consideration of this to the reader. At times one would like to consider the "weighted occupation time" $Z_t := \int_0^t Y_s ds$. Following the computations of [10], we get the characterization of the joint law of Y_t and Z_t,

$$Q_\mu \exp\{-\langle Y_t, f\rangle - \langle Z_t, g\rangle\} = \exp\left\{ -\langle \mu, u_t\rangle - \int_0^t \langle \lambda, u_s\rangle ds \right\}, \tag{3.11}$$

where u_t is the solution of

$$u_t + \int_0^t \Pi_{t-s}(u_s)^2 ds = \Pi_t f + \int_0^t \Pi_s g ds, \quad t \geq 0. \tag{3.12}$$

A remarkable property of the process Z_t is that for each $\mu \in M_p(R^d)$ and bounded Borel $B \subset R^d$,

$$2t^{-2} Z_t(B) \to \lambda(B) \quad (t \to \infty) \tag{3.13}$$

almost surely w.r.t. Q_μ, which is proved by similar means as Theorem 1 of [11].

4. The (ξ, ϕ)-superprocess

In this section, we discuss the long-term behavior of the (ξ, ϕ)-*superprocess* (X_t) defined by (2.2) and (3.2). The observations in paragraph 3.1 shows that usually we need start the process with an infinite initial state to get interesting results (cf. [3]).

4.1. We fix some *strictly* positive reference function $\rho \in B(E)^+$ and introduce the following assumptions

2.A) For each $T > 0$, there exists $C_T > 0$ such that $\Pi_t \rho \leq C_T \rho$ for all $0 \leq t \leq T$.

2.B) The branching mechanism ϕ given by (3.1) is subcritical, i.e., $b \geq 0$.

Then the solution $w_t(f)$ of (3.2) satisfies $w_t(\rho f) \leq \text{const.}\rho$, $f \in B(E)^+$, on each finite interval $0 \leq t \leq T$. Thus we can assume the state space of (X_t) is $M^\rho := \{\rho^{-1}\mu : \;\mu \in M\}$ [e.g. 1,3,6,14]. M^ρ contains some infinite measures unless ρ is bounded away from zero.

Theorem 4.1. *If $m \in M^\rho$ is Π_t-invariant, then*

$$P_m \exp\langle X_t, -f\rangle \;\to\; \exp\left\{-\langle m, f\rangle + \int_0^\infty \langle m, \phi(w_s)\rangle ds\right\} \quad (t \to \infty) \tag{4.1}$$

uniformly on the set $\{0 \le f \le a\rho\}$ for every finite $a \ge 0$, and the right hand side of the above formula defines the Laplace functional of a equilibrium distribution Λ_m of X such that $\int \langle \mu, \rho\rangle \Lambda_m(d\mu) < \infty$.

Proof. By (2.2), (3.2) and the Π_t-invariance of m,

$$\begin{aligned} 1 \ge\, & P_\mu \exp\langle X_t, -f\rangle \\ &= \exp\left\{\left\langle m, -\Pi_t f + \int_0^t \Pi_{t-s}\phi(w_s)ds\right\rangle\right\} \\ &= \exp\left\{\langle m, -f\rangle + \int_0^t \langle m, \phi(w_s)\rangle ds\right\}. \end{aligned}$$

Choosing $f(x) = a\rho(x)$ and letting $t \to \infty$, we have

$$\int_0^\infty \langle m, \phi(w_s(a\rho))\rangle ds \le \langle m, a\rho\rangle < \infty.$$

If $f(x) \le a\rho(x)$, then $\langle m, \phi(w_s(f))\rangle \le \langle m, \phi(w_s(a\rho))\rangle$. Thus

$$\int_0^t \langle m, \phi(w_s(f))\rangle ds \;\to\; \int_0^\infty \langle m, \phi(w_s(f))\rangle ds \quad (t \to \infty)$$

uniformly on the set $\{0 \le f \le a\rho\}$, and the convergence (4.1) follows. By Lemma 2.1 of Dynkin [3], the r.h.s. of (4.1) is the Laplace functional of a probability measure Λ_m on M^ρ. Λ_m is clearly an invariant measure of X, so to finish the proof it is sufficient to observe

$$\begin{aligned} & \int \Lambda_m(d\mu)\langle \mu, f\rangle \\ =& \lim_{\beta \to 0^+} \int \Lambda_m(d\mu)\beta^{-1}\left(1 - e^{-\beta\langle \mu, f\rangle}\right) \\ =& \lim_{\beta \to 0^+} \beta^{-1}\left[1 - \exp\left\{-\langle m, \beta f\rangle + \int_0^\infty \langle m, \phi(w_s(\beta f))\rangle ds\right\}\right] \\ \le& \lim_{\beta \to 0^+} \beta^{-1}\left(1 - e^{-\langle m, \beta f\rangle}\right) \\ =& \langle m, f\rangle < \infty \end{aligned}$$

for any $0 \le f \le a\rho$. $\square$

Theorem 3.1 was clearly inspired by the work of Dynkin [3], where the invariant measures of the superprocess (X_t) was studied completely when $\phi(x,z)=$const.z^2. Suppose Λ is an invariant measure of (X_t) such that $\int\langle\mu,\rho\rangle\Lambda(d\mu)<\infty$. One proves easily that

$$\langle m,f\rangle=\int_{M^\rho}\langle\mu,f\rangle\Lambda(d\mu) \tag{4.2}$$

defines a measure $m\in M^\rho$ that is invariant under the subMarkov semigroup (Π_t^b) :

$$\Pi_t^b f(x)=\Pi_x f(\xi_t)\exp\big\{-\textstyle\int_0^t b(\xi_s)ds\big\}.$$

4.2. Using the martingale characterization, El Karoui and Roelly-Coppoletta [6] showed that the class of (ξ,ϕ)-superprocesses is stable under some spatial transformations. In this paragraph we shall see that this stableness can also be derived easily from (2.2) and (3.2). We shall not assume the transformations to be one to one.

Suppose γ is a measurable surjective map from $(E,\mathcal{B}(E))$ onto another space $(\widetilde{E}$, $\mathcal{B}(\widetilde{E}))$. We assume

4.C) ρ is $\gamma^{-1}\mathcal{B}(\widetilde{E})$-measurable;

4.D) if f is $\gamma^{-1}\mathcal{B}(\widetilde{E})$-measurable, so is $\Pi_t f$ for all $t\geq 0$;

4.E) for each fixed $z\geq 0$, $\phi(\cdot,z)$ is $\gamma^{-1}\mathcal{B}(\widetilde{E})$-measurable.

Let γ^* be the map from $B(\widetilde{E})^+$ to $B(E)^+$ defined by $\gamma^*\widetilde{f}(x)=\widetilde{f}(\gamma x)$. By 4.D), $\widetilde{\xi}=(\gamma\xi_t,t\geq 0)$ is a Markov process in $\widetilde{E}$ with the semigroup $(\widetilde{\Pi}_t)$ determined by $\gamma^*\widetilde{\Pi}_t=\Pi_t\gamma^*$ (cf. [2, p325] and [17, p66]). Put $\widetilde{\phi}(\widetilde{x},z)=\phi(x,z)$ for any $\gamma x=\widetilde{x}$. Operating the equation

$$\widetilde{w}_t+\int_0^t\widetilde{\Pi}_{t-s}\widetilde{\phi}(\widetilde{w}_s)ds=\widetilde{\Pi}_t\widetilde{f}$$

with γ^* gives

$$\gamma^*\widetilde{w}_t+\int_0^t\Pi_{t-s}\phi(\gamma^*\widetilde{w}_s)ds=\Pi_t\gamma^*\widetilde{f}.$$

By the uniqueness of the solution to (3.2), we get

$$w_t(\gamma^*\widetilde{f})=\gamma^*\widetilde{w}_t(\widetilde{f}). \tag{4.3}$$

Let $X=(X_t,t\geq 0)$ be a (ξ,ϕ)-superprocess, and let $\widetilde{X}_t(\widetilde{B})=X_t\circ\gamma^{-1}(\widetilde{B})$, $\widetilde{B}\in\mathcal{B}(\widetilde{E})$. By (2.2) and (4.3), $\widetilde{X}=(\widetilde{X}_t,t\geq 0)$ is a Markov process[2,17] in $\widetilde{M^\rho}$, the space of σ-finite measures ν on $(\widetilde{E},\mathcal{B}(\widetilde{E}))$ satisfying $\langle\nu,\widetilde{\rho}\rangle<\infty$, with transition probabilities $\widetilde{P}_\nu$ determined by

$$\widetilde{P}_\nu\exp\langle\widetilde{X}_t,-\widetilde{f}\rangle=\exp\langle\nu,-\widetilde{w}_t(\widetilde{f})\rangle,$$

i.e., $\widetilde{X}$ is a $(\widetilde{\xi},\widetilde{\phi})$-superprocess.

Example 4.2. Let ξ be a symmetric stable process in R^d. Suppose that $\phi(x, z) \equiv \phi(z)$ is independent of $x \in R^d$. Let γ_x be the spatial translation operator by $x \in R^d$. If the (ξ, ϕ)-superprocess X has initial value λ, the Lebesgue measure on R^d, then by the preceding result, X_t and $X_t \circ \gamma_x^{-1}$ has the same distribution. Letting $t \to \infty$, we see that the equilibrium distribution Λ of X from λ is translation invariant.

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