BRANCHING PARTICLE SYSTEMS WITH IMMIGRATION*

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In this note, we introduce a model of branching particle systems with immigration and show a result on the weak convergence of such systems to measure-valued Markov processes.

1. Notations.

Suppose that E is a topological Lusin space with the Borel σ-algebra $\mathcal{B}(E)$. Let

$B^+(E) = \{$ bounded positive $\mathcal{B}(E)$-measurable functions on E $\}$,
$B_a^+(E) = \{f : f \in B^+(E)$ and $f \leq a\}$, $a \geq 0$,
$C^+(E) = \{f : f \in B^+(E)$ is continuous $\}$,
$C^{++}(E) = \{f : f \in C^+(E)$ is *strictly* positive $\}$,
$M = \{$ finite measures on $(E, \mathcal{B}(E))$ $\}$,
$M_0 = \{\pi : \pi \in M$ and $\pi(E) = 1\}$,
$M_1 = \{\sigma : \sigma \in M$ is integer-valued $\}$,
$M_k = \{k^{-1}\sigma : \sigma \in M_1\}$, $k = 2, 3, \cdots$,
$\delta_x =$ unit mass concentrated at x, $x \in E$.

We topologize M, and hence $M_k, k = 0, 1, 2, \cdots$, with the weak convergence topology. When E is a compact metric space, M is locally compact and metrizable. In this particular case, $D([0, \infty), M)$ denotes the space of cadlag functions from $[0, \infty)$ to M with the Skorohod topology.

For a $\mathcal{B}(E)$-measurable function f and a measure $\mu \in M$, note $\langle \mu, f \rangle = \int f d\mu$.

Finally, G denotes the totality of subcritical probability generating functions g, i.e.,

$$g(z) = \sum_{i=0}^{\infty} p_i z^i \qquad (p_i \geq 0,\ \Sigma p_i = 1,\ \Sigma i p_i \leq 1),$$

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and $\mathcal{G}$ is the σ-algebra on G generated by the mappings $F_z : g \mapsto g(z)$ as z runs over the unit interval $[0,1]$.

2. The particle system.

Suppose $\xi = (\xi_t, \mathcal{F}_t, \Pi_x)$ is a Borel right Markov process with state space E, $\gamma = \gamma(x) \in B^+(E)$ and $F = F(x, dg, d\pi)$ is a Markov kernel from E to $\mathcal{G} \times \mathcal{B}(M_0)$. A branching particle system with parameters (ξ, γ, F) is described by the following properties:

2.A) the particles in E move according to the law of the process ξ;

2.B) for a particle α which is alive at time r and follows the path $(\xi_t, t \geq r)$, the conditional probability of survival during the time interval $[r, t)$ is $\exp\{-\int_r^t \gamma(\xi_s) ds\}$;

2.C) when α dyes at point $x \in E$, it gives birth to a random number of offspring according to a generating function g^α, and the offspring are displaced in E from x according to a probability measure π^α, where (g^α, π^α), depending on the conditions at x, is a random variable in $G \times M_0$ with distribution $F(x, dg, d\pi)$.

It is assumed that the motions, life times and branchings of the particles and the first locations of the offspring are independent of each other.

For $t \geq 0$, let $X_t(B)$ be the number of particles of the system in set $B \in \mathcal{B}(E)$ at time t. Properties 2.A $-$ C) imply that $X = (X_t, t \geq 0)$ is a Markov process in space M_1 with transition probabilities P_σ defined by the Laplace functionals,

$$P_\sigma \exp\langle X_t, -f\rangle = \exp\langle \sigma, -v_t\rangle, \quad f \in B^+(E), \sigma \in M_1, t \geq 0, \tag{2.1}$$

where

$$v_t(x) \equiv v_t(x, f) = -\log P_{\delta_x} \exp\langle X_t, -f\rangle$$

is the unique positive solution of

$$\begin{aligned} e^{-v_t(x)} =& \Pi_x e^{-f(\xi_t) - \int_0^t \gamma(\xi_s) ds} \\ &+ \Pi_x \int_0^t e^{-\int_0^s \gamma(\xi_u) du} \gamma(\xi_s) \iint_{G \times M_0} g(\langle \pi, e^{-v_{t-s}}\rangle) F(\xi_s, dg, d\pi) ds. \end{aligned} \tag{2.2}$$

Equation (2.2) arises as follows: If we start one particle at time 0 at location x, this particle moves following a path of ξ and does not branch before time t (first term on the right hand side), or it splits at time $s \in [0, t]$ with probability

$$\exp\Big\{ -\int_0^s \gamma(\xi_u) du \Big\} \gamma(\xi_s) ds$$

according to $F(\xi_s, dg, d\pi)$ and all the offspring evolve independently after birth in the same fashion (second term). A rigorous construction for process X can be given as for the branching particle system studied in [2].

Let $\Pi_t, t \geq 0$, be the transition semigroup of ξ. By Lemma 2.3 of [2], equation (2.2) can be written in an abbreviative form:

$$\int_0^t \Pi_s \gamma \big[e^{-v_{t-s}} - \iint_{G \times M_0} g(\langle \pi, e^{-v_{t-s}} \rangle) F(dg, d\pi) \big] ds = \Pi_t e^{-f} - e^{-v_t}. \tag{2.3}$$

Suppose that η is a finite measure on $(M_0, \mathcal{B}(M_0))$ and that $h(\pi, z)$ is a $\mathcal{B}(M_0 \times [0,1])$-measurable function such that for each $\pi \in M_0$, $h(\pi, \cdot)$ belongs to G. A *branching particle system with immigration* with parameters $(\xi, \gamma, F, \eta, h)$ is described by properties 2.A $-$ C) and the following

2.D) the entry times and distributions of new particles immigrating to E are governed by a Poisson random measure with intensity $ds \times \eta(d\pi)$;

2.E) $h(\pi, \cdot)$ gives the distribution of the number of new particles entering E with distribution π.

Here we assume that the immigration is independent of the evolution of the inner population and that the immigrants behave as the natives after entering E. The particle distribution process $(Y_t, t \geq 0)$ of the branching system with immigration is a Markov process with state space M_1. Properties 2.A $-$ E) lead through a calculation to the Laplace functional:

$$\begin{aligned} &Q_\sigma \exp\langle Y_t, -f \rangle \\ &= \exp\left\{ -\langle \sigma, v_t \rangle - \int_0^t ds \int_{M_0} [1 - h(\pi, \langle \pi, e^{-v_s} \rangle)] \eta(d\pi) \right\}, \\ &\qquad f \in B^+(E), \sigma \in M_1, t \geq 0, \end{aligned} \tag{2.4}$$

where Q_σ denotes the transition probability from σ of Y_t and v_t is defined by equation (2.3).[6]

3. A limit theorem.

3.1. Let $Y(k) = \{Y_t(k), t \geq 0\}$ be a sequence of branching particle systems with immigration with parameters $(\xi, \gamma, F_k, k\eta, h_k), k = 1, 2, \cdots$. Then for each k,

$$Y^{(k)} = \{Y_t^{(k)} \equiv k^{-1} Y_t(k), t \geq 0\}$$

is a Markov process in M_k with transition probabilities $Q_{\sigma_k}^{(k)}$ determined by

$$\begin{aligned} &Q_{\sigma_k}^{(k)} \exp\langle Y_t^{(k)}, -f \rangle \\ &= \exp\left\{ -\langle \sigma_k, k v_t^{(k)} \rangle - \int_0^t ds \int_{M_0} k[1 - h_k(\pi, \langle \pi, e^{-v_s^{(k)}} \rangle)] \eta(d\pi) \right\}, \\ &\qquad f \in B^+(E), \sigma_k \in M_k, t \geq 0, \end{aligned} \tag{3.1}$$

where $v_t^{(k)}(x) \equiv v_t^{(k)}(x,f)$ satisfies (2.3) with F and f replaced by F_k and $k^{-1}f$, respectively.

If we assume $Y_0^{(k)}$ has the same distribution with $\mu_k := k^{-1}\langle k\mu\rangle$, where μ belongs to M and $\langle k\mu\rangle$ is a Poisson random measure with intensity $k\mu$, then

$$\begin{aligned} &Q_{\mu_k}^{(k)} \exp\langle Y_t^{(k)}, -f\rangle \\ &\quad= \exp\left\{-\langle \mu, w_t^{(k)}\rangle - \int_0^t ds \int_{M_0} \psi_k(\pi, \langle \pi, w_s^{(k)}\rangle)\eta(d\pi)\right\}, \end{aligned} \tag{3.2}$$

where

$$w_t^{(k)}(x) \equiv w_t^{(k)}(x,f) = k[1 - e^{-v_t^{(k)}(x,f)}], \tag{3.3}$$

$$\psi_k(\pi,\lambda) = k[1 - h_k(\pi, 1-\lambda/k)], \quad 0 \le \lambda \le k. \tag{3.4}$$

For simplicity, we assume that

$$F_k(x, dg, d\pi) = \zeta_k(x,\pi,dg)\tau(x,d\pi), \quad k = 1,2,\cdots,$$

where ζ_k and τ are Markov kernels from $E\times M_0$ to $\mathcal{G}$ and from E to $\mathcal{B}(M_0)$, respectively. Let

$$\phi_k(x,\pi,\lambda) = k\left[1 - \int_G g(1-\lambda/k)\zeta_k(x,\pi,dg)\right], \quad 0 \le \lambda \le k, \tag{3.5}$$

and let

$$\varphi_k(x,f) = \int_{M_0} \phi_k(x,\pi,\langle \pi, f\rangle)\tau(x,d\pi). \tag{3.6}$$

It is easy to check that $w_t^{(k)}$ satisfies

$$w_t^{(k)} + \int_0^t \Pi_s \gamma[w_{t-s}^{(k)} - \varphi_k(w_{t-s}^{(k)})]ds = \Pi_t k(1 - e^{-f/k}), \quad t \ge 0. \tag{3.7}$$

If $\lim_{k\to\infty} \psi_k = \psi$ and $\lim_{k\to\infty} \phi_k = \phi$, then the limit functions have representations

$$\psi(\pi,\lambda) = \int_0^\infty (1 - e^{-\lambda u})\frac{1}{u} n(\pi, du), \tag{3.8}$$

$$\phi(x,\pi,\lambda) = \int_0^\infty (1 - e^{-\lambda u})\frac{1}{u} m(x,\pi,du), \tag{3.9}$$

where n and m are subMarkov kernels from M_0 and from $E\times M_0$ to $\mathcal{B}([0,\infty))$, respectively, and the values of the integrands at $u=0$ are defined as λ.[5] Conversely, it is easy to show that, for ψ and ϕ given by (3.8) and (3.9), there exist sequences ψ_k and ϕ_k in forms (3.4) and (3.5), respectively, such that

$$\begin{aligned} &\psi_k(\pi,\lambda) = \psi(\pi,\lambda), \\ &\phi_k(x,\pi,\lambda) = \phi(x,\pi,\lambda), \quad x \in E, \pi \in M_0, 0 \le \lambda \le k. \end{aligned}$$

Lemma 3.1. *Suppose that*

3.A) $\phi_k(x,\pi,\lambda) \to \phi(x,\pi,\lambda)$ $(k\to\infty)$ *uniformly on the set* $E\times M_0\times[0,l]$ *for every* $l\geq 0$.

Then $w_t^{(k)}(x,f)$, *and hence* $kv_t^{(k)}(x,f)$, *converge boundedly and uniformly on each set* $[0,l]\times E\times B_a^+(E)$ *of* (t,x,f) *to the unique bounded positive solution of the evolution equation*

$$w_t + \int_0^t \Pi_s\gamma[w_{t-s}-\varphi(w_{t-s})]ds = \Pi_t f, \quad t\geq 0, \tag{3.10}$$

where

$$\varphi(x,f) = \int_{M_0}\phi(x,\pi,\langle\pi,f\rangle)\tau(x,d\pi), \quad f\in B^+(E), x\in E. \tag{3.11}$$

The previous lemma can be proved in the same way as Lemma 3.3 of [2] (see also [6]). It follows that, for $(\xi,\gamma,\varphi,\eta,\psi)$ given as above, formula

$$\begin{aligned} &Q_\mu \exp\langle Y_t,-f\rangle \\ &= \exp\left\{-\langle\mu,w_t\rangle - \int_0^t ds\int_{M_0}\psi(\pi,\langle\pi,w_s\rangle)\eta(d\pi)\right\}, \\ &\qquad\qquad f\in B^+(E),\mu\in M, t\geq 0, \end{aligned} \tag{3.12}$$

defines the transition probabilities Q_μ of a Markov process $Y=(Y_t,t\geq 0)$ in space M (see Lemma 3.1 of [2]). If 3.A) holds and if

3.B) $\psi_k(\pi,\lambda)\to\psi(\pi,\lambda)$ $(k\to\infty)$ uniformly on the set $M_0\times[0,l]$ for every $l\geq 0$, then

$$Q_{\mu_k}^{(k)}\exp\langle Y_t^{(k)},-f\rangle \to Q_\mu\exp\langle Y_t,-f\rangle \;\; (k\to\infty) \tag{3.13}$$

uniformly in $f\in B_a^+(E)$ for every fixed $\mu\in M, t\geq 0$ and $a\geq 0$.

Suppose that each particle in the kth system is weighted k^{-1}, then (3.13) states that the mass distributions of the particle systems approximate process Y when the single masses are small and the particle populations are large. We call Y a $(\xi,\gamma,\varphi,\eta,\psi)$-*superprocess.*

If A denotes the infinitesimal operator of ξ, then (3.10) is formally equivalent to

$$\frac{dw_t}{dt} = Aw_t + \gamma[\varphi(w_t)-w_t], \;\; w_0 = f. \tag{3.14}$$

An equation of this form has been considered earlier by S. Watanabe [9].

3.2. Under suitable hypotheses, the $(\xi,\gamma,F_k,k\eta,h_k)$-system $Y^{(k)}$ converges to the $(\xi,\gamma,\varphi,\eta,\psi)$-superprocess Y weakly in $D([0,\infty),M)$. We now assume that

– E is a compact metric space and $\gamma\in C^+(E)$;

– $\xi=(\xi_t,\Pi_x)$ is a Markov process in E with strongly continuous Feller transition semigroup Π_t;

– $\psi(\pi,\lambda)$ is given by (3.8);

– $\varphi(\pi,f)$ is given by (3.9) and (3.11), and $\varphi(\cdot,f)\in C^{++}(E)$ for every $f\in C^{++}(E)$.

In the terminology of Watanabe [9], φ is a Ψ-function, and hence the solution $w_t(\cdot, f)$ to (3.10) defines a Ψ-semigroup, in particular, $w_t(\cdot, f) \in C^{++}(E)$ for $f \in C^{++}(E)$.

Consider the renormalized sequence of branching particle systems with immigration $Y^{(k)} = (Y_t^{(k)}, Q_{\sigma_k}^{(k)})$ defined in paragraph 3.1. Assume that

– $\varphi_k(\cdot, f) \in C^+(E)$ for each $f \in C^+(E)$.

It follows from (3.7) that $w_t^{(k)}(\cdot, f)$, and hence $kv_t^{(k)}(\cdot, f)$, belong to $C^+(E)$ for $f \in C^+(E)$.[3,7] If 3.A) holds, then by Lemma 3.1, when k is sufficiently large, $kv_t^{(k)}(\cdot, f) \in C^{++}(E)$ for $f \in C^{++}(E)$. Thus we see from (3.1) and (3.12) that $Y^{(k)}$ and Y have strongly continuous Feller transition semigroups. Since M is locally compact and separable, we can (and do) assume $Y^{(k)}$ and Y have sample paths in $D([0, \infty), M)$.[1,4]

Theorem 3.2. *If conditions 3.A – B) are satisfied and if $Y_0^{(k)} \to Y_0$ $(k \to \infty)$ in distribution, then $Y^{(k)}$ converges to Y weakly in $D([0, \infty), M)$.*

Proof. By Theorem 2.11 of [4, p172], it suffices to show

$$\sup_{\sigma_k \in M_k} \left| Q_{\sigma_k}^{(k)} \exp\langle Y_t^{(k)}, -f\rangle - Q_{\sigma_k} \exp\langle Y_t, -f\rangle \right| \ \to\ 0 \quad (k \to \infty)$$

for every $f \in C^{++}(E)$ and $t \geq 0$, which follows from (3.1), (3.12) and Lemma 3.1.

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