In this note, we introduce a model of branching particle systems with immigration
and show a result on the weak convergence of such systems to measure-valued Markov
processes.

1. Notations.
Suppose that $E$ is a topological Lusin space with the Borel σ-algebra $\mathcal{B}(E)$. Let
$B^+(E) = \{ \text{bounded positive } \mathcal{B}(E)\text{-measurable functions on } E \}$,
$B^+_a(E) = \{ f : f \in B^+(E) \text{ and } f \leq a \}, \quad a \geq 0$,
$C^+(E) = \{ f : f \in B^+(E) \text{ is continuous } \}$,
$C^{++}(E) = \{ f : f \in C^+(E) \text{ is strictly positive } \}$,
$M = \{ \text{finite measures on } (E, \mathcal{B}(E)) \}$,
$M_0 = \{ \pi : \pi \in M \text{ and } \pi(E) = 1 \}$,
$M_1 = \{ \sigma : \sigma \in M \text{ is integer-valued } \}$,
$M_k = \{ k^{-1}\sigma : \sigma \in M_1 \}, \quad k = 2, 3, \ldots$,
$\delta_x = \text{unit mass concentrated at } x, \quad x \in E$.

We topologize $M$, and hence $M_k, k = 0, 1, 2, \ldots$, with the weak convergence topology.
When $E$ is a compact metric space, $M$ is locally compact and metrizable. In this particular case, $D([0, \infty), M)$ denotes the space of cadlag functions from $[0, \infty)$ to $M$
with the Skorohod topology.

For a $\mathcal{B}(E)$-measurable function $f$ and a measure $\mu \in M$, note $\langle \mu, f \rangle = \int f d\mu$.

Finally, $G$ denotes the totality of subcritical probability generating functions $g$, i.e.,

$$g(z) = \sum_{i=0}^{\infty} p_i z^i \quad (p_i \geq 0, \Sigma p_i = 1, \Sigma i p_i \leq 1),$$
and $G$ is the $\sigma$-algebra on $G$ generated by the mappings $F_z : g \mapsto g(z)$ as $z$ runs over the unit interval $[0, 1]$.

2. The particle system.

Suppose $\xi = (\xi_t, F_t, \Pi_x)$ is a Borel right Markov process with state space $E$, $\gamma = \gamma(x) \in B^+(E)$ and $F = F(x, dg, d\pi)$ is a Markov kernel from $E$ to $G \times B(M_0)$. A branching particle system with parameters $(\xi, \gamma, F)$ is described by the following properties:

2.A) the particles in $E$ move according to the law of the process $\xi$;

2.B) for a particle $\alpha$ which is alive at time $r$ and follows the path $(\xi_t, t \geq r)$, the conditional probability of survival during the time interval $[r, t)$ is $\exp\{-\int_r^t \gamma(\xi_s)ds\}$;

2.C) when $\alpha$ dyes at point $x \in E$, it gives birth to a random number of offspring according to a generating function $g^\alpha$, and the offspring are displaced in $E$ from $x$ according to a probability measure $\pi^\alpha$, where $(g^\alpha, \pi^\alpha)$, depending on the conditions at $x$, is a random variable in $G \times M_0$ with distribution $F(x, dg, d\pi)$.

It is assumed that the motions, life times and branchings of the particles and the first locations of the offspring are independent of each other.

For $t \geq 0$, let $X_t(B)$ be the number of particles of the system in set $B \in B(E)$ at time $t$. Properties 2.A $-$ C imply that $X = (X_t, t \geq 0)$ is a Markov process in space $M_1$ with transition probabilities $P_\sigma$ defined by the Laplace functionals,

$$P_\sigma \exp\langle X_t, -f \rangle = \exp\langle \sigma, -v_t \rangle, \quad f \in B^+(E), \sigma \in M_1, t \geq 0,$$

(2.1)

where

$$v_t(x) \equiv v_t(x, f) = -\log P_{\delta_x} \exp\langle X_t, -f \rangle$$

is the unique positive solution of

$$e^{-v_t(x)} = \Pi_x e^{-f(\xi_t)} - \int_0^t \gamma(\xi_s)ds$$

$$+ \Pi_x \int_0^t e^{-\int_0^s \gamma(\xi_u)du} \gamma(\xi_s) \int G \times M_0 g(\langle \pi, e^{-v_t-s} \rangle) F(\xi_s, dg, d\pi)ds.$$

(2.2)

Equation (2.2) arises as follows: If we start one particle at time 0 at location $x$, this particle moves following a path of $\xi$ and does not branch before time $t$ (first term on the right hand side), or it splits at time $s \in [0, t]$ with probability

$$\exp\{-\int_0^s \gamma(\xi_u)du\} \gamma(\xi_s)ds$$

according to $F(\xi_s, dg, d\pi)$ and all the offspring evolve independently after birth in the same fashion (second term). A rigorous construction for process $X$ can be given as for the branching particle system studied in [2].
Let $\Pi_t, t \geq 0$, be the transition semigroup of $\xi$. By Lemma 2.3 of [2], equation (2.2) can be written in an abbreviative form:

$$\int_0^t \Pi_s \gamma \left[ e^{-v_t s} - \int G \times M_0 g(\langle \pi, e^{-v_t s} \rangle) F(dg, d\pi) \right] ds = \Pi_t e^{-f} - e^{-v_t}.$$  (2.3)

Suppose that $\eta$ is a finite measure on $(M_0, B(M_0))$ and that $h(\pi, z)$ is a $B(M_0 \times [0, 1])$-measurable function such that for each $\pi \in M_0$, $h(\pi, \cdot)$ belongs to $G$. A branching particle system with immigration with parameters $(\xi, \gamma, F, \eta, h)$ is described by properties 2.A – C) and the following

2.D) the entry times and distributions of new particles immigrating to $E$ are governed by a Poisson random measure with intensity $ds \times \eta(d\pi)$;
2.E) $h(\pi, \cdot)$ gives the distribution of the number of new particles entering $E$ with distribution $\pi$.

Here we assume that the immigration is independent of the evolution of the inner population and that the immigrants behave as the natives after entering $E$. The particle distribution process $(Y_t, t \geq 0)$ of the branching system with immigration is a Markov process with state space $M_1$. Properties 2.A – E) lead through a calculation to the Laplace functional:

$$Q_{\sigma} \exp\langle Y_t, -f \rangle$$

$$= \exp \left\{ -\langle \sigma, v_t \rangle - \int_0^t ds \int_{M_0} [1 - h(\pi, \langle \pi, e^{-v_s} \rangle)] \eta(d\pi) \right\},$$

where $Q_{\sigma}$ denotes the transition probability from $\sigma$ of $Y_t$ and $v_t$ is defined by equation (2.3).

3. A limit theorem.

3.1. Let $Y(k) = \{Y_t(k), t \geq 0\}$ be a sequence of branching particle systems with immigration with parameters $(\xi, \gamma, F_k, k\eta, h_k), k = 1, 2, \cdots$. Then for each $k$,

$$Y^{(k)} = \{Y_t^{(k)} \equiv k^{-1}Y_t(k), t \geq 0\}$$

is a Markov process in $M_k$ with transition probabilities $Q_{\sigma_k}^{(k)}$ determined by

$$Q_{\sigma_k}^{(k)} \exp(Y_t^{(k)}, -f)$$

$$= \exp \left\{ -\langle \sigma_k, kv^{(k)}_t \rangle - \int_0^t ds \int_{M_0} k[1 - h_k(\pi, \langle \pi, e^{-v^{(k)}_s} \rangle)] \eta(d\pi) \right\},$$

$$f \in B^+(E), \sigma_k \in M_k, t \geq 0,$$
where \( v_t^{(k)}(x) \equiv v_t^{(k)}(x, f) \) satisfies (2.3) with \( F \) and \( f \) replaced by \( F_k \) and \( k^{-1}f \), respectively.

If we assume \( Y_0^{(k)} \) has the same distribution with \( \mu_k := k^{-1}\langle k\mu \rangle \), where \( \mu \) belongs to \( M \) and \( \langle k\mu \rangle \) is a Poisson random measure with intensity \( k\mu \), then

\[
Q_{\mu_k}^{(k)} \exp\{Y_t^{(k)}, -f\} = \exp\left\{ -\langle \mu, w_t^{(k)} \rangle - \int_0^t ds \int_{M_0} \psi_k(\pi, \langle \pi, w_s^{(k)} \rangle) \eta(d\pi) \right\},
\]

where

\[
w_t^{(k)}(x) \equiv w_t^{(k)}(x, f) = k[1 - e^{-v_t^{(k)}(x,f)}],
\]

\[
\psi_k(\pi, \lambda) = k[1 - h_k(\pi, 1 - \lambda/k)], \quad 0 \leq \lambda \leq k.
\]

For simplicity, we assume that

\[
F_k(x, dg, d\pi) = \zeta_k(x, \pi, dg)\tau(x, d\pi), \quad k = 1, 2, \cdots,
\]

where \( \zeta_k \) and \( \tau \) are Markov kernels from \( E \times M_0 \) to \( G \) and from \( E \) to \( B(M_0) \), respectively. Let

\[
\phi_k(x, \pi, \lambda) = k \left[ 1 - \int_G g(1 - \lambda/k)\zeta_k(x, \pi, dg) \right], \quad 0 \leq \lambda \leq k,
\]

and let

\[
\varphi_k(x, f) = \int_{M_0} \phi_k(x, \pi, \langle \pi, f \rangle)\tau(x, d\pi).
\]

It is easy to check that \( w_t^{(k)} \) satisfies

\[
w_t^{(k)} + \int_0^t \Pi_s \gamma[w_{t-s}^{(k)} - \varphi_k(w_{t-s}^{(k)})] ds = \Pi_t k(1 - e^{-f/k}), \quad t \geq 0.
\]

If \( \lim_{k \to \infty} \psi_k = \psi \) and \( \lim_{k \to \infty} \phi_k = \phi \), then the limit functions have representations

\[
\psi(\pi, \lambda) = \int_0^\infty (1 - e^{-\lambda u}) \frac{1}{u} n(\pi, du),
\]

\[
\phi(x, \pi, \lambda) = \int_0^\infty (1 - e^{-\lambda u}) \frac{1}{u} m(x, \pi, du),
\]

where \( n \) and \( m \) are subMarkov kernels from \( M_0 \) and from \( E \times M_0 \) to \( B([0, \infty)) \), respectively, and the values of the integrands at \( u = 0 \) are defined as \( \lambda \).\(^5\) Conversely, it is easy to show that, for \( \psi \) and \( \phi \) given by (3.8) and (3.9), there exist sequences \( \psi_k \) and \( \phi_k \) in forms (3.4) and (3.5), respectively, such that

\[
\psi_k(\pi, \lambda) = \psi(\pi, \lambda),
\]

\[
\phi_k(x, \pi, \lambda) = \phi(x, \pi, \lambda), \quad x \in E, \pi \in M_0, 0 \leq \lambda \leq k.
\]
Lemma 3.1. Suppose that
3.A) $\phi_k(x, \pi, \lambda) \to \phi(x, \pi, \lambda)$ \((k \to \infty)\) uniformly on the set \(E \times M_0 \times [0, l]\) for every \(l \geq 0\).

Then \(w_t^{(k)}(x, f)\), and hence \(k \nu_t^{(k)}(x, f)\), converge boundedly and uniformly on each set \([0, l] \times E \times B_a^+(E)\) of \((t, x, f)\) to the unique bounded positive solution of the evolution equation

\[
\frac{dw_t}{dt} + \int_0^t \Pi_s \gamma \left[w_{t-s} - \varphi(w_{t-s})\right] ds = \Pi_t f, \quad t \geq 0,
\]

where
\[
\varphi(x, f) = \int_{M_0} \phi(x, \pi, \langle \pi, f \rangle) \tau(x, d\pi), \quad f \in B^+(E), x \in E.
\]

The previous lemma can be proved in the same way as Lemma 3.3 of [2] (see also [6]). It follows that, for \((\xi, \gamma, \phi, \eta, \psi)\) given as above, formula

\[
Q_{\mu} \exp\langle Y_t, -f \rangle = \exp \left\{ -\langle \mu, w_t \rangle - \int_0^t \int_{M_0} \psi(\pi, \langle \pi, w_s \rangle) \eta(d\pi) \right\},
\]

defines the transition probabilities \(Q_{\mu}\) of a Markov process \(Y = (Y_t, t \geq 0)\) in space \(M\) (see Lemma 3.1 of [2]). If 3.A) holds and if

3.B) \(\psi_k(\pi, \lambda) \to \psi(\pi, \lambda)\) \((k \to \infty)\) uniformly on the set \(M_0 \times [0, l]\) for every \(l \geq 0\), then

\[
Q_{\phi_k} \exp\langle Y_t^{(k)}, -f \rangle \to Q_{\mu} \exp\langle Y_t, -f \rangle \quad (k \to \infty)
\]

uniformly in \(f \in B_a^+(E)\) for every fixed \(\mu \in M, t \geq 0\) and \(a \geq 0\).

Suppose that each particle in the \(k\)th system is weighted \(k^{-1}\), then (3.13) states that the mass distributions of the particle systems approximate process \(Y\) when the single masses are small and the particle populations are large. We call \(Y\) a \((\xi, \gamma, \phi, \eta, \psi)\)-superprocess.

If \(A\) denotes the infinitesimal operator of \(\xi\), then (3.10) is formally equivalent to

\[
\frac{dw_t}{dt} = Aw_t + \gamma[\varphi(w_t) - w_t], \quad w_0 = f.
\]

An equation of this form has been considered earlier by S. Watanabe [9].

3.2. Under suitable hypotheses, the \((\xi, \gamma, F_k, k\eta, h_k)\)-system \(Y^{(k)}\) converges to the \((\xi, \gamma, \varphi, \eta, \psi)\)-superprocess \(Y\) weakly in \(D([0, \infty), M)\). We now assume that

\- \(E\) is a compact metric space and \(\gamma \in C^+(E)\);
\- \(\xi = (\xi_t, \Pi_x)\) is a Markov process in \(E\) with strongly continuous Feller transition semigroup \(\Pi_t\);
\- \(\psi(\pi, \lambda)\) is given by (3.8);
\- \(\varphi(\pi, f)\) is given by (3.9) and (3.11), and \(\varphi(\cdot, f) \in C^{++}(E)\) for every \(f \in C^{++}(E)\).
In the terminology of Watanabe [9], ϕ is a Ψ-function, and hence the solution $w_t(\cdot, f)$ to (3.10) defines a Ψ-semigroup, in particular, $w_t(\cdot, f) \in C^{++}(E)$ for $f \in C^{++}(E)$.

Consider the renormalized sequence of branching particle systems with immigration $Y(k) = (Y_t^{(k)}, Q^{(k)}_\sigma)$ defined in paragraph 3.1. Assume that

- $-\varphi_k(\cdot, f) \in C^+(E)$ for each $f \in C^+(E)$.

It follows from (3.7) that $w_t^{(k)}(\cdot, f)$, and hence $kv_t^{(k)}(\cdot, f)$, belong to $C^+(E)$ for $f \in C^+(E)$.[3,7] If 3.A) holds, then by Lemma 3.1, when $k$ is sufficiently large, $kv_t^{(k)}(\cdot, f) \in C^{++}(E)$ for $f \in C^{++}(E)$. Thus we see from (3.1) and (3.12) that $Y^{(k)}$ and $Y$ have strongly continuous Feller transition semigroups. Since $M$ is locally compact and separable, we can (and do) assume $Y^{(k)}$ and $Y$ have sample paths in $D([0, \infty), M)$.[1,4]

**Theorem 3.2.** If conditions 3.A − B) are satisfied and if $Y^{(k)}_0 \to Y_0 (k \to \infty)$ in distribution, then $Y^{(k)}$ converges to $Y$ weakly in $D([0, \infty), M)$.

**Proof.** By Theorem 2.11 of [4, p172], it suffices to show

$$\sup_{\sigma_k \in M_k} \left| Q^{(k)}_{\sigma_k} \exp\langle Y_t^{(k)}, -f \rangle - Q_{\sigma_k} \exp\langle Y_t, -f \rangle \right| \to 0 \quad (k \to \infty)$$

for every $f \in C^{++}(E)$ and $t \geq 0$, which follows from (3.1), (3.12) and Lemma 3.1.

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**REFERENCES**