Chinese version published in: *Chinese Science Bulletin* (Chinese Edition) **37** (1992), 1541–1543.

BRANCHING PARTICLE SYSTEMS IN RANDOM ENVIRONMENTS[1]

LI Zeng-hu

Department of Mathematics, Beijing Normal University
Beijing 100875, People's Republic of China

Keywords branching particle system, random environments, superprocess

Based on the results obtained in [3], we give the construction of a class of superprocesses by taking the rescaled limits of some branching particles systems in random environments. The evolution equation characterizing this class of superprocesses is essentially more general than the one we have known before, and is thus helpful in understanding the connection between measure-valued processes and nonlinear equations.

1. A model of branching particles

Suppose that E is a topological Lusin space with the Borel σ-algebra denoted by $\mathcal{B}(E)$. Let

$M = \{$ finite Borel measures on E $\}$,
$M_0 = \{\pi \in M : \pi(E) = 1\}$,
$G = \{$ probability generating functions $g(z)$ $(0 \leq z \leq 1)$ satisfying $g'(1) < \infty\}$.
$B(E)^+ = \{$ bounded nonnegative Borel functions on E $\}$,
$B(E)^+_a = \{f \in B(E)^+ : f(x) \leq a\}$ for $a \geq 0$,

We topologize M and M_0 by the usual weak convergence, and G by the pointwise convergence. Suppose that $\xi = (\xi_t, \mathit{\Pi}_x)$ is a Borel right Markov process on E, $\gamma = \gamma(x) \in B(E)^+$ and $F = F(x, \mathrm{d}\pi, \mathrm{d}g)$ is a Markov kernel from E to $G \times M_0$. A branching particle system in random environments with parameters (ξ, γ, F) is described as follows:

(i) The particles in E move according to the transition law of ξ.

(ii) For a particle α that is alive at time r and follows the path $(\xi_s, s \geq r)$, the conditional probability of survival during the time interval $[r, t]$ is $\exp\{-\int_r^t \gamma(\xi_s)\mathrm{d}s\}$.

(iii) When the particle α dies at point x, it gives birth to a random number of offspring according to probability generating function g^α and the offspring are displaced in E according to probability measure π^α, where (π^α, g^α) are random variables in $M_0 \times G$ with joint distribution $F(x, \mathrm{d}\pi, \mathrm{d}g)$.

The use of the term "random environments" is meant to suggest the randomness of (π^α, g^α) in property (iii). Let $X_t(B)$ denote the number of particles in set $B \in \mathcal{B}(E)$ that

[1]Supported by the NNSF of China. (This is the English translation of the Chinese paper.)

are alive at time $t \geq 0$. If $X_0(E) < \infty$, then $(X_t, t \geq 0)$ form a Markov process on M_1, the subspace of M comprising integer-valued measures on E. A rigorous construction of this process has essentially been given as in [1]. Let P_σ denote the conditional law of (X_t) given $X_0 = \sigma$. Properties (i), (ii) and (iii) yield a characterization of the Laplace functional:

$$P_\sigma \exp\langle X_t, -f\rangle = \exp\langle \sigma, -v_t\rangle, \qquad f \in B(E)^+, \sigma \in M_1, t \geq 0, \tag{1.1}$$

where $\langle \sigma, f\rangle = \int f \mathrm{d}\sigma$ and $v_t(x) \equiv v_t(x, f)$ is the unique positive solution of[2]

$$\begin{aligned} \mathrm{e}^{-v_t(x)} &= \mathit{\Pi}_x \mathrm{e}^{-f(\xi_t) - \int_0^t \gamma(\xi_s)\mathrm{d}s} \\ &\quad + \mathit{\Pi}_x \int_0^t \mathrm{e}^{-\int_0^s \gamma(\xi_u)\mathrm{d}u} \gamma(\xi_s) \int_G \int_{M_0} g(\langle \pi, \mathrm{e}^{-v_{t-s}}\rangle) F(\xi_s, \mathrm{d}\pi, \mathrm{d}g)\mathrm{d}s. \end{aligned}$$

This equation follows as we think about that if a particle starts moving from point x at time 0, it follows a path of ξ and does not branch before time t, or it splits at time $s \in (0, t]$. As in [1] it can be proved that the above equation is equivalent to

$$\begin{aligned} \mathrm{e}^{-v_t(x)} - \mathit{\Pi}_x \mathrm{e}^{-f(\xi_t)} + \int_0^t \mathit{\Pi}_x \gamma(\xi_s) \mathrm{e}^{-v_{t-s}(\xi_s)}\mathrm{d}s \\ = \int_0^t \mathrm{d}s \mathit{\Pi}_x \gamma(\xi_s) \int_G \int_{M_0} g(\langle \pi, \mathrm{e}^{-v_{t-s}}\rangle) F(\xi_s, \mathrm{d}\pi, \mathrm{d}g). \end{aligned} \tag{1.2}$$

2. Measure-valued processes

Let $\{X_t(k), t \geq 0\}$ be the mass distributions of a sequence of branching particle systems in random environments with parameters (ξ, γ_k, F_k), $k = 1, 2, \cdots$. Then for each k,

$$X^{(k)} = \{X_t^{(k)} := k^{-1} X_t(k), t \geq 0\} \tag{2.1}$$

defines a Markov process on $M_k := \{k^{-1}\sigma : \sigma \in M_1\}$. Let $\sigma(k\mu)$ be a Poisson random measure with intensity $k\mu \in M$, and let $P_{\mu_k}^{(k)}$ denote the conditional law of the process $(X_t^{(k)}, t \geq 0)$ given $X_0^{(k)} = k^{-1}\sigma(k\mu)$. Then by (1.1) and (1.2) we get

$$P_{\mu_k}^{(k)} \exp\langle X_t^{(k)}, -f\rangle = \exp\langle \mu, -w_t^{(k)}\rangle, \qquad f \in B(E)^+, t \geq 0, \tag{2.2}$$

where

$$w_t^{(k)}(x) \equiv w_t^{(k)}(x, f) = k[1 - \exp\{-v_t^{(k)}(x)\}] \tag{2.3}$$

[2] Li, Zenghu, 2nd Sino-French Math. Meeting (Wuhan), Sep.24–Oct.11, 1990

satisfies the equation

$$w_t^{(k)}(x) - \mathit{\Pi}_x k[1-\mathrm{e}^{-f(\xi_t)/k}] + \int_0^t \mathit{\Pi}_x \gamma(\xi_s) w_{t-s}^{(k)}(\xi_s)\mathrm{d}s \tag{2.4}$$
$$= \int_0^t \mathrm{d}s \mathit{\Pi}_x \gamma(\xi_s) \int_G \int_{M_0} k[1-g(1-\langle \pi, w_{t-s}^{(k)}/k\rangle)] F_k(\xi_s, \mathrm{d}\pi, \mathrm{d}g).$$

For each k, let $a_k, b_k \in B(E)^+$ be strictly positive, and let $g_k \in B(E\times[0,1])^+$ be such that $g_k(x,\cdot) \in G$ for all x. Suppose that $\tau(x,\mathrm{d}\pi)$ and $\tau_k(x,\pi,\mathrm{d}g)$ are Markov kernels from E to M_0 and from $E\times M_0$ to G, respectively. Let

$$\begin{aligned}\gamma_k(x) =& a_k(x) + b_k(x),\\ F_k(x,\mathrm{d}\pi,\mathrm{d}g) =& a_k(x)\gamma_k^{-1}(x)\delta_k(x,\mathrm{d}\pi,\mathrm{d}g)\\ &+ b_k(x)\gamma_k^{-1}(x)\tau_k(x,\pi,\mathrm{d}g)\tau(x,\mathrm{d}\pi),\end{aligned}$$

where $\delta_k(x,\mathrm{d}\pi,\mathrm{d}g)$ denotes the unit mass concentrated on point $(\delta_x, g_k(x,\cdot))$ in space $M_0\times G$. That is, the randomness of the environments at x only makes a partial influence, represented by the second term, on the branching mechanism at this point. Then we let

$$\phi_k(x,z) = b_k(x)z + ka_k(x)[g_k(x,1-z/k) - (1-z/k)], \quad 0\le z\le k, \tag{2.5}$$

$$\zeta_k(x,\pi,z) = kb_k(x)\int_G [1-g(1-z/k)]\tau_k(x,\pi,\mathrm{d}g), \quad 0\le z\le k, \tag{2.6}$$

$$\varphi_k(x,f) = \int_{M_0} \zeta_k(x,\pi,\langle\pi,f\rangle)\tau(x,\mathrm{d}\pi). \tag{2.7}$$

Now we assume the following conditions:

(1) For each $l>0$, the sequence $\phi_k(x,z)$ is uniformly Lipschitz in z on $E\times[0,l]$;
(2) For each $l>0$, $\phi_k(x,z)\to\phi(x,z)$ uniformly on $E\times[0,l]$;
(3) For each $l>0$, the sequence $\zeta_k(x,\pi,z)$ is uniformly Lipschitz in z on $E\times M_0\times[0,l]$;
(4) For each $l>0$, $\zeta_k(x,\pi,z)\to\zeta(x,\pi,z)$ uniformly on $E\times M_0\times[0,l]$.

Using the results of [3] one shows that ϕ and ζ have representations

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu}-1+zu)m(x,\mathrm{d}u), \quad z\ge 0, \tag{2.8}$$

$$\zeta(x,\pi,z) = d(x,\pi)z + \int_0^\infty (1-\mathrm{e}^{-zu})n(x,\pi,\mathrm{d}u), \quad z\ge 0, \tag{2.9}$$

where $c\ge 0$, $d\ge 0$ and b are bounded Borel functions, m and n are kernels from E and from $E\times M_0$ to $(0,\infty)$, respectively, such that

$$\int_0^\infty u\wedge u^2 m(\cdot,\mathrm{d}u) \quad \text{and} \quad \int_0^\infty u\; n(\cdot,\pi,\mathrm{d}u)$$

are bounded. Conversely, to any ϕ and ζ given by (2.8) and (2.9) there correspond sequences ϕ_k and ζ_k in forms (2.5) and (2.6), respectively, such that (1) – (4) are satisfied; see [4]. Let

$$\varphi(x,f) = \int_{M_0} \zeta(x,\pi,\langle\pi,f\rangle)\tau(x,\mathrm{d}\pi). \tag{2.10}$$

Under hypotheses (1) – (4), it can be proved that $w_t^{(k)}(x,f)$ and $kv_t^{(k)}(x,f)$ converge boundedly and uniformly on each set $[0,l]\times E\times B(E)_a^+$ of (t,x,f) to the unique positive solution to the equation

$$w_t(x) + \int_0^t \mathit{\Pi}_x[\phi(\xi_s, w_{t-s}(\xi_s)) - \varphi(\xi_s, w_{t-s})]\mathrm{d}s = \mathit{\Pi}_x f(\xi_t), \quad t \geq 0. \tag{2.11}$$

Then we have

Theorem *For (ξ,ϕ,φ) as the above, there exists a Markov process $X = (X_t, P_\mu)$ on M with transition probabilities defined by*

$$P_\mu \exp\langle X_t, -f\rangle = \exp\langle\mu, -w_t\rangle, \qquad \mu\in M, f\in B(E)^+, t\geq 0, \tag{2.12}$$

where $w_t(x)$ is given by (2.11).

We may call X a (ξ,ϕ,φ)-superprocess. In particular, if $b_k(x) = 0$ $(k = 1,2,\cdots)$, the randomness of the environments disappears and $\varphi = \lim\varphi_k = 0$. In this case, (2.11) degenerates to the familiar equation; see [1,5]. In the general case, since $\varphi(\xi_s, w_{t-s})$ depends on the values of w_{t-s} outside ξ_s, one cannot write it into $\phi(\xi_s, w_{t-s}(\xi_s))$. For this reason, equation (2.11) is essentially more general than the equation we met before. This observation might serve as the base of a probability approach to some nonlinear boundary value problems. Moreover, using this model we may deduce the existence of a class of multitype measure-valued branching processes generalizing the results of [2], which will be discussed elsewhere.

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