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A NOTE ON THE MULTITYPE MEASURE BRANCHING PROCESS

Zeng-Hu LI* *Beijing Normal University*

Abstract. The existence of a class of multitype measure branching processes is deduced from a single-type model introduced by Li [8], which extends the work of Gorostiza and Lopez-Mimbela [5] and shows that the study of a multitype process can sometimes be reduced to that of a single-type one.

measure branching process; multitype; single-type; superprocess; homeomorphism

1. Introduction

Let E be a topological Lusin space with the Borel σ-algebra denoted by $\mathcal{B}(E)$ and let

$M = \{$ finite Borel measures on E $\}$,

$B(E)^+ = \{$ bounded nonnegative Borel functions on E $\}$.

We endow M with the usual weak convergence topology. Suppose that $\xi = (\xi_t, \mathit{\Pi}_x)$ is a Borel right Markov process and that $\tau = \tau(x, dy)$ is a Markov kernel on the state space $(E, \mathcal{B}(E))$. Let ϕ and φ be given by

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty \big(e^{-zu} - 1 + zu\big)m(x,du), \tag{1.1}$$

$$\varphi(x,z) = d(x)z + \int_0^\infty \big(1 - e^{-zu}\big)n(x,du), \qquad x \in E, z \geq 0, \tag{1.2}$$

where $b, c, d \in B(E)^+$ and m, n are kernels from E to $\mathcal{B}((0,\infty))$ with $\int_0^\infty u \wedge u^2 m(\cdot, du) + \int_0^\infty u\, n(\cdot, du)$ bounded on E.

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*Postal address: Department of Mathematics, Beijing Normal University, Beijing 100875, P. R. China.

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A Markov process $X = (X_t, P_\mu)$ on the state space M is called a $(\xi, \phi, \varphi, \tau)$-*superprocess* in this note if its transition probabilities are determined by

$$P_\mu \exp\langle X_t, -f\rangle = \exp\langle \mu, -w_t\rangle, \quad f \in B(E)^+, \tag{1.3}$$

where $\langle \mu, f\rangle = \int f d\mu$, P_μ denotes the conditional expectation given $X_0 = \mu$ and $w_t = w_t(x)$ is the unique bounded positive solution of the evolution equation

$$\mathit{\Pi}_x f(\xi_t) - w_t(x) = \int_0^t \mathit{\Pi}_x\big[\phi(\xi_s, w_{t-s}(\xi_s)) - \varphi(\xi_s, \langle \tau(\xi_s), w_{t-s}\rangle)\big]ds, \quad t \geq 0. \tag{1.4}$$

The existence of the $(\xi, \phi, \varphi, \tau)$-superprocess can be established by considering a high density limit of the branching particle model introduced by Li [8] and using the results of [6,7]; see also [3]. Note that the term $\varphi(\langle \tau, \cdot\rangle)$ in (1.4) has emerged as we allowed a part of the offspring born at x to be displaced randomly into the space according to the distribution $\tau(x, \cdot)$. (Studying a branching model of this type was suggested by P. J. Fitzsimmons.)

The purpose of this note is to deduce the existence of a class of multitype measure branching processes (MMB-processes) from the above model, which generalizes the results of Gorostiza and Lopez-Mimbela [5] and shows that the study of a multitype process can sometimes be reduced to that of a single-type one.

2. The multitype measure branching process

Let $I = \{1, \cdots, k\}$ and let $\big(\xi^{(i)}, \phi^{(i)}, \varphi^{(i)}\big)$ depending on $i \in I$ be a family of parameters as described in section 1. Assume that the product space $I \times E$ carries the relative topology inherited from $R \times E$. Let $\xi = (\xi_t, \mathit{\Pi}_{(i,x)})$ be the Borel right Markov process on $I \times E$ defined by

$$\mathit{\Pi}_{(i,x)} f(\xi_t) = \mathit{\Pi}_x^{(i)} f(i, \xi_t^{(i)}), \qquad f \in B(I \times E)^+.$$

Set $\phi((i,x), z) = \phi^{(i)}(x, z)$ and $\varphi((i,x), z) = \varphi^{(i)}(x, z)$. Suppose $\tau((i,x), \cdot)$ is a Markov kernel on $I \times E$ with the decomposition

$$\tau((i,x), \cdot) = \sum_{j=1}^k p_j^{(i)}(x)\delta_{(j,x)}(\cdot),$$

where $\delta_{(j,x)}$ denotes the unit mass at (j,x), $p_j^{(i)}(x) \geq 0$ and $\sum\limits_{j=1}^k p_j^{(i)}(x) \equiv 1$.

Let $X = (X_t, P_\mu)$ denote the $(\xi, \phi, \varphi, \tau)$-superprocess. Then X is a Markov process taking values in $M(I \times E)$, the space of finite Borel measures on $I \times E$. For each $\mu \in M(I \times E)$ we define $\mu^{(i)} \in M$ $(i \in I)$ by $\mu^{(i)}(B) = \mu(\{i\} \times B)$, $B \in \mathcal{B}(E)$. The map $\gamma : \mu \mapsto (\mu^{(1)}, \cdots, \mu^{(k)})$ is clearly a homeomorphism between $M(I \times E)$ and the k dimensional product space M^k. It follows from Theorem 10.13 of Dynkin [2] that $\{(X_t^{(1)}, \cdots, X_t^{(k)}),\ t \geq 0\}$ is a Markov process on M^k with transition probabilities $P_{(\mu^{(1)},\cdots,\mu^{(k)})}$ determined by

$$P_{(\mu^{(1)},\cdots,\mu^{(k)})} \exp \sum_{i=1}^{k} \langle X_t^{(i)}, -f^{(i)} \rangle = \exp \sum_{i=1}^{k} \langle \mu^{(i)}, -w_t^{(i)} \rangle, \qquad f^{(i)} \in B(E)^+, \tag{2.1}$$

where $w_t^{(i)} = w_t^{(i)}(x)$ is the solution of

$$\begin{aligned} &\Pi_x^{(i)} f^{(i)}(\xi_t^{(i)}) - w_t^{(i)}(x) \\ &= \int_0^t \Pi_x^{(i)} \Big[\phi^{(i)}\big(\xi_s^{(i)}, w_{t-s}^{(i)}(\xi_s^{(i)})\big) - \varphi^{(i)}\big(\xi_s^{(i)}, \sum_{j=1}^{k} p_j^{(i)}(\xi_s^{(i)}) w_{t-s}^{(j)}(\xi_s^{(i)})\big)\Big] ds, \\ &\qquad t \geq 0,\ i = 1, \cdots, k. \end{aligned} \tag{2.2}$$

Note that if $A^{(i)}$ denotes the infinitesimal generator of $\xi^{(i)}$, then (2.2) is formally equivalent to

(2.3)

$$\begin{aligned} &\frac{\partial}{\partial t} w_t^{(i)}(x) = A^{(i)} w_t^{(i)}(x) - \phi^{(i)}(x, w_t^{(i)}(x)) + \varphi^{(i)}\big(x, \sum_{j=1}^{k} p_j^{(i)}(x) w_t^{(j)}(x)\big), \\ &w_0^{(i)}(x) = f^{(i)}(x), \qquad t \geq 0,\ i = 1, \cdots, k. \end{aligned}$$

We shall call $\big((X_t^{(1)}, \cdots, X_t^{(k)}), P_{(\mu^{(1)},\cdots,\mu^{(k)})}\big)$ an *MMB-process with parameters* $\big(\xi^{(i)}, \phi^{(i)}, \varphi^{(i)}, (p_j^{(i)});\ i \in I\big)$.

Heuristically, $\xi^{(i)}$ gives the law of the migration of the ith type 'particles', $\phi^{(i)}(x, \cdot)$ describes the amount of the ith type offspring born when an ith type parent dies at point x, $\varphi^{(i)}(x, \cdot)$ describes the amount of the offspring born by this parent that change into new types randomly according to the discrete distribution $\{p_1^{(i)}(x), \cdots, p_k^{(i)}(x)\}$. All these offspring start migrating from the death site x of the parent. It is assumed as usual that the migrations, the lifetimes and the branchings of the particles and the mutations of the offspring are independent of each other.

Our model generalizes the one of Gorostiza and Lopez-Mimbela [5], who constructed the MMB-process in the special case where the $\xi^{(i)}$ were symmetric stable processes in R^d, $\phi^{(i)}(x, z) = c^{(i)}(x) z^2$ and $\varphi^{(i)}(x, z) = d^{(i)}(x) z$. These restrictions have made it possible for the two authors to extend the states of the MMB-process to some infinite measures. It is obvious that the immigration model of [5] can also be generalized by modifying the results of [7,8].

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