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MEASURE-VALUED BRANCHING PROCESSES WITH IMMIGRATION

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Abstract. Starting from the cumulant semigroup of a measure-valued branching process, we construct the transition probabilities of some Markov process $Y^{(\beta)} = (Y_t^{(\beta)}, t \in R)$, which we call a measure-valued branching process with discrete immigration of unit β. The immigration of $Y^{(\beta)}$ is governed by a Poisson random measure ρ on the time-distribution space and a probability generating function h, both depending on β. It is shown that, under suitable hypotheses, $Y^{(\beta)}$ approximates to a Markov process $Y = (Y_t, t \in R)$ as $\beta \to 0^+$. The latter is the one we call a measure-valued branching process with immigration. The convergence of branching particle systems with immigration is also studied.

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measure-valued branching process $*$ immigration $*$ particle system $*$ superprocess $*$ weak convergence

1. Introduction

Let M be the totality of finite measures on a measurable space $(E, \mathcal{E})$. Suppose that $X = (X_t, t \in R)$ is a Markov process in M with transition function $P(r, \mu; t, d\nu)$. X is called a measure-valued branching process (MB-process) if

$$P(r, \mu_1 + \mu_2; t, \cdot) = P(r, \mu_1; t, \cdot) * P(r, \mu_2; t, \cdot), \quad \mu_1, \mu_2 \in M,\ r \le t, \tag{1.1}$$

where "$*$" denotes the convolution operation [cf. Dawson (1977), Dawson and Ivanoff (1978), Watanabe (1968), etc]. When E is reduced to one point, X takes values in $R^+ := [0, \infty)$ and is called a continuous state branching process (CB-process).

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Continuous state branching processes with immigration (CBI-processes) were first introduced by Kawazu and Watanabe (1971). Several authors have also studied measure-valued branching processes with immigration (MBI-processes); see Dynkin (1991ab), Konno and Shiga (1988), etc.

In the present paper, we study a general class of MBI-processes that covers the models of the previous authors and can be regarded as the measure-valued counterpart of the one of CBI-processes proposed by Kawazu and Watanabe. Section 2 contains some preliminaries. The general definition for an MBI-process is given in section 3, followed by the model of a measure-valued branching process with discrete immigration (MBDI-process). The heuristic meanings of the latter are clear. It is shown that the MBI-process is in fact an approximation for the MBDI-process with high rate and small unit of immigration. In section 4, we study the convergence of branching particle systems with immigration to MBI-processes. A branching system of particles with immigration is not an MBDI-process in the terminology of this paper. The concluding section 5 contains a brief discussion of MBI-processes with σ-finite values whose study can be reduced to that of the class with finite values studied in sections 3 and 4.

2. Preliminaries

2.1. We first introduce some notation. If F is a topological space, then $\mathcal{B}(F)$ denotes the σ-algebra of F generated by all open sets, and

$B(F) = \{$ bounded $\mathcal{B}(F)$-measurable functions on F $\}$,

$C(F) = \{f : f \in B(F)$ is continuous $\}$,

$B(F)_a = \{f : f \in B(F)$ and $\|f\| \leq a\}$ for $a \geq 0$.

Here "$\|\cdot\|$"denotes the supremum norm. In the case F is locally compact,

$C_0(F) = \{f : f \in C(F)$ vanishes at infinity $\}$.

The subsets of nonnegative members of the function spaces are denoted by the superscript "+", and those of strictly positive members by "++"; e.g., $B(F)^+$, $C(F)^{++}$. If F is a metric space, then $D(R^+, F)$ stands for the space of cadlag functions from R^+ to F equipped with the Skorohod topology. Finally, δ_x denotes the unit mass concentrated at x, and for a function f and a measure μ, $\langle \mu, f\rangle = \int f d\mu$.

2.2. Suppose that E is a topological Lusin space, i.e., a homeomorph of a Borel subset of some compact metric space. Let

$M = \{$ finite measures on $(E, \mathcal{B}(E))$ $\}$,

$M_0 = \{\pi : \pi \in M$ and $\pi(E) = 1\}$,

$M_1 = \{\sigma : \sigma \in M$ is integer-valued $\}$,

$M_k = \{k^{-1}\sigma : \sigma \in M_1\}$ for $k = 2, 3, \cdots$.

We topologize M, and hence M_k, $k = 0, 1, 2, \cdots$, with the weak convergence topology. It is well known that M is locally compact and separable when E is a compact metric space.

The Laplace functional of a probability measure P on M is defined as

$$L_P(f) = \int_M e^{-\langle \mu, f\rangle} P(d\mu), \quad f \in B(E)^+. \tag{2.1}$$

P is said to be infinitely divisible if for each integer $m > 0$, there is a probability measure P_m on M such that $L_P(f) = [L_{P_m}(f)]^m$.

We say a functional w on $B(E)^+$ belongs to the *class* $\mathcal{W}$ if it has the representation

$$w(f) = \iint\limits_{R^+ \times M_0} \left(1 - e^{-u\langle \pi, f\rangle}\right) \frac{1+u}{u} G(du, d\pi), \quad f \in B(E)^+. \tag{2.2}$$

where G is a finite measure on $R^+ \times M_0$ and the value of the integrand at $u = 0$ is defined as $\langle \pi, f\rangle$. The following result coincides with Theorem 1.2 of Watanabe (1968) since $(E, \mathcal{B}(E))$ is isomorphic to a compact metric space with the Borel σ-algebra.

Proposition 2.1. *A probability measure P on M is infinitely divisible if and only if $-\log L_P(\cdot) \in \mathcal{W}$.* Q.E.D.

A family of operators $W_t^r : f \mapsto w_t^r(\cdot, f)$ $(r \leq t \in R)$ on $B(E)^+$ is called a *cumulant semigroup* provided

2.A) for every fixed $r \leq t$ and x, $w_t^r(x, \cdot)$ belongs to $\mathcal{W}$;

2.B) for all $r \leq s \leq t$, $W_s^r W_t^s = W_t^r$ and $W_r^r f \equiv f$.

We say the cumulant semigroup is *homogeneous* if $W_t^r = W_{t-r}$ only depends on the difference $t - r \geq 0$. A homogeneous cumulant semigroup W_t, $t \geq 0$, is called a *Ψ-semigroup* provided E is a compact metric space and W_t preserves $C(E)^{++}$ for all $t \geq 0$[cf. Watanabe (1968)].

2.3. **Definition 2.2.** Suppose that $X = (X_t, P_{r,\mu})$ is an MB-process in the space M. Let

$$w_t^r(x) \equiv w_t^r(x, f) = -\log P_{r,\delta_x} \exp\langle X_t, -f\rangle. \tag{2.3}$$

We say X is *regular* if for every $f \in B(E)^+$ and $r \leq t$, the function $w_t^r(\cdot)$ belongs to $B(E)^+$ and

$$P_{r,\mu} \exp\langle X_t, -f\rangle = \exp\langle \mu, -w_t^r\rangle, \quad \mu \in M. \tag{2.4}$$

Here $P_{r,\mu}$ denotes the conditional expectation given $X_r = \mu$.

An easy application of Proposition 2.1 gives the following

Proposition 2.3. *Formula (2.4) defines the transition probabilities of a regular MB-process $X = (X_t, P_{r,\mu})$ if and only if $W_t^r : f \mapsto w_t^r$ is a cumulant semigroup.* Q.E.D.

If $W_t : f \mapsto w_t$ is a homogeneous cumulant semigroup, then

$$P_\mu \exp\langle X_t, -f\rangle = \exp\langle \mu, -w_t\rangle \tag{2.5}$$

determines the transition probabilities of a homogeneous MB-process $X = (X_t, P_\mu)$. In the case E is a compact metric space, Watanabe (1968) showed that a homogeneous MB-process is a Feller process if and only if it is regular and the corresponding cumulant semigroup is a Ψ-semigroup.

2.4. A special form of the MB-process is the "superprocess" that arises as the high density limit of a branching particle system. Suppose that

2.C) $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t^r, \xi_t(\omega), \Pi_{r,x})$ is a Markov process in the space E with right continuous sample paths and Borel measurable transition probabilities, i.e., for every $f \in B(E)$ and $t \in R$ the function $1_{\{r \leq t\}} \Pi_{r,x} f(\xi_t)$ is measurable in (r, x);

2.D) $K = K(\omega, t)$ is a continuous additive functional of ξ such that $\sup_\omega |K(\omega, t)| < \infty$ for every $t \in R$;

2.E) $\phi = \phi^s(x, \lambda)$ is a $\mathcal{B}(R \times E \times R^+)$-measurable function given by

$$\phi^s(x, \lambda) = b^s(x)\lambda + c^s(x)\lambda^2 + \int_0^\infty \left(e^{-\lambda u} - 1 + \lambda u\right) m^s(x, du),$$

where $c^s(x)$ is nonnegative, $m^s(x, \cdot)$ is carried by $(0, \infty)$, and the function

$$|b^s(x)| + c^s(x) + \int_0^\infty u \wedge u^2 m^s(x, du)$$

of (s, x) is bounded on $R \times E$.

A regular MB-process $X = (X_t, P_{r,\mu})$ is called a (ξ, K, ϕ)-*superprocess* if it has the cumulant semigroup $f \mapsto w_t^r$ determined by the evolution equation

$$w_t^r(x) + \Pi_{r,x} \int_r^t \phi^s(\xi_s, w_t^s(\xi_s)) K(ds) = \Pi_{r,x} f(\xi_t), \quad r \leq t. \tag{2.6}$$

The existence and the uniqueness of the solution to the above equation have been proved by Dynkin (1991ab). Note that the hypothesis $\int u \wedge u^2 m(ds) < \infty$ makes things work only for the MB-processes with finite first moments. [Dynkin also assumed $b^s(x) \geq 0$ for 2.E), but this restriction is not essential; see section 4 of this paper.]

3. MBI-processes

3.1. **Definition 3.1.** Let E be a topological Lusin space. Suppose that

3.A) $W_t^r : f \mapsto w_t^r$ $(r \leq t \in R)$ is a cumulant semigroup such that for every $f \in B(E)^+$ and $u \leq t \in R$, the function $w_t^r(x)$ of (r,x) restricted to $[u,t] \times E$ belongs to $B([u,t] \times E)^+$;

3.B) H is a measure on $R \times M_0$ such that $H([u,t] \times M_0) < \infty$ for every $u \leq t \in R$;

3.C) $\psi^s(\pi, \lambda)$ is a $\mathcal{B}(R \times M_0 \times R^+)$-measurable function given by

$$\psi^s(\pi,\lambda) = d^s(\pi)\lambda + \int_0^\infty (1 - e^{-\lambda u}) n^s(\pi, du), \quad s \in R, \pi \in M_0, \lambda \in R^+,$$

where $d^s(\pi)$ is nonnegative, $n^s(\pi, \cdot)$ is carried by $(0, \infty)$, and

$$\sup_{s,\pi} \left[d^s(\pi) + \int_0^\infty 1 \wedge u \; n^s(\pi, du) \right] < \infty.$$

A Markov process $Y = (Y_t, Q_{r,\mu})$ in the space M is called an *MBI-process with parameters* (W, H, ψ) if

$$\begin{aligned} &Q_{r,\mu} \exp\langle Y_t, -f\rangle \\ &\quad = \exp\left\{ -\langle \mu, w_t^r\rangle - \iint_{(r,t]\times M_0} \psi^s(\pi, \langle \pi, w_t^s\rangle) H(ds, d\pi) \right\} \end{aligned} \tag{3.1}$$

for $f \in B(E)^+,\ \mu \in M$ and $r \leq t$.

Remark 3.2. i) That the right hand side of (3.1) is indeed a Laplace transform follows once we observe that the functional is positive definite on semigroup $B(E)^+$. [See Berg et al. (1984) and Fitzsimmons (1988) for details on positive definite functionals.] This fact also follows from the proof of Theorem 3.5 in paragraph 3.3.

ii) We call the MBI-process defined by (3.1) a (ξ, K, ϕ, H, ψ)-*superprocess* if the corresponding cumulant semigroup $f \mapsto w_t^r$ is determined by equation (2.6). Dynkin (1991ab) has studied the (ξ, K, ϕ, H, ψ)-superprocess in the case where H is carried by $R \times \{\delta_x : \ x \in E\}$ and $\psi^s(\pi, \lambda) \equiv \lambda$.

A time homogeneous MBI-process $Y = (Y_t, Q_\mu)$ is determined by three parameters (W, η, ψ) :

$$\begin{aligned} &Q_\mu \exp\langle Y_t, -f\rangle \\ &\quad = \exp\left\{ -\langle \mu, w_t\rangle - \int_0^t ds \int_{M_0} \psi(\pi, \langle \pi, w_s\rangle) \eta(d\pi) \right\}, \end{aligned} \tag{3.2}$$

where $W_t : f \mapsto w_t$ is a homogeneous cumulant semigroup, η is a finite measure on M_0, and $\psi = \psi(\pi, \lambda)$, given by 3.C), does not depend on s. Note that if W_t is a strongly continuous Ψ-semigroup on $C(E)^{++}$, then the process Y has a strongly continuous

Feller semigroup on $C_0(M)$, so it has a version in $D(R^+, M)$ [see, for example, Ethier and Kurtz (1986)].

Example 3.3. When E is reduced to one point, the MBI-process takes values in R^+ and is called a CBI-process. In this case (3.2) becomes

$$Q_\mu e^{-zY_t} = \exp\left\{-\mu w_t - \int_0^t \psi(w_s)ds\right\}, \quad z \geq 0, \mu \geq 0, t \geq 0. \tag{3.3}$$

Kawazu and Watanabe (1971) showed that if the process Y is stochastically continuous for every Q_μ, then w_t satisfies

$$\frac{dw_t}{dt} = -\phi(w_t), \quad w_0 = z, \tag{3.4}$$

for a function ϕ with the representation

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty \left(e^{-\lambda u} - 1 + \frac{\lambda u}{1+u^2}\right) m(du), \tag{3.5}$$

where $c \geq 0$ and $\int_0^\infty 1 \wedge u^2 m(du) < \infty$.

3.2. An MBDI-process $Y = (Y_t, t \in R)$ depends on four parameters (W, H, h, β), where W and H are given by 3.A and B), β is a positive number, and

3.D) $h^s(\pi, z) = \sum\limits_{i=0}^{\infty} q_i^s(\pi) z^i$, for every $(s, \pi) \in R \times M_0$, is a probability generating function with all $q_i = q_i^s(\pi)$ measurable in (s, π).

Such a process is characterized by the following properties:

(i) the evolution of the branch $(X_t, t \geq r)$ of Y with $X_r = \mu$ a.s. is determined by the Laplace functional (2.4);

(ii) the entry times and entry distributions of the immigrants are governed by a Poisson random measure ρ on the product space $R \times M_0$ with intensity $H(ds, d\pi)$;

(iii) the generating function $h^s(\pi, \cdot)$ describes the number of drops, each of those having mass β, entering E at time s with distribution $\pi(dx)$.

We refer to β as the immigration unit. Suppose that different drops of the immigrants land in E independently of each other and that the immigration is independent of the inner population. Then the MBDI-process is a Markov process in space M. Let $Q_{r,\mu}$ denote the conditional law of $(Y_t, t \geq r)$ given $Y_r = \mu$, and let D denote the distribution of the random measure ρ on space

$$\left\{\zeta \equiv \sum_{\alpha=1}^{\zeta(R\times M_0)} \delta_{(s_\alpha, \pi_\alpha)} \ : \ (s_\alpha, \pi_\alpha) \in R \times M_0\right\}.$$

Properties (i)-(iii) lead through a calculation to the Laplace functional:

$$\begin{aligned} & Q_{r,\mu} \exp\langle Y_t, -f\rangle \\ &= \exp\langle \mu, -w_t^r\rangle \int D(d\zeta) \prod_{r<s_\alpha\le t} \sum_{i=0}^{\infty} q_i^{s_\alpha}(\pi_\alpha)\big\langle \pi_\alpha, \exp\{-\beta w_t^{s_\alpha}\}\big\rangle^i \\ &= \exp\langle \mu, -w_t^r\rangle \int D(d\zeta) \exp \iint\limits_{(r,t]\times M_0} \log h^s(\pi, \langle \pi, e^{-\beta w_t^s}\rangle)\zeta(ds, d\pi) \\ &= \exp\left\{ -\langle \mu, w_t^r\rangle - \iint\limits_{(r,t]\times M_0} \left[1 - h^s(\pi, \langle \pi, e^{-\beta w_t^s}\rangle)\right] H(ds, d\pi)\right\}. \end{aligned} \tag{3.6}$$

3.3. Consider a sequence of MBDI-processes $Y^{(k)} = (Y_t^{(k)}, Q_{r,\mu}^{(k)})$ with parameters $(W, \alpha_k H, h_k, k^{-1})$, where $\alpha_k \ge 0,\ k = 1, 2, \cdots$. By (3.6) we have

$$\begin{aligned} & Q_{r,\mu}^{(k)} \exp\langle Y_t^{(k)}, -f\rangle \\ &= \exp\left\{ -\langle \mu, w_t^r\rangle - \iint\limits_{(r,t]\times M_0} \psi_k^s(\pi, \langle \pi, w_t^s(k)\rangle) H(ds, d\pi)\right\}, \end{aligned} \tag{3.7}$$

where

$$w_t^s(k, x) = k\left[1 - \exp\{-k^{-1} w_t^s(x)\}\right], \tag{3.8}$$

and

$$\psi_k^s(\pi, \lambda) = \alpha_k\left[1 - h_k^s(\pi, 1 - \lambda/k)\right], \quad 0 \le \lambda \le k. \tag{3.9}$$

Since $w_t^s(k) \to w_t^s$ as $k \to \infty$, it is natural to assume the sequence ψ_k to converge if one hopes to obtain $Y_t = \lim\limits_{k\to\infty} Y_t^{(k)}$ in some sense.

Lemma 3.4. *i) Suppose that*

3.E) $\psi_k^s(\pi, \lambda) \to \psi^s(\pi, \lambda)$ $(k \to \infty)$ boundedly and uniformly on the set $R \times M_0 \times [0, l]$ of (s, π, λ) for each $l \ge 0$.

Then $\psi^s(\pi, \lambda)$ has the representation 3.C).

ii) To each function ψ given by 3.C) there corresponds a sequence in form (3.9) such that

$$\psi_k^s(\pi, \lambda) = \psi^s(\pi, \lambda), \quad s \in R, \pi \in M_0, 0 \le \lambda \le k.$$

Proof. Assertion i) was proved in Li (1991). To get ii) one can set

$$\alpha_k = 1 + \sup_{s,\pi} \left[kd^s(\pi) + \int_0^\infty (1 - e^{-ku}) n^s(\pi, du) \right]$$

and

$$h_k^s(\pi, z) = 1 + k\alpha_k^{-1} d^s(\pi)(z-1) + \alpha_k^{-1} \int_0^\infty (e^{ku(z-1)} - 1) n^s(\pi, du). \qquad Q.E.D.$$

Condition 3.E) usually implies $\alpha_k \to \infty$. Thus the following theorem shows that the MBI-process is an approximation for the MBDI-process with high rate and small unit of immigration.

Theorem 3.5. *i) Let $Y^{(k)}$ be as above, and let Y be the MBI-process defined by (3.1). If 3.E) holds, then for every $\mu \in M, r \leq t_1 < \cdots < t_n \in R$ and $a \geq 0$,*

$$Q_{r,\mu}^{(k)} \exp \sum_{i=1}^n \langle Y_{t_i}^{(k)}, -f_i \rangle \;\to\; Q_{r,\mu} \exp \sum_{i=1}^n \langle Y_{t_i}, -f_i \rangle \quad (k \to \infty) \tag{3.10}$$

uniformly in $f_1, \cdots, f_n \in B(E)_a^+$.

ii) For each MBI-process Y defined by (3.1), there is a sequence of MBDI-processes $Y^{(k)}$ such that (3.10) is satisfied.

Proof. It suffices to show assertion i) since ii) follows immediately from i) and Lemma 3.4. We do this by induction in n.

Fix $a \geq 0$ and $r \leq t \in R$. By 3.A) and (3.8), $w_t^s(k, x, f) \to w_t^s(x, f)$ $(k \to \infty)$ boundedly and uniformly in $(s, x, f) \in [r, t] \times E \times B(E)_a^+$. Thus 3.E) yields

$$\iint_{(r,t] \times M_0} \psi_k^s(\pi, \langle \pi, w_t^s(k) \rangle) H(ds, d\pi)$$
$$\to \iint_{(r,t] \times M_0} \psi^s(\pi, \langle \pi, w_t^s \rangle) H(ds, d\pi) \quad (k \to \infty) \tag{3.11}$$

uniformly in $f \in B(E)_a^+$. To see that the right hand side of (3.1) is indeed the Laplace functional of a probability measure we appeal to the following

Lemma 3.6 (Kallenberg,1983; Dynkin,1991a). *Suppose that $P_k, k = 1, 2, \cdots$, are probability measures on M. If $L_{P_k}(f) \to L(f)$ $(k \to \infty)$ uniformly in $f \in B(E)_a^+$ for every $a \geq 0$, then L is the Laplace functional of a probability measure on M.* Q.E.D.

Then it follows immediately that (3.1) really defines the transition probabilities of a Markov process Y in space M and that (3.10) holds for $n = 1$.

Now assuming (3.10) is true for $n = m$, we show the fact for $n = m+1$. Let $r \leq t_1 < \cdots < t_{m+1} \in R$ and $f_1, \cdots, f_{m+1} \in B(E)^+$. Then

$$
\begin{aligned}
& Q^{(k)}_{r,\mu} \exp \sum_{i=1}^{m+1} \langle Y^{(k)}_{t_i}, -f_i \rangle \\
=& Q^{(k)}_{r,\mu} Q^{(k)}_{r,\mu} \left\{ \prod_{i=1}^{m+1} \exp \langle Y^{(k)}_{t_i}, -f_i \rangle \middle| Y^{(k)}_t, t \leq t_m \right\} \\
=& Q^{(k)}_{r,\mu} \prod_{i=1}^{m} \exp \langle Y^{(k)}_{t_i}, -f_i \rangle Q^{(k)}_{r,\mu} \left\{ \exp \langle Y^{(k)}_{t_{m+1}}, -f_{m+1} \rangle \middle| Y^{(k)}_{t_m} \right\}
\end{aligned}
$$

$$
\begin{aligned}
=& Q^{(k)}_{r,\mu} \prod_{i=1}^{m} \exp \langle Y^{(k)}_{t_i}, -f_i \rangle \cdot \exp \langle Y^{(k)}_{t_m}, -w^{t_m}_{t_{m+1}}(f_{m+1}) \rangle \\
& \cdot \exp \left\{ - \iint\limits_{(t_m, t_{m+1}] \times M_0} \psi^s_k \left(\pi, \langle \pi, w^s_{t_{m+1}}(k, f_{m+1}) \rangle \right) H(ds, d\pi) \right\} \\
=& Q^{(k)}_{r,\mu} \prod_{i=1}^{m-1} \exp \langle Y^{(k)}_{t_i}, -f_i \rangle \cdot \exp \langle Y^{(k)}_{t_m}, -f_m - w^{t_m}_{t_{m+1}}(f_{m+1}) \rangle \\
& \cdot \exp \left\{ - \iint\limits_{(t_m, t_{m+1}] \times M_0} \psi^s_k \left(\pi, \langle \pi, w^s_{t_{m+1}}(k, f_{m+1}) \rangle \right) H(ds, d\pi) \right\}.
\end{aligned}
$$

By (3.11) and the induction hypothesis we have

$$
\begin{aligned}
& \lim_{k \to \infty} Q^{(k)}_{r,\mu} \exp \sum_{i=1}^{m+1} \langle Y^{(k)}_{t_i}, -f_i \rangle \\
=& Q_{r,\mu} \prod_{i=1}^{m-1} \exp \langle Y_{t_i}, -f_i \rangle \cdot \exp \langle Y_{t_m}, -f_m - w^{t_m}_{t_{m+1}}(f_{m+1}) \rangle \\
& \cdot \exp \left\{ - \iint\limits_{(t_m, t_{m+1}] \times M_0} \psi^s \left(\pi, \langle \pi, w^s_{t_{m+1}}(f_{m+1}) \rangle \right) H(ds, d\pi) \right\} \\
=& Q_{r,\mu} \exp \sum_{i=1}^{m+1} \langle Y_{t_i}, -f_i \rangle,
\end{aligned}
$$

and the convergence is uniform in $f_1, \cdots, f_{m+1} \in B(E)^+_a$.   Q.E.D.

If E is a compact metric space, then the n-dimensional product topological space $M^n = \{(\mu_1, \cdots, \mu_n) : \ \mu_1, \cdots, \mu_n \in M\}$ is locally compact and separable, and the function class

$$F(\mu_1, \cdots, \mu_n) = \exp \sum_{i=1}^{n} \langle \mu_i, -f_i \rangle, \quad f_i \in C(E)^{++},$$

is convergence determining. Thus (3.10) implies that $Y^{(k)}$ converges to Y in finite dimensional distributions.

3.4. In this paragraph, we prove a result on the weak convergence in space $D(R^+, M)$ of homogeneous MBDI-processes. Let $Y^{(k)} = (Y_t^{(k)}, t \geq 0)$ be a sequence of MBDI-processes with parameters $(W^{(k)}, \alpha_k \eta_k, h_k, k^{-1})$, where for each k,

– $W_t^{(k)} : f \mapsto w_t^{(k)}$ is a strongly continuous Ψ-semigroup on $C(E)^{++}$;

– α_k is a positive number;

– η_k is a finite measure on M_0;

– $h_k(\pi, \cdot)$, for every $\pi \in M_0$, is a probability generating function with $h_k(\pi, z)$ jointly continuous in (π, z).

The transition probabilities $Q_\mu^{(k)}$ of $Y^{(k)}$ are defined by

$$\begin{aligned} &Q_\mu^{(k)} \exp\langle Y_t^{(k)}, -f \rangle \\ &= \exp\left\{ -\langle \mu, w_t^{(k)} \rangle - \int_0^t ds \int_{M_0} \psi_k(\pi, \langle \pi, w_s(k) \rangle) \eta_k(d\pi) \right\}, \end{aligned} \tag{3.12}$$

with

$$w_t(k, x) = k\left[1 - \exp\{-k^{-1} w_t^{(k)}(x)\}\right] \tag{3.13}$$

and

$$\psi_k(\pi, \lambda) = \alpha_k \left[1 - h_k(\pi, 1 - \lambda/k)\right], \quad 0 \leq \lambda \leq k. \tag{3.14}$$

Clearly $Y^{(k)}$ has a strongly continuous Feller semigroup on $C_0(M)$, so we can assume it has sample paths in $D(R^+, M)$.

Theorem 3.7. *Let $W_t : f \mapsto w_t(f)$ be a strongly continuous Ψ-semigroup on $C(E)^{++}$, and let $Y = (Y_t, t \geq 0)$ be an MBI-process in $D(R^+, M)$ with parameters (W, η, ψ) with initial distribution Λ. Suppose that*

3.F) for every $f \in C(E)^{++}$, $w_t^{(k)}(x, f) \to w_t(x, f)$ $(k \to \infty)$ uniformly in (t, x) on each set $[0, l] \times E$;

3.G) $\eta_k \to \eta$ weakly;

3.H) $\psi_k(\pi, \lambda) \to \psi(\pi, \lambda)$ uniformly in (π, λ) on each set $M_0 \times [0, l]$;

3.I) $Y_0^{(k)}$ has limiting distribution Λ.

Then $Y^{(k)}$ converges weakly to Y in the space $D(R^+, M)$ as $k \to \infty$.

Proof. By Theorem 2.5 of Ethier and Kurtz (1986, p167), it is sufficient to prove

$$\sup_{\mu\in M}\left|Q_\mu^{(k)}\exp\langle Y_t^{(k)},-f\rangle - Q_\mu\exp\langle Y_t,-f\rangle\right| \;\to\; 0 \quad (k\to\infty) \tag{3.15}$$

for every fixed $f\in C(E)^{++}$ and $t\geq 0$. Let $2a=\inf_x w_t(x)$. By 3.F), there is a k_1 such that $w_t^{(k)}\geq a$ for all $k>k_1$.

Suppose $0<\varepsilon<1$. If $\mu(E)\geq a^{-1}\log\varepsilon^{-1}$, we have

$$\begin{aligned}&\left|Q_\mu^{(k)}\exp\langle Y_t^{(k)},-f\rangle - Q_\mu\exp\langle Y_t,-f\rangle\right|\\ &\qquad\qquad\leq e^{-\langle\mu,w_t^{(k)}\rangle}+e^{-\langle\mu,w_t\rangle}<2\varepsilon \;\text{ for } k>k_1.\end{aligned}$$

If $\mu(E)<a^{-1}\log\varepsilon^{-1}$, then

$$\begin{aligned}&\left|Q_\mu^{(k)}\exp\langle Y_t^{(k)},-f\rangle - Q_\mu\exp\langle Y_t,-f\rangle\right|\\ \leq&\left|\langle\mu,w_t^{(k)}\rangle-\langle\mu,w_t\rangle\right|\\ &+\left|\int_0^t ds\int_{M_0}\psi_k(\pi,\langle\pi,w_s(k)\rangle)\eta_k(d\pi)-\int_0^t ds\int_{M_0}\psi(\pi,\langle\pi,w_s\rangle)\eta(d\pi)\right|\\ \leq& a^{-1}\|w_t^{(k)}-w_t\|\log\varepsilon^{-1}+\varepsilon_1(k)+\varepsilon_2(k)+\varepsilon_3(k),\end{aligned}$$

where

$$\begin{aligned}\varepsilon_1(k)&=\int_0^t ds\int_{M_0}|\psi_k(\pi,\langle\pi,w_s(k)\rangle)-\psi(\pi,\langle\pi,w_s(k)\rangle)|\eta_k(d\pi),\\ \varepsilon_2(k)&=\int_0^t ds\int_{M_0}|\psi(\pi,\langle\pi,w_s(k)\rangle)-\psi(\pi,\langle\pi,w_s\rangle)|\eta_k(d\pi),\\ \varepsilon_3(k)&=\int_0^t\left|\int_{M_0}\psi(\pi,\langle\pi,w_s\rangle)\eta_k(d\pi)-\int_{M_0}\psi(\pi,\langle\pi,w_s\rangle)\eta(d\pi)\right|ds.\end{aligned}$$

By 3.F), there exits k_2 such that

$$a^{-1}\|w_t^{(k)}-w_t\|\log\varepsilon^{-1}<\varepsilon \;\text{ for } k>k_2.$$

3.F) also implies $w_t(k,x)\to w_t(x)$ boundedly and uniformly on each set $[0,l]\times E$. Then 3.G) and 3.H) yield the existence of k_3 such that $\varepsilon_1(k)+\varepsilon_2(k)<\varepsilon$ for $k>k_3$. By 3.G) and the dominated convergence theorem there is a k_4 such that $\varepsilon_3(k)<\varepsilon$ for $k>k_4$. Thus (3.15) follows. Q.E.D.

4. Particle systems and superprocesses

4.1. As usual, let E be a topological Lusin space. Suppose we have ξ, K, H and h given by 2.C), 2.D), 3.B) and 3.D) respectively. Assume that

4.A) $g^s(x,z) = \sum_{i=0}^{\infty} p_i^s(x) z^i$, for every $(s,x) \in R \times E$, is a probability generating function with the $p_i^s(x)$ and $\sum_{i=1}^{\infty} i p_i^s(x)$ belonging to $B(R \times E)^+$.

A branching particle system with immigration with parameters (ξ, K, g, H, h) is described as follows:

(i) The particles in E move according to the law of ξ.

(ii) For a particle which is alive at time r and follows the path $(\xi_t, t \geq r)$, the conditional probability of survival during the time interval $[r,s)$ is $e^{-K(r,s)}$.

(iii) When a particle dies at time s at point $x \in E$, it gives birth to a random number of offspring at the death site according to the generating function $g^s(x,\cdot)$.

(iv) The entry times and distributions of new particles immigrating to E are governed by a Poisson random measure with intensity $H(ds, d\pi)$.

(v) The generating function $h^s(\pi,\cdot)$ gives the distribution of the number of new particles entering E at time s with distribution π.

For $t \in R$, let $Y_t(B)$ be the number of particles of the system in set $B \in \mathcal{B}(E)$ at time t. Under standard independence hypotheses, $(Y_t, t \in R)$ form a Markov process in space M_1. [Note that the state space of the particle system is different from that of the MBDI-process.] The rigorous construction of the process can be reduced to constructing a branching particle system with parameters (ξ, K, g) generated by a single particle, which was given in Dynkin (1991a). The transition probabilities $Q_{r,\sigma}$ of $(Y_t, t \in R)$ are determined by the Laplace functionals [cf. (3.6)]:

$$\begin{aligned} &Q_{r,\sigma} \exp\langle Y_t, -f\rangle \\ &\quad = \exp\left\{ -\langle \sigma, v_t^r\rangle - \iint_{(r,t]\times M_0} \left[1 - h^s(\pi, \langle \pi, e^{-v_t^s}\rangle)\right] H(ds, d\pi) \right\}, \\ &\qquad\qquad f \in B(E)^+, \sigma \in M_1, r \leq t, \end{aligned} \tag{4.1}$$

where $v_t^r(x) \equiv v_t^r(x,f)$ is the unique positive solution of

$$\begin{aligned} e^{-v_t^r(x)} =& \mathit{\Pi}_{r,x} e^{-f(\xi_t) - K(r,t)} \\ &+ \mathit{\Pi}_{r,x} \int_r^t e^{-K(r,s)} g^s(\xi_s, e^{-v_t^s(\xi_s)}) K(ds). \end{aligned} \tag{4.2}$$

This equation arises as follows: If we start one particle at time r at point x, this particle moves following a path of ξ and does not branch before time t with probability $e^{-K(r,t)}$ [first term on the right hand side], or it splits at time $s \in (r,t]$ with probability $e^{-K(r,s)}K(ds)$ according to $g^s(\xi_s,\cdot)$ and all the offspring evolve independently after birth in the same fashion [second term]. By Lemma 2.3 of Dynkin (1991a), (4.2) is equivalent to

$$\begin{aligned} &\Pi_{r,x} e^{-f(\xi_t)} - e^{-v_t^r(x)} \\ &\qquad = \Pi_{r,x} \int_r^t \left[e^{-v_t^s(\xi_s)} - g^s(\xi_s, e^{-v_t^s(\xi_s)}) \right] K(ds). \end{aligned} \tag{4.3}$$

4.2. Let $Y(k) = \{Y_t(k), t \in R\}$ be a sequence of branching particle systems with immigration with parameters $(\xi, \gamma_k K, g_k, \alpha_k H, h_k)$, where $\alpha_k \geq 0, \gamma_k \geq 0, k = 1, 2, \cdots$. Then

$$Y^{(k)} = \{Y_t^{(k)} := k^{-1} Y_t(k),\ t \in R\}$$

is a Markov process in space M_k with transition probabilities $Q_{r,\sigma_k}^{(k)}$ determined by

$$\begin{aligned} &Q_{r,\sigma_k}^{(k)} \exp\langle Y_t^{(k)}, -f\rangle \\ &= \exp\left\{ -\langle \sigma_k, k v_t^r(k)\rangle - \iint_{(r,t]\times M_0} \psi_k^s(\pi, \langle \pi, w_t^s(k)\rangle) H(ds, d\pi) \right\}, \\ &\qquad\qquad f \in B(E)^+, \sigma_k \in M_k, r \leq t, \end{aligned} \tag{4.4}$$

where $\psi_k^s(\pi, \lambda)$ is given by (3.9), $v_t^r(k,x) \equiv v_t^r(k,x,f)$ satisfies

$$\begin{aligned} &\Pi_{r,x} e^{-f(\xi_t)/k} - e^{-v_t^r(k,x)} \\ &\qquad = \Pi_{r,x} \int_r^t \gamma_k \left[e^{-v_t^s(k,\xi_s)} - g_k^s(\xi_s, e^{-v_t^s(k,\xi_s)}) \right] K(ds) \end{aligned} \tag{4.5}$$

and

$$w_t^s(k,x) \equiv w_t^s(k,x,f) = k[1 - e^{-v_t^s(k,x,f)}]. \tag{4.6}$$

Let $Q_{r,\mu_k}^{(k)}$ denote the conditional law of $(Y_t^{(k)}, t \geq r)$ given $Y_r^{(k)} = k^{-1}\sigma(k\mu)$, where μ belongs to M and $\sigma(k\mu)$ is a Poisson random measure with intensity $k\mu$. Then

$$\begin{aligned} &Q_{r,\mu_k}^{(k)} \exp\langle Y_t^{(k)}, -f\rangle \\ &= \exp\left\{ -\langle \mu, w_t^r(k)\rangle - \iint_{(r,t]\times M_0} \psi_k^s(\pi, \langle \pi, w_t^s(k)\rangle) H(ds, d\pi) \right\}. \end{aligned} \tag{4.7}$$

It is easy to check that $w_t^r(k)$ satisfies

$$w_t^r(k,x) + \Pi_{r,x} \int_r^t \phi_k^s(\xi_s, w_t^s(k,\xi_s)) K(ds) = \Pi_{r,x} k[1 - e^{-f(\xi_t)/k}] \tag{4.8}$$

with

$$\phi_k^s(x,\lambda) = k\gamma_k \left[g_k^s(x, 1-\lambda/k) - (1-\lambda/k)\right], \quad 0 \leq \lambda \leq k. \tag{4.9}$$

For the sequence (4.9) we note

$$\bar{b}_k = \sup_{s,x} \left| \frac{d}{d\lambda}\phi_k^s(x,\lambda)\right|_{\lambda=0}. \tag{4.10}$$

Lemma 4.1. *i) Suppose that*

4.B) $\phi_k^s(x,\lambda) \to \phi^s(x,\lambda)$ $(k\to\infty)$ *uniformly on each set* $R\times E\times[0,l]$;

4.C) $\phi^s(x,\lambda)$ *is Lipschitz in* λ *uniformly on each set* $R\times E\times[0,l]$.

Then $\phi^s(x,\lambda)$ *has the representation 2.E).*

ii) If $\phi^s(x,\lambda)$ *is given by 2.E), then it satisfies 4.C) and there is a sequence* $\phi_k^s(x,\lambda)$ *in form (4.9) such that 4.B) holds and*

$$\frac{d}{d\lambda}\phi_k^s(x,\lambda)\Big|_{\lambda=0} = b^s(x), \quad s\in R, x\in E. \tag{4.11}$$

Proof. Assertion i) follows easily by a result of Li (1991), so we shall prove ii) only. Suppose that $\phi^s(x,\lambda)$ is given by 2.E). 4.C) holds clearly. Let

$$\gamma_{1,k} = 1 + \sup_{s,x}\int_0^\infty u(1-e^{-ku})m^s(x,du)$$

and

$$g_{1,k}^s(x,z) = z + k^{-1}\gamma_{1,k}^{-1}\int_0^\infty \left[e^{ku(z-1)} - 1 + ku(1-z)\right] m^s(x,du).$$

It is easy to check that

$$\begin{aligned}\phi_{1,k}^s(x,\lambda) :&= k\gamma_{1,k}\left[g_{1,k}^s(x,1-\lambda/k) - (1-\lambda/k)\right] \\ &= \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)m^s(x,du).\end{aligned}$$

Let

$$\bar{b} = \sup_{s,x}|b^s(x)|, \quad \bar{c} = \sup_{s,x} c^s(x).$$

Assuming $\gamma_{2,k} := \bar{b} + 2k\bar{c} > 0$ and setting

$$g_{2,k}^s(x,z) = \begin{cases} z + \gamma_{2,k}^{-1}\left[b^s(x)(1-z) + kc^s(x)(1-z)^2\right] & \text{if } b^s(x)\geq 0 \\ \gamma_{2,k}^{-1}\left[\frac{1}{2}\bar{b}(1+z^2) + \frac{1}{2}b^s(x)(1-z^2)\right. & \\ \qquad\qquad \left.+ kc^s(x)(1-z)^2 + 2k\bar{c}z\right] & \text{if } b^s(x) < 0,\end{cases}$$

we have

$$\phi^s_{2,k}(x,\lambda) := k\gamma_{2,k}\left[g^s_{2,k}(x,1-\lambda/k)-(1-\lambda/k)\right] = \begin{cases} b^s(x)\lambda + c^s(x)\lambda^2 & \text{if } b^s(x)\geq 0 \\ b^s(x)\lambda + c^s(x)\lambda^2 + (2k)^{-1}[\bar{b}-b^s(x)]\lambda^2 & \text{if } b^s(x)<0. \end{cases}$$

Finally we let

$$\gamma_k = \gamma_{1,k}+\gamma_{2,k} \quad \text{and} \quad g_k = \gamma_k^{-1}(\gamma_{1,k}g_{1,k}+\gamma_{2,k}g_{2,k}).$$

Then the sequence $\phi^s_k(x,\lambda)$ defined by (4.9) is equal to $\phi^s_{1,k}(x,\lambda)+\phi^s_{2,k}(x,\lambda)$ that satisfies 4.B) and (4.11).   Q.E.D.

Lemma 4.2. *If conditions 4.B and C) are fulfilled and if*
4.D) $\sup_k \bar{b}_k < \infty$,
then $w^r_t(k,x,f)$, *and hence* $kv^r_t(k,x,f)$, *converge boundedly and uniformly on each set* $[u,t]\times E\times B(E)^+_a$ *of* (r,x,f) *to the unique bounded positive solution of equation (2.6).*

Proof. Since $-\frac{d}{d\lambda}\phi^s_k(x,\lambda)\leq \bar{b}_k$, (4.8) implies that

$$w^r_t(k,x) \leq \|f\| + \bar{b}_k \mathit{\Pi}_{r,x}\int_r^t w^s_t(k,\xi_s)K(ds). \tag{4.12}$$

By the generalized Gronwall's inequality proved by Dynkin (1991a), we get

$$w^r_t(k,x) \leq \|f\|\mathit{\Pi}_{r,x}e^{\bar{b}_k K(r,t)}. \tag{4.13}$$

Using (4.8) and (4.13), the convergence of $w^r_t(k)$ is proved in the same way as Lemma 3.3 of Dynkin (1991a). The convergence of $kv^r_t(k)$ follows by (4.6).   Q.E.D.

Based on Lemmas 4.1 and 4.2, the following result can be obtained similarly as Theorem 3.5.

Theorem 4.3. *i) Let* $Y^{(k)}$ *be the sequence of renormalized branching particle systems with immigration defined by (4.7), and let* Y *be the* (ξ,K,ϕ,H,ψ)*-superprocess. Assume that conditions 3.E) and 4.B,C,D) are satisfied. Then for every* $\mu\in M,\ r\leq t_1<\cdots< t_n$ *and* $a\geq 0$,

$$Q^{(k)}_{r,\mu_k}\exp\sum_{i=1}^n\langle Y^{(k)}_{t_i},-f_i\rangle \ \to\ Q_{r,\mu}\exp\sum_{i=1}^n\langle Y_{t_i},-f_i\rangle \quad (k\to\infty) \tag{4.14}$$

uniformly in $f_1,\cdots,f_n\in B(E)^+_a$.

ii) To each (ξ, K, ϕ, H, ψ)-superprocess Y there corresponds a sequence of branching particle systems with immigration $Y^{(k)}$ satisfying (4.14). Q.E.D.

Suppose that each particle in the kth system is weighted k^{-1}. (4.14) states that the mass distribution of the particle system approximates to the process Y when the single mass becomes small and the particle population becomes large. Typically, $\gamma_k \to \infty$ and $\alpha_k \to \infty$, which mean that the rates of the branching and the immigration get high. It is also possible to prove a result on the weak convergence in space $D(R^+, M)$ of the branching system of particles with immigration. The discussions are similar to those of section 3 and left to the reader.

5. Transformations of the measure space

By transformations of the state space M, large classes of MBI-processes that may take infinite (but σ-finite) values can be obtained from the MBI-processes with finite values that we have discussed in sections 3 and 4.

For $\rho \in B(E)^{++}$ we let

$M^\rho = \{\mu :\ \mu$ is a measure on $(E, \mathcal{B}(E))$ such that $\langle \mu, \rho\rangle < \infty\}$,

$M_0^\rho = \{\tau :\ \tau \in M^\rho$ and $\langle \tau, \rho\rangle = 1\}$.

Suppose that $W_t^r : f \mapsto w_t^r(f)$ is a cumulant semigroup on $B(E)^+$ such that

5.A) for every $f \in B(E)^+$ and $u \leq t \in R$, the function $\rho^{-1}(x) w_t^r(x, \rho f)$ of (r, x) restricted to $[u, t] \times E$ belongs to $B([u, t] \times E)^+$.

We define the operators $\widehat{W}_t^r : f \mapsto \widehat{w}_t^r(f)$ on $B(E)^+$ by

$$\widehat{w}_t^r(f) = \rho^{-1} w_t^r(\rho f) \tag{5.1}$$

[cf. El Karoui and Roelly-Coppoletta (1989)]. It is easy to see that $\widehat{W}_t^r,\ r \leq t \in R$, also form a cumulant semigroup. If $(\widehat{Y}_t, t \in R)$ is an MBI-process in M with parameters $(\widehat{W}, H, \psi)$ [Definition 3.1], then

$$Y = (Y_t := \rho^{-1}\widehat{Y}_t,\ t \in R) \tag{5.2}$$

is an MBI-process in the space M^ρ with transition probabilities $Q_{r,\mu}$ determined by

$$\begin{aligned} &Q_{r,\mu} \exp\langle Y_t, -f\rangle \\ &= \exp\left\{ -\langle \mu, w_t^r\rangle - \iint\limits_{(r,t]\times M_0^\rho} \psi_\rho^s(\tau, \langle \tau, w_t^s\rangle) H_\rho(ds, d\tau)\right\}, \\ &\qquad\qquad f \in B(E)^+, \mu \in M^\rho, r \leq t, \end{aligned} \tag{5.3}$$

where

$$H_\rho(ds, d\tau) = H(ds, d\rho\tau) \quad \text{and} \quad \psi_\rho^s(\tau, \lambda) = \psi^s(\rho\tau, \lambda).$$

Example 5.1. Suppose that $0 < \beta \leq 1$ and that Π_t is the semigroup of the d-dimensional Brownian motion. Then equation

$$w_t + \int_0^t \Pi_{t-s}(w_s)^{1+\beta} ds = \Pi_t f, \quad t \geq 0, \tag{5.4}$$

defines a homogeneous cumulant semigroup $W_t : f \mapsto w_t$. For $p > d$, let $\rho(x) = (1+|x|^p)^{-1}, \ x \in R^d$, and let

$M_p(R^d) = \{\mu : \ \mu \text{ is a Borel measure on } R^d \text{ such that } \langle \mu, \rho \rangle < \infty\}.$

Iscoe (1986) showed that W_t satisfies condition 5.A). Assume $0 < \theta \leq 1$ and $\lambda \in M_p(R^d)$. Then formula

$$Q_\mu \exp\langle Y_t, -f \rangle = \exp\left\{ -\langle \mu, w_t \rangle - \int_0^t \langle \lambda, w_s \rangle^\theta ds \right\} \tag{5.5}$$

defines an MBI-process $Y = (Y_t, Q_\mu)$ in the space $M_p(R^d)$. When $\beta = \theta = 1$, Y has continuous sample paths almost surely [in a suitable topology in $M_p(R^d)$; see Konno and Shiga (1988)].

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