MEASURE-VALUED BRANCHING PROCESSES
WITH IMMIGRATION

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Abstract. Starting from the cumulant semigroup of a measure-valued branching pro-
cess, we construct the transition probabilities of some Markov process $Y^{(β)} = (Y_t^{(β)}, t \in R)$, which we call a measure-valued branching process with discrete immigration of unit $β$. The immigration of $Y^{(β)}$ is governed by a Poisson random measure $ρ$ on the time-distribution space and a probability generating function $h$, both depending on $β$. It is shown that, under suitable hypotheses, $Y^{(β)}$ approximates to a Markov process $Y = (Y_t, t \in R)$ as $β \to 0^+$. The latter is the one we call a measure-valued branching process with immigration. The convergence of branching particle systems with immigration is also studied.

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measure-valued branching process * immigration * particle system * superprocess * weak convergence

1. Introduction

Let $M$ be the totality of finite measures on a measurable space $(E, \mathcal{E})$. Suppose that $X = (X_t, t \in R)$ is a Markov process in $M$ with transition function $P(r, \mu; t, d\nu)$. $X$ is called a measure-valued branching process (MB-process) if

$$P(r, \mu_1 + \mu_2; t, \cdot) = P(r, \mu_1; t, \cdot) * P(r, \mu_2; t, \cdot), \quad \mu_1, \mu_2 \in M, \; r \leq t, \quad (1.1)$$

where “*” denotes the convolution operation [cf. Dawson (1977), Dawson and Ivanoff (1978), Watanabe (1968), etc]. When $E$ is reduced to one point, $X$ takes values in $R^+ := [0, \infty)$ and is called a continuous state branching process (CB-process).
Continuous state branching processes with immigration (CBI-processes) were first introduced by Kawazu and Watanabe (1971). Several authors have also studied measure-valued branching processes with immigration (MBI-processes); see Dynkin (1991ab), Konno and Shiga (1988), etc.

In the present paper, we study a general class of MBI-processes that covers the models of the previous authors and can be regarded as the measure-valued counterpart of the one of CBI-processes proposed by Kawazu and Watanabe. Section 2 contains some preliminaries. The general definition for an MBI-process is given in section 3, followed by the model of a measure-valued branching process with discrete immigration (MBDI-process). The heuristic meanings of the latter are clear. It is shown that the MBI-process is in fact an approximation for the MBDI-process with high rate and small unit of immigration. In section 4, we study the convergence of branching particle systems with immigration to MBI-processes. A branching system of particles with immigration is not an MBDI-process in the terminology of this paper. The concluding section 5 contains a brief discussion of MBI-processes with $\sigma$-finite values whose study can be reduced to that of the class with finite values studied in sections 3 and 4.

2. Preliminaries

2.1. We first introduce some notation. If $F$ is a topological space, then $B(F)$ denotes the $\sigma$-algebra of $F$ generated by all open sets, and

$B(F) = \{ \text{bounded } B(F)\text{-measurable functions on } F \}$,

$C(F) = \{ f : f \in B(F) \text{ is continuous } \}$,

$B(F)_a = \{ f : f \in B(F) \text{ and } \|f\| \leq a \} \text{ for } a \geq 0$.

Here $\| \cdot \|$ denotes the supremum norm. In the case $F$ is locally compact,

$C_0(F) = \{ f : f \in C(F) \text{ vanishes at infinity } \}$.

The subsets of nonnegative members of the function spaces are denoted by the superscript “+”, and those of strictly positive members by “++”; e.g., $B(F)^+ \text{, } C(F)^{++}$. If $F$ is a metric space, then $D(R^+, F)$ stands for the space of cadlag functions from $R^+$ to $F$ equipped with the Skorohod topology. Finally, $\delta_x$ denotes the unit mass concentrated at $x$, and for a function $f \text{ and a measure } \mu$, $\langle \mu, f \rangle = \int f d\mu$.

2.2. Suppose that $E$ is a topological Lusin space, i.e., a homeomorph of a Borel subset of some compact metric space. Let

$M = \{ \text{finite measures on } (E, B(E)) \}$,

$M_0 = \{ \pi : \pi \in M \text{ and } \pi(E) = 1 \}$,

$M_1 = \{ \sigma : \sigma \in M \text{ is integer-valued } \}$,

$M_k = \{ k^{-1}\sigma : \sigma \in M_1 \} \text{ for } k = 2, 3, \cdots$.

We topologize $M$, and hence $M_k, k = 0, 1, 2, \cdots$, with the weak convergence topology. It is well known that $M$ is locally compact and separable when $E$ is a compact metric space.
The Laplace functional of a probability measure $P$ on $M$ is defined as
\[
L_P(f) = \int_M e^{-\langle \mu, f \rangle} P(d\mu), \quad f \in B(E)^+.
\] (2.1)

$P$ is said to be infinitely divisible if for each integer $m > 0$, there is a probability measure $P_m$ on $M$ such that $L_P(f) = \left[L_{P_m}(f)\right]^m$.

We say a functional $w$ on $B(E)^+$ belongs to the class $W$ if it has the representation
\[
w(f) = \iint_{R^+ \times M_0} \left(1 - e^{-u \langle \pi, f \rangle}\right) \frac{1 + u}{u} G(du, d\pi), \quad f \in B(E)^+.
\] (2.2)

where $G$ is a finite measure on $R^+ \times M_0$ and the value of the integrand at $u = 0$ is defined as $\langle \pi, f \rangle$.

The following result coincides with Theorem 1.2 of Watanabe (1968) since $(E, B(E))$ is isomorphic to a compact metric space with the Borel $\sigma$-algebra.

**Proposition 2.1.** A probability measure $P$ on $M$ is infinitely divisible if and only if $-\log L_P(\cdot) \in W$. Q.E.D.

A family of operators $W^r_t : f \mapsto w^r_t(\cdot, f)$ ($r \leq t \in R$) on $B(E)^+$ is called a **cumulant semigroup** provided

2.A) for every fixed $r \leq t$ and $x$, $w^r_t(x, \cdot)$ belongs to $W$;

2.B) for all $r \leq s \leq t$, $W^r_s W^s_t = W^r_t$ and $W^r_t f \equiv f$.

We say the cumulant semigroup is **homogeneous** if $W^r_t = W^t_{t-r}$ only depends on the difference $t - r \geq 0$. A homogeneous cumulant semigroup $W_t$, $t \geq 0$, is called a $\Psi$-**semigroup** provided $E$ is a compact metric space and $W_t$ preserves $C(E)^{++}$ for all $t \geq 0$[cf. Watanabe (1968)].

**2.3. Definition 2.2.** Suppose that $X = (X_t, P_r, \mu)$ is an MB-process in the space $M$. Let
\[
w^r_t(x) \equiv w^r_t(x, f) = -\log P_{r,\delta_x} \exp\langle X_t, -f \rangle.
\] (2.3)

We say $X$ is **regular** if for every $f \in B(E)^+$ and $r \leq t$, the function $w^r_t(\cdot)$ belongs to $B(E)^+$ and
\[
P_{r,\mu} \exp\langle X_t, -f \rangle = \exp\langle \mu, -w^r_t \rangle, \quad \mu \in M.
\] (2.4)

Here $P_{r,\mu}$ denotes the conditional expectation given $X_r = \mu$.

An easy application of Proposition 2.1 gives the following

**Proposition 2.3.** Formula (2.4) defines the transition probabilities of a regular MB-process $X = (X_t, P_r, \mu)$ if and only if $W^r_t : f \mapsto w^r_t$ is a cumulant semigroup. Q.E.D.
If $W_t : f \mapsto w_t$ is a homogeneous cumulant semigroup, then
\[ P_\mu \exp(X_t, -f) = \exp(\mu, -w_t) \] (2.5)
determines the transition probabilities of a homogeneous MB-process $X = (X_t, P_\mu)$. In the case $E$ is a compact metric space, Watanabe (1968) showed that a homogeneous MB-process is a Feller process if and only if it is regular and the corresponding cumulant semigroup is a $\Psi$-semigroup.

2.4. A special form of the MB-process is the “superprocess” that arises as the high density limit of a branching particle system. Suppose that

2.C) $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t(\omega), \Pi_{r,x})$ is a Markov process in the space $E$ with right continuous sample paths and Borel measurable transition probabilities, i.e., for every $f \in B(E)$ and $t \in \mathbb{R}$ the function $1_{\{r \leq t\}} \Pi_{r,x} f(\xi_t)$ is measurable in $(r,x)$;

2.D) $K = K(\omega, t)$ is a continuous additive functional of $\xi$ such that $\sup_{\omega} |K(\omega, t)| < \infty$ for every $t \in \mathbb{R}$;

2.E) $\phi = \phi^s(x, \lambda)$ is a $B(R \times E \times R^+)$-measurable function given by
\[ \phi^s(x, \lambda) = b^s(x)\lambda + c^s(x)\lambda^2 + \int_{0}^{\infty} (e^{-\lambda u} - 1 + \lambda u) m^s(x, du), \]
where $c^s(x)$ is nonnegative, $m^s(x, \cdot)$ is carried by $(0, \infty)$, and the function
\[ |b^s(x)| + c^s(x) + \int_{0}^{\infty} u \wedge u^2 m^s(x, du) \]
of $(s,x)$ is bounded on $R \times E$.

A regular MB-process $X = (X_t, P_{r,\mu})$ is called a $(\xi, K, \phi)$-superprocess if it has the cumulant semigroup $f \mapsto w^s_r$ determined by the evolution equation
\[ w^s_r(x) + \Pi_{r,x} \int_{r}^{t} \phi^s(\xi_s, w^s_{t}(\xi_s))K(ds) = \Pi_{r,x} f(\xi_t), \quad r \leq t. \] (2.6)
The existence and the uniqueness of the solution to the above equation have been proved by Dynkin (1991ab). Note that the hypothesis $\int u \wedge u^2 m(ds) < \infty$ makes things work only for the MB-processes with finite first moments. [Dynkin also assumed $b^s(x) \geq 0$ for 2.E), but this restriction is not essential; see section 4 of this paper.]

3. MBI-processes

3.1. Definition 3.1. Let $E$ be a topological Lusin space. Suppose that
3.A) \( W_t^r : f \mapsto w_t^r \) \((r \leq t \in R)\) is a cumulant semigroup such that for every \( f \in B(E)^+ \) and \( u \leq t \in R \), the function \( w_t^r(x) \) of \((r, x)\) restricted to \([u, t] \times E\) belongs to \(B([u, t] \times E)^+\);

3.B) \( H \) is a measure on \( R \times M_0 \) such that \( H([u, t] \times M_0) < \infty \) for every \( u \leq t \in R \);

3.C) \( \psi^s(\pi, \lambda) \) is a \( \mathcal{B}(R \times M_0 \times R^+)-\)measurable function given by

\[
\psi^s(\pi, \lambda) = d^s(\pi)\lambda + \int_0^s (1 - e^{-\lambda u})n^s(\pi, du), \quad s \in R, \pi \in M_0, \lambda \in R^+,
\]

where \( d^s(\pi) \) is nonnegative, \( n^s(\pi, \cdot) \) is carried by \((0, \infty)\), and

\[
\sup_{s,\pi} \left[ d^s(\pi) + \int_0^\infty 1 \wedge u n^s(\pi, du) \right] < \infty.
\]

A Markov process \( Y = (Y_t, Q_{r,\mu}) \) in the space \( M \) is called an \textit{MBI-process with parameters} \((W, H, \psi)\) if

\[
Q_{r,\mu} \exp\langle Y_t, -f \rangle = \exp \left\{ -\langle \mu, w_t^r \rangle - \int_{(r,t] \times M_0} \psi^s(\pi, \langle \pi, w_s^r \rangle)H(ds, d\pi) \right\}
\]

for \( f \in B(E)^+ \), \( \mu \in M \) and \( r \leq t \).

\textbf{Remark 3.2.} i) That the right hand side of (3.1) is indeed a Laplace transform follows once we observe that the functional is positive definite on semigroup \( B(E)^+ \). [See Berg et al. (1984) and Fitzsimmons (1988) for details on positive definite functionals.] This fact also follows from the proof of Theorem 3.5 in paragraph 3.3.

ii) We call the MBI-process defined by (3.1) a \((\xi, K, \phi, H, \psi)-\text{superprocess}\) if the corresponding cumulant semigroup \( f \mapsto w_t^r \) is determined by equation (2.6). Dynkin (1991ab) has studied the \((\xi, K, \phi, H, \psi)-\text{superprocess}\) in the case where \( H \) is carried by \( R \times \{ \delta_x : x \in E \} \) and \( \psi^s(\pi, \lambda) \equiv \lambda \).

A time homogeneous MBI-process \( Y = (Y_t, Q_{\mu}) \) is determined by three parameters \((W, \eta, \psi)\) :

\[
Q_{\mu} \exp\langle Y_t, -f \rangle = \exp \left\{ -\langle \mu, w_t \rangle - \int_0^t ds \int_{M_0} \psi(\pi, \langle \pi, w_s \rangle)\eta(d\pi) \right\},
\]

where \( W_t : f \mapsto w_t \) is a homogeneous cumulant semigroup, \( \eta \) is a finite measure on \( M_0 \), and \( \psi = \psi(\pi, \lambda) \), given by 3.C), does not depend on \( s \). Note that if \( W_t \) is a strongly continuous \( \Psi \)-semigroup on \( C(E)^{++} \), then the process \( Y \) has a strongly continuous
Feller semigroup on $C_0(M)$, so it has a version in $D(R^+, M)$ [see, for example, Ethier and Kurtz (1986)].

**Example 3.3.** When $E$ is reduced to one point, the MBI-process takes values in $R^+$ and is called a CBI-process. In this case (3.2) becomes

$$Q_\mu e^{-zY_{t}} = \exp \left\{ -\mu w_t - \int_0^t \psi(w_s) ds \right\}, \quad z \geq 0, \mu \geq 0, t \geq 0. \quad (3.3)$$

Kawazu and Watanabe (1971) showed that if the process $Y$ is stochastically continuous for every $Q_\mu$, then $w_t$ satisfies

$$\frac{dw_t}{dt} = -\phi(w_t), \quad w_0 = z, \quad (3.4)$$

for a function $\phi$ with the representation

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_0^{\infty} \left( e^{-\lambda u} - 1 + \frac{\lambda u}{1 + u^2} \right) m(du), \quad (3.5)$$

where $c \geq 0$ and $\int_0^{\infty} 1 \wedge u^2 m(du) < \infty$.

3.2. An MBDI-process $Y = (Y_t, t \in R)$ depends on four parameters $(W, H, h, \beta)$, where $W$ and $H$ are given by 3.A and B), $\beta$ is a positive number, and

3.D) $h^s(\pi, z) = \sum_{i=0}^{\infty} q_i^s(\pi) z^i$, for every $(s, \pi) \in R \times M_0$, is a probability generating function with all $q_i = q_i^s(\pi)$ measurable in $(s, \pi)$.

Such a process is characterized by the following properties:

(i) the evolution of the branch $(X_t, t \geq r)$ of $Y$ with $X_r = \mu$ a.s. is determined by the Laplace functional (2.4);

(ii) the entry times and entry distributions of the immigrants are governed by a Poisson random measure $\rho$ on the product space $R \times M_0$ with intensity $H(ds, d\pi)$;

(iii) the generating function $h^s(\pi, \cdot)$ describes the number of drops, each of those having mass $\beta$, entering $E$ at time $s$ with distribution $\pi(dx)$.

We refer to $\beta$ as the immigration unit. Suppose that different drops of the immigrants land in $E$ independently of each other and that the immigration is independent of the inner population. Then the MBDI-process is a Markov process in space $M$. Let $Q_{r, \mu}$ denote the conditional law of $(Y_t, t \geq r)$ given $Y_r = \mu$, and let $D$ denote the distribution of the random measure $\rho$ on space

$$\left\{ \zeta \equiv \sum_{\alpha=1}^{\zeta(R \times M_0)} \delta_{(s_\alpha, \pi_\alpha)} : (s_\alpha, \pi_\alpha) \in R \times M_0 \right\}.$$
Properties (i)-(iii) lead through a calculation to the Laplace functional:

\[
Q_{r,\mu} \exp \langle Y_t, -f \rangle
= \exp \langle \mu, -w^r_t \rangle \int D(d\zeta) \prod_{r < s \leq t} \sum_{i=0}^{\infty} q^s_i(\pi, \exp \{-\beta w^s_i\})^i
= \exp \langle \mu, -w^r_t \rangle \int D(d\zeta) \exp \left( \int \int_{(r,t] \times M_0} \log h^s(\pi, \langle \pi, e^{-\beta w^s_t} \rangle) \zeta(ds, d\pi) \right)
= \exp \left\{ -\langle \mu, w_t^r \rangle - \int \int_{(r,t] \times M_0} \left[ 1 - h^s(\pi, \langle \pi, e^{-\beta w^s_t} \rangle) \right] H(ds, d\pi) \right\}.
\tag{3.6}
\]

3.3. Consider a sequence of MBDI-processes \( Y^{(k)} = (Y^{(k)}_t, Q^{(k)}_{r,\mu}) \) with parameters \((W, \alpha_k H, k^{-1})\), where \(\alpha_k \geq 0\), \(k = 1, 2, \ldots\). By (3.6) we have

\[
Q^{(k)}_{r,\mu} \exp \langle Y^{(k)}_t, -f \rangle
= \exp \left\{ -\langle \mu, w_t^r \rangle - \int \int_{(r,t] \times M_0} \psi^s_k(\pi, \langle \pi, w_t^s(k) \rangle) H(ds, d\pi) \right\},
\tag{3.7}
\]

where

\[
w_t^s(k, x) = k \left[ 1 - \exp\{-k^{-1}w^s_t(x)\} \right], \tag{3.8}
\]

and

\[
\psi^s_k(\pi, \lambda) = \alpha_k \left[ 1 - h^s_k(\pi, 1 - \lambda/k) \right], \quad 0 \leq \lambda \leq k. \tag{3.9}
\]

Since \(w_t^s(k) \to w_t^s\) as \(k \to \infty\), it is natural to assume the sequence \(\psi_k\) to converge if one hopes to obtain \(Y_t = \lim_{k \to \infty} Y^{(k)}_t\) in some sense.

Lemma 3.4. i) Suppose that

3.E) \(\psi^s_k(\pi, \lambda) \to \psi^s(\pi, \lambda) \quad (k \to \infty) \) boundedly and uniformly on the set \(R \times M_0 \times [0, l]\) of \((s, \pi, \lambda)\) for each \(l \geq 0\).

Then \(\psi^s(\pi, \lambda)\) has the representation 3.C).

ii) To each function \(\psi\) given by 3.C) there corresponds a sequence in form (3.9) such that

\[
\psi^s_k(\pi, \lambda) = \psi^s(\pi, \lambda), \quad s \in R, \pi \in M_0, 0 \leq \lambda \leq k.
\]

Proof. Assertion i) was proved in Li (1991). To get ii) one can set
\[ \alpha_k = 1 + \sup_{s, \pi} \left[ kd_s^k(\pi) + \int_0^\infty (1 - e^{-ku})n_s^k(\pi, du) \right] \]

and

\[ h_k^s(\pi, z) = 1 + k\alpha_k^{-1} d_s^k(\pi)(z - 1) + \alpha_k^{-1} \int_0^\infty (e^{kuz} - 1)n_s^k(\pi, du). \quad Q.E.D. \]

Condition 3.E) usually implies \( \alpha_k \to \infty \). Thus the following theorem shows that the MBI-process is an approximation for the MBDI-process with high rate and small unit of immigration.

**Theorem 3.5.** i) Let \( Y^{(k)} \) be as above, and let \( Y \) be the MBI-process defined by (3.1). If 3.E) holds, then for every \( \mu \in M, r \leq t_1 < \cdots < t_n \in R \) and \( a \geq 0 \),

\[ Q^{(k)}_{r, \mu} \exp \sum_{i=1}^n \langle Y^{(k)}_{t_i}, -f_i \rangle \to Q_{r, \mu} \exp \sum_{i=1}^n \langle Y_{t_i}, -f_i \rangle \quad (k \to \infty) \quad (3.10) \]

uniformly in \( f_1, \cdots, f_n \in B(E)_a^+ \).

ii) For each MBI-process \( Y \) defined by (3.1), there is a sequence of MBDI-processes \( Y^{(k)} \) such that (3.10) is satisfied.

**Proof.** It suffices to show assertion i) since ii) follows immediately from i) and Lemma 3.4. We do this by induction in \( n \).

Fix \( a \geq 0 \) and \( r \leq t \in R \). By 3.A) and (3.8), \( w_t^k(k, x, f) \to w_t^*(x, f) \quad (k \to \infty) \) boundedly and uniformly in \( (s, x, f) \in [r, t] \times E \times B(E)_a^+ \). Thus 3.E) yields

\[ \int\int_{(r,t) \times M_0} \psi^*_k(\pi, \langle \pi, w_t^*(k) \rangle) H(ds, d\pi) \]

\[ \to \int\int_{(r,t) \times M_0} \psi^*(\pi, \langle \pi, w_t^* \rangle) H(ds, d\pi) \quad (k \to \infty) \quad (3.11) \]

uniformly in \( f \in B(E)_a^+ \). To see that the right hand side of (3.1) is indeed the Laplace functional of a probability measure we appeal to the following

**Lemma 3.6** (Kallenberg,1983; Dynkin,1991a). Suppose that \( P_k, k = 1, 2, \cdots \), are probability measures on \( M \). If \( L_{P_k}(f) \to L(f) \quad (k \to \infty) \) uniformly in \( f \in B(E)_a^+ \) for every \( a \geq 0 \), then \( L \) is the Laplace functional of a probability measure on \( M \). \quad Q.E.D.

Then it follows immediately that (3.1) really defines the transition probabilities of a Markov process \( Y \) in space \( M \) and that (3.10) holds for \( n = 1 \).
Now assuming (3.10) is true for \( n = m \), we show the fact for \( n = m + 1 \). Let 
\[ r \leq t_1 < \cdots < t_m + 1 \in \mathbb{R} \] 
and \( f_1, \cdots, f_m + 1 \in B(E)^+ \). Then

\[
Q^{(k)}_{r,\mu} \exp \sum_{i=1}^{m+1} \langle Y^{(k)}_{t_i} , -f_i \rangle
\]

\[
=Q^{(k)}_{r,\mu} Q^{(k)}_{r,\mu} \left\{ \prod_{i=1}^{m+1} \exp \langle Y^{(k)}_{t_i} , -f_i \rangle \right\}
\]

\[
=Q^{(k)}_{r,\mu} \prod_{i=1}^{m} \exp \langle Y^{(k)}_{t_i} , -f_i \rangle Q^{(k)}_{r,\mu} \left\{ \exp \langle Y^{(k)}_{t_m+1} , -f_{m+1} \rangle \right\}
\]

\[
=Q^{(k)}_{r,\mu} \prod_{i=1}^{m} \exp \langle Y^{(k)}_{t_i} , -f_i \rangle \cdot \exp \langle Y^{(k)}_{t_m} , -w_{t_m+1}^m (f_{m+1}) \rangle
\]

\[
\cdot \exp \left\{ - \int \int_{(t_m, t_{m+1}] \times M_0} \psi^s_K \left( \pi, \langle \pi, w_{t_m+1}^m (k, f_{m+1}) \rangle \right) H(ds, d\pi) \right\}
\]

\[
=Q^{(k)}_{r,\mu} \prod_{i=1}^{m-1} \exp \langle Y^{(k)}_{t_i} , -f_i \rangle \cdot \exp \langle Y^{(k)}_{t_m} , -f_m - w_{t_m+1}^m (f_{m+1}) \rangle
\]

\[
\cdot \exp \left\{ - \int \int_{(t_m, t_{m+1}] \times M_0} \psi^s_K \left( \pi, \langle \pi, w_{t_m+1}^m (k, f_{m+1}) \rangle \right) H(ds, d\pi) \right\} .
\]

By (3.11) and the induction hypothesis we have

\[
\lim_{k \to \infty} Q^{(k)}_{r,\mu} \exp \sum_{i=1}^{m+1} \langle Y^{(k)}_{t_i} , -f_i \rangle
\]

\[
=Q_{r,\mu} \prod_{i=1}^{m-1} \exp \langle Y_{t_i} , -f_i \rangle \cdot \exp \langle Y_{t_m} , -f_m - w_{t_m+1}^m (f_{m+1}) \rangle
\]

\[
\cdot \exp \left\{ - \int \int_{(t_m, t_{m+1}] \times M_0} \psi^s \left( \pi, \langle \pi, w_{t_m+1}^m (f_{m+1}) \rangle \right) H(ds, d\pi) \right\}
\]

\[
=Q_{r,\mu} \exp \sum_{i=1}^{m+1} \langle Y_{t_i} , -f_i \rangle,
\]

and the convergence is uniform in \( f_1, \cdots, f_{m+1} \in B(E)^+ \). Q.E.D.
If $E$ is a compact metric space, then the $n$-dimensional product topological space $M^n = \{(\mu_1, \cdots, \mu_n) : \mu_1, \cdots, \mu_n \in M\}$ is locally compact and separable, and the function class
\[ F(\mu_1, \cdots, \mu_n) = \exp \sum_{i=1}^{n} \langle \mu_i, -f_i \rangle, \quad f_i \in C(E)^{++}, \]
is convergence determining. Thus (3.10) implies that $Y^{(k)}$ converges to $Y$ in finite dimensional distributions.

3.4. In this paragraph, we prove a result on the weak convergence in space $D(R^+, M)$ of homogeneous MBDI-processes. Let $Y^{(k)} = (Y^{(k)}_t, t \geq 0)$ be a sequence of MBDI-processes with parameters $(W^{(k)}, \alpha_k \eta_k, h_k, k^{-1})$, where for each $k$,
- $W^{(k)}_t : f \mapsto w^{(k)}_t(f)$ is a strongly continuous $\Psi$-semigroup on $C(E)^{++}$;
- $\alpha_k$ is a positive number;
- $\eta_k$ is a finite measure on $M_0$;
- $h_k(\pi, \cdot)$, for every $\pi \in M_0$, is a probability generating function with $h_k(\pi, z)$ jointly continuous in $(\pi, z)$.

The transition probabilities $Q^{(k)}_{\mu}$ of $Y^{(k)}$ are defined by
\[ Q^{(k)}_{\mu} \exp \langle Y^{(k)}_t, -f \rangle = \exp \left\{ -\langle \mu, w^{(k)}_t \rangle - \int_0^t ds \int_{M_0} \psi_k(\pi, \langle \pi, w_s(k) \rangle) \eta_k(d\pi) \right\}, \quad (3.12) \]
with
\[ w_t(k, x) = k \left[ 1 - \exp\{-k^{-1}w^{(k)}_t(x)\} \right] \quad (3.13) \]
and
\[ \psi_k(\pi, \lambda) = \alpha_k \left[ 1 - h_k(\pi, 1 - \lambda/k) \right], \quad 0 \leq \lambda \leq k. \quad (3.14) \]
Clearly $Y^{(k)}$ has a strongly continuous Feller semigroup on $C_0(M)$, so we can assume it has sample paths in $D(R^+, M)$.

**Theorem 3.7.** Let $W_t : f \mapsto w_t(f)$ be a strongly continuous $\Psi$-semigroup on $C(E)^{++}$, and let $Y = (Y_t, t \geq 0)$ be an MBI-process in $D(R^+, M)$ with parameters $(W, \eta, \psi)$ with initial distribution $\Lambda$. Suppose that
- 3.F) for every $f \in C(E)^{++}$, $w^{(k)}_t(x, f) \to w_t(x, f)$ ($k \to \infty$) uniformly in $(t, x)$ on each set $[0, l] \times E$;
- 3.G) $\eta_k \to \eta$ weakly;
- 3.H) $\psi_k(\pi, \lambda) \to \psi(\pi, \lambda)$ uniformly in $(\pi, \lambda)$ on each set $M_0 \times [0, l]$;
- 3.I) $Y^{(k)}_0$ has limiting distribution $\Lambda$.

Then $Y^{(k)}$ converges weakly to $Y$ in the space $D(R^+, M)$ as $k \to \infty$. 
Proof. By Theorem 2.5 of Ethier and Kurtz (1986, p167), it is sufficient to prove

\[ \sup_{\mu \in M} \left| Q^{(k)}_\mu \exp\langle Y^{(k)}_t, -f \rangle - Q_\mu \exp\langle Y_t, -f \rangle \right| \rightarrow 0 \quad (k \rightarrow \infty) \quad (3.15) \]

for every fixed \( f \in C(E)^+ \) and \( t \geq 0 \). Let \( 2a = \inf_x w_t(x) \). By 3.F), there is a \( k_1 \) such that \( w^{(k)}_t \geq a \) for all \( k > k_1 \).

Suppose \( 0 < \varepsilon < 1 \). If \( \mu(E) \geq a^{-1} \log \varepsilon^{-1} \), we have

\[
\left| Q^{(k)}_\mu \exp\langle Y^{(k)}_t, -f \rangle - Q_\mu \exp\langle Y_t, -f \rangle \right| \\
\leq e^{-\langle \mu, w^{(k)}_t \rangle} + e^{-\langle \mu, w_t \rangle} < 2\varepsilon \quad \text{for } k > k_1.
\]

If \( \mu(E) < a^{-1} \log \varepsilon^{-1} \), then

\[
\left| Q^{(k)}_\mu \exp\langle Y^{(k)}_t, -f \rangle - Q_\mu \exp\langle Y_t, -f \rangle \right| \\
= \left| \langle \mu, w^{(k)}_t \rangle - \langle \mu, w_t \rangle \right| \\
+ \left| \int_0^t ds \int_{M_0} \psi_k(\pi, \langle \pi, w_s(k) \rangle) \eta_k(d\pi) - \int_0^t ds \int_{M_0} \psi(\pi, \langle \pi, w_s \rangle) \eta(\pi) \right| \\
\leq a^{-1} \|w^{(k)}_t - w_t\| \log \varepsilon^{-1} + \varepsilon_1(k) + \varepsilon_2(k) + \varepsilon_3(k),
\]

where

\[
\varepsilon_1(k) = \int_0^t ds \int_{M_0} \left| \psi_k(\pi, \langle \pi, w_s(k) \rangle) - \psi(\pi, \langle \pi, w_s \rangle) \right| \eta_k(d\pi),
\]

\[
\varepsilon_2(k) = \int_0^t ds \int_{M_0} \left| \psi(\pi, \langle \pi, w_s(k) \rangle) - \psi(\pi, \langle \pi, w_s \rangle) \right| \eta_k(d\pi),
\]

\[
\varepsilon_3(k) = \int_0^t \left| \int_{M_0} \psi(\pi, \langle \pi, w_s \rangle) \eta_k(d\pi) - \int_{M_0} \psi(\pi, \langle \pi, w_s \rangle) \eta(d\pi) \right| ds.
\]

By 3.F), there exits \( k_2 \) such that

\[
a^{-1} \|w^{(k)}_t - w_t\| \log \varepsilon^{-1} < \varepsilon \quad \text{for } k > k_2.
\]

3.F) also implies \( w_t(k, x) \rightarrow w_t(x) \) boundedly and uniformly on each set \([0, l] \times E \). Then 3.G) and 3.H) yield the existence of \( k_3 \) such that \( \varepsilon_1(k) + \varepsilon_2(k) < \varepsilon \) for \( k > k_3 \). By 3.G) and the dominated convergence theorem there is a \( k_4 \) such that \( \varepsilon_3(k) < \varepsilon \) for \( k > k_4 \). Thus (3.15) follows. Q.E.D.

4. Particle systems and superprocesses
4.1. As usual, let $E$ be a topological Lusin space. Suppose we have $\xi$, $K$, $H$ and $h$ given by 2.C), 2.D), 3.B) and 3.D) respectively. Assume that

4.A) $g^s(x, z) = \sum_{i=0}^{\infty} p_i^s(x) z^i$, for every $(s, x) \in R \times E$, is a probability generating function with the $p_i^s(x)$ and $\sum_{i=1}^{\infty} i p_i^s(x)$ belonging to $B(R \times E)^+$.

A branching particle system with immigration with parameters $(\xi, K, g, H, h)$ is described as follows:

(i) The particles in $E$ move according to the law of $\xi$.

(ii) For a particle which is alive at time $r$ and follows the path $(\xi_t, t \geq r)$, the conditional probability of survival during the time interval $[r, s)$ is $e^{-K(r,s)}$.

(iii) When a particle dies at time $s$ at point $x \in E$, it gives birth to a random number of offspring at the death site according to the generating function $g^s(x, \cdot)$.

(iv) The entry times and distributions of new particles immigrating to $E$ are governed by a Poisson random measure with intensity $H(ds, d\pi)$.

(v) The generating function $h^s(\pi, \cdot)$ gives the distribution of the number of new particles entering $E$ at time $s$ with distribution $\pi$.

For $t \in R$, let $Y_t(B)$ be the number of particles of the system in set $B \in \mathcal{B}(E)$ at time $t$. Under standard independence hypotheses, $(Y_t, t \in R)$ form a Markov process in space $M_1$. [Note that the state space of the particle system is different from that of the MBDI-process.] The rigorous construction of the process can be reduced to constructing a branching particle system with parameters $(\xi, K, g)$ generated by a single particle, which was given in Dynkin (1991a). The transition probabilities $Q_{r,\sigma}$ of $(Y_t, t \in R)$ are determined by the Laplace functionals [cf. (3.6)]:

$$Q_{r,\sigma} \exp \langle Y_t, -f \rangle = \exp \left\{ -\langle \sigma, v_t^r \rangle - \int_{(r,t) \times M_0} \left[ 1 - h^s(\pi, \langle \pi, e^{-v_t^s} \rangle) \right] H(ds, d\pi) \right\},$$

$$f \in B(E)^+, \sigma \in M_1, r \leq t, \quad (4.1)$$

where $v_t^r(x) \equiv v_t^r(x, f)$ is the unique positive solution of

$$e^{-v_t^r(x)} = \Pi_{r,x} e^{-f(\xi_t)-K(r,t)}$$

$$+ \Pi_{r,x} \int_r^t e^{-K(r,s)} g^s(\xi_s, e^{-v_t^s(\xi_s)}) K(ds). \quad (4.2)$$

This equation arises as follows: If we start one particle at time $r$ at point $x$, this particle moves following a path of $\xi$ and does not branch before time $t$ with probability $e^{-K(r,t)}$ [first term on the right hand side], or it splits at time $s \in (r,t]$ with probability $e^{-K(r,s)} K(ds)$ according to $g^s(\xi_s, \cdot)$ and all the offspring evolve independently after birth.
in the same fashion [second term]. By Lemma 2.3 of Dynkin (1991a), (4.2) is equivalent to

\[ \Pi_r x e^{-f(\xi_t)} - e^{-v^r_t(x)} = \Pi_r x \int_r^t \left[ e^{-v^r_t(\xi_s)} - g^s(\xi_s, e^{-v^r_s(\xi_s)}) \right] K(ds). \quad (4.3) \]

4.2. Let \( Y(k) = \{Y_t(k), t \in \mathbb{R}\} \) be a sequence of branching particle systems with immigration with parameters \((\xi, \gamma_k, g_k, \alpha_k H, h_k)\), where \( \alpha_k \geq 0, \gamma_k \geq 0, k = 1, 2, \ldots \). Then

\[ Y^{(k)} = \{Y^{(k)}_t := k^{-1}Y_t(k), t \in \mathbb{R}\} \]

is a Markov process in space \( M_k \) with transition probabilities \( Q^{(k)}_{r, \sigma_k} \) determined by

\[ Q^{(k)}_{r, \sigma_k} \exp \langle Y^{(k)}_t, -f \rangle \]
\[ = \exp \left\{ -\langle \sigma_k, kv^r_t(k) \rangle - \int \int_{(r,t) \times M_0} \psi^s_k(\pi, \langle \pi, w^s_t(k) \rangle) H(ds, d\pi) \right\}, \]
\[ f \in B(E)^+, \sigma_k \in M_k, r \leq t, \quad (4.4) \]

where \( \psi^s_k(\pi, \lambda) \) is given by (3.9), \( v^r_t(k, x) \equiv v^r_t(k, x, f) \) satisfies

\[ \Pi_r x e^{-f(\xi_t)/k} - e^{-v^r_t(k, x)} = \Pi_r x \int_r^t \gamma_k \left[ e^{-v^r_t(k, \xi_s)} - g^s_k(\xi_s, e^{-v^r_s(k, \xi_s)}) \right] K(ds) \quad (4.5) \]

and

\[ w^s_t(k, x) \equiv w^s_t(k, x, f) = k[1 - e^{-v^r_t(k, x, f)}]. \quad (4.6) \]

Let \( Q^{(k)}_{r, \mu_k} \) denote the conditional law of \( \langle Y^{(k)}_t, t \geq r \rangle \) given \( Y^{(k)}_r = k^{-1}\sigma(k\mu) \), where \( \mu \) belongs to \( M \) and \( \sigma(k\mu) \) is a Poisson random measure with intensity \( k\mu \). Then

\[ Q^{(k)}_{r, \mu_k} \exp \langle Y^{(k)}_t, -f \rangle \]
\[ = \exp \left\{ -\langle \mu, w^r_t(k) \rangle - \int \int_{(r,t) \times M_0} \psi^s_k(\pi, \langle \pi, w^s_t(k) \rangle) H(ds, d\pi) \right\}. \quad (4.7) \]

It is easy to check that \( w^r_t(k) \) satisfies

\[ w^r_t(k, x) + \Pi_r x \int_r^t \phi^s_k(\xi_s, w^s_t(k, \xi_s)) K(ds) = \Pi_r x k[1 - e^{-f(\xi_t)/k}] \quad (4.8) \]
with
\[ \phi^s_k(x, \lambda) = k \gamma_k \left[ g^s_k(x, 1 - \lambda/k) - (1 - \lambda/k) \right], \quad 0 \leq \lambda \leq k. \] (4.9)

For the sequence (4.9) we note
\[ b_k = \sup_{s,x} \left| \frac{d}{d\lambda} \phi^s_k(x, \lambda) \right|_{\lambda=0}. \] (4.10)

**Lemma 4.1.** i) Suppose that
4.B) \( \phi^s_k(x, \lambda) \to \phi^s(x, \lambda) \) \( (k \to \infty) \) uniformly on each set \( R \times E \times [0, l] \);
4.C) \( \phi^s(x, \lambda) \) is Lipschitz in \( \lambda \) uniformly on each set \( R \times E \times [0, l] \).
Then \( \phi^s(x, \lambda) \) has the representation 2.E).

ii) If \( \phi^s(x, \lambda) \) is given by 2.E), then it satisfies 4.C) and there is a sequence \( \phi^s_k(x, \lambda) \) in form (4.9) such that 4.B) holds and
\[ \frac{d}{d\lambda} \phi^s_k(x, \lambda) \bigg|_{\lambda=0} = b^s(x), \quad s \in R, x \in E. \] (4.11)

**Proof.** Assertion i) follows easily by a result of Li (1991), so we shall prove ii) only. Suppose that \( \phi^s(x, \lambda) \) is given by 2.E). 4.C) holds clearly. Let
\[ \gamma_{1,k} = 1 + \sup_{s,x} \int_0^\infty u(1 - e^{-ku})m^s(x, du) \]
and
\[ g^s_{1,k}(x, z) = z + k^{-1} \gamma_{1,k}^{-1} \int_0^\infty \left[ e^{ku(z-1)} - 1 + ku(1-z) \right] m^s(x, du). \]

It is easy to check that
\[ \phi^s_{1,k}(x, \lambda) : = k \gamma_{1,k} \left[ g^s_{1,k}(x, 1 - \lambda/k) - (1 - \lambda/k) \right] \]
\[ = \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)m^s(x, du). \]

Let
\[ \overline{b} = \sup_{s,x} |b^s(x)|, \quad \overline{c} = \sup_{s,x} c^s(x). \]

Assuming \( \gamma_{2,k} := \overline{b} + 2k\overline{c} > 0 \) and setting
\[ g^s_{2,k}(x, z) = \begin{cases} 
  z + \gamma_{2,k}^{-1} \left[ b^s(x)(1-z) + kc^s(x)(1-z)^2 \right] & \text{if } b^s(x) \geq 0 \\
  \gamma_{2,k}^{-1} \left[ \frac{1}{2} b^s(x)(1+z^2) + \frac{1}{2} b^s(x)(1-z)^2 \right] + kc^s(x)(1-z)^2 + 2k\overline{c}z & \text{if } b^s(x) < 0,
\end{cases} \]
we have
\[ \phi_{2,k}^s(x, \lambda) := k \gamma_{2,k} \left[ g_{2,k}^s(x, 1 - \lambda/k) - (1 - \lambda/k) \right] \]
\[ = \begin{cases} 
  b^s(x) \lambda + c^s(x) \lambda^2 & \text{if } b^s(x) \geq 0 \\
  b^s(x) \lambda + c^s(x) \lambda^2 + (2k)^{-1} \left[ b - b^s(x) \right] \lambda^2 & \text{if } b^s(x) < 0.
\end{cases} \]

Finally we let
\[ \gamma_k = \gamma_{1,k} + \gamma_{2,k} \quad \text{and} \quad g_k = \gamma_k^{-1} (\gamma_{1,k} g_{1,k} + \gamma_{2,k} g_{2,k}). \]

Then the sequence \( \phi_k^s(x, \lambda) \) defined by (4.9) is equal to \( \phi_{1,k}^s(x, \lambda) + \phi_{2,k}^s(x, \lambda) \) that satisfies 4.B) and (4.11). Q.E.D.

**Lemma 4.2.** If conditions 4.B and C) are fulfilled and if
4.D) \( \sup_k \overline{b}_k < \infty \),

then \( w^r_t(k, x, f) \), and hence \( k w^r_t(k, x, f) \), converge boundedly and uniformly on each set \( [u, t] \times E \times B(E)^+_a \) of \( (r, x, f) \) to the unique bounded positive solution of equation (2.6).

**Proof.** Since \(-\frac{d}{d\lambda} \phi_k^s(x, \lambda) \leq \overline{b}_k\), (4.8) implies that
\[ w^r_t(k, x) \leq \|f\| + \overline{b}_k \Pi_{r,x} \int_r^t w^s_t(k, \xi_s) K(ds). \] (4.12)

By the generalized Gronwall’s inequality proved by Dynkin (1991a), we get
\[ w^r_t(k, x) \leq \|f\| \Pi_{r,x} \overline{b}_K(r, t). \] (4.13)

Using (4.8) and (4.13), the convergence of \( w^r_t(k) \) is proved in the same way as Lemma 3.3 of Dynkin (1991a). The convergence of \( k w^r_t(k) \) follows by (4.6). Q.E.D.

Based on Lemmas 4.1 and 4.2, the following result can be obtained similarly as Theorem 3.5.

**Theorem 4.3.** i) Let \( Y^{(k)} \) be the sequence of renormalized branching particle systems with immigration defined by (4.7), and let \( Y \) be the \((\xi, K, \phi, H, \psi)\)-superprocess. Assume that conditions 3.E) and 4.B,C,D) are satisfied. Then for every \( \mu \in M, \ r \leq t_1 < \cdots < t_n \) and \( a \geq 0 \),
\[ Q_{r,\mu}^{(k)} \exp \sum_{i=1}^n \langle Y_{t_i}^{(k)}(i) - f_i \rangle \to Q_r \mu \exp \sum_{i=1}^n \langle Y_{t_i} - f_i \rangle \ (k \to \infty) \] (4.14)

uniformly in \( f_1, \cdots, f_n \in B(E)^+_a \).
ii) To each \((\xi, K, \phi, H, \psi)\)-superprocess \(Y\) there corresponds a sequence of branching particle systems with immigration \(Y^{(k)}\) satisfying (4.14). Q.E.D.

Suppose that each particle in the \(k\)th system is weighted \(k^{-1}\). (4.14) states that the mass distribution of the particle system approximates to the process \(Y\) when the single mass becomes small and the particle population becomes large. Typically, \(\gamma_k \to \infty\) and \(\alpha_k \to \infty\), which mean that the rates of the branching and the immigration get high. It is also possible to prove a result on the weak convergence in space \(D(R^+, M)\) of the branching system of particles with immigration. The discussions are similar to those of section 3 and left to the reader.

5. Transformations of the measure space

By transformations of the state space \(M\), large classes of MBI-processes that may take infinite (but \(\sigma\)-finite) values can be obtained from the MBI-processes with finite values that we have discussed in sections 3 and 4.

For \(\rho \in B(E)^{++}\) we let

\[
M^\rho = \{ \mu : \mu \text{ is a measure on } (E, B(E)) \text{ such that } \langle \mu, \rho \rangle < \infty \},
\]

\[
M^\rho_0 = \{ \tau : \tau \in M^\rho \text{ and } \langle \tau, \rho \rangle = 1 \}.
\]

Suppose that \(W_r^\tau : f \mapsto w^r_t(f)\) is a cumulant semigroup on \(B(E)^+\) such that \(5.A)\) for every \(f \in B(E)^+\) and \(u \leq t \in \mathbb{R}\), the function \(\rho^{-1}(x)w^r_t(x, \rho f)\) of \((r, x)\) restricted to \([u, t] \times E\) belongs to \(B([u, t] \times E)^+\).

We define the operators \(\hat{W}^r_t : f \mapsto \hat{w}^r_t(f)\) on \(B(E)^+\) by

\[
\hat{w}^r_t(f) = \rho^{-1}w^r_t(\rho f) \quad (5.1)
\]

[cf. El Karoui and Roelly-Coppoletta (1989)]. It is easy to see that \(\hat{W}^r_t, r \leq t \in \mathbb{R}\), also form a cumulant semigroup. If \((\hat{Y}_t, t \in \mathbb{R})\) is an MBI-process in \(M\) with parameters \((\hat{W}, H, \psi)\) [Definition 3.1], then

\[
Y = (Y_t := \rho^{-1}\hat{Y}_t, \ t \in \mathbb{R}) \quad (5.2)
\]

is an MBI-process in the space \(M^\rho\) with transition probabilities \(Q_{r, \mu}\) determined by

\[
Q_{r, \mu} \exp\langle Y_t, -f \rangle = \exp \left\{ -\langle \mu, w^r_t \rangle - \int \int_{(r, t] \times M^\rho_0} \psi^s(\tau, \langle \tau, w^r_t \rangle)H^\rho(ds, d\tau) \right\},
\]

\[
f \in B(E)^+, \mu \in M^\rho, r \leq t, \quad (5.3)
\]

where \(H^\rho(ds, d\tau) = H(ds, d\rho \tau)\) and \(\psi^s(\tau, \lambda) = \psi^s(\rho \tau, \lambda)\).
Example 5.1. Suppose that $0 < \beta \leq 1$ and that $\Pi_t$ is the semigroup of the $d$-dimensional Brownian motion. Then equation

$$w_t + \int_0^t \Pi_{t-s}(w_s)^{1+\beta} ds = \Pi_t f, \quad t \geq 0, \quad (5.4)$$

defines a homogeneous cumulant semigroup $W_t : f \mapsto w_t$. For $p > d$, let $\rho(x) = (1 + |x|^p)^{-1}$, $x \in \mathbb{R}^d$, and let

$$M_p(\mathbb{R}^d) = \{\mu : \mu \text{ is a Borel measure on } \mathbb{R}^d \text{ such that } \langle \mu, \rho \rangle < \infty\}.$$

Iscoe (1986) showed that $W_t$ satisfies condition 5.A). Assume $0 < \theta \leq 1$ and $\lambda \in M_p(\mathbb{R}^d)$. Then formula

$$Q_\mu \exp\langle Y_t, -f \rangle = \exp \left\{ -\langle \mu, w_t \rangle - \int_0^t \langle \lambda, w_s \rangle^\theta ds \right\} \quad (5.5)$$

defines an MBI-process $Y = (Y_t, Q_\mu)$ in the space $M_p(\mathbb{R}^d)$. When $\beta = \theta = 1$, $Y$ has continuous sample paths almost surely [in a suitable topology in $M_p(\mathbb{R}^d)$; see Konno and Shiga (1988)].

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