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INTEGRAL REPRESENTATIONS OF CONTINUOUS FUNCTIONS[1]

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In this note, we give a necessary and sufficient condition for a continuous function $\sigma(\lambda)$, $\lambda \geq 0$, to have the integral representation

$$\sigma(\lambda) = \sum_{i=0}^{n-1} a_i\lambda^i + \int_0^\infty \left[\mathrm{e}^{-\lambda u} - 1 - (1+u^n)^{-1}\sum_{i=1}^{n-1}\frac{(-\lambda u)^i}{i!}\right](1-\mathrm{e}^{-u})^{-n}G(\mathrm{d}u), \tag{0.1}$$

where n is a positive integer, a_i are constants ($i = 0, 1, \cdots, n-1$), G is a finite measure on $[0,\infty)$ and the value of the integrand at $u = 0$ is defined by continuity as $(-\lambda)^n/n!$. Using this condition, we get the general description of some characters of superprocesses.

1. Lemmas

Recall that the Laplace transform of a finite measure G on $[0,\infty)$ is defined as

$$\theta(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda u}G(\mathrm{d}u), \quad \lambda \geq 0, \tag{1.1}$$

which determines G uniquely.

Given a function $\theta(\lambda)$, $\lambda \geq 0$, we define the difference operator Δ_h by

$$\Delta_h\theta(\lambda) = \theta(\lambda+h) - \theta(\lambda), \quad \lambda \geq 0, h \geq 0. \tag{1.2}$$

Let $\Delta_h^m = \Delta_h \cdots \Delta_h$ ($m-1$ times). We have

$$\Delta_h^m\theta(\lambda) = (-1)^m\sum_{i=0}^m \binom{m}{i}(-1)^i\theta(\lambda+ih). \tag{1.3}$$

We say that θ is completely monotone if it satisfies

$$(-1)^i\Delta_h^i\theta(\lambda) \geq 0, \quad \lambda \geq 0, h \geq 0, i = 0, 1, 2, \cdots. \tag{1.4}$$

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The Bernstein polynomials of a continuous function $f(s)$, $0 \le s \le 1$, are given by

$$B_{f,m}(s) = \sum_{i=0}^{m} \left(\begin{array}{c} m \\ i \end{array} \right) \Delta^i_{1/m} f(0) s^i, \quad 0 \le s \le 1, m = 1, 2, \cdots. \tag{1.5}$$

It is well-known that

$$B_{f,m}(s) \to f(s) \tag{1.6}$$

uniformly on $[0, 1]$ as $m \to \infty$ (see [1]).

Lemma 1. *A continuous function $\theta(\lambda)$, $\lambda \ge 0$, is the Laplace transform of a finite measure G on $[0, \infty)$ if and only if it is completely monotone.*

Proof. The necessity is an immediate consequence of formula (1.3). Assume (1.4) holds. For fixed $a > 0$, we let $\gamma_a(s) = \theta(a - as)$, $0 \le s \le 1$. The complete monotonicity of θ implies

$$\Delta^i_h \gamma_a(0) \ge 0, \quad i = 0, 1, \cdots, m, h = m^{-1}.$$

Then the Bernstein polynomial $B_{\gamma_a,m}(s)$ has non-negative coefficients, and the function $B_{\gamma_a,m}(\mathrm{e}^{-\lambda/a})$, $\lambda \ge 0$, is the Laplace transform of a finite measure $G_{a,m}$ on $[0, \infty)$. By the continuity theorem [1], it follows that the function

$$\theta(\lambda) = \lim_{a \to \infty} \lim_{m \to \infty} B_{\gamma_a,m}(\mathrm{e}^{-\lambda/a}), \quad \lambda \ge 0,$$

is the Laplace transform of a finite measure G on $[0, \infty)$. Q.E.D.

Lemma 2. *A continuous function $\eta(\lambda)$, $\lambda \ge 0$, is a polynomial of degree less than n if and only if $\Delta^n_h \eta(0) = 0$ for all $h \ge 0$.*

Proof. The necessity of the condition is obvious. Assume $\Delta^n_h \eta(0) = 0$, $h \ge 0$. For fixed $a > 0$, let $\eta_a(s) = \eta(as)$, $0 \le s \le 1$. Since

$$\Delta^n_h \eta_a(0) = 0, \quad 0 \le h \le n^{-1},$$

the polynomials of η_a have degree less than n:

$$B_{\eta_a,m}(s) = \sum_{i=0}^{n-1} b_i^{(m)} s^i, \quad m = n, n+1, \cdots.$$

Here the $b_i^{(m)}$ can be represented as the linear combinations of

$$B_{\eta_a,m}(1/n), B_{\eta_a,m}(2/n), \cdots, B_{\eta_a,m}(n/n).$$

By (1.6) the limits

$$\lim_{m \to \infty} b_i^{(m)} = b_i, \quad i = 0, 1, \cdots, n-1,$$

exist and $\eta_a(s) = \sum_{i=0}^{n-1} b_i s^i$, $0 \le s \le 1$. Setting $a_i = a^{-1} b_i$, we get

$$\eta(s) = \sum_{i=0}^{n-1} b_i s^i, \quad 0 \le s \le a.$$

Clearly, this formula in fact holds for all $\lambda \ge 0$. Q.E.D.

2. The Main Theorem

Theorem. *A continuous function $\sigma(\lambda)$, $\lambda \ge 0$, has representation (0.1) if and only if for every $c \ge 0$ the function*

$$\theta_c(\lambda) := (-1)^n \Delta^n_c \sigma(\lambda), \quad \lambda \ge 0, \tag{2.1}$$

is completely monotone.

Proof. If σ is given by (0.1), then by (1.3) we get

$$\theta_c(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda u}(1-\mathrm{e}^{-cu})^n(1-\mathrm{e}^{-u})^{-n}G(\mathrm{d}u).$$

Thus θ_c is the Laplace transform of a finite measure on $[0,\infty)$, and by Lemma 1 θ_c is completely monotone. Conversely, assume that θ_c is completely monotone. By Lemma 1 we have

$$\theta_c(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda u}G_c(\mathrm{d}u), \quad \lambda \geq 0, \tag{2.2}$$

where G_c is a finite measure on $[0,\infty)$. From (1.3) and the relation

$$(-1)^n \varDelta_1^n \theta_c(\lambda) = \varDelta_1^n \varDelta_c^n \sigma(\lambda) = \varDelta_c^n \varDelta_1^n \sigma(\lambda) = (-1)^n \varDelta_c^n \theta_1(\lambda)$$

it follows that

$$\int_0^\infty \mathrm{e}^{-\lambda u}(1-\mathrm{e}^{-u})^n G_c(\mathrm{d}u) = \int_0^\infty \mathrm{e}^{-\lambda u}(1-\mathrm{e}^{-cu})^n G(\mathrm{d}u),$$

where $G = G_1$. Therefore

$$G_c(\mathrm{d}u) = (1-\mathrm{e}^{-cu})^n(1-\mathrm{e}^{-u})^{-n}G(\mathrm{d}u), \quad 0 < u < \infty. \tag{2.3}$$

Let

$$\sigma_0(\lambda) = \int_0^\infty \left[\mathrm{e}^{-\lambda u} - 1 - (1+u^n)^{-1}\sum_{i=1}^{n-1}\frac{(-\lambda u)^i}{i!}\right](1-\mathrm{e}^{-u})^{-n}G(\mathrm{d}u), \tag{2.4}$$

The function $\eta(\lambda) := \sigma(\lambda) - \sigma_0(\lambda)$, $\lambda \geq 0$, is continuous and

$$\begin{aligned}
(-1)^n \varDelta_c^{n+1}\eta &= (-1)^n \varDelta_c[\varDelta_c^n \sigma(\lambda) - \varDelta_c^n \sigma_0(\lambda)] \\
&= \varDelta_c\left[\int_0^\infty \mathrm{e}^{-\lambda u}G_c(\mathrm{d}u) - \int_0^\infty \mathrm{e}^{-\lambda u}(1-\mathrm{e}^{-cu})^n(1-\mathrm{e}^{-u})^{-n}G(\mathrm{d}u)\right] \\
&= \varDelta_c[G_c(\{0\}) - c^n G(\{0\})] = 0.
\end{aligned}$$

By Lemma 2, $\eta(\lambda)$ is a polynomial of degree less than $n+1$, say $\eta(\lambda) = \sum_{i=0}^n a_i\lambda^i$. By (2.1) and (2.4), we have

$$\begin{aligned}
n!a_n &= \varDelta_1^n \eta(\lambda) = \varDelta_1^n \sigma(\lambda) - \varDelta_1^n \sigma_0(\lambda) \\
&= (-1)^n\left[\theta_1(\lambda) - \int_0^\infty \mathrm{e}^{-\lambda u}G(\mathrm{d}u)\right] = 0.
\end{aligned}$$

Therefore $\sigma(\lambda) = \sum_{i=0}^{n-1} a_i\lambda^i + \sigma_0(\lambda)$. Q.E.D.

3. Corollaries

Let $\{g_k\}$ be a sequence of (possibly defective) probability generating functions, i.e.

$$g_k(s) = \sum_{i=0}^\infty p_i^{(k)} s^i, \quad 0 \leq s \leq 1, \tag{3.1}$$

where $p_i^{(k)} \geq 0$ and $\sum_{i=0}^\infty p_i^{(k)} \leq 1$. Let $\{\alpha_k\}$ and $\{\beta_k\}$ be two sequences of non-negative numbers and $\beta_k \to 0^+$ as $k \to \infty$. In the study of superprocesses, we sometimes need to consider function sequences as follows:

$$\phi_k(\lambda) := \alpha_k[g_k(1-\beta_k\lambda) - (1-\beta_k\lambda)], \quad 0 \leq \lambda \leq \beta_k^{-1}, \tag{3.2}$$

$$\psi_k(\lambda) := \alpha_k[1 - g_k(1 - \beta_k\lambda)], \qquad 0 \le \lambda \le \beta_k^{-1}. \tag{3.3}$$

Using Theorem 1, we can give the general representations of the limit functions of (3.2) and (3.3). Some special forms of Corollary 1 in the following have been achieved earlier by E.B. Dynkin and Z.B. Li.

Corollary 1. *If the sequence (3.2) converges to a continuous function $\phi(\lambda)$, $\lambda \ge 0$, then the limit function has the representation*

$$\phi(\lambda) = a + b\lambda + \int_0^\infty \left(\mathrm{e}^{-\lambda u} - 1 + \frac{\lambda u}{1+u^2}\right)(1 - \mathrm{e}^{-u})^{-2} F(\mathrm{d}u), \tag{3.4}$$

where $a \le 0$ and b are constants, and F is a finite measure on $[0, \infty)$. The value of the integrand at $u = 0$ is defined as $\lambda^2/2$.

Proof. Clearly, (3.4) is the special case of (0.1) when $n = 2$. For $c \ge 0$, we let $\theta_c(\lambda) = \Delta_c^2\phi(\lambda)$. It is easy to check that

$$\Delta_c^2\phi_k(\lambda) = \alpha_k \Delta_c^2 g_k(1 - \beta_k\cdot)(\lambda).$$

Since $g_k^{(n)}$ is a power series with non-negative coefficients, so is $\Delta_h^m g_k^{(n)}$ for every $m \ge 1$. In particular, using (3.2) we have

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \Delta_c^2\phi_k(\lambda) = \alpha_k\beta_k^n \Delta_c^2 g_k^{(n)}(1 - \beta_k\cdot)(\lambda) \ge 0.$$

By mean-value theorem, it is simple to show inductively that $(-1)^n\Delta_h^n\Delta_c^2\phi_k(\lambda) \ge 0$. Letting $k \to \infty$ we obtain $(-1)^n\Delta_h^n\Delta_c^2\phi(\lambda) \ge 0$. That is, $\theta_c(\lambda)$ is a completely monotone function of $\lambda \ge 0$. By Theorem 1, $\phi(\lambda)$ has representation (3.4). Clearly, $a = \phi(0) = \lim_{k\to\infty}\phi_k(0) \le 0$. Q.E.D.

Corollary 2. *If the sequence (3.3) converges to a continuous function $\psi(\lambda)$, $\lambda \ge 0$, then ψ has the representation*

$$\psi(\lambda) = d + \int_0^\infty (1 - \mathrm{e}^{-\lambda u})(1 - \mathrm{e}^{-u})^{-1} G(\mathrm{d}u), \tag{3.5}$$

where $d \ge 0$ is a constant, G is a finite measure on $[0, \infty)$, and the value of the integrand at $u = 0$ is defined as λ.

Proof. This is similar to the proof of Corollary 1. Q.E.D.

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