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Moments of continuous-state branching processes with or without immigration [1]

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Abstract

For a positive continuous function f satisfying some standard conditions, we study the f-moments of continuous-state branching processes with or without immigration. The main results give criteria for the existence of the f-moments. The characterization of the processes in terms of stochastic equations plays an essential role in the proofs.

Keywords and phrases: branching process; continuous-state; immigration; moments; stochastic equation.

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1 Introduction

Branching processes in discrete state space were introduced as probabilistic models for the stochastic evolution of populations. For the basic theory of those processes we refer to Athreya and Ney (1972) and Harris (1965). Jiřina (1958) defined continuous-state branching processes (CB-processes) in both discrete and continuous times. Those processes with continuous times were obtained in Lamperti (1967a) as weak limits of rescaled discrete branching processes. Lamperti (1967b) showed that they are in one-to-one correspondence with spectrally Lévy processes via simple random time changes. Continuous-state branching processes with immigration (CBI-processes) are more general population models taking into consideration the influence of the environments. They were introduced by Kawazu and Watanabe (1971) as rescaled limits of discrete branching processes with immigration; see also Aliev (1985). The approach of stochastic equations for CB- and CBI-processes have been developed by Dawson and Li (2006, 2012), Fu and Li (2010) and Li (2011) with some applications.

Moment properties play important roles in the study of limit theorems of branching processes. The integer-moments for the processes can be easily represented thanks to the simple forms of the generating functions or Laplace transforms of the distributions. The

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characterization of general function moments is usually more difficult. Suppose that f is a positive continuous function on $[0, \infty)$ satisfying the following:

Condition A. There exist constants $c \geq 0$ and $K > 0$ such that

(A1) f is convex on $[c, \infty)$;

(A2) $f(xy) \leq Kf(x)f(y)$ for all $x, y \in [c, \infty)$.

A typical example satisfying the above condition is the function $f(x) = x^\alpha |\log x|^\beta$ $(\alpha \geq 1, \beta \geq 0)$. For a branching process with continuous time and discrete state space it was proved in Athreya (1969) that the existence of the f-moment is equivalent to that of its offspring distribution; see also Athreya and Ney (1972). The proof of Athreya (1969) was essentially based on a construction of the process from two sequences of random variables giving the split times and the progeny numbers. The result was generalized in Bingham (1976) to a CB-process for the function $f(x) = x^n$ with integer $n \geq 2$, which corresponds to integer-moments. A recursive formula for integer-moments of multi-type CBI-processes was given recently by Barczy et al. (2015). As far as we know, the result of Athreya (1969) has not been extended to the general f-moment in the continuous-state setting. The difficulty of such an extension lies in the fact that the CB-process cannot be constructed in the simple way as the discrete-state process in Athreya (1969). We notice that a result on the f-moment of the CB-process for $f(x) = x \log x$ was presented in Section 5 of Grey (1974). It was mentioned there the topic would be studied elsewhere, but we could not find the subsequent work in the literature.

The purpose of this paper is to study general f-moments of CB- and CBI-processes with continuous time. Our two main theorems are stated in Section 2, giving criteria for the existence of the f-moments. The results yield immediately those of Bingham (1976) and Grey (1974). The proofs of the main theorems are given in Sections 3 and 4. Our strategy for the proofs is to use the characterization of the CB- and CBI-processes as strong solutions of stochastic equations established in Dawson and Li (2006, 2012). We here do not assume the existence of the first moment of the branching Lévy measure, so we shall need to generalize slightly their results. Throughout the paper, we make the convention that, for $a \leq b \in \mathbb{R}$,

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}.$$

2 Main Results

We first review some basic facts on CB- and CBI-processes with continuous time. The reader may refer to Kawazu and Watanabe (1971) for the details; see also Kyprianou (2014) and Li (2011). A *branching mechanism* is a continuous function ϕ on $[0, \infty)$ with the representation

$$\phi(\lambda) = \beta\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty \big(e^{-z\lambda} - 1 + z\lambda 1_{\{z\leq 1\}}\big) m(\mathrm{d}z), \qquad \lambda \geq 0, \tag{2.1}$$

where $\beta \in \mathbb{R}$ and $\sigma \geq 0$ are constants, and $m(\mathrm{d}z)$ is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge z^2) m(\mathrm{d}z) < \infty.$$

The above integrability condition is weaker than that assumed in Li (2011). Throughout this paper, we assume

$$\int_{0+} \frac{1}{|\phi(\lambda)|} \mathrm{d}\lambda = \infty. \tag{2.2}$$

(This is a correction of (1.21) in Kawazu and Watanabe (1971).) By Kawazu and Watanabe (1971, Theorem 1.2), the *CB-process* with branching mechanism ϕ is a conservative Markov process on $[0, \infty)$ with transition semigroup $(Q_t)_{t\geq 0}$ defined by

$$\int_{[0,\infty)} e^{-\lambda y} Q_t(x, \mathrm{d}y) = \exp\{-x v_t(\lambda)\}, \qquad \lambda, x \geq 0, \tag{2.3}$$

where $t \to v_t(\lambda)$ is the unique positive solution of

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda)) \mathrm{d}s, \qquad \lambda, t \geq 0. \tag{2.4}$$

A generalization of the CB-process can be defined as follows. The reader may refer to Li (2006) for the details. By an *immigration mechanism* we mean a continuous positive function ψ on $[0, \infty)$ given by

$$\psi(\lambda) = h\lambda + \int_0^\infty (1 - e^{-\lambda z}) n(\mathrm{d}z), \tag{2.5}$$

where $h \geq 0$ is a constant and $n(\mathrm{d}z)$ is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge z) n(\mathrm{d}z) < \infty.$$

Then there is a family $(\gamma_s)_{s\geq 0}$ of infinitely divisible probability measures on $[0, \infty)$ so that

$$\int_{[0,\infty)} e^{-\lambda y} \gamma_t(\mathrm{d}y) = \exp\bigg\{ - \int_0^t \psi(v_s(\lambda)) \mathrm{d}s \bigg\}, \qquad \lambda \geq 0.$$

A Markov process on $[0, \infty)$ is called *CBI-process* with branching mechanism ϕ and immigration mechanism ψ if it has transition semigroup $(Q_t^\gamma)_{t\geq 0}$ given by

$$\int_{[0,\infty)} e^{-\lambda y} Q_t^\gamma(x, \mathrm{d}y) = \exp\bigg\{ - x v_t(\lambda) - \int_0^t \psi(v_s(\lambda)) \mathrm{d}s \bigg\}, \qquad \lambda, x \geq 0. \tag{2.6}$$

The main results of this paper are the following:

Theorem 2.1. *Suppose that f satisfies Condition A. Let $\{X_t : t \geq 0\}$ be CB-processes with $\mathbf{P}(X_0 > 0) > 0$. Then for any $t > 0$ we have $\mathbf{P} f(X_t) < \infty$ if and only if $\mathbf{P} f(X_0) < \infty$ and $\int_1^\infty f(z) m(\mathrm{d}z) < \infty$.*

Theorem 2.2. *Suppose that f satisfies Condition A. Let $\{Y_t : t \geq 0\}$ be a CBI-process. Then for every $t > 0$ we have $\mathbf{P} f(Y_t) < \infty$ if and only if $\int_1^\infty f(z)(m+n)(\mathrm{d}z) < \infty$ and $\mathbf{P} f(Y_0) < \infty$.*

The proofs of the above theorems are given in the next two sections. We shall need the following proposition, which is a special case of Theorem 4.3 in Barczy et al. (2015):

Proposition 2.3. *Let $\{Y_t : t \geq 0\}$ be a CBI-process with $\mathbf{P}(Y_0^r) < \infty$. Suppose that $\int_1^\infty z^r (m+n)(\mathrm{d}z) < \infty$ for some integer $r \geq 1$. Then $\mathbf{P}(Y_t^r) < \infty$ for every $t \geq 0$. Moreover, we have*

$$\mathbf{P}(Y_t) = e^{-\tilde{b}t}\mathbf{P}(Y_0) + \frac{\tilde{h}}{\tilde{b}}(1 - e^{-\tilde{b}t}),$$

where

$$b = \beta - \int_1^\infty z m(\mathrm{d}z), \quad \tilde{h} = h + \int_0^\infty z n(\mathrm{d}z).$$

For $2 \leq k \leq r$ we have the recursion formula:

$$\begin{aligned} \mathbf{P}(Y_t^k) = {} & e^{-kbt}\mathbf{P}(Y_0^k) + k\Big[\tilde{h} + \frac{\sigma^2}{2}(k-1)\Big] \int_0^t e^{-kb(t-s)}\mathbf{P}(Y_s^{k-1})\mathrm{d}s \\ & + \sum_{j=0}^{k-2} \binom{k}{j} \int_0^\infty z^{k-j} m(\mathrm{d}z) \int_0^t e^{-kb(t-s)}\mathbf{P}(Y_s^{j+1})\mathrm{d}s \\ & + \sum_{j=0}^{k-2} \binom{k}{j} \int_0^\infty z^{k-j} n(\mathrm{d}z) \int_0^t e^{-kb(t-s)}\mathbf{P}(Y_s^{j})\mathrm{d}s. \end{aligned} \tag{2.7}$$

For continuous-time branching processes and age dependent branching processes in discrete state space, some similar results as the above were established by Athreya (1969); see also Athreya and Ney (1972, p.153). By taking $f(x) = x^n$ or $f(x) = x \log x$ in Theorem 2.1, we obtain the results of Theorem 6.1 of Bingham (1976) and Section 5 of Grey (1974), respectively. The result of Theorem 2.1 should also be compared with that of Theorem 25.3 in Sato (1999) for Lévy processes.

3 Moments of CB-processes

In this section, we discuss the f-moment of the CB-process with branching mechanism ϕ given by (2.1). We shall first give a construction of the process in terms of a stochastic equation. This construction generalizes slightly the results of Dawson and Li (2006, 2012) and plays an important role in the study of the f-moment.

Let $(\Omega, \mathscr{G}, \mathbf{P})$ be a complete probability space with the augmented filtration $(\mathscr{G}_t)_{t\geq 0}$. Let $W(\mathrm{d}s, \mathrm{d}u)$ be a $(\mathscr{G}_t)$-time-space Gaussian white noise on $(0,\infty)^2$ based on the Lebesgue measure $\mathrm{d}s\mathrm{d}u$. Let $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be a $(\mathscr{G}_t)$-time-space Poisson random measure on

$(0,\infty)^3$ with intensity $\mathrm{d}s m(\mathrm{d}z)\mathrm{d}u$. Let $\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ denote the compensated measure of $M(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$. For any given $\mathscr{G}_0$-measurable positive random variable X_0, we consider the stochastic integral equation

$$\begin{aligned} X_t = X_0 + \sigma \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s,\mathrm{d}u) + \int_0^t \int_0^1 \int_0^{X_{s-}} z\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ - \beta \int_0^t X_{s-}\mathrm{d}s + \int_0^t \int_1^\infty \int_0^{X_{s-}} zM(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned} \tag{3.1}$$

Theorem 3.1. *There is a unique positive strong solution to* (3.1) *and the solution* $(X_t)_{t\geq 0}$ *is a CB-process with transition semigroup* $(Q_t)_{t\geq 0}$ *defined by* (2.3).

Proof. In the special case where $(z \wedge z^2)m(\mathrm{d}z)$ is a finite measure on $(0,\infty)$, we can let $b = \beta - \int_1^\infty zm(\mathrm{d}z)$ and rewrite (3.1) into

$$X_t = X_0 - b\int_0^t X_{s-}\mathrm{d}s + \sigma \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s,\mathrm{d}u) + \int_0^t \int_0^\infty \int_0^{X_{s-}} z\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).$$

Then the theorem holds in this case by Theorem 2.5 or Theorem 3.1 in Dawson and Li (2012); see also Dawson and Li (2006). In the general case, for each integer $k \geq 1$ there is a unique positive strong solution $\{X_t^{(k)} : t \geq 0\}$ to the stochastic equation

$$\begin{aligned} X_t = X_0 + \sigma \int_0^t \int_0^{X_{s-}} W(\mathrm{d}s,\mathrm{d}u) + \int_0^t \int_0^1 \int_0^{X_{s-}} z\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ - \beta \int_0^t X_{s-}\mathrm{d}s + \int_0^t \int_1^\infty \int_0^{X_{s-}} (z\wedge k)M(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \end{aligned} \tag{3.2}$$

In view of (3.2), we have $X_t^{(k+1)} = X_t^{(k)}$ for $0 \leq t < S_k$ and $k \geq 1$, where $S_k = \inf\{t > 0 : X_t^{(k)} - X_{t-}^{(k)} \geq k\}$. It is easy to see that the process $t \mapsto X_t := \lim_{k\to\infty} X_t^{(k)}$ is a solution to (3.1). The pathwise uniqueness of the solution of (3.1) follows from that of (3.2). By Theorem 3.1 of Dawson and Li (2012) one sees that $\{X_t^{(k)} : t \geq 0\}$ is a CB-process with branching mechanism ϕ_k defined by

$$\phi_k(\lambda) = \beta\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_0^\infty (e^{-\lambda(z\wedge k)} - 1 + \lambda z 1_{\{z\leq 1\}})m(\mathrm{d}z). \tag{3.3}$$

The transition semigroup $(Q_t^{(k)})_{t\geq 0}$ of this process is determined by

$$\int_{[0,\infty)} e^{-\lambda y} Q_t^{(k)}(x,\mathrm{d}y) = \exp\{-xv_t^{(k)}(\lambda)\}, \qquad \lambda, x \geq 0,$$

where $t \mapsto v_t^{(k)}(\lambda)$ is the unique positive solution of

$$v_t^{(k)}(\lambda) = \lambda - \int_0^t \phi_k(v_s^{(k)}(\lambda))\mathrm{d}s, \qquad \lambda, t \geq 0. \tag{3.4}$$

By comparison theorem we see $v_t^{(k)}(\lambda) \leq v_t^{(k+1)}(\lambda) \leq v_t(\lambda)$, where $t \mapsto v_t(\lambda)$ is the unique positive solution to (2.4). It follows that $v_t^{(k)}(\lambda) \to v_t(\lambda)$ increasingly as $k \to \infty$. Then $(X_t)_{t\geq 0}$ is a CB-process with branching mechanism ϕ. □

Let $\{X_t(x) : t \geq 0\}$ be the solution of (3.1) with $X_0(x) = x \geq 0$. Then $\{X_t(x) : t \geq 0\}$ is a CB-process with transition semigroup $(Q_t)_{t\geq 0}$.

Theorem 3.2. *The path-valued process $x \mapsto \{X_t(x) : t \geq 0\}$ has positive and independent increments. Furthermore, for any $y \geq x \geq 0$ the difference $\{X_t(y) - X_t(x) : t \geq 0\}$ is a CB-process with initial value $y - x$.*

Proof. When $(z \wedge z^2)m(\mathrm{d}z)$ is a finite measure on $(0, \infty)$, the theorem is a consequence of Theorems 3.2 and 3.3 in Dawson and Li (2012). In the general case, it follows from the approximation of the solution given in the proof of Theorem 3.1. □

We next study the existence of the f-moment of the CB-process. Instead of Condition A, we here introduce the following more convenient condition:

Condition B. There exists a constant $K > 0$ such that

(B1) $f(x)$ is convex and nondecreasing on $[0, \infty)$;

(B2) $f(xy) \leq Kf(x)f(y)$ for all $x, y \in [0, \infty)$;

(B3) $f(x) > 1$ for all $x \in [0, \infty)$.

This replacement of the condition is not essential. Indeed, as observed in Athreya and Ney (1972, p.154), for any unbounded function f on $[0, \infty)$ satisfying Condition A there is a constant $a \geq 0$ so that the function $x \mapsto f_a(x) := f(a \vee x)$ satisfies Condition B. Of course, a probability measure on $[0, \infty)$ has finite f-moment if and only if it has finite f_a-moment.

Let $\tau_0(x) = 0$ and for $k \geq 1$ let $\tau_k(x)$ denote the kth jump time with jump size in $(1, \infty)$ of $\{X_t(x) : t \geq 0\}$.

Proposition 3.3. *Suppose that f satisfies Condition B. Then for any $t \geq 0$ and $y \geq x > 0$ we have*

$$\mathbf{P}[f(X_t(y))1_{\{t<\tau_k(y)\}}] \leq Kf(1 + y/x)\mathbf{P}[f(X_t(x))1_{\{t<\tau_k(x)\}}]. \tag{3.5}$$

Proof. Let $X_t^{(i)}(x) = X_t(ix) - X_t((i-1)x)$. By Theorem 3.2, $\{X_t^{(i)}(x) : t \geq 0\}$, $i = 1, 2, \ldots$ are i.i.d. CB-processes with $X_0^{(i)}(x) = x$. Let $\tau_k^{(i)}(x)$ and σ_k denote the kth jump times of $\{X_t^{(i)}(x) : t \geq 0\}$ and $\{X_t(y) - X_t(y - x) : t \geq 0\}$ with jump size in $(1, \infty)$, respectively. Let $\lfloor x \rfloor$ denote the largest integer smaller than or equal to $x \geq 0$. Since $\tau_k^{(i)}(x) \geq \tau_k(y)$ and $\sigma_k \geq \tau_k(y)$, by Condition B we have

$$\begin{aligned}
\mathbf{P}[f(X_t(y))1_{\{t<\tau_k(y)\}}] &= \mathbf{P}\bigg[f\bigg(\sum_{i=1}^{\lfloor y/x \rfloor} X_t^{(i)}(x) + X_t(y) - X_t(\lfloor y/x \rfloor x)\bigg)1_{\{t<\tau_k(y)\}}\bigg] \\
&\leq \mathbf{P}\bigg[f\bigg(\sum_{i=1}^{\lfloor y/x \rfloor} X_t^{(i)}(x) + X_t(y) - X_t(y - x)\bigg)1_{\{t<\tau_k(y)\}}\bigg] \\
&\leq Kf(\lfloor y/x \rfloor + 1)\mathbf{P}\bigg\{f\bigg(\frac{1}{\lfloor y/x \rfloor + 1}\bigg[\sum_{i=1}^{\lfloor y/x \rfloor} X_t^{(i)}(x)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + X_t(y) - X_t(y-x)\Big]\Big)1_{\{t<\tau_k(y)\}}\Big\} \\
&\le Kf(\lfloor y/x\rfloor+1)\mathbf{P}\Big\{\frac{1}{\lfloor y/x\rfloor+1}\Big[\sum_{i=1}^{\lfloor y/x\rfloor} f(X_t^{(i)}(x))1_{\{t<\tau_k^{(i)}(x)\}} \\
&\qquad\qquad + f(X_t(y)-X_t(y-x))1_{\{t<\sigma_k\}}\Big]\Big\} \\
&\le Kf(y/x+1)\mathbf{P}[f(X_t(x))1_{\{t<\tau_k(x)\}}].
\end{aligned}
$$

That proves (3.5). □

Corollary 3.4. *Suppose that f satisfies Condition B. Then for any $t \ge 0$ and $y \ge x > 0$ we have*

$$\mathbf{P}f(X_t(y)) \le Kf(1+y/x)\mathbf{P}f(X_t(x)). \tag{3.6}$$

Consequently, we have $\mathbf{P}f(X_t(y)) < \infty$ if and only if $\mathbf{P}f(X_t(x)) < \infty$.

Proof. By letting $n \to \infty$ in (3.5) we obtain the first result. The second one is then an immediate consequence. □

Proposition 3.5. *Suppose that f satisfies Condition B. Let $\{X_t : t \ge 0\}$ be a CB-process with branching mechanism ϕ and arbitrary initial distribution. Then we have*

$$\mathbf{P}f(X_t) \le \frac{1}{2}K^2 f(2)\big[f(1) + \mathbf{P}f(X_0)\big]\mathbf{P}f(X_t(1)), \qquad t \ge 0. \tag{3.7}$$

Proof. Without loss of generality, we may assume $\{X_t : t \ge 0\}$ solves the stochastic equation (3.1). By Corollary 3.4, the Markov property and the convexity of f we have

$$\mathbf{P}[f(X_t)|\mathscr{G}_0] \le Kf(1+X_0)\mathbf{P}f(X_t(1)) \le \frac{1}{2}K^2 f(2)\big[f(1)+f(X_0)\big]\mathbf{P}f(X_t(1)).$$

When $X_0 < 1$, the first inequality follows from Theorem 3.2. Then we get (3.7) by taking the expectation. □

Proposition 3.6. *Suppose that f satisfies Condition B. Let $\{X_t : t \ge 0\}$ be a CB-process with branching mechanism ϕ and arbitrary initial distribution. If $\mathbf{P}f(X_t) < \infty$ for some $t \ge 0$, then $\mathbf{P}f(X_0) < \infty$.*

Proof. As in the proof of Proposition 3.3, let $\lfloor x\rfloor$ denote the largest integer smaller than or equal to $x \ge 0$. By Condition B we have

$$\mathbf{P}f(X_0) \le \mathbf{P}f(\lfloor X_0\rfloor + 1) \le \frac{1}{2}Kf(2)\{\mathbf{P}f(\lfloor X_0\rfloor) + f(1)\}.$$

Then it suffices to show $\mathbf{P}f(\lfloor X_0\rfloor) < \infty$. From Proposition 3.1 in Li (2011) and the proof of Theorem 3.1, we see $v_t(\lambda) > 0$ for any $\lambda > 0$. Let $X_t^{(i)} = X_t(i) - X_t(i-1)$. Then $\{X_t^{(i)} : t \ge 0\}$, $i = 1, 2, \cdots$ are i.i.d. CB-processes with $X_0^{(i)} = 1$. From (2.3) we see $\mathbf{P}(X_t(1) \in (0,\infty)) = Q_t(1,(0,\infty)) > 0$. Then there is $\epsilon > 0$ so that $\mathbf{P}(X_t^{(i)} \geq \epsilon) = \mathbf{P}(X_t(1) \geq \epsilon) \in (0,1)$. Now define the sequence of i.i.d. random variables $\{\delta_1, \delta_2, \dots\}$ by

$$\delta_i = \begin{cases} 1, \text{ if } X_t^{(i)} \geq \epsilon; \\ 0, \text{otherwise.} \end{cases}$$

Then $\mathbf{P}(\delta_i = 1) = \mathbf{P}(X_t^{(i)} \geq \epsilon) \in (0,1)$. Observe that

$$\sum_{i=1}^{\lfloor X_0 \rfloor} \delta_i \leq \epsilon^{-1} \sum_{i=1}^{\lfloor X_0 \rfloor} X_t^{(i)} \leq \epsilon^{-1} X_t.$$

By Condition B we have

$$\mathbf{P}f\bigg(\sum_{i=1}^{\lfloor X_0 \rfloor} \delta_i\bigg) \leq \mathbf{P}f(\epsilon^{-1} X_t) \leq K f(\epsilon^{-1}) \mathbf{P}f(X_t) < \infty.$$

By the property of independent increments of the noises in (3.1), the $\mathscr{G}_0$-measurable random variable X_0 is independent of $\{X_t^{(i)} : t \geq 0\}$, $i = 1, 2, \dots$. Then $\lfloor X_0 \rfloor$ is independent of the sequence $\{\delta_1, \delta_2, \dots\}$. Under Condition B we have $\lim_{x\to\infty} f(x) = \infty$. By Lemmas 4 and 5 of Athreya and Ney (1972, pp.156–157) we have $\mathbf{P}f(\lfloor X_0 \rfloor) < \infty$. $\square$

Lemma 3.7. *Suppose that f satisfies Condition B and $\int_1^\infty z^r m(\mathrm{d}z) < \infty$ for every $r \geq 1$. Then for any $x \geq 0$ the function $t \mapsto \mathbf{P}f(X_t(x))$ is locally bounded on $[0,\infty)$.*

Proof. It is easy to see that the function $z \mapsto g(z) := f(e^z)$ is convex and nondecreasing on $[0,\infty)$. By Condition B, there exists a constant $K > 0$, such that

$$g(z+y) = f(e^z e^y) \leq K f(e^z) f(e^y) = K g(z) g(y), \qquad z, y \geq 0.$$

By Lemma 25.5 of Sato (1999, p.160), there is some $c > 0$ and some integer $r \geq 1$ so that $g(z) \leq c e^{rz}$ for $z \geq 0$. It follows that $f(z) \leq c z^r$ for $z \geq 1$, so $f(z) \leq f(1) + c z^r$ for $z \geq 0$. By Theorem 6.1 of Bingham (1976) or Proposition 2.3 in Section 2, we can get $\mathbf{P}(X_t(x)^r) < \infty$. Then

$$\mathbf{P}f(X_t(x)) \leq f(1) + c\mathbf{P}[X_t(x)^r] < \infty.$$

Since $\int_1^\infty z m(\mathrm{d}z) < \infty$, we can rewrite (2.1) into

$$\phi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty \big(e^{-\lambda z} - 1 + \lambda z\big) m(\mathrm{d}z), \qquad \lambda \geq 0, \tag{3.8}$$

where

$$b = \beta - \int_1^\infty z m(\mathrm{d}z).$$

In this case, we have

$$\int_{[0,\infty)} y Q_t(x, \mathrm{d}y) = x e^{-bt}, \qquad t, x \geq 0.$$

See Li (2011, Chapter 3). Then the Markov property implies that $t \mapsto W_t(x) := e^{bt} X_t(x)$ is a martingale, and hence $t \mapsto f(W_t(x))$ is a positive sub-martingale. For $t \in [0, T]$ we have

$$\begin{aligned} \mathbf{P} f(X_t(x)) &= \mathbf{P} f(e^{-bt} W_t(x)) \leq K f(e^{-bt}) \mathbf{P} f(W_t(x)) \\ &\leq K f(e^{-bt}) \mathbf{P} f(W_T(x)) \leq K f(1 \vee e^{-bT}) \mathbf{P} f(e^{bT} X_T(x)) \\ &\leq K^2 f(1 \vee e^{-bT}) f(e^{bT}) \mathbf{P} f(X_T(x)). \end{aligned}$$

Then $t \mapsto \mathbf{P} f(X_t(x))$ is a locally bounded function. □

Recall that $\tau_k(x)$ is the kth jump time with jump size in $(1, \infty)$ of the process $\{X_t(x) : t \geq 0\}$. Let $G_x(\mathrm{d}t) = \mathbf{P}(\tau_1(x) \in \mathrm{d}t)$ and $\mu_k(t) = \mathbf{P}(f(X_t(1)); t < \tau_k(1))$ for $t \geq 0$. Notice that $\mu_0(t) = 0$ for $t \geq 0$ since $\tau_0(1) = 0$. A characterization of the distribution $G_x(\mathrm{d}t)$ can be derived from Theorem 3.1 of He and Li (2016).

Proposition 3.8. *Suppose that f satisfies Condition B and $\int_1^\infty f(z) m(\mathrm{d}z) < \infty$. Then for every $T > 0$ there are constants $c_1(T) \geq 0$ and $c_2(T) \geq 0$ so that*

$$\mu_k(t) \leq c_1(T) + c_2(T) \int_0^t \mu_{k-1}(t-u) G_1(\mathrm{d}u), \qquad t \in (0, T],\ k \geq 1. \tag{3.9}$$

Proof. To avoid triviality, we assume $m(1, \infty) > 0$. Recall that $\{X_t(x) : t \geq 0\}$ is the strong solution of (3.1) with $X_0(x) = x \geq 0$. On the same probability space, let $\{Z_t(x) : t \geq 0\}$ be the strong solution of the stochastic equation

$$\begin{aligned} Z_t(x) = x - \beta \int_0^t Z_{s-}(x) \mathrm{d}s + \sigma \int_0^t \int_0^{Z_{s-}(x)} W(\mathrm{d}s, \mathrm{d}u) \\ + \int_0^t \int_0^1 \int_0^{Z_{s-}(x)} z \tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{3.10}$$

Then $\{Z_t(x) : t \geq 0\}$ is a CB-process with branching mechanism

$$\phi_1(\lambda) = \beta\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^1 (e^{-\lambda z} - 1 + \lambda z) m(\mathrm{d}z), \qquad \lambda \geq 0.$$

Let D denote the space of all càdlàg paths $t \mapsto x(t)$ from $[0, \infty)$ to itself equipped with the Skorokhod topology. Let $\mathscr{F} = \sigma\{x(s) : s \geq 0\}$ and $\mathscr{F}_t = \sigma\{x(s) : 0 \leq s \leq t\}$, $t \geq 0$ be the natural σ-algebras on D. Let $\mathbf{P}_x$ denote the distribution of $\{X_t(x) : t \geq 0\}$ on D. Then $(D, \mathscr{F}, \mathscr{F}_t, \mathbf{P}_x)$ is the canonical realization of the CB-process with transition semigroup $(Q_t)_{t \geq 0}$. Let σ_k denote the kth jump time of $\{x(t) : t \geq 0\}$ with jump size in $(1, \infty)$. In view of (3.1) and (3.10), we can use the notation in the theory of Markov processes to write

$$\begin{aligned} \mu_k(t) &= \mathbf{P}[f(X_t(1)) 1_{\{t < \tau_1(1)\}}] + \mathbf{P}[f(X_t(1)) 1_{\{\tau_1(1) \leq t < \tau_k(1)\}}] \\ &= \mathbf{P}[f(Z_t(1)) 1_{\{t < \tau_1(1)\}}] + \mathbf{P}\{1_{\{\tau_1(1) \leq t\}} \mathbf{P}[f(X_t(1)) 1_{\{t < \tau_k(1)\}} | \mathscr{G}_{\tau_1(1)}]\} \\ &\leq \mathbf{P} f(Z_t(1)) + \mathbf{P}\{1_{\{\tau_1(1) \leq t\}} (\mathbf{P}_{X_s(1)}[f(x(t-s)) 1_{\{t-s < \sigma_{k-1}\}}]) |_{s = \tau_1(1)}\} \\ &= \mathbf{P} f(Z_t(1)) + \mathbf{P}\{1_{\{\tau_1(1) \leq t\}} \mathbf{P}_{Z_{\tau_1(1)}(1) + \Delta X_{\tau_1(1)}(1)}[f(x(t - \tau_1(1))) 1_{\{t - \tau_1(1) < \sigma_{k-1}\}}]\}. \end{aligned}$$

From the stochastic equation (3.1) we see $\mathbf{P}(\tau_1(1) \in \mathrm{d}s, \Delta X_{\tau_1(1)}(1) \in \mathrm{d}z) = G_1(\mathrm{d}s)\hat{m}_1(\mathrm{d}z)$, where $\hat{m}_1(\mathrm{d}z) = m(1,\infty)^{-1}1_{\{z>1\}}m(\mathrm{d}z)$. Then, by Proposition 3.3,

$$\begin{aligned}\mu_k(t) &\le \mathbf{P}f(Z_t(1)) + \int_0^t G_1(\mathrm{d}s)\int_1^\infty \mathbf{P}\{\mathbf{P}_{Z_s(1)+z}[f(x(t-s))1_{\{t-s<\sigma_{k-1}\}}]\}\hat{m}_1(\mathrm{d}z)\\ &\le c_1(T) + K\int_0^t \mu_{k-1}(t-s)G_1(\mathrm{d}s)\int_1^\infty \mathbf{P}f(Z_s(1)+z+1)\hat{m}_1(\mathrm{d}z),\end{aligned}$$

where $c_1(T) = \sup_{0\le t\le T}\mathbf{P}f(Z_t(1)) < \infty$ by Lemma 3.7 and

$$\begin{aligned}&\int_1^\infty \mathbf{P}f(Z_s(1)+z+1)\hat{m}_1(\mathrm{d}z)\\ &\qquad\le Kf(3)\int_1^\infty \mathbf{P}f\Big(\frac{1}{3}[Z_s(1)+z+1]\Big)\hat{m}_1(\mathrm{d}z)\\ &\qquad\le \frac{1}{3}Kf(3)\Big[\mathbf{P}f(Z_s(1)) + \int_1^\infty f(z)\hat{m}_1(\mathrm{d}z) + f(1)\Big]\\ &\qquad\le \frac{1}{3}Kf(3)\Big[c_1(T) + \int_1^\infty f(z)\hat{m}_1(\mathrm{d}z) + f(1)\Big]\\ &\qquad=: c(T) < \infty.\end{aligned}$$

Then we get (3.9) with $c_2(T) = Kc(T)$. □

Proposition 3.9. *Suppose that f satisfies Condition B and $\int_1^\infty f(z)m(\mathrm{d}z) < \infty$. Then for any $x \ge 0$ the function $t \mapsto \mathbf{P}f(X_t(x))$ is locally bounded on $[0,\infty)$.*

Proof. Let $c_1(T) \ge 0$ and $c_2(T) \ge 0$ be provided by Proposition 3.8. By Lemma 2 of Athreya and Ney (1972, p.145) there is a bounded positive function $t \mapsto \mu(t)$ on $[0,T]$ satisfying

$$\mu(t) = c_1(T) + c_2(T)\int_0^t \mu(t-u)\mathrm{d}G_1(u), \qquad 0 \le t \le T. \tag{3.11}$$

In view of (3.9) and (3.11), one can show by induction that $\mu_k(t) \le \mu(t)$ for all $0 \le t \le T$ and $n \ge 1$. Since $\sigma_k \to \infty$ as $n \to \infty$ we have $\mathbf{P}f(X_t(1)) = \lim_{n\to\infty}\mu_k(t) \le \mu(t)$. Then the result follows by Corollary 3.4. □

Proof of Theorem 2.1. Without loss of generality, we may assume $\{X_t : t \ge 0\}$ solves the stochastic equation (3.1). Suppose that $\mathbf{P}f(X_0) < \infty$ and $\int_1^\infty f(z)m(\mathrm{d}z) < \infty$. Then $\mathbf{P}f(X_t(1)) < \infty$ by Proposition 3.9, and so $\mathbf{P}f(X_t) < \infty$ by Proposition 3.5. Conversely, suppose that $\mathbf{P}f(X_t) < \infty$ for some $t > 0$. By Proposition 3.6 we have $\mathbf{P}f(X_0) < \infty$. Let τ_1 denote the first jump time of $\{X_t : t \ge 0\}$ with jump size $\Delta X_{\tau_1} := X_{\tau_1} - X_{\tau_1-} \in (1,\infty)$ and let $G(\mathrm{d}t) = \mathbf{P}(\tau_1 \in \mathrm{d}t)$. To avoid triviality, we assume $m(1,\infty) > 0$, so (3.1) implies that $t \mapsto G(0,t]$ is strictly increasing on $[0,\infty)$. Using the notation introduced in the proof of Proposition 3.8, we have

$$\begin{aligned}\mathbf{P}f(X_t) &\ge \mathbf{P}[f(X_t)1_{\{\tau_1\le t\}}] = \mathbf{P}\{1_{\{\tau_1\le t\}}\mathbf{P}_{X_{\tau_1}}f(x(t-\tau_1))\}\\ &\qquad\ge \mathbf{P}\{1_{\{\tau_1\le t\}}(\mathbf{P}_{\Delta X_s}f(x(t-s))|_{s=\tau_1}\} = \int_0^t \mathbf{P}f(\xi_{t-s})G(\mathrm{d}s),\end{aligned}$$

where $\{\xi_t : t \geq 0\}$ is a CB-process with initial distribution $\mathbf{P}(\xi_0 \in \mathrm{d}z) = \hat{m}_1(\mathrm{d}z)$. Then there must be some $s \in (0, t]$ so that $\mathbf{P}f(\xi_{t-s}) < \infty$. By Proposition 3.6 we have

$$\mathbf{P}f(\xi_0) = \int_1^\infty f(z)\hat{m}_1(\mathrm{d}z) < \infty.$$

Then $\int_1^\infty f(z)m(\mathrm{d}z) < \infty$. That proves the theorem. □

4 Moments of CBI-processes

In this section, we discuss the f-moment of the CBI-process. As in the previous section, we first give a construction of the process in terms of a stochastic equation.

Let $(\Omega, \mathscr{G}, \mathbf{P})$ be a complete probability space with the augmented filtration $(\mathscr{G}_t)_{t\geq 0}$. Let $W(\mathrm{d}s, \mathrm{d}u)$ be a $(\mathscr{G}_t)$-time-space Gaussian white noise on $(0,\infty)^2$ based on the Lebesgue measure $\mathrm{d}s\mathrm{d}u$. Let $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $N(\mathrm{d}s, \mathrm{d}z)$ be $(\mathscr{G}_t)$-time-space Poisson random measures on $(0,\infty)^3$ and $(0,\infty)^2$ with intensities $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$ and $\mathrm{d}sn(\mathrm{d}z)$, respectively. Suppose that $W(\mathrm{d}s, \mathrm{d}u)$, $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ and $N(\mathrm{d}s, \mathrm{d}z)$ are independent of each other. Let $\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ denote the compensated measure of $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$. For any given $\mathscr{G}_0$-measurable positive random variable Y_0, we consider the stochastic integral equation

$$\begin{aligned} Y_t = Y_0 &+ \sigma \int_0^t \int_0^{Y_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t \int_0^1 \int_0^{Y_{s-}} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^t (h - \beta Y_s)\mathrm{d}s + \int_0^t \int_1^\infty \int_0^{Y_{s-}} zM(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &+ \int_0^t \int_0^\infty zN(\mathrm{d}s, \mathrm{d}z). \end{aligned} \tag{4.1}$$

Theorem 4.1. *There is a unique positive strong solution to* (4.1) *and the solution* $(Y_t)_{t\geq 0}$ *is a CBI-process with transition semigroup* $(Q_t^\gamma)_{t\geq 0}$ *defined by* (2.6).

Theorem 4.2. *For any* $x \geq 0$ *let* $\{Y_t(x) : t \geq 0\}$ *be the solution to* (4.1) *with* $Y_0(x) = x \geq 0$. *Then the path-valued process* $x \mapsto \{Y_t(x) : t \geq 0\}$ *has positive and independent increments. Furthermore, for any* $y \geq x \geq 0$ *the difference* $\{Y_t(y) - Y_t(x) : t \geq 0\}$ *is a CB-process with initial value* $y - x$.

The above theorems generalize the results of Dawson and Li (2012). We here omit their proofs since the arguments are quite similar to those for the corresponding results in Section 3.

Proposition 4.3. *Suppose that* f *satisfies Condition B. Let* $\{X_t : t \geq 0\}$ *be a CB-process and* $\{Y_t : t \geq 0\}$ *a CBI-process with* $X_0 \overset{d}{=} Y_0$. *Then*

$$\mathbf{P}f(Y_t) \leq \frac{1}{2}Kf(2)\big[\mathbf{P}f(Y_t(0)) + \mathbf{P}f(X_t)\big], \qquad t \geq 0. \tag{4.2}$$

Proof. Without loss of generality, we assume $\{Y_t : t \geq 0\}$ and $\{X_t : t \geq 0\}$ are solutions of (4.1) and (3.1), respectively, with $Y_0 = X_0$. Since f satisfies Condition B, we have

$$\begin{aligned}
\mathbf{P}f(Y_t) &= \mathbf{P}f(Y_t(0) + Y_t - Y_t(0)) \\
&\leq Kf(2)\mathbf{P}f\Big(\frac{1}{2}[Y_t(0) + Y_t - Y_t(0)]\Big) \\
&\leq \frac{1}{2}Kf(2)\big[\mathbf{P}f(Y_t(0)) + \mathbf{P}f(Y_t - Y_t(0))\big] \\
&= \frac{1}{2}Kf(2)\big[\mathbf{P}f(Y_t(0)) + \mathbf{P}f(X_t)\big],
\end{aligned}$$

where the last equality follows by Theorem 4.2. □

Lemma 4.4. *Suppose that f satisfies Condition B and $\int_1^\infty z^r(m+n)(\mathrm{d}z) < \infty$ for every $r \geq 1$. Then for any $x \geq 0$ the function $t \to \mathbf{P}f(Y_t(x))$ is locally bounded on $[0,\ \infty)$.*

Proof. By differentiating both sides of (2.6) we obtain the moment formula

$$\int_{[0,\infty)} yQ_t^\gamma(x, \mathrm{d}y) = xe^{-bt} + \Big[h + \int_0^\infty un(\mathrm{d}u)\Big]\int_0^t \mathrm{e}^{-bs}\mathrm{d}s.$$

It is then easy to check that the process $t \to e^{bt}Y_t(x)$ is a sub-martingale. Based on those, the result follows in the same way as in the proof of Lemma 3.7. We leave the details to the reader. □

Let $\zeta_0(x) = 0$ and let $\zeta_k(x)$ be the kth jump time of $\{Y_t(x) : t \geq 0\}$ with jump size in $(1,\infty)$. Let $H(\mathrm{d}t) = \mathbf{P}(\zeta_1(0) \in \mathrm{d}t)$ and $\nu_k(t) = \mathbf{P}(f(Y_t(0)); t < \zeta_k(0))$ for $t \geq 0$. A characterization of the distribution $H(\mathrm{d}t)$ was given by He and Li (2016).

Proposition 4.5. *Suppose that f satisfies Condition B and $\int_1^\infty f(z)(m+n)(\mathrm{d}z) < \infty$. Then for every $T > 0$ there is a constant $0 \leq c_3(T) < \infty$ so that*

$$\nu_k(t) \leq c_3(T) + \frac{1}{2}Kf(2)\int_0^t \nu_{k-1}(t-s)H(\mathrm{d}s), \qquad 0 \leq t \leq T,\ k \geq 1. \tag{4.3}$$

Proof. Let $(D, \mathscr{F}, \mathscr{F}_t, x(t))$ and $\{\sigma_k : k = 1, 2, \dots\}$ be as in the proof of Proposition 3.8. Let $\mathbf{P}_x$ and $\mathbf{P}_x^\gamma$ denote the laws on $(D, \mathscr{F})$ of $\{X_t(x) : t \geq 0\}$ and $\{Y_t(x) : t \geq 0\}$, respectively. Then $(D, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{P}_x)$ is a canonical realization of the CB-process with transition semigroup $(Q_t)_{t\geq 0}$ and $(D, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{P}_x^\gamma)$ is a canonical realization of the CBI-process with transition semigroup $(Q_t^\gamma)_{t\geq 0}$. Let us also consider the stochastic equation

$$\begin{aligned}
Z_t = Z_0 &+ \sigma\int_0^t\int_0^{Z_{s-}} W(\mathrm{d}s, \mathrm{d}u) + \int_0^t\int_0^1\int_0^{Z_{s-}} z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\
&+ \int_0^t (h - \beta Z_{s-})\mathrm{d}s + \int_0^t\int_0^1 zN(\mathrm{d}s, \mathrm{d}z).
\end{aligned} \tag{4.4}$$

Let $\{Z_t(x) : t \geq 0\}$ denote the solution with $Z_0(x) = x \geq 0$. In view of (4.1) and (4.4), we have

$$\nu_k(t) = \mathbf{P}[f(Y_t(0))1_{\{t<\zeta_1(0)\}}] + \mathbf{P}[f(Y_t(0))1_{\{\zeta_1(0)\leq t<\zeta_k(0)\}}]$$

$$
\begin{aligned}
&= \mathbf{P}[f(Z_t(0))1_{\{t<\zeta_1(0)\}}] + \mathbf{P}\{1_{\{\zeta_1(0)\le t\}}\mathbf{P}[f(Y_t(0))1_{\{t<\zeta_k(0)\}}|\mathscr{G}_{\zeta_1(0)}]\} \\
&\le \mathbf{P}f(Z_t(0)) + \mathbf{P}\{1_{\{\zeta_1(0)\le t\}}(\mathbf{P}^\gamma_{Y_s(0)}[f(x(t-s))1_{\{t-s<\sigma_{k-1}\}}])|_{s=\zeta_1(0)}\} \\
&= \mathbf{P}f(Z_t(0)) + \mathbf{P}\{1_{\{\zeta_1(0)\le t\}}(\mathbf{P}^\gamma_{Z_s(0)+\Delta Y_s(0)}[f(x(t-s))1_{\{t-s<\sigma_{k-1}\}}])|_{s=\zeta_1(0)}\} \\
&\le c_0(T) + \mathbf{P}\bigg\{\int_0^t H(\mathrm{d}s)\int_1^\infty \mathbf{P}^\gamma_{Z_s(0)+z}[f(x(t-s))1_{\{t-s<\sigma_{k-1}\}}]\eta_s(\mathrm{d}z)\bigg\},
\end{aligned}
$$

where $c_0(T) = \sup_{0\le t\le T}\mathbf{P}f(Z_t(0)) < \infty$ by Lemma 4.4 and

$$
\eta_s(\mathrm{d}z) = 1_{\{Y_{s-}(0)m(1,\infty)+n(1,\infty)>0\}}\frac{Y_{s-}(0)m(\mathrm{d}z)+n(\mathrm{d}z)}{Y_{s-}(0)m(1,\infty)+n(1,\infty)}.
$$

Observe that $\eta_s(\mathrm{d}z) \le (\hat{m}_1+\hat{n}_1)(\mathrm{d}z)$, where $\hat{m}_1(\mathrm{d}z) = m(1,\infty)^{-1}1_{\{z>1\}}m(\mathrm{d}z)$ and $\hat{n}_1(\mathrm{d}z) = n(1,\infty)^{-1}1_{\{z>1\}}n(\mathrm{d}z)$. By Theorem 4.2 and Corollary 3.4,

$$
\begin{aligned}
\nu_k(t) &\le c_0(T) + \frac{1}{2}Kf(2)\mathbf{P}\bigg\{\int_0^t H(\mathrm{d}s)\int_1^\infty \mathbf{P}_{Z_s(0)+z}[f(x(t-s))1_{\{t-s<\sigma_{k-1}\}}]\eta_s(\mathrm{d}z)\bigg\} \\
&\qquad + \frac{1}{2}Kf(2)\mathbf{P}\bigg\{\int_0^t H(\mathrm{d}s)\int_1^\infty \mathbf{P}^\gamma_0[f(x(t-s))1_{\{t-s<\sigma_{k-1}\}}]\eta_s(\mathrm{d}z)\bigg\} \\
&\le c_0(T) + \frac{1}{2}K^2f(2)\int_0^t \mu_{k-1}(t-s)H(\mathrm{d}s)\int_1^\infty \mathbf{P}f(Z_s(0)+z+1)\eta_s(\mathrm{d}z) \\
&\qquad + \frac{1}{2}Kf(2)\int_0^t \mathbf{P}^\gamma_0[f(x(t-s))1_{\{t-s<\sigma_{k-1}\}}]H(\mathrm{d}s) \\
&\le c_0(T) + \int_0^t \mu(t-s)h_0(s)H(\mathrm{d}s) + \frac{1}{2}Kf(2)\int_0^t \nu_{k-1}(t-s)H(\mathrm{d}s),
\end{aligned}
$$

where $s\mapsto\mu(s)$ is defined in the proof of Proposition 3.9 and

$$
\begin{aligned}
h_0(s) &= \frac{1}{2}K^2f(2)\int_1^\infty \mathbf{P}f(Z_s(0)+z+1)(\hat{m}_1+\hat{n}_1)(\mathrm{d}z) \\
&\le \frac{1}{6}K^3f(2)f(3)\bigg\{2c_0(T) + \int_1^\infty f(z)(\hat{m}_1+\hat{n}_1)(\mathrm{d}z) + 2f(1)\bigg\} =: c_4(T).
\end{aligned}
$$

It is easy to see that

$$
c_3(T) := c_0(T) + c_4(T)\sup_{0\le t\le T}\int_0^t \mu(t-s)H(\mathrm{d}s) < \infty.
$$

Then we have (4.3). □

Proof of Theorem 2.2. Suppose that $\mathbf{P}f(Y_0) < \infty$ and $\int_1^\infty f(z)(m+n)(\mathrm{d}z) < \infty$. Using Proposition 4.5 we see as in the proof of Proposition 3.9 that $\mathbf{P}f(Y_t(0)) = \lim_{k\to\infty}\nu_k(t) < \infty$. Then $\mathbf{P}f(Y_t) < \infty$ by Theorem 2.1 and Proposition 4.3. Conversely, suppose that $\mathbf{P}f(Y_t) < \infty$ for some $t > 0$. Let $\{X_t : t \ge 0\}$ be a solution of (3.1) with $Y_0 = X_0$. By Theorem 4.2 we see

$$
\mathbf{P}f(X_t) = \mathbf{P}f(Y_t - Y_t(0)) \le \mathbf{P}f(Y_t) < \infty.
$$

Then Theorem 2.1 implies $\mathbf{P}f(Y_0) = \mathbf{P}f(X_0) < \infty$. Moreover, let ζ_1 be the first jump time of $\{Y_t : t \geq 0\}$ with jump size in $(1,\infty)$ and $\tilde{H}(\mathrm{d}t) = \mathbf{P}(\zeta_1 \in \mathrm{d}t)$. Using the notation introduced in the proof of Proposition 4.5, we have

$$\begin{aligned}\mathbf{P}f(Y_t) &\geq \mathbf{P}[f(Y_t)1_{\{\zeta_1 \leq t\}}] = \mathbf{P}\{1_{\{\zeta_1 \leq t\}}\mathbf{P}[f(Y_t)|\mathscr{G}_{\zeta_1}]\} \\ &= \mathbf{P}\{1_{\{\zeta_1 \leq t\}}(\mathbf{P}^{\gamma}_{Y_s}f(x(t-s)))|_{s=\zeta_1}\} \\ &\geq \mathbf{P}\{1_{\{\zeta_1 \leq t\}}(\mathbf{P}_{\Delta Y_s}f(x(t-s)))|_{s=\zeta_1}\} \\ &= \int_0^t \mathbf{P}\bigg\{\int_1^\infty \mathbf{P}_z f(x(t-s))\tilde{\eta}_s(\mathrm{d}z)\bigg\}\tilde{H}(\mathrm{d}s),\end{aligned}$$

where

$$\tilde{\eta}_s(\mathrm{d}z) = 1_{\{Y_{s-}m(1,\infty)+n(1,\infty)>0\}}\frac{Y_{s-}m(\mathrm{d}z) + n(\mathrm{d}z)}{Y_{s-}m(1,\infty) + n(1,\infty)}.$$

To avoid triviality, in the following we assume $(m+n)(1,\infty) > 0$. From (4.1) we see $t \mapsto \tilde{H}(0,t]$ is strictly increasing on $[0,\infty)$. Then there must be some $s \in (0,t]$ so that, a.s.,

$$\int_1^\infty \mathbf{P}_z f(x(t-s))\tilde{\eta}_s(\mathrm{d}z) < \infty.$$

Since $\{Y_t : t \geq 0\}$ is a Hunt process, we have $\mathbf{P}(Y_{s-} = Y_s) = 1$. Let $\{X_t : t \geq 0\}$ be the solution of (3.1) with $X_0 = Y_0$. By comparison we have a.s. $Y_s \geq X_s$. Then Theorem 3.5 of Li (2011, p.59) implies that $\mathbf{P}(Y_{s-} > 0) = \mathbf{P}(Y_s > 0) \geq \mathbf{P}(X_s > 0) > 0$. It follows that

$$\int_1^\infty \mathbf{P}_z f(x(t-s))(\hat{m}_1 + \hat{n}_1)(\mathrm{d}z) < \infty.$$

This means $\mathbf{P}(\xi_{t-s}) < \infty$ for a CB-process $\{\xi_t : t \geq 0\}$ with initial distribution $\mathbf{P}(\xi_0 \in \mathrm{d}z) = 2^{-1}(\hat{m}_1 + \hat{n}_1)(\mathrm{d}z)$. By Proposition 3.6 we have

$$\mathbf{P}f(\xi_0) = \frac{1}{2}\int_1^\infty f(z)(\hat{m}_1 + \hat{n}_1)(\mathrm{d}z) < \infty.$$

Then $\int_1^\infty f(z)(m+n)(\mathrm{d}z) < \infty$. That proves the theorem. $\square$

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