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A conditioned continuous-state branching process with applications

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Abstract: A supercritical CB-process conditioned on explosion is again a Markov process. We characterize the transition semigroup of the conditioned process by its Laplace transform and by a Doob's h-transform. The conditioned CB-process is constructed as the strong solution of a stochastic integral equation. We use the distribution of the conditioned process to construct directly the canonical Kuznetsov measure of the CB-process. The later is reconstructed from positive paths picked up by a Poisson random measure based on the canonical Kuznetsov measure.

Key words and phrases: conditioned CB-process; Doob's h-transform; entrance rule; stochastic integral equation; canonical Kuznetsov measure.

1 Introduction

Continuous-state branching processes (CB-processes) are positive Markov processes introduced by Jiřina (1958) to model the random evolution of population dynamics. Those processes can be obtained as rescaling limits of discrete Galton-Watson processes; see, e.g., Lamperti (1967a), Li (2011) and Grimvall (1974). A representation of the processes was given by Lamperti (1967b) as time-changed spectrally positive Lévy processes. The continuous-state branching processes with immigration (CBI-processes) introduced by Kawazu and Watanabe (1971) are the natural generalizations of CB-processes. Stochastic integral equations for general CBI-processes were established by Dawson and Li (2006, 2012) and Fu and Li (2010).

Conditioned branching processes and conditional limit theorems have been studied in both discrete and continuous state spaces by many authors. It is well-known that a non-degenerate branching goes either extinction or explosion. It was showed in Li (2000, 2011) that a supercritical CB-process conditioned on extinction is equivalent to a subcritical one and a critical or subcritical CB-process conditioned on distant extinction time is equivalent to a special CBI-process. Li (2000, 2011) also showed the later can be defined by a Doob's h-transform related to the first moment; see also Lambert (2007). Bertoin et al.

(2008) proved that the number of individuals with infinite lines of descents in a supercritical CB-process evolves as an immortal supercritical Galton-Watson process. The analogous results of those for Galton-Watson processes are classical; see, e.g., Athreya and Ney (1972, pp.47–60). However, as far as we know, there are still no results on the characterizations and properties of the processes obtained by conditioning supercritical branching processes on explosion.

The purpose of this note is to study the basic structures of supercritical CB-processes conditioned on explosion. We here consider general conservative CB-processes which do not necessarily have finite first moments. In Section 2, we recall some preliminary results. In Section 3, we show that a conditioned supercritical CB-process is a Markov process and characterize the transition semigroup by its Laplace transform and by a Doob's h-transform. In Section 4, we give a construction of the conditioned process as the unique strong solution of a stochastic integral equation, which shows the conditioned process can be identified as a generalized CBI-process with dependent immigration. In Section 5, two applications of the conditioned process are given. We first use the distribution of the conditioned CB-process to construct directly the canonical Kuznetsov measure of the original CB-process, which extends the construction of Li (2019a) under the first moment assumption. Then we give a reconstruction of CB-process from positive paths picked up by a Poisson random measure based on the canonical Kuznetsov measure.

2 Preliminaries

We consider a conservative CB-process which is a $[0,\infty)$-Markov process with Feller transition semigroup $(Q_t)_{t\geq 0}$ satisfying the *branching property*:

$$Q_t(x_1+x_2,\cdot) = Q_t(x_1,\cdot) * Q_t(x_2,\cdot), \qquad t\in(0,\infty),\ x_1,x_2\in[0,\infty),$$

where "$*$" denotes the convolution operation. The process is characterized by its *branching mechanism*, which is a function ϕ on $[0,\infty)$ with the following representation:

$$\phi(\lambda) = b\lambda + c\lambda^2 + \int_{(0,\infty)} (\mathrm{e}^{-\lambda z} - 1 + \lambda z \mathbf{1}_{\{z\leq 1\}}) m(\mathrm{d}z), \qquad \lambda\geq 0, \tag{2.1}$$

where $b\in\mathbb{R}$ and $c\geq 0$ are constants, and $(1\wedge z^2)m(\mathrm{d}z)$ is a finite measure on $(0,\infty)$. We assume ϕ is not linear and satisfies

$$\int_{0+} \frac{1}{|\phi(\lambda)|}\mathrm{d}\lambda = \infty.$$

For every $t,\lambda,x\in[0,\infty)$, we have

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t(x,\mathrm{d}y) = \mathrm{e}^{-xv_t(\lambda)}, \tag{2.2}$$

where the *cumulant semigroup* $(v_t)_{t\geq 0}$ is defined by

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda))\mathrm{d}s, \qquad t,\lambda\geq 0. \tag{2.3}$$

Let $C_0^2([0,\infty))$ be the set of twice continuously differentiable functions with their derivatives up to the second order vanish at ∞. It is known that $(Q_t)_{t\geq 0}$ has the generator A defined by

$$\begin{aligned} Af(x) &= cxf''(x) + x\int_{(0,1]} [f(x+z) - f(x) - zf'(x)] m(\mathrm{d}z) \\ &\quad - bxf'(x) + x\int_{(1,\infty)} [f(x+z) - f(x)] m(\mathrm{d}z), \end{aligned} \tag{2.4}$$

where $f \in C_0^2([0,\infty))$. The branching property entails that $(v_t)_{t\geq 0}$ can be expressed canonically as

$$v_t(\lambda) = h_t\lambda + \int_{(0,\infty)} (1 - \mathrm{e}^{-\lambda z}) l_t(\mathrm{d}z), \qquad t, \lambda \geq 0, \tag{2.5}$$

where $h_t \geq 0$ and $(1 \wedge z)l_t(\mathrm{d}z)$ is a finite measure on $(0,\infty)$. For every $t \geq 0$ the function $\lambda \mapsto v_t(\lambda)$ is strictly increasing on $[0,\infty)$. Therefore the limit $\bar{v}_t :=\uparrow \lim_{\lambda\to\infty} v_t(\lambda)$ exists in $(0,\infty]$ for every $t \geq 0$. It was proved in Grey (1974) that $\bar{v}_t < \infty$ for some $t > 0$ (and then for all $t > 0$) if and only if the following condition (*Grey's condition*) holds: $\phi(\theta) > 0$ and $\int_\theta^\infty \phi(z)^{-1}\mathrm{d}z < \infty$ for large enough $\theta > 0$.

3 The conditioned CB-process

Since ϕ is not linear, the function $\lambda \to \phi(\lambda)$ is strictly convex and there exists $\lambda > 0$ such that $\phi(\lambda) \neq 0$. Therefore the equation $\phi(\lambda) = 0$ has at most one root in $(0,\infty)$. Let $\theta_0 = \inf\{\lambda \geq 0 : \phi(\lambda) > 0\}$ with the convention $\inf\emptyset = \infty$. Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, x(t), \mathbf{Q}_x)$ be a Hunt realization of the CB-process with branching mechanism ϕ.

Proposition 3.1 *For any $x \in (0,\infty)$, the limit $x(\infty) = \lim_{t\to\infty} x(t)$ exists $\mathbf{Q}_x$-a.s. in $[0,\infty]$ and*

$$\mathbf{Q}_x\{x(\infty) = 0\} = \mathrm{e}^{-x\theta_0}, \quad \mathbf{Q}_x\{x(\infty) = \infty\} = 1 - \mathrm{e}^{-x\theta_0}. \tag{3.1}$$

Proof. As in the proof of Corollary 3.3 in Li (2019a) one can show that, by weak convergence of probability measures on $[0,\infty]$,

$$\lim_{t\to\infty} Q_t(x,\cdot) = \mathrm{e}^{-x\theta_0}\delta_0 + (1 - \mathrm{e}^{-x\theta_0})\delta_\infty. \tag{3.2}$$

We consider separately the cases $\theta_0 = \infty$, $\theta_0 = 0$ and $\theta_0 \in (0,\infty)$. (1) When $\theta_0 = \infty$, the process $\{x(t) : t \geq 0\}$ is distributed identically with a random time change of a subordinator with Laplace exponent ϕ; see, e.g., Theorem 12.2 in Kyprianou (2014). Then $\{x(t) : t \geq 0\}$ is $\mathbf{Q}_x$-a.s. increasing. Then (3.2) implies that $\mathbf{Q}_x\{x(\infty) = \infty\} = 1$ for $x \in (0,\infty)$. (2) When $\theta_0 = 0$, we have $\phi'(0) \in [0,\infty)$ and the CB-process is critical or subcritical. In this case, it is well-known that $t \mapsto \mathrm{e}^{\phi'(0)t}x(t)$ is a positive martingale. By (3.2) and martingale convergence theorem we see $\mathbf{Q}_x\{x(\infty) = 0\} = 1$ for $x \in [0,\infty)$. (3) When $\theta_0 \in (0,\infty)$, we have $\phi(\theta_0) = 0$ and hence $v_t(\theta_0) = \theta_0$ for $t \geq 0$. Then $t \mapsto \mathrm{e}^{-\theta_0 x(t)}$ is a positive martingale. By martingale convergence theorem we see $\lim_{t\to\infty} \mathrm{e}^{-\theta_0 x(t)}$ exists $\mathbf{Q}_x$-a.s., so the limit $x(\infty) = \lim_{t\to\infty} x(t)$ exists $\mathbf{Q}_x$-a.s. in $[0,\infty]$. By (3.2), $\mathbf{Q}_x[\mathrm{e}^{-\lambda x(\infty)}] = \mathrm{e}^{-x\theta_0}$ and $\mathbf{Q}_x[1 - \mathrm{e}^{-\lambda x(\infty)}] = 1 - \mathrm{e}^{-x\theta_0}$ for $\lambda > 0$. Then let $\lambda \downarrow 0$, we have (3.1) for $x \in [0,\infty)$. $\square$

It is well-known that zero is a trap for the CB-process. Let $(Q_t^\circ)_{t\geq 0}$ denote the restriction of its transition semigroup $(Q_t)_{t\geq 0}$ to $(0,\infty)$. Let $h(\theta, x) = 1 - \mathrm{e}^{-\theta x}$ for $\theta, x \in [0,\infty)$. In the special case $\theta_0 \in (0,\infty)$, it is easy to see that $h_0(x) := \theta_0^{-1}h(\theta_0, x)$ is an invariant function for $(Q_t^\circ)_{t\geq 0}$. In this case, set

$$H_t(x,\lambda) = \begin{cases} h(\theta_0, x)^{-1}h(v_t(\lambda+\theta_0) - v_t(\lambda), x) & \text{if } x > 0, \\ \theta_0^{-1}(v_t(\lambda+\theta_0) - v_t(\lambda)) & \text{if } x = 0. \end{cases} \tag{3.3}$$

Theorem 3.2 *Suppose that $\theta_0 \in (0,\infty)$. Then we can define a Feller transition semigroup $(Q_t^\infty)_{t\geq 0}$ on $[0,\infty)$ by*

$$\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q_t^\infty(x, \mathrm{d}y) = \mathrm{e}^{-xv_t(\lambda)} H_t(x,\lambda). \tag{3.4}$$

Moreover, we have $Q_t^\infty(x,\mathrm{d}y) = h_0(x)^{-1}h_0(y)Q_t(x,\mathrm{d}y)$ *for* $x > 0$ *and*

$$Q_t^\infty(0,\mathrm{d}y) = h_t\delta_0(\mathrm{d}y) + h_0(y)l_t(\mathrm{d}y), \qquad t, y \geq 0. \tag{3.5}$$

Proof. Since h_0 is an invariant function for $(Q_t^\circ)_{t\geq 0}$, we can define a Markov transition semigroup $(Q_t^\infty)_{t\geq 0}$ on $(0,\infty)$ by $Q_t^\infty(x,\mathrm{d}y) := h_0(x)^{-1}h_0(y)Q_t^\circ(x,\mathrm{d}y)$. By (2.2) we have

$$\begin{aligned}\int_{(0,\infty)} \mathrm{e}^{-\lambda y}Q_t^\infty(x,\mathrm{d}y) &= \theta_0^{-1}h_0(x)^{-1}\int_{(0,\infty)} \mathrm{e}^{-\lambda y}(1-\mathrm{e}^{-\theta_0 y})Q_t^\circ(x,\mathrm{d}y) \\ &= \theta_0^{-1}h_0(x)^{-1}\int_{[0,\infty)} (\mathrm{e}^{-\lambda y}-\mathrm{e}^{-(\lambda+\theta_0)y})Q_t(x,\mathrm{d}y) \\ &= \theta_0^{-1}h_0(x)^{-1}(\mathrm{e}^{-xv_t(\lambda)}-\mathrm{e}^{-xv_t(\lambda+\theta_0)}).\end{aligned}$$

Then (3.4) holds for $x > 0$. By the continuity of the function $(t,x)\mapsto H_t(x,\lambda)$, we can extend $(Q_t^\infty)_{t\geq 0}$ to a Feller transition semigroup on $[0,\infty)$ with Laplace transform given by (3.4). The relation (3.5) follows by (2.5) and (3.4). □

Remark 3.3 *In the critical case* $\phi'(0) = 0$, *we have* $\theta_0 = 0$ *and* $h_0(x) := x$ *is an invariant function for* $(Q_t^\circ)_{t\geq 0}$. *Note that*

$$\phi'(\lambda) = b + 2c\lambda + \int_{(0,\infty)} (\mathbf{1}_{\{z\leq 1\}} - \mathrm{e}^{-\lambda z})zm(\mathrm{d}z), \qquad \lambda \geq 0.$$

The results of Theorem 3.2 remain valid in this case if we understand

$$H_t(x,\lambda) \equiv \exp\bigg\{-\int_0^t \phi'(v_s(\lambda))\mathrm{d}s\bigg\},$$

which is actually independent of $x \geq 0$. *In this case, the transition semigroup* $(Q_t^\infty)_{t\geq 0}$ *corresponds to a special CBI-process; see, e.g., Li (2011, 2019a).*

In the case $\theta_0 \in (0,\infty)$, if a positive Markov process has transition semigroup $(Q_t^\infty)_{t\geq 0}$, we call it a *supercritical CB-process conditioned on explosion.* This terminology is justified by the next result.

Theorem 3.4 *Suppose that* $\theta_0 \in (0,\infty)$. *Then for any* $x > 0$ *the process* $\{x(t) : t \geq 0\}$ *under the conditional probability* $\mathbf{Q}_x^\infty := \mathbf{Q}_x(\cdot|x(\infty) = \infty)$ *is a Markov process with transition semigroup* $(Q_t^\infty)_{t\geq 0}$.

Proof. Let $t \geq r \geq 0$ and let F be a bounded $\mathscr{F}_r$-measurable random variable. For any $\lambda \geq 0$ we can use the Markov property to see that

$$\begin{aligned}\mathbf{Q}_x^\infty[F\mathrm{e}^{-\lambda x(t)}] &= \mathbf{Q}_x\big\{x(\infty)=\infty\big\}^{-1}\mathbf{Q}_x\Big[F\mathrm{e}^{-\lambda x(t)}1_{\{x(\infty)=\infty\}}\Big] \\ &= \mathbf{Q}_x\big\{x(\infty)=\infty\big\}^{-1}\mathbf{Q}_x\Big[\mathbf{Q}_x\Big(F\mathrm{e}^{-\lambda x(t)}1_{\{x(\infty)=\infty\}}\Big|\mathscr{F}_t\Big)\Big] \\ &= \mathbf{Q}_x\big\{x(\infty)=\infty\big\}^{-1}\mathbf{Q}_x\Big[F\mathrm{e}^{-\lambda x(t)}\mathbf{Q}_{x(t)}\{x(\infty)=\infty\}\Big] \\ &= h_0(x)^{-1}\mathbf{Q}_x\Big[F\mathrm{e}^{-\lambda x(t)}h_0(x(t))\Big].\end{aligned} \tag{3.6}$$

Using the relation $Q_t^\infty(x,\mathrm{d}y) = h_0(x)^{-1}h_0(y)Q_t(x,\mathrm{d}y)$ we can continue the calculations:

$$\mathbf{Q}_x^\infty[F\mathrm{e}^{-\lambda x(t)}] = h_0(x)^{-1}\mathbf{Q}_x\Big[F\int_{(0,\infty)} \mathrm{e}^{-\lambda y}h_0(y)Q_{t-r}(x(r),\mathrm{d}y)\Big]$$

$$
\begin{aligned}
&= h_0(x)^{-1}\mathbf{Q}_x\Big[F h_0(x(r))\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q^\infty_{t-r}(x(r),\mathrm{d}y)\Big] \\
&= \theta_0^{-1} h_0(x)^{-1}\mathbf{Q}_x\Big[F \mathbf{Q}_{x(r)}\{x(\infty)=\infty\}\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q^\infty_{t-r}(x(r),\mathrm{d}y)\Big] \\
&= \mathbf{Q}_x\{x(\infty)=\infty\}^{-1}\mathbf{Q}_x\Big[F 1_{\{x(\infty)=\infty\}}\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q^\infty_{t-r}(x(r),\mathrm{d}y)\Big] \\
&= \mathbf{Q}^\infty_x\Big[F\int_{[0,\infty)} \mathrm{e}^{-\lambda y} Q^\infty_{t-r}(x(r),\mathrm{d}y)\Big].
\end{aligned}
$$

Then $\{x(t): t\geq 0\}$ under $\mathbf{Q}^\infty_x$ is a Markov process with transition semigroup $(Q^\infty_t)_{t\geq 0}$. □

Corollary 3.5 *Suppose that $\theta_0\in(0,\infty)$. Then for any $T>0$ and $x>0$ we have $\mathbf{Q}^\infty_x(\mathrm{d}\omega) = h_0(x)^{-1}h_0(x(T,\omega))\mathbf{Q}_x(\mathrm{d}\omega)$ on $\mathscr{F}_T$.*

Proof. This is a consequence of (3.6) with $r=T$ and $\lambda=0$. □

Theorem 3.6 *Suppose that $\theta_0\in(0,\infty)$ and Grey's condition holds. Let $\tau_0=\inf\{s\geq 0: x(s)=0\}$ be the extinction time of the CB-process. Then $\mathbf{Q}^\infty_x(\cdot)=\mathbf{Q}_x(\cdot|\tau_0=\infty)$.*

Proof. By Proposition 3.1 we can remove a null set from Ω so that $\Omega=\{x(\infty)=0\}\cup\{x(\infty)=\infty\}$. The strong Markov property implies $\mathbf{Q}_x\{x(\tau_0+t)>0 \text{ for some } t\geq 0\}=0$. Then, by remove another null set from Ω, we have $x(t)=x(t\wedge\tau_0)$ for all $t\geq 0$. It follows that $\{\tau_0<\infty\}\subset\{x(\infty)=0\}$ and so $\{x(\infty)=\infty\}\subset\{\tau_0=\infty\}$. By the arguments in the proof of Theorem 3.7 in Li (2019a) one can see that $\mathbf{Q}_x\{\tau_0<\infty\}=\mathrm{e}^{-x\theta_0}$. Then $\mathbf{Q}_x\{\tau_0<\infty\}=\mathbf{Q}_x\{x(\infty)=0\}$ by Proposition 3.1. By removing a null set from Ω again, we get $\{\tau_0<\infty\}=\{x(\infty)=0\}$, so $\{\tau_0=\infty\}=\{x(\infty)=0\}^c$. Let $\triangle$ denote the symmetric difference of events. It follows that

$$
\mathbf{Q}_x(\{x(\infty)=\infty\}\triangle\{\tau_0=\infty\})=\mathbf{Q}_x(\{x(\infty)=0\}^c\triangle\{\tau_0=\infty\})=0.
$$

Then $\mathbf{Q}^\infty_x(\cdot)=\mathbf{Q}_x(\cdot|x(\infty)=\infty)=\mathbf{Q}_x(\cdot|\tau_0=\infty)$. □

Corollary 3.7 *If $\theta_0\in(0,\infty)$ and Grey's condition holds, then for any $F\in\mathscr{F}$ we have $\mathbf{Q}^\infty_x(F) = \lim_{T\to\infty}\mathbf{Q}_x(F|\tau_0\geq T)$.*

4 Construction by a stochastic equation

Throughout this section, we assume $\theta_0\in(0,\infty)$ and so $\phi'(0+)\in[-\infty,0)$. Suppose that $(\Omega,\mathscr{G},\mathscr{G}_t,\mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Let $\{B(t): t\geq 0\}$ be a $(\mathscr{G}_t)$-Brownian motion. Let $M_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ be a $(\mathscr{G}_t)$-Poisson random measure on $(0,\infty)^3$ with intensity $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$, where $m(\mathrm{d}z)$ is defined in (2.1) and $\tilde{M}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ the compensated measure. Let $M_1(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ be a $(\mathscr{G}_t)$-Poisson random measure on $(0,\infty)^3$ with intensity $\mathrm{d}sn(\mathrm{d}z)\mathrm{d}u$, where $n(\mathrm{d}z)=h_0(z)m(\mathrm{d}z)=\theta_0^{-1}(1-\mathrm{e}^{-\theta_0 z})m(\mathrm{d}z)$. Assume those random elements are independent of each other. Let Y_0 be a non-negative $\mathscr{G}_0$-measurable random variable. We consider the stochastic integral equation

$$
\begin{aligned}
Y_t &= Y_0+\int_0^t(2cg(Y_{s-})-bY_{s-})\mathrm{d}s+\int_0^t\sqrt{2cY_{s-}}\mathrm{d}B_s+\int_0^t\int_0^1\int_0^{Y_{s-}} z\tilde{M}_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\
&\quad +\int_0^t\int_1^\infty\int_0^{Y_{s-}} zM_0(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)+\int_0^t\int_0^\infty\int_0^{g(Y_{s-})} zM_1(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u), \qquad (4.1)
\end{aligned}
$$

where $g(x) = x\mathrm{e}^{-\theta_0 x} h_0(x)^{-1} = x\theta_0(\mathrm{e}^{\theta_0 x} - 1)^{-1}$ with $g(0) = 1$ by continuity. Here we make the conventions

$$\int_a^b = \int_{(a,b]} \qquad \text{and} \qquad \int_a^\infty = \int_{(a,\infty)}$$

for any real numbers $a \le b$.

Theorem 4.1 *The SDE (4.1) has a unique non-negative strong solution and the solution is a conservative Markov process in $[0,\infty)$ with the transition semigroup $(Q_t^\infty)_{t\ge 0}$.*

Proof. Step 1. Let $E = \{1,2\}$, $U_0 = (0,1] \times (0,\infty)$ and $U_1 = (0,\infty)^2 \times E$. Let $\pi(\mathrm{d}y) = \delta_1(\mathrm{d}y) + \delta_2(\mathrm{d}y)$ for $y \in E$. Then $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) := \mathbf{1}_{(0,1]}(z) M_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson random measure on $(0,\infty) \times U_0$ with intensity $\mathrm{d}s \mathbf{1}_{(0,1]}(z) m(\mathrm{d}z)\mathrm{d}u$ and

$$N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}y) := \mathbf{1}_{(1,\infty)}(z) M_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\delta_1(\mathrm{d}y) + M_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)\delta_2(\mathrm{d}y)$$

is a Poisson random measure on $(0,\infty)\times U_1$ with intensity $\mathrm{d}s\mathbf{1}_{(1,\infty)}(z)m(\mathrm{d}z)\mathrm{d}u\delta_1(\mathrm{d}y) + \mathrm{d}s n(\mathrm{d}z)\mathrm{d}u\delta_2(\mathrm{d}y)$. Clearly, N_0 and N_1 are independent. Let $b(x) = 2cg(x) - bx$ and $\sigma(x) = \sqrt{2cx}$ for $x \in [0,\infty)$. Let $g_0(x,z,u) = z\mathbf{1}_{(0,x]}(u)$ for $(x,z,u) \in (0,\infty) \times (0,1] \times (0,\infty)$ and

$$g_1(x,z,u,y) = z\mathbf{1}_{(1,\infty)}(z)\mathbf{1}_{(0,x]}(u)\mathbf{1}_{\{y=1\}} + z\mathbf{1}_{(0,g(x)]}(u)\mathbf{1}_{\{y=2\}}$$

for $(x,z,u,y) \in (0,\infty)^3 \times E$. Note that

$$g'(x) = \theta_0 \mathrm{e}^{\theta_0 x}(1 - \mathrm{e}^{-\theta_0 x} - \theta_0 x)(\mathrm{e}^{\theta_0 x} - 1)^{-2} \le 0, \qquad x \ge 0.$$

Moreover, we have $g''(x) \ge 0$ and $|g(x) - g(y)| \le \frac{1}{2}\theta_0|x-y|$ for $x, y \ge 0$. By Proposition 1 in Palau and Pardo (2018), there is a unique $[0,\infty]$-valued strong solution $\{Y_t : t \ge 0\}$ to

$$\begin{aligned} Y_t &= Y_0 + \int_0^t b(Y_{s-})\mathrm{d}s + \int_0^t \sigma(Y_{s-})\mathrm{d}B_s + \int_0^t \int_{U_0} g_0(Y_{s-}, z, u)\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &\quad + \int_0^t \int_{U_1} g_1(Y_{s-}, z, u, y) N_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}y), \end{aligned} \tag{4.2}$$

which is a reformulation of (4.1). Then the SDE (4.1) has a unique $[0,\infty]$-valued strong solution $\{Y_t : t \ge 0\}$. More precisely, we have, for each $n \ge 1$ and $t \ge 0$,

$$\begin{aligned} Y_{t\wedge\tau_n} &= Y_0 + \int_0^{t\wedge\tau_n} (2cg(Y_{s-}) - bY_{s-})\mathrm{d}s + \int_0^{t\wedge\tau_n} \sqrt{2cY_{s-}}\mathrm{d}B_s \\ &\quad + \int_0^{t\wedge\tau_n} \int_0^1 \int_0^{Y_{s-}} z\tilde{M}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^{t\wedge\tau_n} \int_1^\infty \int_0^{Y_{s-}} zM_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \\ &\quad + \int_0^{t\wedge\tau_n} \int_0^\infty \int_0^{g(Y_{s-})} zM_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \end{aligned} \tag{4.3}$$

where $\tau_n = \inf\{t \ge 0 : Y_t \ge n\}$, and $Y_t = \infty$ for $t \ge \tau_\infty := \lim_{n\to\infty} \tau_n$.

Step 2. For $f \in C_0^2([0,\infty))$, we can use (4.3) and Itô's formula to see

$$f(Y_t) = f(Y_0) + \int_0^t A_\infty f(Y_s)\mathrm{d}s + \text{local mart.}, \tag{4.4}$$

where

$$A_\infty f(x) = Af(x) + 2cg(x)f'(x) + g(x)\int_0^\infty [f(x+z) - f(x)]n(\mathrm{d}z). \tag{4.5}$$

For $T \geq 0$ let $C^{1,2}([0,T] \times [0,\infty))$ denote the set of functions $(t,x) \mapsto G(t,x)$ which are C^1 with respect to t and C_0^2 with respect to x. For any $G \in C^{1,2}([0,T] \times [0,\infty))$, we can deduce from (4.4) by standard arguments that

$$G(t,Y_t) = G(0,Y_0) + \int_0^t [G_t'(s,Y_s) + A_\infty G(s,Y_s)]\mathrm{d}s + \text{local mart.}, \tag{4.6}$$

where G_t' is the derivative of $G(t,x)$ with respect to t and A_∞ acts on the function $x \mapsto G(s,x)$.

Step 3. For any $T \geq 0$ and $\lambda > 0$ we apply (4.6) to the function

$$G(t,x) = (\mathrm{e}^{-v_{T-t}(\lambda)x} - \mathrm{e}^{-v_{T-t}(\lambda+\theta_0)x})(1 - \mathrm{e}^{-\theta_0 x})^{-1}.$$

Here we make the convention that $\mathrm{e}^{-\lambda\cdot\infty} = 0$. Set $\mathrm{e}_\lambda(x) = \mathrm{e}^{-\lambda x}$ for $x \geq 0$ and $\lambda > 0$. It is elementary to check that $A_\infty G(s,x) = -G_t'(s,x)$, so $t \mapsto G(t,Y_t)$ is a martingale. Then $\mathbf{P}[G(T,Y_T)|\mathscr{G}_t] = G(t,Y_t)$, which implies $\mathbf{P}[\mathrm{e}^{-\lambda Y_T}|\mathscr{G}_t] = Q_{T-t}^\infty \mathrm{e}_\lambda(Y_t)$ for $\lambda > 0$ and $0 \leq t \leq T$. That shows $\{Y_t : t \geq 0\}$ is a Markov process with transition semigroup $(Q_t^\infty)_{t\geq 0}$. It follows that $\mathbf{P}\{Y_t < \infty \text{ for all } t \geq 0\} = 1$. Then we get the result. □

Remark 4.2 *In view of (4.1) and (4.5), we can also think of $\{Y_t : t \geq 0\}$ as a generalized CBI-process with dependent immigration determined by the function $g(x) = x\theta_0(\mathrm{e}^{\theta_0 x} - 1)^{-1}$. In the critical case $\phi'(0) = 0$, we have $\theta_0 = 0$ and the process reduces to a special CBI-process, which is a CB-process conditioned on large extinction times; see, e.g., Li (2011, Theorem 3.25). More general stochastic equations of the type of (4.2) have been studied in Dawson and Li (2012) and Fu and Li (2010). A construction of CB-processes with dependent immigration was given in Li (2019b) by solving a stochastic equation driven by Poisson random measures on the space of positive paths.*

5 Applications

Let $D[0,\infty)$ denote the space of $[0,\infty)$-càdlàg paths on $[0,\infty)$. For any $w \in D[0,\infty)$ let $\alpha(w) = \inf\{s \geq 0 : w(s) > 0\}$ and $\beta(w) = \sup\{s \geq 0 : w(s) > 0\}$. Let $W = \{w \in D[0,\infty) : w(t) > 0 \text{ for } t \in (\alpha(w),\beta(w))$ and $w(t) = 0$ for $t \in [\alpha(w),\beta(w))^c\}$. On the space W, we define the σ-algebras $\mathscr{W} = \sigma(w(s) : s \in [0,\infty))$ and $\mathscr{W}_t = \sigma(w(s) : s \in [0,t])$ for $t \geq 0$. Let $[0] \in W$ be the path which is constantly zero. For notation convenience, we extend the definition of each $w \in W$ by setting $w(s) = 0$ for $s < 0$.

Theorem 5.1 *There is a unique σ-finite measure $\mathbf{N}_0$ on $(W,\mathscr{W})$ satisfying $\mathbf{N}_0(\{[0]\}) = 0$ and*

$$\begin{aligned} &\mathbf{N}_0(\alpha(w) \leq t_1, w(t_1) \in \mathrm{d}x_1, w(t_2) \in \mathrm{d}x_2, \ldots, w(t_n) \in \mathrm{d}x_n, t_n < \beta(w)) \\ &\quad = l_{t_1}(\mathrm{d}x_1)Q_{t_2-t_1}^\circ(x_1,\mathrm{d}x_2)\cdots Q_{t_{n-1}-t_{n-2}}^\circ(x_{n-2},\mathrm{d}x_{n-1})Q_{t_n-t_{n-1}}^\circ(x_{n-1},\mathrm{d}x_n), \end{aligned} \tag{5.1}$$

for $\{t_1 < t_2 < \cdots < t_n\} \subset (0,\infty)$ and $\{x_1,x_2,\cdots,x_n\} \subset (0,\infty)$. Moreover for $t \geq r > 0$, $\lambda \geq 0$ and a positive $\mathscr{W}_r$-measurable function F on W,

$$\mathbf{N}_0[F(w)(1 - \mathrm{e}^{-\lambda w(t)})] = \mathbf{N}_0[F(w)(1 - \mathrm{e}^{-v_{t-r}(\lambda)w(r)})] + F([0])[h_r v_{t-r}(\lambda) - h_t\lambda]. \tag{5.2}$$

Proof. To construct a measure $\mathbf{N}_0$ satisfying (5.1) we may consider separately the cases $\phi'(\infty) = \infty$ and $\phi'(\infty) < \infty$. (1) In the case $\phi'(\infty) = \infty$, we have $h_t = 0$ for every $t > 0$ and $(l_t)_{t>0}$ is an entrance law for $(Q_t^\circ)_{t\geq 0}$. For $\theta_0 = 0$, we have $\phi'(0) \in [0,\infty)$ and a construction of $\mathbf{N}_0$ was given in Theorem 6.1 of Li

(2019a). The arguments there can be modified for $\theta_0 \in (0,\infty)$ as follows. For any $T > 0$ we first construct a probability measure $\mathbf{P}_0^T$ on $(W, \mathscr{W})$ so that $\{w(s) : s \in [0,T]\}$ under this measure is a Markov process with initial state $w(0) = 0$ and transition semigroup $(Q_t^\infty)_{t\geq 0}$ and $\{w(s) : s \in [T,\infty)\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. By Theorem 3.2 we have $\mathbf{P}_0^T(w(s) = 0) = Q_s^\infty(0,\{0\}) = 0$ for every $0 < s \leq T$. Let $\mathbf{N}_0^T(\mathrm{d}w) = h_0(w(T))^{-1}1_{\{w(T)>0\}}\mathbf{P}_0^T(\mathrm{d}w)$. The increasing limit $\mathbf{N}_0 := \lim_{T\to 0}\mathbf{N}_0^T$ exists and defines a σ-finite measure on $(W, \mathscr{W})$ satisfying (5.1). In this case, the measure $\mathbf{N}_0$ is supported by $\{w \in W : \alpha(w) = 0, w(0) = 0\}$. (2) In the case of $\delta := \phi'(\infty) < \infty$, we have $h_t = \mathrm{e}^{-\delta t}$ for every $t \geq 0$ and $(l_t)_{t>0}$ is an entrance rule for $(Q_t^\circ)_{t\geq 0}$. Indeed, as in the proof of Theorem 3.15 of Li (2019a) one can see

$$v_t(\lambda) = \mathrm{e}^{-\delta t}\lambda + \int_0^t \mathrm{e}^{-\delta s}\mathrm{d}s\int_{(0,\infty)}(1 - \mathrm{e}^{-uv_{t-s}(\lambda)})m(\mathrm{d}u).$$

Then $\mathbf{N}_0$ can be constructed as in the proof of Theorem 2.2 of Li (2019b). In this case, the measure is supported by $\{w \in W : \alpha(w) > 0, w(\alpha(w)) > 0\}$. The uniqueness of the measure satisfying (5.1) is a consequence of the measure extension theorem. The relation (5.2) can be proved similarly as Theorem 2.3 in Li (2019b). $\square$

The family $(l_t)_{t>0}$ of σ-finite measures on $(0,\infty)$ in the canonical representation (2.5) is an *entrance rule* for the semigroup $(Q_t^\circ)_{t\geq 0}$ in the following sense: $l_rQ_{t-r}^\circ \leq l_t$ for all $t > r > 0$ and $l_rQ_{t-r}^\circ \to l_t$ as $r \to t$. The σ-finite measure $\mathbf{N}_0$ determined by (5.1) is referred to as the *canonical Kuznetsov measure* associated with $(l_t)_{t>0}$. The existence of $\mathbf{N}_0$ follows also from general results in the theory of Markov processes; see Kuznetsov (1974) and Getoor and Glover (1987). The reader can refer to Li (2019a, 2019b) for constructions of the measure under the first moment assumption; see also Duquesne and Labbé (2014).

Theorem 5.2 *Let $x \geq 0$ and let $N(\mathrm{d}w)$ be a Poisson random measure on W with intensity $x\mathbf{N}_0(\mathrm{d}w)$. Define $X_0 = x$ and*

$$X_t = xh_t + \int_W w(t)N(\mathrm{d}w), \qquad t > 0.$$

Define $\mathscr{N}_t = \sigma\{N(A) : A \in \mathscr{W}_t\}$. Then $\{(X_t, \mathscr{N}_t) : t \geq 0\}$ is a CB-process with transition semigroup $(Q_t)_{t\geq 0}$.

Proof. It is obvious that $\{X_t\}$ is adapted to the filtration $\{\mathscr{N}_t\}$. Note that the random variable X_t has distribution $Q_t(x,\cdot)$. Indeed, for $t > 0$ and $\lambda \geq 0$,

$$\begin{aligned}\mathbf{P}[\mathrm{e}^{-\lambda X_t}] &= \exp\{-\lambda xh_t\}\cdot\mathbf{P}\bigg[\exp\bigg\{-\lambda\int_W w(t)N(\mathrm{d}w)\bigg\}\bigg] \\ &= \exp\{-\lambda xh_t\}\cdot\exp\Big\{-x\mathbf{N}_0(1 - \mathrm{e}^{-\lambda w(t)})\Big\} \\ &= \exp\{-\lambda xh_t\}\cdot\exp\bigg\{-\int_{(0,\infty)}(1 - \mathrm{e}^{-\lambda z})xl_t(\mathrm{d}z)\bigg\} = \mathrm{e}^{-xv_t(\lambda)}.\end{aligned}$$

Let $t > r > 0$ and let h be a bounded positive function on $\mathscr{W}_r$ with $h([0]) = 0$. For any $\lambda \geq 0$, by (2.5) and (5.2),

$$\begin{aligned}&\mathbf{P}\bigg[\exp\bigg\{-\int_W h(w)N(\mathrm{d}w) - \lambda X_t\bigg\}\bigg] \\ &\quad = \mathbf{P}\bigg[\exp\bigg\{-xh_t\lambda - \int_W[h(w) + \lambda w(t)]N(\mathrm{d}w)\bigg\}\bigg] \\ &\quad = \exp\Big\{-xh_t\lambda - x\mathbf{N}_0(1 - \mathrm{e}^{-h(w)-\lambda w(t)})\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \exp\Big\{ - xh_t\lambda - x\mathbf{N}_0(1 - \mathrm{e}^{-h(w)})\Big\} \\
&\qquad \cdot \exp\Big\{ - x\mathbf{N}_0[\mathrm{e}^{-h(w)}(1 - \mathrm{e}^{-\lambda w(t)})]\Big\} \\
&= \exp\Big\{ - xh_t\lambda - x\mathbf{N}_0(1 - \mathrm{e}^{-h(w)}) - x[h_r v_{t-r}(\lambda) - h_t\lambda]\Big\} \\
&\qquad \cdot \exp\Big\{ - x\mathbf{N}_0[\mathrm{e}^{-h(w)}(1 - \mathrm{e}^{-v_{t-r}(\lambda)w(r)})]\Big\} \\
&= \exp\Big\{ - xh_r v_{t-r}(\lambda) - x\mathbf{N}_0(1 - \mathrm{e}^{-h(w)-v_{t-r}(\lambda)w(r)})\Big\} \\
&= \mathbf{P}\Big[\exp\Big\{ - \int_W h(w)N(\mathrm{d}w) - v_{t-r}(\lambda)X_r\Big\}\Big].
\end{aligned}
$$

Then $\{(X_t, \mathscr{N}_t) : t \geq 0\}$ is a CB-process with transition semigroup $(Q_t)_{t\geq 0}$. □

Remark 5.3 *The reconstruction of the CB-process given in Theorem 5.2 shows that the population can be divided into two parts. One part develops deterministically following the path $t \mapsto xh_t$, the other part consists of sample paths picked from W by the Poisson random measure $N(\mathrm{d}w)$. For similar constructions, see Duquesne and Labbé (2014), Li (2011, 2019a, 2019b) and the references therein.*

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