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Asymptotic results for exponential functionals of Lévy processes

Zenghu Li and Wei Xu

School of Mathematical Sciences, Beijing Normal University,
Beijing 100875, People's Republic of China
E-mails: lizh@bnu.edu.cn, xuwei@mail.bnu.edu.cn

Abstract. The asymptotic behavior of expectations of some exponential functionals of a Lévy process is studied. The key point is the observation that the asymptotics only depend on the sample paths with slowly decreasing local infimum. We give not only the convergence rate but also the expression of the limiting coefficient. The latter is given in terms of some transformations of the Lévy process based on its renewal function. As an application, we give an exact evaluation of the decay rate of the survival probability of a continuous-state branching process in random environment with stable branching mechanism.

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1 Introduction

The study of exponential functionals of random walks and Lévy processes has drawn the attention of many researchers in recent years. Those functionals play important roles in the study of probabilistic models in random environments among their other applications. Let $\xi = \{\xi(t) : t \geq 0\}$ be a one-dimensional Lévy process. Given a constant $\alpha > 0$, we define the exponential functional:

$$A_t^\alpha(\xi) = \int_0^t e^{-\alpha\xi(s)} ds, \qquad 0 \leq t \leq \infty. \tag{1.1}$$

Here we allow $A_\infty^\alpha(\xi) = \infty$. When ξ is a Brownian motion with drift, a characterization of the distribution of $A_t^\alpha(\xi)$ was obtained by Yor (1992, Proposition 2). For an exponentially distributed random variable T, positive and negative moments of $A_T^\alpha(\xi)$ were calculated by Carmona et al. (1994, 1997). By a result of Bertoin and Yor (2005), we have $A_\infty^\alpha(\xi) < \infty$ a.s. if and only if $\lim_{t\to\infty} \xi(t) = \infty$ a.s. In this case, Bertoin and Yor (2005) gave some characterizations for the distribution of $A_\infty^\alpha(\xi)$ and Pardo et al. (2012) established a Wiener-Hopf type factorization for this functional. Let $z \mapsto F(z)$ be a positive decreasing function on $(0, \infty)$ that vanishes as $z \to \infty$ at a certain rate. In the case of $A_\infty^\alpha(\xi) = \infty$, a natural problem is to evaluate the decay rate as $t \to \infty$ of the expectation:

$$\mathbf{P}[F(A_t^\alpha(\xi))] = \mathbf{P}\Big[F\Big(\int_0^t e^{-\alpha\xi(s)} ds\Big)\Big]. \tag{1.2}$$

In the special case where $F(z) = a(a+z)^{-1}$ and $\{\xi(t) : t \geq 0\}$ is a Brownian motion with drift, the problem was studied by Kawazu and Tanaka (1993) in their work on the tail behavior of a diffusion process in random environment. Other specific forms of the function F arising from applications were discussed in Carmona et al. (1994, 1997).

Let $\{Z_\alpha(t) : t \geq 0\}$ be a spectrally positive $(\alpha+1)$-stable process with $0 < \alpha \leq 1$ and $\{L(t) : t \geq 0\}$ a Lévy process with no jump less than -1. Let $c \geq 0$ be another constant. Given the initial value $x \geq 0$, we consider the following stochastic integral equation:

$$X(t) = x + \int_0^t \sqrt[1+\alpha]{(1+\alpha)cX(s-)}dZ_\alpha(s) + \int_0^t X(s-)dL(s). \tag{1.3}$$

By Theorem 6.2 in Fu and Li (2010), there exists a unique positive strong solution $\{X(t) : t \geq 0\}$ to (1.3). The solution is called a *continuous-state branching process in random environment (CBRE-process)* with *stable branching mechanism*. Here the random environment is modeled by the Lévy process $\{L(t) : t \geq 0\}$. The reader may refer to He et al. (2016) and Palau and Pardo (2015b) for discussions of more general CBRE-processes and to Bansaye et al. (2013) for a special case. We shall see that there is another Lévy process $\{\xi(t) : t \geq 0\}$ determined by the environment so that the *survival probability* of the CBRE-process up to time $t \geq 0$ is given by

$$\mathbf{P}(X(t) > 0) = \mathbf{P}\Big[1 - \exp\big\{ - x(c\alpha)^{-1/\alpha} A_t^\alpha(\xi)^{-1/\alpha}\big\}\Big]. \tag{1.4}$$

Clearly, the right-hand side of (1.4) is a special case of (1.2). Based on the above expression, the asymptotic behavior of the survival probability as $t \to \infty$ were studied by Böinghoff and Hutzenthaler (2012) for the case where $\alpha = 1$ and the environment process is a Brownian motion with drift. Their results were extended recently to the case $1 < \alpha \leq 1$ by Palau and Pardo (2015a). The main strategy of Böinghoff and Hutzenthaler (2012) and Palau and Pardo (2015a) is the formula of Yor (1992) for the distribution of the exponential functional of the Brownian motion with drift; see also Matsumoto and Yor (2003). Bansaye et al. (2013) studied the problem in the case where the environment is given by a Lévy process with bounded variations and showed some interesting applications of the results to a cell infection model. The key step in their proof is to study the expectation (1.2) for $F(x) = (1+x)^{-1/\beta}[1 + (1+x)^{-\gamma}h(x)]$, where $0 < \beta \leq 1$ and $\gamma \geq 1$ are constants and h is a bounded Lipschitz function. The asymptotics of survival probabilities for classical Galton-Watson branching processes in random environment (GWRE-processes) were studied earlier by Afanasy'ev et al. (2005), Dyakonova et al. (2004), Geiger and Kersting (2002), Geiger et al. (2003), Guivarc'h and Liu (2001), Kozlov (1976), Liu (1996) and Vatutin et al. (2013) among others. Roughly speaking, for critical branching the survival probability decays at a polynomial rate and for subcritical branching it decays at an exponential rate with three different polynomial modifying factors, which classify the processes into weakly subcritical, intermediately subcritical and strongly subcritical ones. Those results play important roles in the study of various conditional limit theorems of the CBRE- and GWRE-processes. Unfortunately, in most of the results established before, the limiting coefficients were not explicitly identified except in very special cases; see, e.g., Böinghoff and Hutzenthaler (2012).

The purpose of this paper is to study the asymptotic behavior of the expectation in (1.2) for a general function F and a general Lévy process ξ. Under natural assumptions, we prove some accurate results for asymptotics of the expectation as $t \to \infty$. We shall see that five regimes arise for the convergence rate. We also apply the results to study the survival probability of the CBRE-process defined by (1.3). The feature of this work is that we give not only the convergence rate but also the expression of the limiting coefficient in all regimes. The key of the results is the observation that the asymptotics of (1.2) only depends on the charge of probability on sample paths of the Lévy process whose local infimum decreases slowly. This makes it possible for us to determine the limiting coefficients by extensions of the conditional limit theorems of Hirano (2001). The constants are expressed in terms of some transformations based on the renewal functions associated with the ladder processes of ξ and its dual process. The main results of the paper are presented in Section 2. The proofs for recurrent and transient Lévy processes are given in Sections 3 and 4, respectively. The applications of the results to CBRE-processes are discussed in Section 5.

We use some standard notations from the theory of Markov processes; see, for example, Sharpe (1988). In particular, we use the symbol, say $\mathbf{P}$, for a probability measure to denote the corresponding expectation. For an event A and a random variable X, we write $\mathbf{P}(X|A)$ for the conditional expectation of X given A and write $\mathbf{P}(X;A)$ for the expectation $\mathbf{P}(X1_A)$, where 1_A is the indicator of A.

After putting the first version of this paper to Arxiv, we noticed the interesting work of Palau et al. (2016), where some results for the asymptotics of exponential functionals of Lévy processes were obtained and the results were also applied to study the survival probability of the CBRE-process. But, as in most of the references mentioned above, they did not identify the limiting coefficients.

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2 Asymptotics of exponential functionals

In this section, we present our main results on the asymptotic behavior of the expectation (1.2). To this end, we first introduce some basic notations and establish some preliminary results which are helpful for the understanding of the main theorem.

Let $\varPsi(\lambda)$ be the *characteristic exponent* of an infinitely divisible probability measure on the one-dimensional Euclidean space $\mathbb{R}$ given by

$$\varPsi(\lambda) = ia\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x)\nu(dx), \qquad \lambda \in \mathbb{R}, \tag{2.1}$$

where $a \in \mathbb{R}$ and $\sigma \geq 0$ are constants and $\nu(dx)$ is a σ-finite measure on $\mathbb{R}$ supported by $\mathbb{R} \setminus \{0\}$ and satisfying

$$\int_{\mathbb{R}} (|x| \wedge |x|^2)\nu(dx) < \infty. \tag{2.2}$$

We also need to consider the *Laplace exponent* $\varPhi$ defined by

$$\varPhi(\lambda) = -a\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{\mathbb{R}} (e^{\lambda x} - 1 - \lambda x)\nu(dx), \qquad \lambda \in \mathbb{R}. \tag{2.3}$$

Of course, we may have $\Phi(\lambda) = \infty$ for some $\lambda \in \mathbb{R}$. Let $\mathscr{D}(\Phi) = \{\lambda \in \mathbb{R} : \Phi(\lambda) < \infty\}$. Then $\mathscr{D}(\Phi)$ is necessarily an interval containing the origin. Let $\mathscr{D}_+(\Phi) = \mathscr{D}(\Phi) \cap [0, \infty)$. Let $\mathscr{D}^\circ(\Phi)$ and $\mathscr{D}^\circ_+(\Phi)$ denote the interior sets of $\mathscr{D}(\Phi)$ and $\mathscr{D}_+(\Phi)$, respectively.

Let Ω be the set of all càdlàg paths from $[0, \infty)$ to $\mathbb{R}$. For $t \geq 0$ and $\omega \in \Omega$ let $\xi_t(\omega) = \omega(t)$ denote the *coordinate process*. Let $\mathscr{F} = \sigma(\{\xi_s : s \geq 0\})$ and $\mathscr{F}_t = \sigma(\{\xi_s : 0 \leq s \leq t\})$ be the natural σ-algebras. For each $x \in \mathbb{R}$ there is a probability measure $\mathbf{P}_x$ on $(\Omega, \mathscr{F})$ so that $\{(\xi_t, \mathscr{F}_t) : t \geq 0\}$ under this measure is a process with independent and stationary increments and

$$\mathbf{P}_x[\exp\{i\lambda\xi_t\}] = \exp\{i\lambda x - t\Psi(\lambda)\}, \qquad t \geq 0, \lambda \in \mathbb{R}. \tag{2.4}$$

Then $\xi = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \mathbf{P}_x)$ is the *canonical realization* of the Lévy process with characteristic exponent Ψ. Let $\hat{\mathbf{P}}_x$ denote the law of $\{-\xi_t : t \geq 0\}$ under $\mathbf{P}_{-x}$. Then $\hat{\xi} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \hat{\mathbf{P}}_x)$ is the *dual process* of ξ, which is also a Lévy process. For simplicity, write $\mathbf{P} = \mathbf{P}_0$ and $\hat{\mathbf{P}} = \hat{\mathbf{P}}_0$. It is well-known that under the integrability condition (2.2) we have

$$\mathbf{P}_0(\xi_t) = -\hat{\mathbf{P}}_0(\xi_t) = -\Phi'(0)t = -at, \qquad t \geq 0.$$

To avoid triviality, in the sequel of the paper we make the following:

Assumption 1 *The function Φ is strictly convex, so that the process ξ is not a deterministic motion.*

For notational convenience, we may write $\xi(t)$ instead of ξ_t for $t \geq 0$. It is known that:

(2.a) If $a < 0$, then $\mathbf{P}(\lim_{t\to\infty} \xi(t) = -\infty) = 1$.

(2.b) If $a > 0$, then $\mathbf{P}(\lim_{t\to\infty} \xi(t) = \infty) = 1$.

(2.c) If $a = 0$, then $\mathbf{P}(\limsup_{t\to\infty} \xi(t) = -\liminf_{t\to\infty} \xi(t) = \infty) = 1$.

In cases (2.a) and (2.b), the process ξ is *transient*, and in case (2.c) it is *recurrent*. See, e.g., Kyprianou (2014, pp.204–205) and Sato (1999, p.237, p.248 and p.255).

For any $\theta \in \mathscr{D}(\Phi)$, it is easy to see that $t \mapsto e^{-\theta x + \theta\xi(t) - \Phi(\theta)t}$ is a $\mathbf{P}_x$-martingale. Then, using the Esscher transform, we can define the probability measure $\mathbf{P}^{(\theta)}_x$ on $(\Omega, \mathscr{F})$ by

$$\mathbf{P}^{(\theta)}_x(A) = e^{-\theta x} \int_A e^{\theta\xi(t) - \Phi(\theta)t} d\mathbf{P}_x, \qquad A \in \mathscr{F}_t, t \geq 0. \tag{2.5}$$

It is known that $\xi^{(\theta)} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \mathbf{P}^{(\theta)}_x)$ is a Lévy process with Laplace exponent $\Phi_\theta(\lambda) := \Phi(\lambda+\theta) - \Phi(\theta)$; see, e.g., Theorem 3.9 in Kyprianou (2014, p.83). Let $\hat{\xi}^{(\theta)} = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi(t), \hat{\mathbf{P}}^{(\theta)}_x)$ be its dual process. For simplicity, write $\mathbf{P}^{(\theta)} = \mathbf{P}^{(\theta)}_0$ and $\hat{\mathbf{P}}^{(\theta)} = \hat{\mathbf{P}}^{(\theta)}_0$.

We define the *supremum process* $S := (S(t) : t \geq 0)$ by $S(t) = \sup_{s\in[0,t]} \xi(s)$. Let $S - \xi := \{S(t) - \xi(t) : t \geq 0\}$ be the *reflected process*, which is a Markov process with Feller transition semigroup; see, e.g., Proposition 1 in Bertoin (1996, p.156). Let $L = \{L(t) : t \geq 0\}$ be the local time at zero of $S - \xi$ in the sense of Bertoin (1996, p.109). The *inverse local time process* $L^{-1} = \{L^{-1}(t) : t \geq 0\}$ is defined by

$$L^{-1}(t) = \begin{cases} \inf\{s > 0 : L(s) > t\}, & t < L(\infty); \\ \infty, & \text{otherwise.} \end{cases}$$

The *ladder height process* $H = \{H(t) : t \geq 0\}$ of ξ is defined by

$$H(t) = \begin{cases} \xi(L^{-1}(t)), & t < L(\infty); \\ \infty, & \text{otherwise.} \end{cases}$$

Note that $H(t) = S(L^{-1}(t))$ when $L^{-1}(t) < \infty$. By Lemma 2 in Bertoin (1996, p.157), the two-dimensional process (L^{-1}, H) is a Lévy process (possibly killed at an exponential rate). This is known as the *ladder process* of ξ and is characterized by

$$\mathbf{P}[\exp\{-\lambda_1 L^{-1}(t) - \lambda_2 H(t)\}] = \exp\{-t\kappa(\lambda_1, \lambda_2)\}, \qquad \lambda_1, \lambda_2 \geq 0,$$

where the *bivariate exponent* $\kappa(\lambda_1, \lambda_2)$ is given by

$$\kappa(\lambda_1, \lambda_2) \; = \; k \exp\Big\{ \int_0^\infty \frac{dt}{t} \int_{[0,\infty)} (e^{-t} - e^{-\lambda_1 t - \lambda_2 x}) \mathbf{P}(\xi(t) \in dx) \Big\}.$$

In particular, both L^{-1} and H are (possibly killed) subordinators. The constant $k > 0$ here is determined by the normalization of the local time; see Corollary 10 in Bertoin (1996, pp.165–166). In this work, we choose the normalization suitably so that $k = 1$; see also Hirano (2001, p.293). The *renewal function* V associated with the ladder height process H is defined by

$$V(x) = \int_0^\infty \mathbf{P}(H(t) \leq x) dt = \mathbf{P}\Big(\int_{[0,\infty)} 1_{\{S(t) \leq x\}} dL(t) \Big), \qquad x \geq 0; \tag{2.6}$$

see, e.g., Bertoin (1996, p.171) and Chaumont and Doney (2005, p.950). Let $\hat{V}$ and $\hat{\kappa}(\lambda_1, \lambda_2)$ be defined similarly as the above from the dual process $\hat{\xi}$.

For $x \in \mathbb{R}$ define the stopping time $\tau_x^- = \inf\{t \geq 0 : \xi(t) < x\}$. Let $\mathrm{b}\mathscr{B}[0,\infty)$ denote the set of bounded Borel functions on $[0,\infty)$. We can define a transition semigroup $(q_t)_{t \geq 0}$ on $[0,\infty)$ by

$$q_t f(x) = \mathbf{P}_x\Big[f(\xi(t)) \mathbf{1}_{\{\tau_0^- > t\}} \Big], \qquad x \geq 0, f \in \mathrm{b}\mathscr{B}[0,\infty).$$

If $\mathbf{P}(\limsup_{t\to\infty} \xi(t) = \infty) = 1$, then $\hat{V}$ is an invariant function for $(q_t)_{t\geq 0}$ by Lemma 1 of Chaumont and Doney (2005, p.951). It follows that $t \mapsto \hat{V}(\xi(t)) \mathbf{1}_{\{\tau_0^- > t\}}$ is a $\mathbf{P}_x$-martingale for each $x \geq 0$; see also Bertoin (1996, p.184) and Hirano (2001, p.293). In this case, we have the conservative Markov process $\varXi = (\varOmega, \mathscr{F}, \mathscr{F}_t, \xi(t), \mathbf{Q}_x)$, where $\mathbf{Q}_x$ is the probability measure on $(\varOmega, \mathscr{F})$ determined by

$$\mathbf{Q}_x(A) = \hat{V}(x)^{-1} \int_A \hat{V}(\xi(t)) \mathbf{1}_{\{\tau_0^- > t\}} d\mathbf{P}_x, \qquad A \in \mathscr{F}_t, t \geq 0.$$

Similarly, in the case $\hat{\mathbf{P}}(\limsup_{t\to\infty} \xi(t) = \infty) = 1$, we have the conservative Markov process $\hat{\varXi} = (\varOmega, \mathscr{F}, \mathscr{F}_t, \xi(t), \hat{\mathbf{Q}}_x)$, where $\hat{\mathbf{Q}}_x$ is the probability measure on $(\varOmega, \mathscr{F})$ determined by

$$\hat{\mathbf{Q}}_x(A) = V(x)^{-1} \int_A V(\xi(t)) \mathbf{1}_{\{\tau_0^- > t\}} d\hat{\mathbf{P}}_x, \qquad A \in \mathscr{F}_t, t \geq 0.$$

Let $V^{(\theta)}$ and $\hat{V}^{(\theta)}$ be the renewal functions associated with the ladder height processes of the Lévy processes $\xi^{(\theta)}$ and $\hat{\xi}^{(\theta)}$, respectively. Let $\varXi^{(\theta)} = (\varOmega, \mathscr{F}, \mathscr{F}_t, \xi(t), \mathbf{Q}_x^{(\theta)})$ and $\hat{\varXi}^{(\theta)} = (\varOmega, \mathscr{F}, \mathscr{F}_t, \xi(t), \hat{\mathbf{Q}}_x^{(\theta)})$ be the resulting Markov processes, respectively.

For $\alpha > 0$ let $\{A_t^\alpha(\xi) : 0 \leq t \leq \infty\}$ be defined by (1.1).

Proposition 2.1 (Carmona et al., 1997) *For any $\alpha > 0$ the following statements are equivalent: (1) $\mathbf{P}[\xi(1)] > 0$; (2) $\mathbf{P}(A_\infty^\alpha(\xi) < \infty) > 0$; (3) $\mathbf{P}(A_\infty^\alpha(\xi) < \infty) = 1$.*

Lemma 2.2 *For any $\alpha > 0$, $t > 0$ and $\beta \in \mathscr{D}_+^\circ(\Phi)$ we have*

$$\mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] \le t^{-\beta/\alpha}\mathbf{P}[e^{\beta S(t)}] \le 4t^{-\beta/\alpha}e^{\beta(a+|a|)t}\mathbf{P}[e^{\beta\xi(t)}].$$

Proof. By applying Doob's inequality to the submartingale $t \mapsto e^{\beta[\xi(t)+at]/2}$ we have

$$\begin{aligned}\mathbf{P}[e^{\beta S(t)}] &\le e^{\beta|a|t}\mathbf{P}\Big[\sup_{0\le s\le t} e^{\beta[\xi(s)+as]}\Big] \le e^{\beta|a|t}\mathbf{P}\Big[\sup_{0\le s\le t}\Big(e^{\beta[\xi(s)+as]/2}\Big)^2\Big] \\ &\le 4e^{\beta|a|t}\mathbf{P}[e^{\beta[\xi(t)+at]}] = 4e^{\beta(a+|a|)t}\mathbf{P}[e^{\beta\xi(t)}],\end{aligned}$$

where the right-hand side is finite since $\beta \in \mathscr{D}_+^\circ(\Phi)$. It is simple to see that

$$\mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] \le \mathbf{P}\Big[\Big(\int_0^t e^{-\alpha S(t)}ds\Big)^{-\beta/\alpha}\Big] = t^{-\beta/\alpha}\mathbf{P}[e^{\beta S(t)}]$$

Then we obtain the result. □

Lemma 2.3 *For any $\alpha > 0$, $t \ge 2$ and $\beta \in \mathscr{D}_+^\circ(\Phi)$ we have*

$$\mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] \le \mathbf{P}\Big[\exp\Big\{\min_{k\le[t]-1}\beta\xi(k)\Big\}\Big]\mathbf{P}[e^{\beta S(1)}],$$

where $[t]$ denotes the integer part of t.

Proof. For any $j = 0, 1, \cdots, [t]-1$, define

$$Z(j) = \log\Big(\int_j^{j+1} e^{-\alpha(\xi(s)-\xi(j))}ds\Big).$$

Then $\{Z(j) : j = 0, 1, \cdots, [t]-1\}$ is a sequence of i.i.d. random variables. It is easy to see that

$$\begin{aligned}\mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] &\le \mathbf{P}\Big[\Big(\int_0^{[t]} e^{-\alpha\xi(s)}ds\Big)^{-\beta/\alpha}\Big] \\ &= \mathbf{P}\Big[\Big(\sum_{j=0}^{[t]-1} e^{-\alpha\xi(j)+Z(j)}\Big)^{-\beta/\alpha}\Big] \le \mathbf{P}[e^{\beta\xi(\kappa)-\beta Z(\kappa)/\alpha}],\end{aligned}$$

where $\kappa = \min\{j \le [t]-1 : \xi(j) = \min_{k\le[t]-1}\xi(k)\}$. Since $Z(\kappa)$ is independent of $\xi(\kappa)$ and κ, we have

$$\begin{aligned}\mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] &\le \sum_{k=0}^{[t]-1}\mathbf{P}(\kappa = k)\mathbf{P}[e^{\beta\xi(\kappa)-\beta Z(\kappa)/\alpha}|\kappa = k] \\ &= \sum_{k=0}^{[t]-1}\mathbf{P}(\kappa = k)\mathbf{P}[e^{\beta\xi(\kappa)}|\kappa = k]\mathbf{P}[e^{-\beta Z(\kappa)/\alpha}] \\ &= \sum_{k=0}^{[t]-1}\mathbf{P}(\kappa = k)\mathbf{P}[e^{\beta\xi(\kappa)}|\kappa = k]\mathbf{P}[e^{-\beta Z(0)/\alpha}]\end{aligned}$$

$$\le \sum_{k=0}^{[t]-1} \mathbf{P}(\kappa = k)\mathbf{P}[e^{\beta\xi(\kappa)}|\kappa = k]\mathbf{P}[e^{\beta S(1)}]$$
$$= \mathbf{P}\Big[\exp\Big\{\min_{k\le[t]-1} \beta\xi(k)\Big\}\Big]\mathbf{P}[e^{\beta S(1)}].$$

Then the desired result follows. □

Lemma 2.4 *If* $\mathbf{P}(\limsup_{t\to\infty} \xi(t) = \infty) = 1$*, then for any* $\alpha > 0$ *and* $x > 0$ *we have* $\mathbf{Q}_x[A^\alpha_\infty(\xi)] < \infty$.

Proof. By the definition of $\mathbf{Q}_x$ and Fubini's theorem, we have

$$\begin{aligned}\mathbf{Q}_x[A^\alpha_\infty(\xi)] &= \hat{V}(x)^{-1} \int_0^\infty \mathbf{P}_x[e^{-\alpha\xi(r)}\hat{V}(\xi(r)); \tau_0^- > r]dr \\ &= \hat{V}(x)^{-1}\mathbf{P}_x\Big[\int_0^{\tau_0^-} e^{-\alpha\xi(r)}\hat{V}(\xi(r))dr\Big] \\ &\le \hat{V}(x)^{-1} \int_0^\infty dV(y) \int_0^x e^{-\alpha(y+x-z)}\hat{V}(y+x-z)d\hat{V}(z),\end{aligned}$$

where the last step follows by Theorem 20 in Bertoin (1996, p.176). By Corollary 5.3 in Kyprianou (2014, p.118) we have $\hat{V}(y) \sim y/\hat{\mathbf{P}}[H(1)]$ as $y \to \infty$. Then we can take $\gamma \in (0, \alpha)$ and $C \ge 0$ so that $e^{-(\alpha-\gamma)y}\hat{V}(y) \le C$ for $y \ge 0$. It follows that

$$\mathbf{Q}_x[A^\alpha_\infty(\xi)] \le C\hat{V}(x)^{-1} \int_0^\infty e^{-\gamma y}dV(y) \int_0^x d\hat{V}(z),$$

The right-hand side is clearly finite. □

In the sequel of the paper, we also make the following:

Assumption 2 *Fix the constants* $\alpha > 0$, $\beta \in \mathscr{D}^\circ_+(\varPhi)$ *and a decreasing and strictly positive function* F *on* $(0,\infty)$*. Assume there exist* $C_0 > 0$ *and* $\beta_0 \in \mathscr{D}_+(\varPhi)$ *so that* $F(z) \le C_0 z^{-\beta_0/\alpha}$ *for* $0 < z \le 1$.

To simplify the presentation of the results, let us state the following conditions:

Condition 2.5 *For each* $\delta > 0$ *there is a constant* $K_\delta > 0$ *so that* $|F(z) - F(y)| \le K_\delta|z - y|$ *for* $z, y \ge \delta$.

Condition 2.6 *There is a constant* $K > 0$ *so that* $F(z) \le Kz^{-\beta/\alpha}$ *for* $z \ge 1$.

Condition 2.7 *There is a constant* $K > 0$ *so that* $F(z) \sim Kz^{-\beta/\alpha}$ *as* $z \to \infty$.

Condition 2.8 *The characteristic exponent of* ξ *satisfies* $\mathbf{Re}\,\varPsi(\lambda) > 0$ *for all* $\lambda \ne 0$.

Let $\varrho \in \mathscr{D}^\circ(\varPhi)$ be the solution of $\varPhi'(\varrho) = 0$. Let $\varXi^{(\varrho)}$ and $\hat{\varXi}^{(\varrho)}$ be defined as the above with $\theta = \varrho$. Let $W = \varOmega \times \varOmega$. For $t \ge 0$ and $w = (\omega_1, \omega_2) \in W$ let $\xi_1(t, w) = \omega_1(t)$ and $\xi_2(t, w) = \omega_2(t)$. Let $\mathscr{G} = \sigma(\{(\xi_1(s), \xi_2(s)) : t \ge 0\})$ and $\mathscr{G}_t = \sigma(\{(\xi_1(s), \xi_2(s)) : 0 \le s \le t\})$. Let $\mathbf{Q}^{(\varrho)}_{(x,y)} = \mathbf{Q}^{(\varrho)}_x \times \mathbf{Q}^{(\varrho)}_y$ for $x \ge 0$ and $y \ge 0$. Then $(W, \mathscr{G}, \mathscr{G}_t, (\xi_1(t), \xi_2(t)), \mathbf{Q}^{(\varrho)}_{(x,y)})$ is the *independent coupling* of $\varXi^{(\varrho)}$ and $\hat{\varXi}^{(\varrho)}$.

The main theorem of this paper is the following:

Theorem 2.9 (1) *If $0 \in \mathscr{D}^\circ(\Phi)$ and $\Phi'(0) > 0$, we have the finite and nonzero limit*

$$\lim_{t\to\infty} \mathbf{P}[F(A_t^\alpha(\xi))] = \mathbf{P}[F(A_\infty^\alpha(\xi))].$$

(2) *Suppose that Conditions 2.5, 2.6 and 2.8 are satisfied. If $0 \in \mathscr{D}^\circ(\Phi)$ and $\Phi'(0) = 0$, then we have the finite and nonzero limit*

$$\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi))] = \sqrt{\frac{2}{\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]D_2(\alpha, F),$$

where

$$D_2(\alpha, F) = \lim_{x\to\infty} \hat{V}(x)\mathbf{Q}_x[F(e^{-\alpha x}A_\infty^\alpha(\xi))]. \tag{2.7}$$

(3) *Suppose that Conditions 2.5, 2.6 and 2.8 are satisfied and that $0 \in \mathscr{D}^\circ(\Phi)$ and $\Phi'(0) < 0 < \Phi'(\beta)$. Let $\varrho \in (0, \beta)$ be the solution of $\Phi'(\varrho) = 0$. Then we have the finite and nonzero limit*

$$\lim_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A_t^\alpha(\xi))] = \frac{c(\varrho)}{\sqrt{2\pi\,\Phi''(\varrho)}}D_3(\alpha, F),$$

where

$$c(\varrho) = \exp\left\{\int_0^\infty (e^{-t} - 1)t^{-1}e^{-t\Phi(\varrho)}\mathbf{P}(\xi(t) = 0)dt\right\}, \tag{2.8}$$

$$D_3(\alpha, F) = \lim_{x\to\infty} e^{\varrho x}\hat{V}^{(\varrho)}(x)\int_0^\infty e^{-\varrho y}V^{(\varrho)}(y)G(x, y)dy \tag{2.9}$$

and

$$G(x, y) = \mathbf{Q}_{(x,y)}^{(\varrho)}\{F(e^{-\alpha x}[A_\infty^\alpha(\xi_1) + A_\infty^\alpha(\xi_2)])\}. \tag{2.10}$$

(4) *Suppose that Conditions 2.7 and 2.8 are satisfied and $\Phi'(\beta) = 0$. Then we have the finite and nonzero limit*

$$\lim_{t\to\infty} t^{1/2}e^{-t\Phi(\beta)}\mathbf{P}[F(A_t^\alpha(\xi))] = K\sqrt{\frac{2}{\pi\,\Phi''(\beta)}}\mathbf{P}^{(\beta)}[H(1)]D_4(\alpha, \beta),$$

where

$$D_4(\alpha, \beta) = \lim_{x\to\infty} V^{(\beta)}(x)\mathbf{Q}_x^{(\beta)}[e^{-\beta x}A_\infty^\alpha(-\xi)^{-\beta/\alpha}]. \tag{2.11}$$

(5) *Suppose that Condition 2.7 is satisfied and $\Phi'(\beta) < 0$. Then we have the finite and nonzero limit*

$$\lim_{t\to\infty} e^{-t\Phi(\beta)}\mathbf{P}[F(A_t^\alpha(\xi))] = K\mathbf{P}^{(\beta)}[A_\infty^\alpha(-\xi)^{-\beta/\alpha}].$$

Remark 2.10 *It is known that Condition 2.8 holds if and only if $\sigma > 0$ or $\nu(\mathbb{R}\backslash\{0, \pm r, \pm 2r, \cdots\}) > 0$ for every $r > 0$; see, e.g., Hirano (2001, p.294). Instead of this condition, if we assume for some $r > 0$ the characteristic exponent has the representation:*

$$\Psi(\lambda) = \sum_{k\in\mathbb{Z}}(1 - e^{ikr\lambda}), \qquad \lambda \in \mathbb{R}, \tag{2.12}$$

the results of regimes (2) and (4) in the above theorem still hold. The proofs are modifications of those given in Sections 3 and 4. However, it seems some extra work is needed to establish the result in regime (3) for the characteristic exponent (2.12).

By using the above theorem we can give some simple derivations of the results of Böinghoff and Hutzenthaler (2012), Carmona et al. (1994, 1997) and Kawazu and Tanaka (1993) on the asymptotics of exponential functionals; see Xu (2016).

3 Recurrent Lévy processes

In this section, we give the proof of Theorem 2.9 in regime (2). Throughout the section, we assume $0 \in \mathscr{D}^\circ(\Phi)$ and $\mathbf{P}[\xi(1)] = \Phi'(0) = 0$. It follows that $0 < \mathbf{P}[\xi(1)^2] = \Phi''(0) < \infty$ by Assumption 1. Then both ξ and $\hat{\xi}$ are recurrent; see Theorem 36.7 in Sato (1999, p.248).

Proposition 3.1 (1) *Let $I(t) = \inf_{0\leq s\leq t}\xi(s)$. Then for $x > 0$ we have, as $t \to \infty$,*

$$\mathbf{P}(\tau_{-x}^- > t) = \mathbf{P}(I(t) > -x) \sim \sqrt{\frac{2}{\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]\hat{V}(x)t^{-1/2},$$

where $0 < \hat{\mathbf{P}}[H(1)] < \infty$.

(2) *Suppose that Condition 2.8 holds. Then for any $x > 0$ and $\alpha > 0$ we have, as $t \to \infty$,*

$$\mathbf{P}(e^{-\alpha\xi(t)}; \tau_{-x}^- > t) \sim \frac{ce^{\alpha x}}{\sqrt{2\pi\,\Phi''(0)}}\hat{V}(x)t^{-3/2}\int_0^\infty e^{-\alpha z}V(z)dz,$$

where

$$c = \exp\Big\{\int_0^\infty (e^{-t} - 1)t^{-1}\mathbf{P}(\xi(t) = 0)dt\Big\}.$$

Proof. Since $\mathbf{P}(I(t) > -x) = \hat{\mathbf{P}}(\sup_{0\leq s\leq t}\xi(s) < x)$, the first result follows from Lemma 11 of Hirano (2001). From Corollary 4 (ii) in Doney (2007, p.31) it follows that $0 < \hat{\mathbf{P}}[H(1)] < \infty$. By the spatial homogeneity of the Lévy process we have

$$\mathbf{P}(e^{-\alpha\xi(t)}; \tau_{-x}^- > t) = \mathbf{P}(e^{-\alpha[x+\xi(t)]}; \tau_{-x}^- > t)e^{\alpha x} = \mathbf{P}_x(e^{-\alpha\xi(t)}; \tau_0^- > t)e^{\alpha x}.$$

Then the second result follows by Lemma A-(a) in Hirano (2001). □

The discrete version of the following proposition for random walks was established in Theorem 1 of Bertoin and Doney (1994). The result extends slightly Theorem 1 in Hirano (2001), who considered the case where $\xi(t) \to -\infty$ as $t \to \infty$. Based on Proposition 3.1(1), its proof goes similarly as that given in Hirano (2001). For $s \geq 0$ let $D[0, s]$ denote the space of càdlàg real functions on $[0, s]$ equipped with the Skorokhod topology.

Proposition 3.2 *Let f be a bounded Borel function on $D[0,s]$. Then for any $s \geq 0$ and $x > 0$ we have*

$$\mathbf{P}[f(\xi(r) : r \in [0,s]) | \tau_{-x}^- > t] \to \mathbf{Q}_x[f(\xi(r) - x : r \in [0,s])], \qquad t \to \infty.$$

Proof. Without loss of generality, we may assume $0 \leq f \leq 1$. By the Markov property of ξ, for any $t > s \geq 0$,

$$\mathbf{P}\big[f(\xi(r) : r \in [0,s]); \tau_{-x}^- > t\big] = \mathbf{P}\big[f(\xi(r) : r \in [0,s])\mathbf{P}_{\xi(s)}(\tau_{-x}^- > t-s); \tau_{-x}^- > s\big].$$

From Proposition 3.1(1), we have, for any $y \geq -x$, as $t \to \infty$,

$$\frac{\mathbf{P}_y(\tau_{-x}^- > t)}{\mathbf{P}(\tau_{-x}^- > t-s)} = \frac{\mathbf{P}(\tau_{-(y+x)}^- > t)}{\mathbf{P}(\tau_{-x}^- > t-s)} \sim \frac{\hat{V}(y+x)}{\hat{V}(x)}. \tag{3.1}$$

Using Fatou's lemma we have

$$\begin{aligned}
&\liminf_{t\to\infty} \mathbf{P}\Big[f(\xi(r) : r \in [0,s]) \big| \tau_{-x}^- > t\Big] \\
&\quad = \liminf_{t\to\infty} \frac{\mathbf{P}\big[f(\xi(r) : r \in [0,s]); \tau_{-x}^- > t\big]}{\mathbf{P}(\tau_{-x}^- > t)} \\
&\quad = \liminf_{t\to\infty} \mathbf{P}\Big[f(\xi(r) : r \in [0,s]) \frac{\mathbf{P}_{\xi(s)}(\tau_{-x}^- > t-s)}{\mathbf{P}(\tau_{-x}^- > t)}; \tau_{-x}^- > s\Big] \\
&\quad \geq \mathbf{P}\Big[f(\xi(r) : r \in [0,s]) \liminf_{t\to\infty} \frac{\mathbf{P}_{\xi(s)}(\tau_{-x}^- > t-s)}{\mathbf{P}(\tau_{-x}^- > t)}; \tau_{-x}^- > s\Big] \\
&\quad = \mathbf{P}\Big[f(\xi(r) : r \in [0,s]) \frac{\hat{V}(\xi(s)+x)}{\hat{V}(x)}; \tau_{-x}^- > s\Big] \\
&\quad = \mathbf{P}_x\Big[f(\xi(r) - x : r \in [0,s]) \frac{\hat{V}(\xi(s))}{\hat{V}(x)}; \tau_0^- > s\Big] \\
&\quad = \mathbf{Q}_x\Big[f(\xi(r) - x : r \in [0,s])\Big].
\end{aligned}$$

By applying the above calculations to $1-f$, we see

$$\begin{aligned}
\limsup_{t\to\infty} \mathbf{P}\Big[f(\xi(r) : r \in [0,s]) \big| \tau_{-x}^- > t\Big] &= 1 - \liminf_{t\to\infty} \mathbf{P}\Big[1 - f(\xi(r) : r \in [0,s]) \big| \tau_{-x}^- > t\Big] \\
&\leq 1 - \mathbf{Q}_x\Big[1 - f(\xi(r) - x : r \in [0,s])\Big] \\
&= \mathbf{Q}_x\Big[f(\xi(r) - x : r \in [0,s])\Big].
\end{aligned}$$

Then we have the result. □

The key of the proof of Theorem 2.9(2) is the observation that the asymptotics of the expectation (1.2) only depends on the sample paths of the Lévy process with slowly decreasing local infimum so that we can use the above two propositions to determine the limiting coefficient. To show clearly the main ideas of the proof, we write the main steps into a series of lemmas.

Lemma 3.3 *There exists a constant $C \geq 0$ such that*

$$\limsup_{t\to\infty} t^{1/2} \mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] \leq C.$$

Proof. By Theorem A in Kozlov (1976), there exists a constant $C = C_\beta \geq 0$ such that, as $t \to \infty$,

$$\mathbf{P}\Big[\exp\Big\{\min_{k \leq [t]-1} \beta\xi(k)\Big\}\Big] \sim C([t]-1)^{-1/2} \sim Ct^{-1/2}.$$

Then the desired result follows from Lemma 2.3. $\square$

Lemma 3.4 *Suppose that Condition 2.8 holds. Then for any $x > 0$ there is a constant $C = C_x \geq 0$ so that*

$$\mathbf{P}[e^{-\alpha\xi(r)}; \tau_{-x}^- > t] \leq C(t-r)^{-1/2}r^{-3/2}, \qquad t > r > 0.$$

Proof. By the Markov property and the spatial homogeneity of the Lévy process,

$$\begin{aligned}\mathbf{P}[e^{-\alpha\xi(r)}; \tau_{-x}^- > t] &= \mathbf{P}[e^{-\alpha\xi(r)}\mathbf{P}_{\xi(r)}(\tau_{-x}^- > t-r); \tau_{-x}^- > r] \\ &= \mathbf{P}[e^{-\alpha\xi(r)}\mathbf{P}(\tau_{-x-\xi(r)}^- > t-r); \tau_{-x}^- > r].\end{aligned}$$

By Corollary 5.3 in Kyprianou (2014, p.118) we have $y^{-1}\hat{V}(y) \to \{\hat{\mathbf{P}}[H(1)]\}^{-1}$ as $y \to \infty$. Since $y \mapsto \hat{V}(y)$ is increasing, we infer that $y \mapsto e^{-\lambda y}\hat{V}(y)$ is a bounded function on $[0,\infty)$ for any $\lambda \in (0,\alpha)$. For $x > x_0 > 0$ we can use Proposition 3.1 to see

$$\begin{aligned}\mathbf{P}[e^{-\alpha\xi(r)}; \tau_{-x}^- > t] &= \mathbf{P}[e^{-\alpha\xi(r)}\mathbf{P}(\tau_{-x-\xi(r)}^- > t-r); \tau_{-x}^- > r] \\ &= \mathbf{P}[e^{-\alpha\xi(r)}\mathbf{P}(\tau_{-x-\xi(r)}^- > t-r); \xi(r) \leq -x_0, \tau_{-x}^- > r] \\ &\quad + \mathbf{P}[e^{-\alpha\xi(r)}\mathbf{P}(\tau_{-x-\xi(r)}^- > t-r); \xi(r) > -x_0, \tau_{-x}^- > r] \\ &\leq \mathbf{P}[e^{-\alpha\xi(r)}\mathbf{P}(\tau_{x_0-x}^- > t-r); \tau_{-x}^- > r] \\ &\quad + C(x)(t-r)^{-1/2}\mathbf{P}[e^{-\alpha\xi(r)}\hat{V}(x+\xi(r)); \xi(r) > -x_0, \tau_{-x}^- > r] \\ &\leq C(x)\hat{V}(x-x_0)(t-r)^{-1/2}\mathbf{P}[e^{-\alpha\xi(r)}; \tau_{-x}^- > r] \\ &\quad + C(x)(t-r)^{-1/2}\mathbf{P}[e^{-(\alpha-\lambda)\xi(r)}e^{\lambda x}; \xi(r) > -x_0, \tau_{-x}^- > r] \\ &\leq C(x)\hat{V}(x-x_0)(t-r)^{-1/2}e^{\alpha x}\hat{V}(x)r^{-3/2} \\ &\quad + C(x)(t-r)^{-1/2}e^{\lambda x}\mathbf{P}[e^{-(\alpha-\lambda)\xi(r)}; \tau_{-x}^- > r] \\ &\leq C(x)[\hat{V}(x-x_0)+1]e^{\alpha x}\hat{V}(x)(t-r)^{-1/2}r^{-3/2}.\end{aligned}$$

That gives the desired result. $\square$

Lemma 3.5 *Suppose that F is a bounded function satisfying Condition 2.6. Then there is a constant $C \geq 0$ so that, for any $x > 0$,*

$$\limsup_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi)); \tau_{-x}^- \leq t] \leq Ce^{-\beta x}[1+\hat{V}(x)].$$

Proof. Since F is bounded and satisfies Condition 2.6, there is a constant $C_1 \geq 0$ such that $F(z) \leq C_1(1 \wedge z^{-\beta/\alpha})$ for all $z > 0$. By Lemma 3.3 we can find an integer $t_0 \geq 3$ and some constant $C \geq 0$ such that

$$t^{1/2}\mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] \leq C, \qquad t \geq t_0.$$

By Proposition 3.1(1), for any $\epsilon > 0$ we have, as $t \to \infty$,

$$\begin{aligned}
\mathbf{P}(\tau_{-x}^- > t)^{-1}\mathbf{P}(t < \tau_{-x}^- \le t+\epsilon) &= \frac{\mathbf{P}(\tau_{-x}^- > t) - \mathbf{P}(\tau_{-x}^- > t+\epsilon)}{\mathbf{P}(\tau_{-x}^- > t)} \\
&= \frac{t^{-1/2} - (t+\epsilon)^{-1/2} + o(t^{-1/2})}{t^{-1/2} + o(t^{-1/2})} \\
&= \frac{t^{-1} - (t+\epsilon)^{-1} + o(t^{-1})}{t^{-1/2}[t^{-1/2} + (t+\epsilon)^{-1/2}] + o(t^{-1})} \\
&= \frac{\epsilon t^{-1}(t+\epsilon)^{-1} + o(t^{-1})}{t^{-1/2}[t^{-1/2} + (t+\epsilon)^{-1/2}] + o(t^{-1})} \\
&= \frac{\epsilon (t+\epsilon)^{-1} + o(1)}{t^{1/2}[t^{-1/2} + (t+\epsilon)^{-1/2}] + o(1)} \sim \epsilon t^{-1}/2.
\end{aligned}$$

It follows that

$$\mathbf{P}(t < \tau_{-x}^- \le t+\epsilon) \sim \frac{1}{\sqrt{2\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]\hat{V}(x)\epsilon t^{-3/2}. \tag{3.2}$$

By the strong Markov property, up to some adjustments of the value of $C \ge 0$, we have

$$\begin{aligned}
\mathbf{P}[F(A_t^\alpha(\xi)); \tau_{-x}^- \le t] &\le C_1 \mathbf{P}\Big[1 \wedge \Big(\int_0^t e^{-\alpha\xi(r)}dr\Big)^{-\beta/\alpha}; \tau_{-x}^- \le t\Big] \\
&\le C_1 \sum_{i=1}^{[t]-t_0} \mathbf{P}\Big[\Big(\int_{\tau_{-x}^-}^{t+\tau_{-x}^- - i} e^{-\alpha\xi(r)}dr\Big)^{-\beta/\alpha}; i-1 < \tau_{-x}^- \le i\Big] \\
&\qquad + C_1\mathbf{P}([t]-t_0 < \tau_{-x}^- \le t) \\
&\le C \sum_{i=1}^{[t]-t_0} \mathbf{P}\Big[e^{\beta\xi(\tau_{-x}^-)}\mathbf{P}_{\xi(\tau_{-x}^-)}[(A_{t-i}^\alpha(\xi)^{-\beta/\alpha}]; i-1 < \tau_{-x}^- \le i\Big] \\
&\qquad + C\hat{V}(x)(t_0+1)([t]-t_0)^{-3/2} \\
&\le Ce^{-\beta x} \sum_{i=1}^{[t]-t_0} (t-i)^{-1/2}\mathbf{P}(i-1 < \tau_{-x}^- \le i) \\
&\qquad + C\hat{V}(x)(t_0+1)([t]-t_0)^{-3/2} \\
&\le Ce^{-\beta x}(t-1)^{-1/2} + C\hat{V}(x)e^{-\beta x}\sum_{i=2}^{[t]-t_0}(t-i)^{-1/2}(i-1)^{-3/2} \\
&\qquad + C\hat{V}(x)(t_0+1)([t]-t_0)^{-3/2}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{i=2}^{[t]-t_0}(t-i)^{-1/2}(i-1)^{-3/2} &\le (t-2)^{-1/2} + \int_3^t (t-s)^{-1/2}(s-2)^{-3/2}ds \\
&\le (t-2)^{-1/2} + (t/2)^{-1/2}\int_3^{t/2}(s-2)^{-3/2}ds \\
&\quad + (t/2-2)^{-3/2}\int_{t/2}^t (t-s)^{-1/2}ds \\
&\le (t-2)^{-1/2} + (t/2)^{-1/2}\int_3^\infty (s-2)^{-3/2}ds \\
&\quad + (t/2-2)^{-3/2}\int_0^t (t-s)^{-1/2}ds
\end{aligned}$$

$$\leq (t-2)^{-1/2} + 2(t/2)^{-1/2} + 2(t/2-2)^{-3/2}t^{1/2}.$$

By combining the above estimates we get desired result. $\square$

Lemma 3.6 *Suppose that Condition 2.8 holds and F is a globally Lipschitz function on $(0,\infty)$. Then for any $x>0$ we have*

$$\lim_{s\to\infty}\limsup_{t\to\infty} t^{1/2}\mathbf{P}[|F(A_s^\alpha(\xi)) - F(A_t^\alpha(\xi))|; \tau_{-x}^- > t] = 0.$$

Proof. Since F is decreasing and globally Lipschitz, there exists a constant $C>0$ such that

$$0 \leq F(A_s^\alpha(\xi)) - F(A_t^\alpha(\xi)) \leq C\int_s^t e^{-\alpha\xi(r)}dr, \qquad t\geq s\geq 0.$$

Then it suffices to prove

$$\lim_{s\to\infty}\limsup_{t\to\infty} t^{1/2}\int_s^t \mathbf{P}[e^{-\alpha\xi(r)}; \tau_{-x}^- > t]dr = 0. \tag{3.3}$$

By Lemma 3.4, for any $s>1$ we have

$$\begin{aligned}
&\limsup_{t\to\infty} t^{1/2}\int_s^t \mathbf{P}[e^{-\alpha\xi(r)}; \tau_{-x}^- > t]dr \\
&\quad\leq \limsup_{t\to\infty} Ct^{1/2}\int_s^t (t-r)^{-1/2}r^{-3/2}dr \\
&\quad\leq \limsup_{t\to\infty} C\Big(\int_s^{t/2} r^{-3/2}dr + t^{-1}\int_{t/2}^t (t-r)^{-1/2}dr\Big) \\
&\quad\leq \limsup_{t\to\infty} C\Big(\int_s^{\infty} r^{-3/2}dr + t^{-1}\int_0^t (t-r)^{-1/2}dr\Big) \leq Cs^{-1/2}.
\end{aligned}$$

That proves (3.3). $\square$

Lemma 3.7 *Suppose that Condition 2.8 holds and F is a globally Lipschitz function on $(0,\infty)$. Then for any $x>0$ we have the finite and nonzero limit*

$$\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi))]); \tau_{-x}^- > t] = \sqrt{\frac{2}{\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]D_2(x,\alpha,F), \tag{3.4}$$

where

$$D_2(x,\alpha,F) = \hat{V}(x)\mathbf{Q}_x[F(e^{-\alpha x}A_\infty^\alpha(\xi))]. \tag{3.5}$$

Furthermore, the function $x\mapsto D_2(x,\alpha,F)$ on $(0,\infty)$ is increasing, strictly positive and bounded.

Proof. From Lemma 2.4 it follows that $\mathbf{Q}_x(A_\infty^\alpha(\xi)<\infty)=1$. Since $F(z)>0$ for each $z>0$, we have $D_2(x,\alpha,F)>0$. By the spatial homogeneity of the Lévy process and Proposition 3.2, we have

$$\begin{aligned}
\lim_{t\to\infty}\mathbf{P}[F(A_s^\alpha(\xi))|\tau_{-x}^- > t] &= \lim_{t\to\infty}\mathbf{P}_x[F(A_s^\alpha(\xi - x))|\tau_0^- > t] \\
&= \lim_{t\to\infty}\mathbf{P}_x[F(e^{-\alpha x}A_s^\alpha(\xi))|\tau_0^- > t] = \mathbf{Q}_x[F(e^{-\alpha x}A_s^\alpha(\xi))].
\end{aligned}$$

Combining this with Proposition 3.1(1) and Lemma 3.6,

$$\begin{aligned}\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi));\tau_{-x}^- > t] &= \lim_{s\to\infty}\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_s^\alpha(\xi));\tau_{-x}^- > t] \\ &= \lim_{s\to\infty}\lim_{t\to\infty} t^{1/2}\mathbf{P}(\tau_{-x}^- > t)\mathbf{P}[F(A_s^\alpha(\xi))|\tau_{-x}^- > t] \\ &= \lim_{s\to\infty}\sqrt{\frac{2}{\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]\hat{V}(x)\mathbf{Q}_x[F(e^{-\alpha x}A_s^\alpha(\xi))] \\ &= \sqrt{\frac{2}{\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]\hat{V}(x)\mathbf{Q}_x[F(e^{-\alpha x}A_\infty^\alpha(\xi))].\end{aligned}$$

Since the left-hand side of (3.4) is increasing in $x > 0$, so is $D_2(x,\alpha,F)$. By Lemma 3.3 this function is bounded on $(0,\infty)$. □

Proof of Theorem 2.9(2). We first consider the special case where F is globally Lipschitz and hence bounded on $(0,\infty)$. By Lemma 3.5 we have

$$\lim_{x\to\infty}\limsup_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi));\tau_{-x}^- \le t] = 0.$$

Then we can use Lemma 3.7 to see

$$\begin{aligned}\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi))] &= \lim_{x\to\infty}\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi));\tau_{-x}^- > t] \\ &\quad + \lim_{x\to\infty}\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi));\tau_{-x}^- \le t] \\ &= \lim_{x\to\infty}\sqrt{\frac{2}{\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]D_2(x,\alpha,F).\end{aligned}$$

By Lemma 3.7, the limit $D_2(\alpha,F) := \lim_{x\to\infty} D_2(x,\alpha,F)$ is finite, strictly positive and given by (2.7). Then the result follows in the special case. In the general case, for $n \ge 1$ let $F_n(y) = F(1/n)1_{\{y\le 1/n\}} + F(y)1_{\{y>1/n\}}$ and $G_n(y) = F(y) - F_n(y)$. Then each F_n is globally Lipschitz, so $D_2(x,\alpha,F_n)$ and $D_2(\alpha,F_n)$ can be defined. Clearly, the limit $D_2(x,\alpha,F) := \lim_{n\to\infty} D_2(x,\alpha,F_n)$ exists and is given by (3.5). It is also easy to see that $D_2(x,\alpha,F)$ is bounded, strictly positive and increasing on $(0,\infty)$. Then the limit $D_2(\alpha,F) := \lim_{x\to\infty} D_2(x,\alpha,F)$ exists and it is finite and strictly positive. Observe that

$$\mathbf{P}[F(A_t^\alpha(\xi))] = \mathbf{P}[G_n(A_t^\alpha(\xi))] + \mathbf{P}[F_n(A_t^\alpha(\xi))].$$

By multiplying this by $t^{1/2}$ and taking the limit we have

$$\lim_{t\to\infty} t^{1/2}\mathbf{P}[F(A_t^\alpha(\xi))] = \lim_{t\to\infty} t^{1/2}\mathbf{P}[G_n(A_t^\alpha(\xi))] + \sqrt{\frac{2}{\pi\,\Phi''(0)}}\hat{\mathbf{P}}[H(1)]D_2(\alpha,F_n). \tag{3.6}$$

Under our Assumption 2, we can find constants $C \ge 0$ and $\beta_1 \in \mathscr{D}_+^\circ(\Phi) \cap (\beta_0,\infty)$ so that

$$\begin{aligned}\lim_{t\to\infty} t^{1/2}\mathbf{P}[G_n(A_t^\alpha(\xi))] &\le C\limsup_{t\to\infty} t^{1/2}\mathbf{P}[A_t^\alpha(\xi)^{-\beta_0/\alpha}; A_t^\alpha(\xi) \le 1/n] \\ &\le Cn^{-(\beta_1-\beta_0)/\alpha}\limsup_{t\to\infty} t^{1/2}\mathbf{P}[A_t^\alpha(\xi)^{-\beta_1/\alpha}] \\ &\le Cn^{-(\beta_1-\beta_0)/\alpha}.\end{aligned}$$

Then for $k \ge n \ge 1$ we have

$$\begin{aligned}
0 \le\ & D_2(x,\alpha,F_k) - D_2(x,\alpha,F_n) \\
=\ & \sqrt{\frac{\pi \Phi''(0)}{2}}\hat{\mathbf{P}}[H(1)]^{-1} \lim_{t\to\infty} t^{1/2}\mathbf{P}[F_k(A_t^\alpha(\xi)) - F_n(A_t^\alpha(\xi)), \tau_{-x}^- > t] \\
\le\ & \sqrt{\frac{\pi \Phi''(0)}{2}}\hat{\mathbf{P}}[H(1)]^{-1} \lim_{t\to\infty} t^{1/2}\mathbf{P}[G_n(A_t^\alpha(\xi)), \tau_{-x}^- > t] \\
\le\ & \sqrt{\frac{\pi \Phi''(0)}{2}}\hat{\mathbf{P}}[H(1)]^{-1} C n^{-(\beta_1-\beta_0)/\alpha}.
\end{aligned}$$

By letting $k \to \infty$ in the above we see

$$0 \le D_2(x,\alpha,F) - D_2(x,\alpha,F_n) \le \sqrt{\frac{\pi \Phi''(0)}{2}}\hat{\mathbf{P}}[H(1)]^{-1} C n^{-(\beta_1-\beta_0)/\alpha},$$

and hence

$$0 \le D_2(\alpha,F) - D_2(\alpha,F_n) \le \sqrt{\frac{\pi \Phi''(0)}{2}}\hat{\mathbf{P}}[H(1)]^{-1} C n^{-(\beta_1-\beta_0)/\alpha}.$$

Then we can let $n \to \infty$ in (3.6) to get the result. □

We remark that for the characteristic exponent given by (2.12) a result similar to Proposition 3.1(2) was established in Lemma A-(b) of Hirano (2001). One may check that all the arguments given above carry over to that case under obvious modifications.

4 Transient Lévy processes

In this section, we give the proof of Theorem 2.9 when $a = \Phi'(0) \neq 0$. In this case, both ξ and $\hat{\xi}$ are transient by Theorem 36.7 in Sato (1999, p.248). In fact, the result of regime (1) is a simple consequence of Proposition 2.1. The proof for regime (3) is based on an extension of Theorem 2 of Hirano (2001), where the exponential functional was approximated by functionals of two independent processes obtained by transformations. The proofs for regimes (4) and (5) are also based on transformations of the underlying Lévy process.

Proof of Theorem 2.9(1). Under the condition, we have $\mathbf{P}[\xi(1)] > 0$. It follows that $\lim_{t\to\infty}\xi(t) = \infty$ and hence $\mathbf{P}(A_\infty^\alpha(\xi) < \infty) = 1$ by Proposition 2.1. Then the result is immediate for any bounded function F. For an unbounded function F, by Assumption 2 we have

$$\mathbf{P}[F(A_t^\alpha(\xi))] \le C_0 \mathbf{P}[A_t^\alpha(\xi)^{-\beta_0/\alpha}] + \mathbf{P}[F(A_t^\alpha(\xi))1_{\{A_t^\alpha(\xi)\ge 1\}}].$$

The right-hand side is finite by Lemma 2.2. By monotone convergence, we get $\mathbf{P}[F(A_t^\alpha(\xi))] \to \mathbf{P}[F(A_\infty^\alpha(\xi))]$ decreasingly as $t \to \infty$. Clearly, the limit is finite and strictly positive. □

Proposition 4.1 *Suppose that Condition 2.8 holds and there exists $\varrho \in \mathscr{D}_+^\circ(\Phi)$ satisfying $\Phi'(\varrho) = 0$. Then for any $x > 0$ and $\theta \in \mathscr{D}(\Phi) \cap (-\varrho, \infty)$ we have, as $t \to \infty$,*

$$\mathbf{P}(\tau_{-x}^- > t) \sim M e^{\varrho x}\hat{V}^{(\varrho)}(x) t^{-3/2} e^{\Phi(\varrho)t} \int_0^\infty e^{-\varrho z} V^{(\varrho)}(z) dz$$

and

$$\mathbf{P}(e^{-\theta\xi(t)};\tau_{-x}^{-}>t)\sim M\hat{V}^{(\varrho)}(x)e^{(\theta+\varrho)x}t^{-3/2}e^{\Phi(\varrho)t}\int_0^\infty e^{-(\theta+\varrho)z}V^{(\varrho)}(z)dz,$$

where

$$M=\frac{1}{\sqrt{2\pi\Phi''(\varrho)}}\exp\Big\{\int_0^\infty(e^{-t}-1)t^{-1}e^{-t\Phi(\varrho)}\mathbf{P}(\xi(t)=0)dt\Big\}.$$

Proof. We only need to show the second result since the first one is its special case with $\theta=0$. By the definition of $\mathbf{P}^{(\varrho)}$ we have

$$e^{-\Phi(\varrho)t}\mathbf{P}(e^{-\theta\xi(t)};\tau_{-x}^{-}>t)=\mathbf{P}^{(\varrho)}(e^{-(\theta+\varrho)\xi(t)};\tau_{-x}^{-}>t).$$

Since $\mathbf{P}^{(\varrho)}[\xi(1)]=\Phi'(\varrho)=0$ and $\mathbf{P}^{(\varrho)}[\xi(1)^2]=\Phi''(\varrho)<\infty$, the desired result follows by Proposition 3.1(2). □

Recall that $D[0,s]$ denotes the space of càdlàg real functions on $[0,s]$ equipped with the Skorokhod topology as specified in Ethier and Kurtz (1986, p.118). Let $C_0(\mathbb{R}_+)$ and $C_0(\mathbb{R}_+^2)$ denote respectively the spaces of continuous function on $\mathbb{R}_+$ and $\mathbb{R}_+^2$ vanishing at infinity. The following proposition is a simple extension of Theorem 2-(a) in Hirano (2001).

Proposition 4.2 *Suppose that Condition 2.8 holds and there exists $\varrho\in\mathscr{D}_+^\circ(\Phi)$ satisfying $\Phi'(\varrho)=0$. Let $H\in C_0(\mathbb{R}_+^2)$ and let f,g be continuous functions on $D[0,s]$. Then for any $x>0$ we have*

$$\begin{aligned}\lim_{t\to\infty}&\mathbf{P}\Big[H(f((\xi(r))_{0\le r\le s}),g((\xi(t-r))_{0\le r\le s})\Big|\tau_{-x}^{-}>t\Big]\\&=\frac{1}{h(\varrho)}\int_0^\infty e^{-\varrho y}V^{(\varrho)}(y)\mathbf{Q}_{(x,y)}^{(\varrho)}[H(f((\xi_1(r)-x)_{0\le r\le s}),g((\xi_2(r)-x)_{0\le r\le s})]dy,\end{aligned}\tag{4.1}$$

where

$$h(\varrho)=\int_0^\infty e^{-\varrho y}V^{(\varrho)}(y)dy.$$

Proof. If $H(x_1,x_2)=G_1(x_1)G_2(x_2)$ for $G_1,G_2\in C_0(\mathbb{R}_+)$, we have (4.1) by Theorem 2-(a) in Hirano (2001). The general result follows by the Stone-Weierstrass Theorem. □

Lemma 4.3 *Suppose that Condition 2.8 holds and there exists $\varrho\in\mathscr{D}_+^\circ(\Phi)$ satisfying $\Phi'(\varrho)=0$. Then for any $\beta\in\mathscr{D}(\Phi)\cap(\varrho,\infty)$ there exists a constant $C=C_\beta>0$ such that*

$$\limsup_{t\to\infty}t^{3/2}e^{-\Phi(\varrho)t}\mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}]\le C.$$

Proof. By Lemma 7(3) in Hirano (1998), under the conditions of the lemma there exists a constant $\eta(\beta)\ge 0$ such that

$$\mathbf{P}\Big[\exp\Big\{\min_{k\le[t]-1}\beta\xi(k)\Big\}\Big]\le\eta(\beta)([t]-1)^{-3/2}e^{\Phi(\varrho)([t]-1)}\le 2^{3/2}\eta(\beta)e^{-\Phi(\varrho)}t^{-3/2}e^{\Phi(\varrho)t}.$$

Then we have the result by Lemmas 2.2 and 2.3. □

Lemma 4.4 *Suppose that Conditions 2.5 and 2.8 hold and there exists $\varrho \in \mathscr{D}^\circ_+(\Phi)$ satisfying $\Phi'(\varrho) = 0$. If F is a bounded function, in addition, then for any $x > 0$ and $\beta \in \mathscr{D}(\Phi) \cap (\varrho, \infty)$ there is a constant $C \geq 0$ so that*

$$\limsup_{t\to\infty} t^{3/2} e^{-t\Phi(\varrho)} \mathbf{P}[F(A^\alpha_t(\xi)); \tau^-_{-x} \leq t - \epsilon] \leq Ce^{-\beta x} + Ce^{-(\beta-\varrho)x}\hat{V}^{(\varrho)}(x),$$

and hence

$$\lim_{x\to\infty} \limsup_{t\to\infty} t^{3/2} e^{-t\Phi(\varrho)} \mathbf{P}[F(A^\alpha_t(\xi)); \tau^-_{-x} \leq t - \epsilon] = 0.$$

Proof. This is a modification of the proof of Lemma 3.5. By Lemma 4.3, there exists a constant $C \geq 0$ so that

$$\mathbf{P}[A^\alpha_t(\xi)^{-\beta/\alpha}] \leq Ct^{-3/2}e^{\Phi(\varrho)t}, \qquad t \geq 2.$$

For any $\eta > 0$, one can use Proposition 4.1 to see as the derivation of (3.2) that, as $t \to \infty$,

$$\mathbf{P}(t < \tau^-_{-x} \leq t + \eta) \sim Me^{\varrho x}\hat{V}^{(\varrho)}(x)(1 - e^{\Phi(\varrho)\eta})t^{-3/2}e^{\Phi(\varrho)t}\int_0^\infty e^{-\varrho z}V^{(\varrho)}(z)dz,$$

where

$$M = \frac{1}{\sqrt{2\pi\Phi''(\varrho)}} \exp\Big\{\int_0^\infty (e^{-t} - 1)t^{-1}e^{-t\Phi(\varrho)}\mathbf{P}(\xi(t) = 0)dt\Big\}.$$

By adjusting the value of $C \geq 0$ we have, for $t \geq 3$ and $0 < \epsilon \leq 1$,

$$\begin{aligned}
\mathbf{P}[F(A^\alpha_t(\xi)); \tau^-_{-x} \leq t - \epsilon] \leq\ & C\sum_{i=1}^{[t]-2}\mathbf{P}\Big[\Big(\int_{\tau^-_{-x}}^{t-i+\tau^-_{-x}} e^{-\alpha\xi(r)}dr\Big)^{-\beta/\alpha}; i - 1 < \tau^-_{-x} \leq i\Big] \\
& + C\mathbf{P}\Big[\Big(\int_{\tau^-_{-x}}^{\tau^-_{-x}+\epsilon} e^{-\alpha\xi(r)}dr\Big)^{-\beta/\alpha}; [t] - 2 < \tau^-_{-x} \leq t - \epsilon\Big] \\
=\ & C\sum_{i=1}^{[t]-2}\mathbf{P}\big\{e^{-\beta\xi(\tau^-_{-x})}\mathbf{P}_{\xi(\tau^-_{-x})}[A^\alpha_{t-i}(\xi)^{-\beta/\alpha}]; i - 1 < \tau^-_{-x} \leq i\big\} \\
& + C\mathbf{P}\big\{e^{-\beta\xi(\tau^-_{-x})}\mathbf{P}_{\xi(\tau^-_{-x})}[A^\alpha_\epsilon(\xi)^{-\beta/\alpha}]; [t] - 2 < \tau^-_{-x} \leq t\big\} \\
\leq\ & Ce^{-\beta x}\sum_{i=1}^{[t]-2}\mathbf{P}\big\{\mathbf{P}_{\xi(\tau^-_{-x})}[A^\alpha_{t-i}(\xi)^{-\beta/\alpha}]; i - 1 < \tau^-_{-x} \leq i\big\} \\
& + Ce^{-\beta x}\mathbf{P}\big\{\mathbf{P}_{\xi(\tau^-_{-x})}[A^\alpha_\epsilon(\xi)^{-\beta/\alpha}]; [t] - 2 < \tau^-_{-x} \leq t\big\} \\
\leq\ & Ce^{-\beta x}\sum_{i=1}^{[t]-2} e^{\Phi(\varrho)(t-i)}(t - i)^{-3/2}\mathbf{P}(i - 1 < \tau^-_{-x} \leq i) \\
& \quad + Ce^{-\beta x}\mathbf{P}([t] - 2 < \tau^-_{-x} \leq t) \\
\leq\ & Ce^{-(\beta-\varrho)x}\hat{V}^{(\varrho)}(-x)e^{\Phi(\varrho)(t-1)}\sum_{i=2}^{[t]-2}(t - i)^{-3/2}(i - 1)^{-3/2} \\
& + Ce^{-\beta x}e^{\Phi(\varrho)(t-1)}(t - 1)^{-3/2} \\
& + Ce^{-(\beta-\varrho)x}\hat{V}^{(\varrho)}(-x)(1 - e^{3\Phi(\varrho)})([t] - 2)^{-3/2}e^{\Phi(\varrho)([t]-2)},
\end{aligned}$$

where

$$\begin{aligned}\sum_{i=2}^{[t]-2}(t-i)^{-3/2}(i-1)^{-3/2} &\leq (t-2)^{-3/2}+\int_2^{t-2}(t-s-1)^{-3/2}(s-1)^{-3/2}ds\\ &\leq (t-2)^{-3/2}+\Big(\frac{t}{2}-1\Big)^{-3/2}\int_2^{t/2}(s-1)^{-3/2}ds\\ &\quad+\Big(\frac{t}{2}-1\Big)^{-3/2}\int_{t/2}^{t-2}(t-s-1)^{-3/2}ds\\ &\leq (t-2)^{-3/2}+4\Big(\frac{t}{2}-1\Big)^{-3/2}.\end{aligned}$$

Then we get the desired result. □

Lemma 4.5 *Suppose that Conditions 2.5 and 2.8 hold and there exists $\varrho\in\mathscr{D}_+^\circ(\Phi)$ satisfying $\Phi'(\varrho)=0$. If F is a bounded function, in addition, then for any $x>0$ we have*

$$\lim_{s\to\infty}\limsup_{t\to\infty}t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}\Big[F\Big(\int_{[0,s]\cup[t-s,t]}e^{-\alpha\xi(r)}dr\Big)-F(A_t^\alpha(\xi));\tau_{-x}^->t\Big]=0.$$

Proof. As in the proof of Lemma 3.4, one can use Proposition 4.1 to see there is a constant $C=C_x\geq 0$ so that

$$e^{-\Phi(\varrho)t}\mathbf{P}\Big[e^{-\alpha\xi(r)};\tau_{-x}^->t\Big]\leq C(t-r)^{-3/2}r^{-3/2}.$$

By elementary analysis we have

$$\int_s^{t-s}(t-r)^{-3/2}r^{-3/2}dr\leq 2(t/2)^{-3/2}\int_s^\infty r^{-3/2}dr=8\sqrt{2}t^{-3/2}s^{-1/2}.$$

Then we can prove the statement as in the proof of Lemma 3.6. □

Lemma 4.6 *Suppose that Conditions 2.5 and 2.8 hold and there exists $\varrho\in\mathscr{D}_+^\circ(\Phi)$ satisfying $\Phi'(\varrho)=0$. If F is a bounded function, in addition, then for any $x>0$ we have*

$$\lim_{t\to\infty}t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A_t^\alpha(\xi));\tau_{-x}^->t]=\frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}D_3(x,\alpha,F),\tag{4.2}$$

where

$$D_3(x,\alpha,F)=e^{\varrho x}\hat{V}^{(\varrho)}(x)\int_0^\infty e^{-\varrho y}V^{(\varrho)}(y)G(x,y)dy,$$

where $G(\cdot,\cdot)$ is defined by (2.10). Moreover, the function $x\mapsto D_3(x,\alpha,F)$ is bounded, increasing and strictly positive in $(0,\infty)$.

Proof. By the definition of the coupled process $(W,\mathscr{G},\mathscr{G}_t,(\xi_1(t),\xi_2(t)),\mathbf{Q}_{(x,y)}^{(\varrho)})$ given before Theorem 2.9 we have

$$\mathbf{Q}_{(x,z)}^{(\varrho)}\big[A_\infty^\alpha(\xi_1)+A_\infty^\alpha(\xi_2)\big]=\mathbf{Q}_x^{(\varrho)}\big[A_\infty^\alpha(\xi)\big]+\hat{\mathbf{Q}}_z^{(\varrho)}\big[A_\infty^\alpha(\xi)\big].$$

The right-hand side is finite by Lemma 2.4. It follows that $\mathbf{Q}^{(\varrho)}_{(x,z)}\{A^\alpha_\infty(\xi_1)+A^\alpha_\infty(\xi_2)<\infty\}=1$. Then $D_3(x,\alpha,F)$ is well-defined and strictly positive. By Lemma 4.5 we have

$$\begin{aligned}&\lim_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi));\tau^-_{-x}>t]\\ &\quad=\lim_{s\to\infty}\lim_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}\Big[F\Big(\int_{[0,s]\cup[t-s,t]} e^{-\alpha\xi(r)}dr\Big);\tau^-_{-x}>t\Big].\end{aligned}$$

By applying Propositions 4.1 and 4.2 with $H(x_1,x_2)=F(x_1+x_2)$, we have, for any $s>0$,

$$\begin{aligned}&\lim_{t\to\infty} t^{3/2}e^{-\Phi(\varrho)t}\mathbf{P}\Big[F\Big(\int_0^s e^{-\alpha\xi(r)}dr+\int_{t-s}^t e^{-\alpha\xi(r)}dr\Big);\tau^-_{-x}>t\Big]\\ &\quad=M(x)\int_0^\infty e^{-\varrho y}V^{(\varrho)}(y)\mathbf{Q}^{(\varrho)}_{(x,y)}\{F(e^{-\alpha x}[A^\alpha_s(\xi_1)+A^\alpha_s(\xi_2)])\}dy,\end{aligned}$$

where

$$M(x)=\frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}e^{\varrho x}\hat V^{(\varrho)}(x).$$

Then we get (4.2) by letting $s\to\infty$. As in the proof of Lemma 3.7 one can see that the function $x\mapsto D_3(x,\alpha,F)$ is bounded and increasing in $(0,\infty)$. □

Proof of Theorem 2.9(3). We here only consider a bounded function F. The general case can be treated similarly as in the proof of Theorem 2.9(2). By Lemma 4.6 we have

$$\begin{aligned}\liminf_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi))] &\geq \lim_{x\to\infty}\lim_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi));\tau^-_{-x}>t]\\ &=\lim_{x\to\infty}\frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}D_3(x,\alpha,F)\\ &=\frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}D_3(\alpha,F).\end{aligned}$$

On the other hand, for any $\epsilon>0$ and $x<0$ we can write

$$\begin{aligned}t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi))] &= t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi));\tau^-_{-x}\leq t-\epsilon]\\ &\quad+t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi));\tau^-_{-x}>t-\epsilon].\end{aligned}$$

By Lemma 4.4, the first term on the right-hand side tends to zero as $t\to\infty$ and $x\to-\infty$. Since $z\mapsto F(z)$ is decreasing, we can use Lemma 4.6 to see

$$\begin{aligned}\limsup_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi))] &\leq \lim_{x\to\infty}\lim_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_{t-\epsilon}(\xi));\tau^-_{-x}>t-\epsilon]\\ &=\lim_{x\to\infty} e^{-\epsilon\Phi(\varrho)}\frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}D_3(x,\alpha,F)\\ &=e^{-\epsilon\Phi(\varrho)}\frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}D_3(\alpha,F).\end{aligned}$$

Since $\epsilon>0$ was arbitrary, we get

$$\limsup_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}[F(A^\alpha_t(\xi))]\leq\frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}D_3(\alpha,F).$$

That gives the desired result. □

Lemma 4.7 *Suppose that $\Phi'(\beta) = 0$. Then for any $\theta \in \mathscr{D}_+^\circ(\Phi) \cap (\beta, \infty)$ we have*

$$\lim_{t\to\infty} t^{1/2} e^{-t\Phi(\beta)} \mathbf{P}[A_t^\alpha(\xi)^{-\theta/\alpha}] = 0.$$

Proof. By the definition of $\mathbf{P}^{(\beta)}$ and the property of independent increments of $\{\xi(t) : t \geq 0\}$ under this probability measure,

$$\begin{aligned} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\theta/\alpha}] &= t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha} A_t^\alpha(\xi)^{-(\theta-\beta)/\alpha}] \\ &= t^{1/2} \mathbf{P}^{(\beta)} \Big[\Big(\int_0^t e^{\beta[\xi(t)-\xi(s)]} ds\Big)^{-\beta/\alpha} A_t^\alpha(\xi)^{-(\theta-\beta)/\alpha}\Big] \\ &\leq t^{1/2} \mathbf{P}^{(\beta)} \Big[\Big(\int_{t/2}^t e^{\beta[\xi(t)-\xi(s)]} ds\Big)^{-\beta/\alpha} A_{t/2}^\alpha(\xi)^{-(\theta-\beta)/\alpha}\Big] \\ &= t^{1/2} \mathbf{P}^{(\beta)} \Big[\Big(\int_{t/2}^t e^{\beta[\xi(t)-\xi(s)]} ds\Big)^{-\beta/\alpha}\Big] \mathbf{P}^{(\beta)}[A_{t/2}^\alpha(\xi)^{-(\theta-\beta)/\alpha}] \\ &= t^{1/2} \mathbf{P}^{(\beta)}[A_{t/2}^\alpha(-\xi)^{-\beta/\alpha}] \mathbf{P}^{(\beta)}[A_{t/2}^\alpha(\xi)^{-(\theta-\beta)/\alpha}], \end{aligned}$$

where we have used the duality relation in the last equality; see, e.g., Lemma 3.4 in Kyprianou (2014, p.77). Since $\{\xi(t) : t \geq 0\}$ is a recurrent Lévy process under $\mathbf{P}^{(\beta)}$, the right-hand side tends to zero as $t \to \infty$ by Lemma 3.3. $\square$

Proof of Theorem 2.9(4). By Assumption 2, for each $y > 0$ there is a constant $C = C_y \geq 0$ so that $F(z) \leq Cz^{-\beta_0/\alpha}$ for $0 < z \leq y$. Upon an adjustment of the value of the constants, we may assume $\beta_0 \in \mathscr{D}_+^\circ(\Phi) \cap (\varrho, \infty)$. By Lemma 4.7 we have

$$\begin{aligned} &\limsup_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[F(A_t^\alpha(\xi)); A_t^\alpha(\xi) < y] \\ &\quad \leq C \limsup_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\beta_0/\alpha}; A_t^\alpha(\xi) < y] \\ &\quad \leq C \limsup_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\beta_0/\alpha}] = 0, \end{aligned}$$

and hence

$$\lim_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[F(A_t^\alpha(\xi))] = \lim_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[F(A_t^\alpha(\xi)); A_t^\alpha(\xi) \geq y]. \tag{4.3}$$

By the duality relation of the Lévy process,

$$A_t^\alpha(\xi)^{-\beta/\alpha} \stackrel{\mathrm{d}}{=} \Big(\int_0^t e^{\alpha[\xi(t-s)-\xi(t)]} ds\Big)^{-\beta/\alpha} \stackrel{\mathrm{d}}{=} e^{\beta\xi(t)} A_t^\alpha(-\xi)^{-\beta/\alpha}.$$

It follows that

$$e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] = e^{-\Phi(\beta)t} \mathbf{P}[e^{\beta\xi(t)} A_t^\alpha(-\xi)^{-\beta/\alpha}] = \mathbf{P}^{(\beta)}[A_t^\alpha(-\xi)^{-\beta/\alpha}]. \tag{4.4}$$

By Condition 2.7, given any $\delta \in (0, 1)$ we can choose sufficiently large $y > 0$ so that

$$(1 - \delta)Kz^{-\beta/\alpha} \leq F(z) \leq (1 + \delta)Kz^{-\beta/\alpha}, \qquad z > y.$$

Then, in view of (4.3) and (4.4),

$$\begin{aligned}
&\limsup_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[F(A_t^\alpha(\xi))] \\
&\qquad \le\ (1+\delta)K \limsup_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}; A_t^\alpha(\xi) \ge y] \\
&\qquad \le\ (1+\delta)K \limsup_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}] \\
&\qquad =\ (1+\delta)K \limsup_{t\to\infty} t^{1/2} \mathbf{P}^{(\beta)}[A_t^\alpha(-\xi)^{-\beta/\alpha}] \\
&\qquad =\ (1+\delta)K \sqrt{\frac{2}{\pi \Phi''(\beta)}} \mathbf{P}^{(\beta)}[H(1)] D_4(\alpha,\beta), \qquad (4.5)
\end{aligned}$$

where the last equality follows by Theorem 2.9(2). On the other hand, by the definition of $\mathbf{P}^{(\beta)}$ we have

$$\begin{aligned}
e^{-\Phi(\beta)t}\mathbf{P}[F(A_t^\alpha(\xi))] &\ge (1-\delta)K e^{-\Phi(\beta)t} \mathbf{P}[A_t^\alpha(\xi)^{-\beta/\alpha}; A_t^\alpha(\xi) \ge y] \\
&\ge (1-\delta)K \mathbf{P}^{(\beta)}\Big[\Big(\int_0^t e^{\alpha[\xi(t)-\xi(s)]} ds\Big)^{-\beta/\alpha}; A_t^\alpha(\xi) \ge y\Big].
\end{aligned}$$

Since $\mathbf{P}^{(\beta)}[\xi(1)] = \Phi'(\beta) = 0$, from Proposition 2.1 it follows that $\mathbf{P}^{(\beta)}(A_\infty^\alpha(\xi) = \infty) = 1$. Then, by dominated convergence and the duality relation,

$$\begin{aligned}
&\liminf_{t\to\infty} t^{1/2} e^{-\Phi(\beta)t} \mathbf{P}[F(A_t^\alpha(\xi))] \\
&\qquad \ge\ (1-\delta)K \liminf_{t\to\infty} t^{1/2} \mathbf{P}^{(\beta)}\Big[\Big(\int_0^t e^{\alpha[\xi(t)-\xi(s)]} ds\Big)^{-\beta/\alpha}\Big] \\
&\qquad =\ (1-\delta)K \liminf_{t\to\infty} t^{1/2} \mathbf{P}^{(\beta)}[A_t^\alpha(-\xi)^{-\beta/\alpha}] \\
&\qquad =\ (1-\delta)K \sqrt{\frac{2}{\pi \Phi''(\beta)}} \mathbf{P}^{(\beta)}[H(1)] D_4(\alpha,\beta), \qquad (4.6)
\end{aligned}$$

where we used Theorem 2.9(2) again for the last equality. Since $\delta \in (0,1)$ was arbitrary, we get the desired result by combining (4.5) and (4.6). □

Proof of Theorem 2.9(5). Under the condition, we can take $\theta \in \mathscr{D}_+^\circ(\Phi) \cap (\beta,\infty)$ satisfying $\Phi(\theta) < \Phi(\beta)$ and $\Phi'(\beta) < \Phi'(\theta) < 0$. Then we have

$$\begin{aligned}
\limsup_{t\to\infty} e^{-\Phi(\beta)t}\mathbf{P}[A_t^\alpha(\xi)^{-\theta/\alpha}] &= \limsup_{t\to\infty} e^{[\Phi(\theta)-\Phi(\beta)]t}\mathbf{P}^{(\theta)}[A_t^\alpha(-\xi)^{-\theta/\alpha}] \\
&= \limsup_{t\to\infty} e^{[\Phi(\theta)-\Phi(\beta)]t}\mathbf{P}^{(\theta)}[A_1^\alpha(-\xi)^{-\theta/\alpha}] = 0.
\end{aligned}$$

The remaining arguments are modifications of those in the proof of Theorem 2.9(4). □

5 Survival probability of the CBRE-process

Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Let $\sigma \ge 0$ and b be real constants. Let $(z \wedge z^2)\nu(dz)$ be a finite measure on $\mathbb{R}$ supported by

$\mathbb{R}\setminus\{0\}$. Let $\{B(t) : t \geq 0\}$ be an $(\mathscr{F}_t)$-Brownian motion and $N(ds, dz)$ an $(\mathscr{F}_t)$-Poisson random measure on $(0,\infty)\times\mathbb{R}$ with intensity $ds\nu(dz)$. Let $\{L(t) : t \geq 0\}$ be an $(\mathscr{F}_t)$-Lévy process with the following Lévy-Itô decomposition:

$$L(t) = \beta t + \sigma B(t) + \int_0^t \int_{[-1,1]} (e^z - 1)\tilde{N}(ds, dz) + \int_0^t \int_{[-1,1]^c} (e^z - 1)N(ds, dz),$$

where $[-1,1]^c = \mathbb{R}\setminus[-1,1]$ and $\tilde{N}(ds, dz) = N(ds, dz) - ds\nu(dz)$. Then $\{L(t) : t \geq 0\}$ has no jump smaller than -1. We can define another Lévy process $\{\xi(t) : t \geq 0\}$ by

$$\xi(t) = a_0 t + \sigma B(t) + \int_0^t \int_{\mathbb{R}} z\tilde{N}(ds, dz),$$

where

$$a_0 = \beta - \frac{\sigma^2}{2} - \int_{[-1,1]} (e^z - 1 - z)\nu(dz) + \int_{[-1,1]^c} z\nu(dz).$$

Clearly, the two processes $\{L(t) : t \geq 0\}$ and $\{\xi(t) : t \geq 0\}$ generate the same filtration.

Let $0 < \alpha \leq 1$ and let $\{Z_\alpha(t) : t \geq 0\}$ be a spectrally positive $(\mathscr{F}_t)$-stable process with index $(1+\alpha)$. Assume that $\{Z_\alpha(t) : t \geq 0\}$ is independent of $\{L(t) : t \geq 0\}$ and $\{\xi(t) : t \geq 0\}$. When $\alpha = 1$, we think of $\{Z_\alpha(t) : t \geq 0\}$ as a Brownian motion. When $0 < \alpha < 1$, we assume $\{Z_\alpha(t) : t \geq 0\}$ has Lévy measure:

$$m(dz) = \frac{\alpha 1_{\{z>0\}} dz}{\Gamma(1-\alpha) z^{2+\alpha}}.$$

Let $c \geq 0$ be another constant and let $\{X(t) : t \geq 0\}$ be the CBRE-process defined by (1.3). Let $\mathbf{P}^\xi$ denote the conditional law given $\{L(t) : t \geq 0\}$ or $\{\xi(t) : t \geq 0\}$. Let $Z(t) = X(t)\exp\{-\xi(t)\}$. By Theorem 1 in Bansaye et al. (2013) or Theorem 3.4 in He et al. (2016) we have

$$\mathbf{P}^\xi[e^{-\lambda Z(t)}|\mathscr{F}_r] = \exp\{-Z(r)u^\xi_{r,t}(\lambda)\}, \qquad \lambda \geq 0,\ t \geq r \geq 0, \tag{5.1}$$

where $r \mapsto u^\xi_{r,t}(\lambda)$ is the solution to

$$\frac{d}{dr}u^\xi_{r,t}(\lambda) = ce^{-\alpha\xi(r)}u^\xi_{r,t}(\lambda)^{1+\alpha}, \qquad u^\xi_{t,t}(\lambda) = \lambda.$$

By solving the above equation, we get

$$u^\xi_{r,t}(\lambda) = \Big(c\alpha\int_r^t e^{-\alpha\xi(s)}ds + \lambda^{-\alpha}\Big)^{-1/\alpha}; \tag{5.2}$$

see the proof of Proposition 4 in Bansaye et al. (2013). From (5.1) and (5.2) we see that the survival probability of the CBRE-process up to time $t \geq 0$ is given by

$$\begin{aligned}\mathbf{P}(X(t) > 0) &= \mathbf{P}(Z(t) > 0) = \lim_{\lambda\to\infty}\mathbf{P}\big[1 - e^{-\lambda Z(t)}\big] \\ &= \lim_{\lambda\to\infty}\mathbf{P}\big[1 - \exp\{-xu^\xi_{0,t}(\lambda)\}\big] = \mathbf{P}\Big[F_x\Big(\int_0^t e^{-\alpha\xi(s)}ds\Big)\Big],\end{aligned} \tag{5.3}$$

where $F_x(z) = 1 - \exp\{-x(c\alpha z)^{-1/\alpha}\}$; see also (3.2) in Bansaye et al. (2013). Let $\Phi(\lambda) = \log\mathbf{P}\exp\{\lambda\xi(1)\}$ denote the Laplace exponent of $\{\xi(t) : t \geq 0\}$.

The following theorem is an immediate consequence of Theorem 2.9. Using the notation introduced there, it gives characterizations of the five regimes of the asymptotics of the survival probability of the CBRE-process:

Theorem 5.1 *Suppose that $\{0,1\} \subset \mathscr{D}^\circ(\Phi)$. Then we have the following five regimes of the survival probability of the CBRE-process:*

(1) (Supercritical case) *If $0 < \Phi'(0)$, we have the finite and nonzero limit*

$$\lim_{t\to\infty} \mathbf{P}(X(t)=0) = \mathbf{P}[F_x(A_\infty^\alpha(\xi))].$$

(2) (Critical case) *Suppose that Condition 2.8 is satisfied and $\Phi'(0) = 0$. Then we have the finite and nonzero limit*

$$\lim_{t\to\infty} t^{1/2}\mathbf{P}(X(t)>0) = \sqrt{\frac{2}{\pi\Phi''(0)}}\hat{\mathbf{P}}[H(1)]D_2(\alpha, F_x).$$

(3) (Weakly subcritical case) *Suppose that Condition 2.8 is satisfied and $\Phi'(0) < 0 < \Phi'(1)$. Let $\varrho \in (0,1)$ be the solution of $\Phi'(\varrho) = 0$. Then we have the finite and nonzero limit*

$$\lim_{t\to\infty} t^{3/2}e^{-t\Phi(\varrho)}\mathbf{P}(X(t)>0) = \frac{c(\varrho)}{\sqrt{2\pi\Phi''(\varrho)}}D_3(\alpha, F_x).$$

(4) (Intermediately subcritical case) *Suppose that Condition 2.8 is satisfied and $\Phi'(1) = 0$. Then we have the finite and nonzero limit*

$$\lim_{t\to\infty} t^{1/2}e^{-t\Phi(1)}\mathbf{P}(X(t)>0) = x(c\alpha)^{-1/\alpha}\sqrt{\frac{2}{\pi\Phi''(\beta)}}\mathbf{P}^{(1)}[H(1)]D_4(\alpha, 1).$$

(5) (Strongly subcritical case) *If $\Phi'(0) < 0$, we have the finite and nonzero limit*

$$\lim_{t\to\infty} e^{-t\Phi(1)}\mathbf{P}(X(t)>0) = x(c\alpha)^{-1/\alpha}\mathbf{P}^{(1)}[A_\infty^\alpha(-\xi)^{-1/\alpha}].$$

If there always exists $\varrho \in \mathscr{D}^\circ(\Phi)$ so that $\Phi'(\varrho) = 0$, the above theorem treats the five regimes $\varrho < 0 < 1$, $0 = \varrho < 1$, $0 < \varrho < 1$, $0 < \varrho = 1$ and $0 < 1 < \varrho$ in the natural order. But, the theorem does not assume the point $\varrho \in \mathscr{D}^\circ(\Phi)$ always exists.

The asymptotics of the survival probability of CBRE-processes have been studied by Böinghoff and Hutzenthaler (2012), Bansaye et al. (2013), Palau and Pardo (2015a) and Palau et al. (2016). The expression for the limiting coefficient was not given explicitly in Bansaye et al. (2013), Palau and Pardo (2015a) and Palau et al. (2016). When the environment process is a Brownian motion with drift, Böinghoff and Hutzenthaler (2012) computed accurately the limiting constant. In the general case, it seems difficult to calculate directly the quantities of the expectations involved in the expression. We would recommend the Mentor Carlo method in practical applications.

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