Published in: *Journal of Theoretical Probability* **31** (2018), 4: 1952–1974.

Continuous-state branching processes in Lévy random environments [1]

Hui He, Zenghu Li and Wei Xu [2]

School of Mathematical Sciences, Beijing Normal University,

Beijing 100875, People's Republic of China

E-mails: hehui@bnu.edu.cn, lizh@bnu.edu.cn, xuwei@mail.bnu.edu.cn

Abstract. A general continuous-state branching processes in random environment (CBRE-process) is defined as the strong solution of a stochastic integral equation. The environment is determined by a Lévy process with no jump less than -1. We give characterizations of the quenched and annealed transition semigroups of the process in terms of a backward stochastic integral equation driven by another Lévy process determined by the environment. The process hits zero with strictly positive probability if and only if its branching mechanism satisfies Grey's condition. In that case, a characterization of the extinction probability is given using a random differential equation with blowup terminal condition. The strong Feller property of the CBRE-process is established by a coupling method. We also prove a necessary and sufficient condition for the ergodicity of the subcricital CBRE-process with immigration.

Mathematics Subject Classification (2010): Primary 60J80, 60K37; Secondary 60H20, 60G51

Key words and phrases: continuous-state branching process, random environment, Lévy process, transition semigroup, backward stochastic equation, survival probability, immigration, ergodicity.

1 Introduction

Galton-Watson processes in random environments (GWRE-processes) were introduced by Smith [29] and Smith and Wilkinson [30] as extensions of classical Galton-Watson processes (GW-processes). Those extensions possess many interesting new properties. For instance, different regimes for the survival probability arise in the subcritical regime. For recent results on the speed of decay of the survival probability, the reader may refer to [1, 2, 12, 32] and the references therein.

[1] Supported by NSFC (No. 11371061, No. 11531001 and No. 11671041).

[2] Corresponding author.

Let $\{W(t) : t \geq 0\}$ be a Brownian motion and $\{S(t) : t \geq 0\}$ a Brownian motion with drift. We assume the two processes are independent of each other. By the Yamada-Watanabe theorem, for any constants $c \geq 0$, $\sigma \geq 0$ and $b \in \mathbb{R}$ there is a unique positive strong solution to the stochastic differential equation:

$$dX(t) = \sqrt{2cX(t)}dW(t) - bX(t)dt + \sigma X(t)dS(t). \tag{1.1}$$

The solution $\{X(t) : t \geq 0\}$ is called a continuous-state branching diffusion in random environment (CBRE-diffusion). The environment here is determined by the process $\{S(t) : t \geq 0\}$. It was proved in Kurtz [19] that the CBRE-diffusion arises as the limit of a sequence of suitably rescaled GWRE-processes; see also Helland [14]. A diffusion approximation of the GWRE-process was actually conjectured by Keiding [18]. It turns out that the CBRE-diffusion is technically more tractable than the GWRE-process. In the particular case of $\sigma = 0$, the CBRE-diffusion reduces to the well-known Feller branching diffusion, which belongs to an important class of positive Markov processes called continuous-state branching processes (CB-processes); see Feller [10], Jiřina [16] and Lamperti [20, 21].

In the work of Böinghoff and Hutzenthaler [5], it was shown that the survival probability of the CBRE-diffusion can be represented explicitly in terms of an exponential functional of the environment process $\{S(t) : t \geq 0\}$. Based on the representation, Böinghoff and Hutzenthaler [5] gave an exact characterization for the decay rate of the survival probability of the CBRE-diffusion in the critical and subcritical cases. The results of [5] are more complete than the corresponding results for the GWRE-processes in the sense that they calculated the accurate limiting constants. In addition, they characterized the CBRE-diffusion conditioned to never go extinct and established a backbone construction for the conditioned process. See also Hutzenthaler [15] for some related results.

Continuous-state branching processes with immigration (CBI-processes), which generalize the CB-processes, were introduced by Kawazu and Watanabe [17] as rescaling limits of Galton-Watson processes with immigration (GWI-processes); see also Aliev [3] and Li [22, 23]. Let $b \in \mathbb{R}$ and $c \geq 0$ be given constants. Let $m(dz)$ be a Radon measure on $(0, \infty)$ satisfying $\int_0^\infty (z \wedge z^2) m(dz) < \infty$. Suppose that $\{W(t) : t \geq 0\}$ is a Brownian motion, $\{\eta(t) : t \geq 0\}$ is an increasing Lévy process with $\eta(0) = 0$ and $\tilde{M}(ds, dz, du)$ is a compensated Poisson random measure on $(0, \infty)^3$ with intensity $ds m(dz) du$. We assume those three noises are independent of each other. By Dawson and Li [6, Theorems 5.1 and 5.2] or Fu and Li [11, Corollary 5.2], there is a unique positive strong solution to

$$\begin{aligned} X(t) = X(0) - b\int_0^t X(s)ds + \int_0^t \sqrt{2cX(s)}dW(s) \\ + \int_0^t \int_0^\infty \int_0^{X(s-)} z\tilde{M}(ds, dz, du) + \eta(t). \end{aligned} \tag{1.2}$$

It was shown in [6, 11] that the solution $\{X(t) : t \geq 0\}$ is a CBI-process. The process $\{\eta(t) : t \geq 0\}$ describes the inputs of the immigrants. Here and in the sequel, we understand $\int_a^b = \int_{(a,b]}$ and $\int_a^\infty = \int_{(a,\infty)}$ for any $a \leq b \in \mathbb{R}$.

A class of continuous-state branching processes in random environments (CBRE-processes) were introduced by Bansaye et al. [4], where the environments were defined by Lévy processes with bounded variation. The authors gave a criticality classification of their CBRE-processes according to the long time behavior of the environmental Lévy process. They also characterized the Laplace exponent of the processes using a backward ordinary differential equation involving the environment process. For stable branching CBRE-processes, Bansaye et al. [4] calculated explicitly the survival probability and characterized its decay rate in the critical and subcritical cases. In addition, they showed some interesting applications of their results to a cell infection model. The results of Bansaye et al. [4] were extended by Palau and Pardo [27] to the case where the environment was given by a Brownian motion with drift.

The CBRE-processes studied in [4, 27] can be generalized to continuous-state branching processes with immigration in random environment (CBIRE-processes). Let $\{L(t) : t \geq 0\}$ be a Lévy process with no jump less than -1 and assume it is independent of the three noises in (1.2). It is natural to define a CBIRE-process $\{Y(t) : t \geq 0\}$ by the stochastic equation

$$\begin{aligned} Y(t) = Y(0) - b\int_0^t Y(s)ds + \int_0^t \sqrt{2cY(s)}dW(s) + \eta(t) \\ + \int_0^t \int_0^\infty \int_0^{Y(s-)} z\tilde{M}(ds, dz, du) + \int_0^t Y(s-)dL(s). \end{aligned} \tag{1.3}$$

Here the influence of the environment is represented by the Lévy process $\{L(t) : t \geq 0\}$. The existence and uniqueness of the positive strong solution to the above equation follow from the results of Dawson and Li [7], Fu and Li [11] and Li and Pu [25]; see Sections 3 and 5 for details. In the very recent work of Palau and Pardo [28], a further generalization of the CBIRE-process was introduced by considering a competition mechanism. For that purpose, those authors first established some general existence and uniqueness results on stochastic equations following the arguments in [7, 11, 25]. The long term behavior of the CBIRE-process was also studied in [28].

The purpose of this paper is to study the basic structures of the CBRE- and CBIRE-processes. In Section 2, we introduce two random cumulant semigroups, which are important tools in the study of those processes. The semigroups are defined in terms of a backward stochastic equations driven by a Lévy process. The existence of them follows from a general result in Li [23] on Dawson-Watanabe superprocesses. In Section 3, a construction of the CBRE-process is given by applying the results of [7, 11, 25] on stochastic equations driven by time-space noises. Then we give characterizations of the quenched and annealed transition probabilities of the CBRE-process. In particular, we show the annealed transition semigroup is a Feller semigroup. In Section 4, we show the CBRE-process hits zero with strictly positive probability if and only if its branching mechanism satisfies Grey's condition. In that case, we give a characterization of the extinction probabilities by the solution of a random differential equation with blowup terminal condition. The strong Feller property of the CBRE-process is established by a coupling method.

Some of the results are extended in Section 5 to CBIRE-processes. In addition, we give a necessary and sufficient condition for the ergodicity of subcritical CBIRE-processes. Most of the results here are obtained or presented using stochastic equations driven by Lévy processes, which are more elegant than those in the classical discrete setting.

We refer to Ethier and Kurtz [9] and Sharpe [31] for the general theory and terminology of Markov processes. For the convenience of the reader, we here review two important concepts. Consider a locally compact and superable metric space E and let $C_0(E)$ denote the class of continuous functions on the space vanishing at infinity. The transition semigroup $(T_t)_{t\geq 0}$ of a Markov process with state space E is called a *Feller semigroup* if the operators $(T_t)_{t\geq 0}$ map $C_0(E)$ into itself and $\lim_{t\to 0+} T_t f(x) = f(x)$ for $x \in E$ and $f \in C_0(E)$. We say a Feller semigroup $(T_t)_{t\geq 0}$ has *strong Feller property* if for each $t > 0$ the operator T_t maps bounded Borel functions into bounded continuous functions. The strong Feller property plays an important role in the study of Markov processes; see, e.g., Hairer and Mattingly [13] and Wang [33].

Acknowledgements. We thank Professors Yueyun Hu and Zhan Shi for helpful discussions on branching processes in random environments. We are grateful to Sandra Palau for pointing out a gap in the original proof of Theorem 3.1 and a referee for a list of comments which helped us in improving the presentation of the results.

2 Random cumulant semigroups

In this section, we introduce some random cumulant semigroups, which generalize that of a classical CB-process. Those semigroups are important tools in the study of the CBRE-process. Let $I \subset \mathbb{R}$ be an interval and $\zeta = \{\zeta(t) : t \in I\}$ a càdlàg function. Let ϕ be a *branching mechanism* given by

$$\phi(z) = bz + cz^2 + \int_0^\infty (e^{-uz} - 1 + uz)m(du), \qquad z \geq 0. \tag{2.1}$$

where $b \in \mathbb{R}$ and $c \geq 0$ are constants and $(z \wedge z^2)m(dz)$ is a finite measure on $(0, \infty)$. From [23, Theorem 6.10] it follows that, for any $t \in I$ and $\lambda \geq 0$, there is a unique positive solution $r \mapsto u^\zeta_{r,t}(\lambda)$ to the integral evolution equation

$$u^\zeta_{r,t}(\lambda) = \lambda - \int_r^t e^{\zeta(s)}\phi(e^{-\zeta(s)}u^\zeta_{s,t}(\lambda))ds, \qquad r \in I \cap (-\infty, t]. \tag{2.2}$$

Moreover, there is an inhomogeneous transition semigroup $(P^\zeta_{r,t})_{t\geq r\in I}$ on $[0, \infty)$ defined by

$$\int_{[0,\infty)} e^{-\lambda y} P^\zeta_{r,t}(x, dy) = e^{-xu^\zeta_{r,t}(\lambda)}, \qquad \lambda \geq 0. \tag{2.3}$$

By a simple transformation, we can define another inhomogeneous transition semigroup $(Q^{\zeta}_{r,t})_{t\geq r\in I}$ on $[0,\infty)$ by

$$\int_{[0,\infty)} e^{-\lambda y} Q^{\zeta}_{r,t}(x,dy) = e^{-xv^{\zeta}_{r,t}(\lambda)}, \qquad \lambda \geq 0, \tag{2.4}$$

where

$$v^{\zeta}_{r,t}(\lambda) = e^{-\zeta(r)} u^{\zeta}_{r,t}(e^{\zeta(t)}\lambda). \tag{2.5}$$

The uniqueness of the solution to (2.2) implies that

$$u^{\zeta}_{r,t}(\lambda) = u^{\zeta}_{r,s} \circ u^{\zeta}_{s,t}(\lambda), \qquad \lambda \geq 0, t \geq s \geq r \in I. \tag{2.6}$$

There is a similar relation for $(v^{\zeta}_{r,t})_{t\geq r\in I}$. By (2.2) and Lebesgue's theorem one can see $r \mapsto u^{\zeta}_{r,t}(\lambda)$ is also the unique positive continuous solution to the differential equation

$$\frac{d}{dr} u^{\zeta}_{r,t}(\lambda) = e^{\zeta(r)} \phi(e^{-\zeta(r)} u^{\zeta}_{r,t}(\lambda)), \qquad \text{a.e. } r \in I \cap (-\infty, t] \tag{2.7}$$

with terminal condition $u^{\zeta}_{t,t}(\lambda) = \lambda$.

Proposition 2.1 *Let $(P^{\zeta}_{r,t})_{t\geq r\in I}$ be defined by (2.3). Then for any $x \geq 0$ and $t \geq r \in I$ we have*

$$\int_{[0,\infty)} y P^{\zeta}_{r,t}(x,dy) = xe^{-b(t-r)}. \tag{2.8}$$

Proof. By differentiating both sides of (2.2) and solving the resulted integral equation we obtain $(d/d\lambda)u^{\zeta}_{r,t}(0+) = e^{-b(t-r)}$. Then we get the desired equality by differentiating both sides of (2.3). □

Proposition 2.2 *For any $t \geq r \in I$, the mapping $\lambda \mapsto u^{\zeta}_{r,t}(\lambda)$ is strictly increasing on $[0,\infty)$.*

Proof. In view of (2.3) and (2.8), we have $u^{\zeta}_{r,t}(\lambda) \leq \lambda e^{-b(t-r)}$ by Jessen's inequality. Fix a bounded interval $J \subset I$ and let $M = \sup_{s\in J}(e^{-\zeta(s)} \vee e^{\zeta(s)})$ and $L = \sup\{|\phi(z)| : 0 \leq z \leq Me^{-b}\}$. By (2.2), for $t \geq r \in J$ with $t - r \leq 1$ we have

$$u^{\zeta}_{r,t}(1) = 1 - \int_r^t e^{\zeta(s)} \phi(e^{-\zeta(s)} u^{\zeta}_{s,t}(1)) ds \geq 1 - (t-r)ML.$$

Then, as $t - r \leq 1 \wedge (ML)^{-1}$, we have $u^{\zeta}_{r,t}(1) > 0$, so (2.3) implies $P^{\zeta}_{r,t}(x,(0,\infty)) > 0$ for any $x > 0$, and hence $\lambda \mapsto u^{\zeta}_{r,t}(\lambda)$ is strictly increasing. Using (2.3) we see $\lambda \mapsto u^{\zeta}_{r,t}(\lambda)$ is strictly increasing for any $t \geq r \in J$. Since $J \subset I$ was arbitrary, the desired result follows. □

Proposition 2.3 *If $b \geq 0$, then $t \mapsto u^\zeta_{r,t}(\lambda)$ is decreasing on $I \cap [r, \infty)$ and $r \mapsto u^\zeta_{r,t}(\lambda)$ is increasing on $I \cap (-\infty, t]$.*

Proof. From (2.3) we see that $\lambda \mapsto u^\zeta_{r,t}(\lambda)$ is increasing. Since $b \geq 0$, we have $\phi(z) \geq 0$ for every $z \geq 0$. Then (2.2) implies $u^\zeta_{r,t}(\lambda) \leq \lambda$. By (2.6) we see $u^\zeta_{r,t}(\lambda) \leq u^\zeta_{s,t}(\lambda)$ and $u^\zeta_{r,t}(\lambda) \leq u^\zeta_{r,s}(\lambda)$ for $r \leq s \leq t \in I$. $\square$

When the function ζ is degenerate ($\zeta(t) = 0$ for all $t \in I$), both $(u^\zeta_{r,t})_{t \geq r \in I}$ and $(v^\zeta_{r,t})_{t \geq r \in I}$ reduce to the cumulant semigroup of a classical CB-process with branching mechanism ϕ; see, e.g., [23, Chapter 3]. In the general case, we may think of $(u^\zeta_{r,t})_{t \geq r \in I}$ as an *inhomogeneous cumulant semigroup* determined by the *time-dependent branching mechanism* $(s, z) \mapsto e^{\zeta(s)}\phi(e^{-\zeta(s)}z)$. The idea of the proof of [23, Theorem 6.10] is to reduce the construction of an inhomogeneous cumulant semigroup to that of a homogeneous one by some time-space processes. The transformation from $(u^\zeta_{r,t})_{t \geq r \in I}$ to $(v^\zeta_{r,t})_{t \geq r \in I}$ is a time-dependent variation of the one used in the proof of [23, Theorem 6.1].

We next consider some randomization of the inhomogeneous cumulant semigroups defined above. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. Let $a \in \mathbb{R}$ and $\sigma \geq 0$ be given constants and $(1 \wedge z^2)\nu(dz)$ a finite measure on $(0, \infty)$. Suppose that $\{B(t) : t \geq 0\}$ is an $(\mathscr{F}_t)$-Brownian motion with $B(0) = 0$ and $N(ds, dz)$ is an $(\mathscr{F}_t)$-Poisson random measure on $(0, \infty) \times \mathbb{R}$ with intensity $ds\nu(dz)$. Let $\{\xi(t) : t \geq 0\}$ be an $(\mathscr{F}_t)$-Lévy process with the following Lévy-Itô decomposition:

$$\xi(t) = at + \sigma B(t) + \int_0^t \int_{[-1,1]} z\tilde{N}(ds, dz) + \int_0^t \int_{[-1,1]^c} zN(ds, dz), \tag{2.9}$$

where $[-1, 1]^c = \mathbb{R} \setminus [-1, 1]$. Let $\{L(t) : t \geq 0\}$ be the $(\mathscr{F}_t)$-Lévy process defined by

$$L(t) = \beta t + \sigma B(t) + \int_0^t \int_{[-1,1]} (e^z - 1)\tilde{N}(ds, dz) + \int_0^t \int_{[-1,1]^c} (e^z - 1)N(ds, dz),$$

where

$$\beta = a + \frac{\sigma^2}{2} + \int_{[-1,1]} (e^z - 1 - z)\nu(dz). \tag{2.10}$$

Clearly, the two processes $\{\xi(t) : t \geq 0\}$ and $\{L(t) : t \geq 0\}$ generate the same filtration.

Let $(u^\xi_{r,t})_{t \geq r \geq 0}$ and $(v^\xi_{r,t})_{t \geq r \geq 0}$ be the *random cumulant semigroups* defined by (2.2) and (2.5) with $\zeta = \xi$. From (2.2) we see that $r \mapsto v^\xi_{r,t}(\lambda)$ is the pathwise unique positive solution to

$$v^\xi_{r,t}(\lambda) = e^{\xi(t)-\xi(r)}\lambda - \int_r^t e^{\xi(s)-\xi(r)}\phi(v^\xi_{s,t}(\lambda))ds, \qquad 0 \leq r \leq t. \tag{2.11}$$

From (2.11) we see that the left-continuous process $\{v^\xi_{t-s,t}(\lambda) : 0 \leq s \leq t\}$ is progressively measurable with respect to the filtration generated by the Lévy process $\{L_t(s) := L(t-) - L((t-s)-) : 0 \leq s \leq t\}$.

Theorem 2.4 *For any $t \geq 0$ and $\lambda \geq 0$, the process $\{v^{\xi}_{r,t}(\lambda) : 0 \leq r \leq t\}$ is the pathwise unique positive solution to*

$$v^{\xi}_{r,t}(\lambda) = \lambda - \int_r^t \phi(v^{\xi}_{s,t}(\lambda))ds + \int_r^t v^{\xi}_{s,t}(\lambda)L(\overleftarrow{ds}), \qquad 0 \leq r \leq t, \tag{2.12}$$

where the backward stochastic integral is defined by

$$\int_r^t v^{\xi}_{s,t}(\lambda)L(\overleftarrow{ds}) = \int_0^{(t-r)-} v^{\xi}_{t-s,t}(\lambda)L_t(ds).$$

Proof. Let $\xi_t(r) = \xi(t-) - \xi((t-r)-)$ and $B_t(r) = B(t) - B(t-r)$ for $0 \leq r \leq t$. Let $N_t(ds, dz)$ be the Poisson random measure defined by

$$N_t([0, r] \times B) = N([t-r, t] \times B), \qquad 0 \leq r \leq t, B \in \mathscr{B}(\mathbb{R}).$$

From (2.9) we have

$$\xi_t(r) = ar + \sigma B_t(r) + \int_0^r \int_{[-1,1]} z\tilde{N}_t(ds, dz) + \int_0^r \int_{[-1,1]^c} zN_t(ds, dz).$$

On the other hand, from (2.11) we have $f_t(r) := v^{\xi}_{(t-r)-,t}(\lambda) = e^{\xi_t(r)}F_t(r)$, where

$$F_t(r) = \lambda - \int_0^r e^{-\xi_t(s)}\phi(f_t(s))ds.$$

By Itô's formula,

$$\begin{aligned} f_t(r) &= \lambda + \int_0^r e^{\xi_t(s-)}F_t(s)\xi_t(ds) + \frac{\sigma^2}{2}\int_0^r e^{\xi_t(s)}F_t(s)ds \\ &\quad + \int_0^r \int_{\mathbb{R}} e^{\xi_t(s-)}F_t(s)(e^z - 1 - z)N_t(ds, dz) + \int_0^r e^{\xi_t(s)}F_t(ds) \\ &= \lambda + \int_0^r f_t(s-)L_t(ds) - \int_0^r \phi(f_t(s))ds. \end{aligned}$$

It follows that

$$\begin{aligned} v^{\xi}_{r,t}(\lambda) &= \lambda + \int_0^{r-} f_t(s-)L_t(ds) - \int_0^r \phi(f_t(s))ds \\ &= \lambda + \int_0^{(t-r)-} v^{\xi}_{t-s,t}(\lambda)L_t(ds) - \int_0^{t-r} \phi(v^{\xi}_{t-s,t}(\lambda))ds \\ &= \lambda + \int_r^t v^{\xi}_{s,t}(\lambda)L(\overleftarrow{ds}) - \int_r^t \phi(v^{\xi}_{t-s,t}(\lambda))ds. \end{aligned}$$

That proves the existence of the solution to (2.12). Conversely, assuming $r \mapsto v^{\xi}_{r,t}(\lambda)$ is a solution to (2.12), one can use similar calculations to see it also solves (2.11). Then the pathwise uniqueness for (2.12) is a consequence of that for (2.11). □

3 Construction of CBRE-processes

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. Let $\{\xi(t) : t \geq 0\}$ and $\{L(t) : t \geq 0\}$ be $(\mathscr{F}_t)$-Lévy processes given as in Section 2. Let $b \in \mathbb{R}$ and $c \geq 0$ be constants and $(z \wedge z^2)m(dz)$ a finite measure on $(0, \infty)$. Suppose that $\{W(t) : t \geq 0\}$ is an $(\mathscr{F}_t)$-Brownian motion and $M(ds, dz, du)$ is an $(\mathscr{F}_t)$-Poisson random measure on $(0, \infty)^3$ with intensity $ds m(dz) du$. We assume both of those are independent of the Lévy process $\{L(t) : t \geq 0\}$. Given a positive $\mathscr{F}_0$-measurable random variable $X(0)$, we consider the following stochastic integral equation:

$$\begin{aligned} X(t) = X(0) - b\int_0^t X(s)ds + \int_0^t \sqrt{2cX(s)}dW(s) \\ + \int_0^t \int_0^\infty \int_0^{X(s-)} z\tilde{M}(ds, dz, du) + \int_0^t X(s-)dL(s), \end{aligned} \tag{3.1}$$

where $\tilde{M}(ds, dz, du) = M(ds, dz, du) - ds m(dz) du$.

Theorem 3.1 *There is a unique positive strong solution $\{X(t) : t \geq 0\}$ to (3.1).*

Proof. Let $E = \{1, 2\}$ and $U_0 = A_0 \cup B_0$, where $A_0 = \{1\} \times (0, \infty)^2$ and $B_0 = \{2\} \times [-1, 1]$. Let $\pi(du) = \delta_1(du) + \delta_2(du)$ for $u \in E$. Then $W(ds, du) := dW(s)\delta_1(du) + dB(s)\delta_2(du)$ is a Gaussian white noise on $(0, \infty) \times E$ with intensity $ds\pi(dz)$. Let $\mu_0(dy, dz, du) = \delta_1(dy)m(dz)du$ for $(y, z, u) \in A_0$ and $\mu_0(dy, dz) = \delta_2(dy)\nu(dz)$ for $(y, z) \in B_0$. Then $N_0(ds, dy, dz, du) := \delta_1(dy)M(ds, dz, du)$ is a Poisson random measure on $(0, \infty) \times A_0$ with intensity $ds\delta_1(dy)m(dz)du$ and $N_0(ds, dy, dz) := \delta_2(dy)N(ds, dz)$ is a Poisson random measure on $(0, \infty) \times B_0$ with intensity $ds\delta_2(dy)\nu(dz)$. Let $b(x) = (\beta - b)x$ for $x \in [0, \infty)$ and $\sigma(x, u) = \sqrt{2cx}1_{\{u=1\}} + \sigma x 1_{\{u=2\}}$ for $(x, u) \in [0, \infty) \times E$. Let $g_0(x, y, z, u) = z1_{\{u \leq x\}}$ for $(x, y, z, u) \in [0, \infty) \times A_0$ and $g_0(x, y, z) = x(e^z - 1)$ for $(x, y, z) \in [0, \infty) \times B_0$. By modifying [25, Theorem 6.1] to the setting of [7, Theorem 2.5], one can see there is a unique positive strong solution to the stochastic equation

$$\begin{aligned} X(t) = X(0) + \int_0^t b(X(s))ds + \int_0^t \int_{A_0} g_0(X(s-), y, z, u)\tilde{N}_0(ds, dy, dz, du) \\ + \int_0^t \int_E \sigma(X(s), u)W(ds, du) + \int_0^t \int_{B_0} g_0(X(s-), y, u)\tilde{N}_0(ds, dy, du). \end{aligned}$$

The above equation can be rewritten into

$$\begin{aligned} X(t) = X(0) + \int_0^t \sqrt{2cX(s)}dW(s) + \int_0^t \int_0^\infty \int_0^{X(s-)} z\tilde{M}(ds, dz, du) \\ + (\beta - b)\int_0^t X(s)ds + \sigma\int_0^t X(s)dB(s) + \int_0^t \int_{[-1,1]} X(s-)(e^z - 1)\tilde{N}(ds, dz). \end{aligned}$$

Since the process

$$t \mapsto \int_0^t \int_{[-1,1]^c} (e^z - 1) N(ds, dz)$$

has at most a finite number of jumps in each bounded time interval, as in the proof of [11, Proposition 2.2], one can see that there is also a unique positive strong solution to

$$\begin{aligned} X(t) = X(0) &+ \int_0^t \sqrt{2cX(s)} dW(s) + \int_0^t \int_0^\infty \int_0^{X(s-)} z\tilde{M}(ds, dz, du) \\ &+ (\beta - b) \int_0^t X(s) ds + \sigma \int_0^t X(s) dB(s) + \int_0^t \int_{[-1,1]} X(s-)(e^z - 1)\tilde{N}(ds, dz) \\ &+ \int_0^t \int_{[-1,1]^c} X(s-)(e^z - 1) N(ds, dz). \end{aligned} \tag{3.2}$$

The above equation is just a reformulation of (3.1). Then we have the result of the theorem. □

Remark 3.2 *A result more general than Theorem 3.1 was given in Corollary 1 of Palau and Pardo [28] as a consequence of their Theorem 1, which was proved by modifying the arguments in [7, 11, 25]. The above proof explains how the results in [7, 11, 25] should be used directly in the current situation. The proof can be modified to construct more complex models with extra structures such as immigration, competition and so on. In fact, since the results on stochastic equations in [7, 11, 25] are formulated in the abstract setting, they are quite flexible for applications.*

We call the solution $\{X(t) : t \geq 0\}$ to (3.1) a *CBRE-process*, which is a càdlàg strong Markov process. Here the random environment is provided by the Lévy process $\{L(t) : t \geq 0\}$. By Itô's formula one can see $\{X(t) : t \geq 0\}$ has strong generator A defined as follows: For $f \in C^2[0, \infty)$,

$$\begin{aligned} Af(x) = (\beta - b)xf'(x) &+ cxf''(x) + x \int_0^\infty [f(x+z) - f(x) - zf'(x)] m(dz) \\ &+ \frac{\sigma^2}{2} x^2 f''(x) + \int_{[-1,1]} [f(xe^z) - f(x) - x(e^z - 1)f'(x)] \nu(dz) \\ &+ \int_{[-1,1]^c} [f(xe^z) - f(x)] \nu(dz). \end{aligned} \tag{3.3}$$

Proposition 3.3 *Let $Z(t) = X(t)e^{-\xi(t)}$ for $t \geq 0$. Then we have*

$$\begin{aligned} Z(t) = X(0) - b \int_0^t e^{-\xi(s)} X(s) ds &+ \int_0^t e^{-\xi(s)} \sqrt{2cX(s)} dW(s) \\ &+ \int_0^t \int_0^\infty \int_0^{X(s-)} z e^{-\xi(s-)} \tilde{M}(ds, dz, du). \end{aligned} \tag{3.4}$$

In particular, the process $\{Z(t) : t \geq 0\}$ is a positive local martingale when $b = 0$.

Proof. Let $f(x,y) = xe^{-y}$. Then $xf'_x(x,y) = -f'_y(x,y) = f(x,y)$ and $-xf''_{xy}(x,y) = f''_{yy}(x,y) = f(x,y)$. Observe that the Poison random measure $N(ds,dz)$ actually does not produce any jump of $t \mapsto Z(t)$. By (3.1) and Itô's formula,

$$
\begin{aligned}
Z(t) = {} & X(0) - b\int_0^t f'_x(X(s),\xi(s))X(s)ds + \int_0^t f'_x(X(s),\xi(s))\sqrt{2cX(s)}dW(s) \\
& + \int_0^t\int_0^\infty\int_0^{X(s-)} f'_x(X(s-),\xi(s-))z\tilde{M}(ds,dz,du) \\
& + \beta\int_0^t f'_x(X(s),\xi(s))X(s)ds + \sigma\int_0^t f'_x(X(s),\xi(s))X(s)dB((s) \\
& + \int_0^t\int_{[-1,1]} f'_x(X(s-),\xi(s-))X(s-)(e^z-1)\tilde{N}(ds,dz) \\
& + \int_0^t\int_{[-1,1]^c} f'_x(X(s-),\xi(s-))X(s-)(e^z-1)N(ds,dz) \\
& + a\int_0^t f'_y(X(s),\xi(s))ds + \int_0^t\int_{[-1,1]} f'_y(X(s-),\xi(s-))z\tilde{N}(ds,dz) \\
& + \sigma\int_0^t f'_y(X(s),\xi(s))dB((s) + \int_0^t\int_{[-1,1]^c} f'_y(X(s-),\xi(s-))zN(ds,dz) \\
& + \sigma^2\int_0^t \Big[f''_{xy}(X(s),\xi(s))X(s) + \frac{1}{2}f''_{yy}(X(s),\xi(s))\Big]ds \\
& + \int_0^t\int_0^\infty\int_0^{X(s-)} [f(X(s-)+z,\xi(s)) - f(X(s-),\xi(s)) \\
& \qquad\qquad - f'_x(X(s-),\xi(s))z]M(ds,dz,du) \\
& + \int_0^t\int_{\mathbb{R}} [f(X(s-)e^z,\xi(s-)+z) - f(X(s-),\xi(s-)) \\
& \qquad - f'_x(X(s-),\xi(s-))X(s-)(e^z-1) - f'_y(X(s-),\xi(s-))z]N(ds,dz) \\
= {} & X(0) - b\int_0^t f'_x(X(s),\xi(s))X(s)ds + \int_0^t f'_x(X(s),\xi(s))\sqrt{2cX(s)}dW(s) \\
& + \int_0^t\int_0^\infty\int_0^{X(s-)} f'_x(X(s-),\xi(s-))z\tilde{M}(ds,dz,du) \\
& + \beta\int_0^t f'_x(X(s),\xi(s))X(s)ds + \sigma\int_0^t f'_x(X(s),\xi(s))X(s)dB((s) \\
& + a\int_0^t f'_y(X(s),\xi(s))ds + \sigma\int_0^t f'_y(X(s),\xi(s))dB((s) \\
& + \sigma^2\int_0^t \Big[f''_{xy}(X(s),\xi(s))X(s) + \frac{1}{2}f''_{yy}(X(s),\xi(s))\Big]ds \\
& - \int_0^t ds\int_{[-1,1]} [f'_x(X(s),\xi(s))X(s)(e^z-1) + f'_y(X(s),\xi(s))z]\nu(dz).
\end{aligned}
$$

By reorganizing the terms on the right-hand side we get the desired equality. □

It is easy to see that the two Lévy processes $\{\xi(t) : t \geq 0\}$ and $\{L(t) : t \geq 0\}$ generate the same σ-algebra. Let $\mathbf{P}^\xi$ denote the quenched law given $\{\xi(t) : t \geq 0\}$ or $\{L(t) : t \geq$

$0\}$. The following theorem shows that the random cumulant semigroups $(u^\xi_{r,t})_{t\geq r\geq 0}$ and $(v^\xi_{r,t})_{t\geq r\geq 0}$ are connected with the processes $\{Z(t) : t \geq 0\}$ and $\{X(t) : t \geq 0\}$, respectively.

Theorem 3.4 *Let $(P^\xi_{r,t})_{t\geq r\geq 0}$ and $(Q^\xi_{r,t})_{t\geq r\geq 0}$ be defined by (2.3) and (2.4), respectively, with $\zeta = \xi$. Then for any $\lambda \geq 0$ and $t \geq r \geq 0$ we have*

$$\mathbf{P}^\xi[e^{-\lambda Z(t)}|\mathscr{F}_r] = \exp\{-Z(r)u^\xi_{r,t}(\lambda)\} = \int_{[0,\infty)} e^{-\lambda y} P^\xi_{r,t}(Z(r), dy) \tag{3.5}$$

and

$$\mathbf{P}^\xi[e^{-\lambda X(t)}|\mathscr{F}_r] = \exp\{-X(r)v^\xi_{r,t}(\lambda)\} = \int_{[0,\infty)} e^{-\lambda y} Q^\xi_{r,t}(X(r), dy). \tag{3.6}$$

Proof. Fix $\lambda \geq 0$. For $t \geq r \geq 0$, let $H_t(r) = \exp\{-Z(r)u^\xi_{r,t}(\lambda)\}$. By (2.2) and Proposition 3.3, given the environment $\{\xi(t) : t \geq 0\}$, we can use Itô formula to see

$$\begin{aligned}
H_t(t) = H_t(r) &- \int_r^t H_t(s)Z(s)e^{\xi(s)}\phi(e^{-\xi(s)}u^\xi_{s,t}(\lambda))ds \\
&- \int_r^t H_t(s-)u^\xi_{s,t}(\lambda)dZ(s) + c\int_r^t H_t(s)u^\xi_{s,t}(\lambda)^2 e^{-2\xi(s)}X(s)ds \\
&+ \int_r^t\int_0^\infty\int_0^{X(s-)} H_t(s-)\Big[\exp\{-ze^{-\xi(s-)}u^\xi_{s,t}(\lambda)\} \\
&\qquad\qquad -1 + ze^{-\xi(s-)}u^\xi_{s,t}(\lambda)\Big]M(ds,dz,du) \\
= H_t(r) &- \int_r^t H_t(s)X(s)\phi(e^{-\xi(s)}u^\xi_{s,t}(\lambda))ds + b\int_r^t H_t(s)u^\xi_{s,t}(\lambda)e^{-\xi(s)}X(s)ds \\
&- \int_r^t H_t(s)u^\xi_{s,t}(\lambda)e^{-\xi(s)}\sqrt{2cX(s)}dW(s) + c\int_r^t H_t(s)e^{-2\xi(s)}u^\xi_{s,t}(\lambda)^2X(s)ds \\
&+ \int_r^t\int_0^\infty\int_0^{X(s-)} H_t(s-)\Big[\exp\{-ze^{-\xi(s-)}u^\xi_{s,t}(\lambda)\} \\
&\qquad\qquad -1 + ze^{-\xi(s-)}u^\xi_{s,t}(\lambda)\Big]M(ds,dz,du) \\
= H_t(r) &- \int_r^t H_t(s-)u^\xi_{s,t}(\lambda)e^{-\xi(s)}\sqrt{2cX(s)}dW(s) \\
&+ \int_r^t\int_0^\infty\int_0^{X(s-)} H_t(s-)\Big[\exp\{-ze^{-\xi(s-)}u^\xi_{s,t}(\lambda)\} \\
&\qquad\qquad -1 + ze^{-\xi(s-)}u^\xi_{s,t}(\lambda)\Big]\tilde{M}(ds,dz,du).
\end{aligned}$$

Since $\{H_t(r) : t \geq r\}$ is a bounded process, by taking the quenched expectation in both sides we get $\mathbf{P}^\xi[H_t(t)|\mathscr{F}_r] = H_t(r)$. That gives (3.5), and as a consequence we get (3.6). $\square$

Corollary 3.5 *If $\mathbf{P}[Z(0)] = \mathbf{P}[X(0)] < \infty$, then $\{e^{bt}Z(t) : t \geq 0\}$ is a martingale.*

Proof. Let $t \geq r \geq 0$ and let F be a bounded random variable measurable with respect to the σ-algebra generated by $\mathscr{F}_r \cup \sigma(\xi)$. By (3.5) and Proposition 2.1, we have

$$\mathbf{P}[Fe^{bt}Z(t)] = \mathbf{P}\{Fe^{bt}\mathbf{P}^\xi[Z(t)|\mathscr{F}_r]\} = \mathbf{P}\left[Fe^{bt}\int_{[0,\infty)} yP^\xi_{r,t}(Z(r),dy)\right] = \mathbf{P}[Fe^{br}Z(r)].$$

Then $\{e^{bt}Z(t) : t \geq 0\}$ is a martingale. $\square$

By Theorem 3.4 we see that $\{Z(t) : t \geq 0\}$ and $\{X(t) : t \geq 0\}$ are actually CB-processes under the quenched law with inhomogeneous cumulant semigroups $(u^\xi_{r,t})_{t\geq r\geq 0}$ and $(v^\xi_{r,t})_{t\geq r\geq 0}$, respectively. The next theorem gives a characterization of the transition semigroup of $\{X(t) : t \geq 0\}$ under the annealed law.

Theorem 3.6 *The Markov process $\{X(t) : t \geq 0\}$ has Feller transition semigroup $(Q_t)_{t\geq 0}$ defined by*

$$\int_{[0,\infty)} e^{-\lambda y}Q_t(x,dy) = \mathbf{P}[e^{-xv^\xi_{0,t}(\lambda)}], \qquad \lambda \geq 0. \tag{3.7}$$

Proof. By (2.4) one can see that (3.7) defines a probability kernel $Q_t(x,dy)$. In view of (3.6), for any bounded $\mathscr{F}_r$-measurable random variable F we have

$$\mathbf{P}[Fe^{-\lambda X(t)}] = \mathbf{P}[F\mathbf{P}^\xi(e^{-\lambda X(t)}|\mathscr{F}_r)] = \mathbf{P}[F\exp\{-X(r)v^\xi_{r,t}(\lambda)\}].$$

The pathwise uniqueness of the solution to (2.12) implies that the random variable $v^\xi_{r,t}(\lambda)$ is measurable with respect to the σ-algebra generated by $\{L(s) - L(t) : r \leq s \leq t\}$, so it is independent of $\{X(s) : 0 \leq s \leq r\}$. Moreover, this random variable is identically distributed with $v^\xi_{0,t-r}(\lambda)$. It follows that

$$\mathbf{P}[Fe^{-\lambda X(t)}] = \mathbf{P}\left[F\int_{[0,\infty)} e^{-y\lambda}Q_{t-r}(X(r),dy)\right].$$

Then $\{X(t) : t \geq 0\}$ has transition semigroup $(Q_t)_{t\geq 0}$. Let $C_0[0,\infty)$ be the set of continuous functions on $[0,\infty)$ vanishing at infinity. Since the right-hand side of (3.7) is continuous in $x \geq 0$, the continuity theorem implies that $Q_t(x,\cdot)$ depends on $x \geq 0$ continuously by weak convergence. By Proposition 2.2, for $t \geq 0$ we have a.s. $v^\xi_{0,t}(1) > 0$, so (3.7) implies $Q_t(x,J) \to 0$ as $x \to \infty$ for any bounded interval $J \subset [0,\infty)$. Then the operator Q_t maps $C_0[0,\infty)$ into itself. Let $\{X(x,t) : t \geq 0\}$ be the solution of (3.1) with $X(x,0) = x \geq 0$. The right continuity of the process implies that $\lim_{t\to 0+} Q_tf(x) = \lim_{t\to 0+}\mathbf{P}[f(X(x,t))] = f(x)$ for any $f \in C_0[0,\infty)$. Then $(Q_t)_{t\geq 0}$ is a Feller semigroup. $\square$

Under the annealed law, the process $\{Z(t) : t \geq 0\}$ usually does not satisfy the Markov property, but $\{(\xi(t), Z(t)) : t \geq 0\}$ is a two-dimensional Markov process. Let $\mathscr{F}_\infty = \sigma(\cup_{t\geq 0}\mathscr{F}_t)$. By Corollary 3.5, there is a probability measure $\tilde{\mathbf{P}}$ on $(\Omega, \mathscr{F}_\infty)$ so that

$\tilde{\mathbf{P}}(F) = \mathbf{P}[Fe^{bt}Z(t)]$ for each bounded $\mathscr{F}_t$-measurable random variable F. Let $\tilde{\mathbf{P}}^\xi$ denote the quenched law under $\tilde{\mathbf{P}}$ given the environment $\{\xi(t) : t \geq 0\}$. By differentiate both sides of (2.4) we obtain

$$\phi'(z) = b + 2cz + \int_0^\infty u(1 - e^{-uz})m(du), \qquad z \geq 0. \tag{3.8}$$

Theorem 3.7 *For any $\lambda \geq 0$ and $t \geq r \geq 0$, we have*

$$\tilde{\mathbf{P}}^\xi[e^{-\lambda X(t)}|\mathscr{F}_r] = \exp\bigg\{ - X(r)v^\xi_{r,t}(\lambda) - \int_r^t \phi_0'(v^\xi_{s,t}(\lambda))ds\bigg\}, \tag{3.9}$$

where $\phi_0'(z) = \phi'(z) - b$.

Proof. From (2.4) one can see that $\lambda \mapsto v^\xi_{r,t}(\lambda)$ is infinitely differentiable in $(0, \infty)$. By differentiating both sides of (2.2) we obtain

$$\frac{d}{d\lambda}u^\xi_{r,t}(\lambda) = 1 - \int_r^t \phi'(e^{-\xi(s)}u^\xi_{s,t}(\lambda))\frac{d}{d\lambda}u^\xi_{s,t}(\lambda)ds.$$

Then we can solve the equation to get

$$\frac{d}{d\lambda}u^\xi_{r,t}(\lambda) = \exp\bigg\{ - \int_r^t \phi'(e^{-\xi(s)}u^\xi_{s,t}(\lambda))ds\bigg\}.$$

Let F be a bounded random variable measurable with respect to the σ-algebra generated by $\mathscr{F}_r \cup \sigma(\xi)$. From (3.5) it follows that

$$\mathbf{P}[Fe^{-\lambda Z(t)}] = \mathbf{P}[F\mathbf{P}^\xi(e^{-\lambda Z(t)}|\mathscr{F}_r)] = \mathbf{P}[F\exp\{-Z(r)u^\xi_{r,t}(\lambda)\}].$$

By differentiating both sides in $\lambda > 0$ we have

$$\mathbf{P}[Fe^{-\lambda Z(t)}Z(t)] = \mathbf{P}\bigg[F\exp\bigg\{ - Z(r)u^\xi_{r,t}(\lambda) - \int_r^t \phi'(e^{-\xi(s)}u^\xi_{s,t}(\lambda))ds\bigg\}Z(r)\bigg],$$

and hence

$$\mathbf{P}[Fe^{-\lambda X(t)}Z(t)] = \mathbf{P}\bigg[F\exp\bigg\{ - X(r)v^\xi_{r,t}(\lambda) - \int_r^t \phi'(v^\xi_{s,t}(\lambda))ds\bigg\}Z(r)\bigg].$$

It follows that

$$\tilde{\mathbf{P}}[Fe^{-\lambda X(t)}] = \tilde{\mathbf{P}}\bigg[F\exp\bigg\{ - X(r)v^\xi_{r,t}(\lambda) - \int_r^t \phi_0'(v^\xi_{s,t}(\lambda))ds\bigg\}\bigg].$$

Then we have (3.9). The extension of the equality to $\lambda \geq 0$ is immediate. □

By Theorem 3.7 one can show as in the proof of Theorem 3.6 that $\{X(t) : t \geq 0\}$ is a Markov process under $\tilde{\mathbf{P}}$ with Feller transition semigroup $(\tilde{Q}_t)_{t\geq 0}$ defined by

$$\int_{[0,\infty)} e^{-\lambda y}\tilde{Q}_t(x, dy) = \mathbf{P}\bigg[\exp\bigg\{ - xv^\xi_{0,t}(\lambda) - \int_0^t \phi_0'(v^\xi_{s,t}(\lambda))ds\bigg\}\bigg], \qquad \lambda \geq 0. \tag{3.10}$$

This is a special case of a larger class of transition semigroups to be given in Section 5.

4 Survival and extinction probabilities

In this section, we assume $X(0) = x > 0$ is a deterministic constant for simplicity. Let $\mathbf{P}$ or $\mathbf{P}_x$ denote the annealed law and $\mathbf{P}^\xi$ or $\mathbf{P}^\xi_x$ the quenched law given $\{\xi(t) : t \geq 0\}$. In addition, we assume $b = \phi'(0) = 0$ so that $z \mapsto \phi(z)$ is positive and increasing on $[0, \infty)$. This restriction is not essential since all the results obtained here can be applied to the general case if one replaces the environment process $\{\xi(t) : t \geq 0\}$ with $\{\xi(t) - bt : t \geq 0\}$. Let $\tau_0 = \inf\{t \geq 0 : X(t) = Z(t) = 0\}$ denote the *extinction time* of the CBRE-process. From (2.4) one can see that $v^\xi_{0,t}(\lambda)$ is increasing in $\lambda \geq 0$. For $t > 0$ let $\bar{v}^\xi_{0,t} := \lim_{\lambda\to\infty} v^\xi_{0,t}(\lambda) \in [0, \infty]$. Then

$$\bar{u}^\xi_{0,t} := \lim_{\lambda\to\infty} u^\xi_{0,t}(\lambda) = \lim_{\lambda\to\infty} v^\xi_{0,t}(e^{-\xi(t)}\lambda) = \bar{v}^\xi_{0,t}. \tag{4.1}$$

By (3.5) and (3.6) we have the following characterizations of the extinction probabilities:

$$\mathbf{P}^\xi_x(\tau_0 \leq t) = \mathbf{P}^\xi_x(Z(t) = 0) = \mathbf{P}^\xi_x(X(t) = 0) = e^{-x\bar{u}^\xi_{0,t}} = e^{-x\bar{v}^\xi_{0,t}} \tag{4.2}$$

and

$$\mathbf{P}_x(\tau_0 \leq t) = \mathbf{P}_x(Z(t) = 0) = \mathbf{P}_x(X(t) = 0) = \mathbf{P}(e^{-x\bar{u}^\xi_{0,t}}) = \mathbf{P}(e^{-x\bar{v}^\xi_{0,t}}). \tag{4.3}$$

We say the branching mechanism ϕ satisfies *Grey's condition* if

$$\int_1^\infty \phi(z)^{-1} dz < \infty. \tag{4.4}$$

Theorem 4.1 *The following statements are equivalent:*

(1) ϕ *satisfies Grey's condition;*

(2) $\mathbf{P}_x(Z(t) = 0) = \mathbf{P}_x(X(t) = 0) > 0$ *for some and hence all* $t > 0$*;*

(3) $\mathbf{P}(\bar{u}^\xi_{0,t} < \infty) = \mathbf{P}(\bar{v}^\xi_{0,t} < \infty) > 0$ *for some and hence all* $t > 0$*;*

(4) $\mathbf{P}(\bar{u}^\xi_{0,t} < \infty) = \mathbf{P}(\bar{v}^\xi_{0,t} < \infty) = 1$ *for some and hence all* $t > 0$*.*

Proof. From (4.3) we see (2)⇔(3)⇐(4). Then we only need to show (3)⇒(1)⇒(4). From (2.7) we have

$$t = \int_0^t \frac{1}{e^{\xi(s)}\phi(e^{-\xi(s)}u^\xi_{s,t}(\lambda))} du^\xi_{s,t}(\lambda). \tag{4.5}$$

Suppose that $\mathbf{P}(\bar{v}^\xi_{0,t} < \infty) = \mathbf{P}(\bar{u}^\xi_{0,t} < \infty) > 0$ for some $t > 0$. Choose the constants $0 < M_1 < M_2 < \infty$ so that the event $A := \{\bar{u}^\xi_{0,t} < \infty\} \cap \{M_1 \leq e^{\xi(s)} \leq M_2 \text{ for } s \in [0, t]\}$ has strictly positive probability. Since $z \mapsto \phi(z)$ is increasing, by (4.5) we have on A that

$$t \geq \int_0^t \frac{1}{M_2\phi(M_1^{-1}u^\xi_{s,t}(\lambda))} du^\xi_{s,t}(\lambda) = \frac{M_1}{M_2} \int_{M_1^{-1}u^\xi_{0,t}(\lambda)}^{M_1^{-1}\lambda} \frac{1}{\phi(z)} dz.$$

By letting $\lambda \to \infty$ we have on A that

$$\frac{M_1}{M_2}\int_{M_2\bar{u}^\xi_{0,t}}^\infty \frac{1}{\phi(z)}dz \leq t.$$

Then (4.4) holds. That proves (3)⇒(1). Now suppose that Grey's condition (4.4) is satisfied. Fix any $t>0$. Choose sufficiently large $n \geq 1$ so that the event $\Omega_n := \{1/n \leq e^{\xi(s)} \leq n \text{ for } s \in [0,t]\}$ has strictly positive probability. By (4.5), on the event Ω_n we have

$$t \leq \int_0^t \frac{n}{\phi(n^{-1}u^\xi_{s,t}(\lambda))}du^\xi_{s,t}(\lambda) = n^2\int_{n^{-1}u^\xi_{0,t}(\lambda)}^{n^{-1}\lambda} \frac{1}{\phi(z)}dz,$$

which implies

$$n^2\int_{n^{-1}\bar{u}^\xi_{0,t}}^\infty \frac{1}{\phi(z)}dz \geq t.$$

It follows that $\bar{u}^\xi_{0,t} = \bar{v}^\xi_{0,t} < \infty$ on Ω_n. Since $\mathbf{P}(\cup_{n=1}^\infty \Omega_n) = 1$, we have $\mathbf{P}(\bar{v}^\xi_{0,t} < \infty) = \mathbf{P}(\bar{u}^\xi_{0,t} < \infty) = 1$. □

Corollary 4.2 *Under Grey's condition, for any $t>0$, the function $r \mapsto u(r) := \bar{u}^\xi_{r,t} = \bar{v}^\xi_{r,t}$ on $[0,t)$ is the minimal positive continuous solution to*

$$\frac{d}{dr}u(r) = e^{\xi(r)}\phi(e^{-\xi(r)}u(r)), \qquad a.e.\ r \in (0,t) \tag{4.6}$$

with blowup terminal condition $u(t-) = \infty$.

Proof. For any $t > s > r > 0$ we have $\bar{u}^\xi_{r,t} = \lim_{\lambda\to\infty} u^\xi_{r,s}(u^\xi_{s,t}(\lambda)) = u^\xi_{r,s}(\bar{u}^\xi_{s,t})$. From (2.7) we see the differential equation in (4.6) is satisfied first for a.e. $r \in (0,s)$ and then for a.e. $r \in (0,t)$. Since $\bar{u}^\xi_{t-,t} \geq u^\xi_{t-,t}(\lambda) = \lambda$ for any $\lambda \geq 0$, we have the terminal property $\bar{u}^\xi_{t-,t} = \infty$. Now suppose that $r \mapsto w(r)$ is another positive continuous solution to (4.6). By the uniqueness of the solution to (2.7) we have $w(r) = u^\xi_{r,s}(w(s))$ for $0 \leq r \leq s < t$. For any $\lambda \geq 0$ we can choose $s \in (r,t)$ so that $w(s) \geq \lambda e^{|b|t}$. By Proposition 2.1 and Jensen's inequality one can see $u^\xi_{s,t}(\lambda) \leq \lambda e^{-b(t-s)} \leq \lambda e^{|b|t}$. From monotonicity of $\lambda \mapsto u^\xi_{r,s}(\lambda)$ we get $w(r) = u^\xi_{r,s}(w(s)) \geq u^\xi_{r,s}(\lambda e^{|b|t}) \geq u^\xi_{r,s}(u^\xi_{s,t}(\lambda)) = u^\xi_{r,t}(\lambda)$. Then $w(r) \geq \bar{u}^\xi_{r,t} = \lim_{\lambda\to\infty} u^\xi_{r,t}(\lambda)$. □

Theorem 4.3 *Let $\bar{v}^\xi :=\downarrow \lim_{t\to\infty} \bar{v}^\xi_{0,t} \in [0,\infty]$ and $\tau_0 := \inf\{t \geq 0 : X(t) = 0\}$. Then*

$$\mathbf{P}_x(\tau_0 < \infty) = \lim_{t\to\infty}\mathbf{P}_x(\tau_0 \leq t) = \lim_{t\to\infty}\mathbf{P}_x(X(t) = 0) = \mathbf{P}[e^{-x\bar{v}^\xi}].$$

Moreover, we have $\bar{v}^\xi < \infty$ if and only if Grey's condition (4.4) holds.

Proof. From $\mathbf{P}_x^\xi(\tau_0 \le t) = \mathbf{P}_x^\xi(X(t) = 0) = e^{-x\bar{v}_{0,t}^\xi}$ we see $t \mapsto \bar{v}_{0,t}^\xi$ is decreasing, so $\bar{v}^\xi$ is well defined. From (4.2) it follows that

$$\mathbf{P}_x(\tau_0 < \infty) = \lim_{t\to\infty} \mathbf{P}_x(\tau_0 \le t) = \lim_{t\to\infty} \mathbf{P}_x(X(t) = 0) = \lim_{t\to\infty} \mathbf{P}[e^{-x\bar{v}_{0,t}^\xi}] = \mathbf{P}[e^{-x\bar{v}^\xi}].$$

The second statement follows immediately from Theorem 4.1. □

Corollary 4.4 *Suppose that* $\liminf_{t\to\infty} \xi(t) = -\infty$ *and Grey's condition (4.4) holds. Then*

$$\mathbf{P}_x(\tau_0 = \infty) = \lim_{t\to\infty} \mathbf{P}_x(\tau_0 > t) = \lim_{t\to\infty} \mathbf{P}_x(X(t) > 0) = 0.$$

Proof. Suppose that $p(x) := \mathbf{P}_x(\tau_0 = \infty) = 1 - \mathbf{P}[e^{-x\bar{v}^\xi}] > 0$. Then we have $\mathbf{P}(\bar{v}^\xi > 0) > 0$, so $x \mapsto p(x)$ is strictly increasing. Under the assumption $\liminf_{t\to\infty} \xi(t) = -\infty$, we have $\liminf_{t\to\infty} X(t) = 0$ as observed by Bansaye et al. [4, Corollary 2]. Then the stopping time $\sigma = \inf\{t > 0 : \xi(t) < x/2\}$ is a.s. finite. By Theorem 4.3 and the strong Markov property, we have

$$p(x) = \mathbf{P}_x[\mathbf{P}_{X(\sigma)}(\tau_0 = \infty)] = \mathbf{P}_x[\mathbf{P}_{x/2}(\tau_0 = \infty)] = p(x/2),$$

which yields a contradiction. Then we must have $p(x) = \mathbf{P}_x(\tau_0 = \infty) = 0$. □

Theorem 4.5 *Under Grey's condition, the transition semigroup* $(Q_t)_{t\ge0}$ *defined by (3.7) has the strong Feller property.*

Proof. We here need to introduce a simple reformulation of the stochastic equation (3.1). By El Karoui and Méléard [8, Theorem III.6], on an extension of the original probability space we can define an $(\mathscr{F}_t)$-Gaussian white noise $W(ds, du)$ on $(0,\infty)^2$ with intensity $dsdu$ so that

$$\int_0^t \sqrt{X(s)}dW(s) = \int_0^t \int_0^{X(s)} W(ds, du).$$

Then we may rewrite (3.1) into

$$\begin{aligned} X(t) = X(0) + \int_0^t X(s-)dL(s) + \sqrt{2c}\int_0^t \int_0^{X(s)} W(ds, du) \\ + \int_0^t \int_0^\infty \int_0^{X(s-)} z\tilde{M}(ds, dz, du). \end{aligned} \tag{4.7}$$

(Recall the condition $b = 0$ introduced at the beginning of this section.) By a modification of the proof of Theorem 3.1, one can see for each $x \ge 0$ there is a unique positive strong solution $\{Y_t(x) : t \ge 0\}$ to (4.7) with $Y_0(x) = x$. Clearly, this solution is also a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. As in the proof of [7, Theorem 3.2], one can show that, for $y \geq x \geq 0$ and $t \geq 0$ we have a.s. $Y_t(y) \geq Y_t(x)$ and the process $\{Y_t(y) - Y_t(x) : t \geq 0\}$ is identically distributed with the solution $\{X(t) : t \geq 0\}$ to (4.7) with $X(0) = y - x$. Let $T(x, y) = \inf\{t \geq 0 : Y_t(x) = Y_t(y)\} = \inf\{t \geq 0 : Y_t(y) - Y_t(x) = 0\}$. Then $Y_t(x) = Y_t(y)$ a.s. for $t \geq T(x, y)$. Let f be a bounded Borel function on $[0, \infty)$. By (4.2), for any $t > 0$ we have

$$\begin{aligned}|Q_t f(x) - Q_t f(y)| &\leq \mathbf{P}[|f(Y_t(x)) - f(Y_t(y))|\mathbf{1}_{\{T(x,y)>t\}}] \\ &\leq 2\|f\|\mathbf{P}(T(x, y) > t) = 2\|f\|\mathbf{P}[1 - e^{-(y-x)\bar{v}^{\xi}_{0,t}}],\end{aligned}$$

where $\|\cdot\|$ denotes the supremum norm. The right-hand side tends to zero as $|x - y| \to 0$. Then $Q_t f$ is a continuous function on $[0, \infty)$. That proves the strong Feller property of $(Q_t)_{t\geq 0}$. □

In view of the result of Corollary 4.4, one may naturally expect a characterization of the decay rate of the survival probability $\mathbf{P}_x(\tau_0 > t)$ as $t \to \infty$. This problem for the CBRE-diffusion was studied by Böinghoff and Hutzenthaler [5]. More recently, Bansaye et al. [4] studied the problem for a CBRE-process with stable branching and finite variation Lévy environment. Palau and Pardo [27] studied the same problem for a CBRE-process with stable branching in a random environment given by a Brownian motion with drift. The decay rate of the survival probability for a CBRE-process with stable branching and a general Lévy environment was studied in Li and Xu [26]. The strong Feller property of classical CBI-processes was proved in Li and Ma [24].

5 CBIRE-processes

In this section, we discuss the CBIRE-process defined by (1.3). Let $h \geq 0$ be a constant and $(1 \wedge u)n(du)$ a finite measure on $(0, \infty)$. Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Let the processes $\{W(t) : t \geq 0\}$ and $\{L(t) : t \geq 0\}$ and the Poisson random measure $M(ds, dz, du)$ be as before. In addition, let $\{\eta(t) : t \geq 0\}$ be an increasing $(\mathscr{F}_t)$-Lévy process with

$$\mathbf{P}(e^{-\lambda\eta(t)}) = e^{-t\psi(\lambda)}, \qquad \lambda \geq 0, \tag{5.1}$$

where

$$\psi(\lambda) = h\lambda + \int_0^\infty (1 - e^{-\lambda u})n(du). \tag{5.2}$$

We assume that all those noises are independent of each other. The construction and basic properties of the CBIRE-process are provided by the following results. We here omit their proofs since they follow by modifications of the arguments in Sections 3 and 4.

Theorem 5.1 *For any positive $\mathscr{F}_0$-measurable random variable $Y(0)$, there is a unique positive strong solution $\{Y(t) : t \geq 0\}$ to (1.3).*

Proposition 5.2 *Let $\{Y(t) : t \geq 0\}$ be defined by (1.3) and $Z(t) = Y(t)\exp\{-\xi(t)\}$ for $t \geq 0$. Then we have*

$$\begin{aligned} Z(t) &= Y(0) - b\int_0^t e^{-\xi(s)}Y(s)ds + \int_0^t e^{-\xi(s)}\sqrt{2cY(s)}dW(s) \\ &\quad + \int_0^t\int_0^\infty\int_0^{Y(s-)} ze^{-\xi(s-)}\tilde{M}(ds,dz,du) + \int_0^t e^{-\xi(s)}d\eta(s). \end{aligned}$$

Theorem 5.3 *Let $\mathbf{P}^\xi$ be the quenched law given $\{\xi(t) : t \geq 0\}$. Then for any $\lambda \geq 0$ and $t \geq r \geq 0$, we have*

$$\mathbf{P}^\xi[e^{-\lambda Y(t)}|\mathscr{F}_r] = \exp\bigg\{ - Y(r)v^\xi_{r,t}(\lambda) - \int_r^t \psi(v^\xi_{s,t}(\lambda))ds\bigg\}.$$

Theorem 5.4 *The Markov process $\{Y(t) : t \geq 0\}$ defined by (1.3) has Feller transition semigroup $(\bar{Q}_t)_{t\geq 0}$ defined by*

$$\int_{[0,\infty)} e^{-\lambda y}\bar{Q}_t(x,dy) = \mathbf{P}\bigg[\exp\bigg\{ - xv^\xi_{0,t}(\lambda) - \int_0^t \psi(v^\xi_{s,t}(\lambda))ds\bigg\}\bigg]. \tag{5.3}$$

Theorem 5.5 *Under Grey's condition, the transition semigroup $(\bar{Q}_t)_{t\geq 0}$ defined by (5.3) has the strong Feller property.*

The transition semigroup $(\bar{Q}_t)_{t\geq 0}$ given by (5.3) generalizes the one defined by (3.10). We can give another useful characterization of this semigroup. For this purpose, let us make an extension of the probability space $(\Omega, \mathscr{F}, \mathbf{P})$ so that an independent copy $\{L'(t) : t \geq 0\}$ of the Lévy process $\{L(t) : t \geq 0\}$ is defined. By the Lévy-Itô decomposition we have

$$L'(t) = \beta t + \sigma B'(t) + \int_0^t\int_{[-1,1]}(e^z - 1)\tilde{N}'(ds,dz) + \int_0^t\int_{[-1,1]^c}(e^z - 1)N'(ds,dz),$$

where $\{B'(t) : t \geq 0\}$ is a Brownian motion and $N'(ds,dz)$ is a Poisson random measure on $(0,\infty) \times \mathbb{R}$ with intensity $ds\nu(dz)$. Let $\{\xi'(t) : t \geq 0\}$ be the Lévy process defined by

$$\xi'(t) = at + \sigma B'(t) + \int_0^t\int_{[-1,1]} z\tilde{N}'(ds,dz) + \int_0^t\int_{[-1,1]^c} zN'(ds,dz).$$

Set $L(t) = -L'(-t-)$ and $\xi(t) = -\xi'(-t-)$ for $t < 0$. Then $\{L(t) : -\infty < t < \infty\}$ and $\{\xi(t) : -\infty < t < \infty\}$ are time homogeneous Lévy processes with $L(0) = \xi(0) = 0$. We can extend (2.12) easily to $r \leq t \in \mathbb{R}$. In particular, for any $\lambda \geq 0$ there is a pathwise unique positive solution $r \mapsto v^\xi_{r,0}(\lambda)$ to

$$v^\xi_{r,0}(\lambda) = \lambda - \int_r^0 \phi(v^\xi_{s,0}(\lambda))ds + \int_r^0 v^\xi_{s,0}(\lambda)L(\overleftarrow{ds}), \qquad r \leq 0.$$

The result of Theorem 2.4 can be extended to $r \le t \in \mathbb{R}$. Then $r \mapsto v^\xi_{r,0}(\lambda)$ is also the pathwise unique positive solution to

$$v^\xi_{r,0}(\lambda) = e^{-\xi(r)}\lambda - \int_r^0 e^{\xi(s)-\xi(r)}\phi(v^\xi_{s,0}(\lambda))ds, \qquad r \le 0. \tag{5.4}$$

It follows that $r \mapsto u^\xi_{r,0}(\lambda) := e^{\xi(r)}v^\xi_{r,0}(\lambda)$ is the pathwise unique positive solution to

$$u^\xi_{r,0}(\lambda) = \lambda - \int_r^0 e^{\xi(s)}\phi(e^{-\xi(s)}u^\xi_{s,t}(\lambda))ds, \qquad r \le 0. \tag{5.5}$$

By the time homogeneity of the Lévy process $\{\xi(t) : -\infty < t < \infty\}$, we have

$$\int_{[0,\infty)} e^{-\lambda y}\bar{Q}_t(x, dy) = \mathbf{P}\bigg[\exp\bigg\{ - xv^\xi_{-t,0}(\lambda) - \int_{-t}^0 \psi(v^\xi_{s,0}(\lambda))ds\bigg\}\bigg]. \tag{5.6}$$

In the subcricital case, a necessary and sufficient condition for the ergodicity of the transition semigroup $(\bar{Q}_t)_{t\ge 0}$ defined by (5.3) is provided by the following theorem:

Theorem 5.6 *Suppose that $a_1 := \mathbf{P}[\xi(1)] < b$. Then there is a probability measure μ on $[0, \infty)$ so that $\bar{Q}_t(x, \cdot) \to \mu$ weakly as $t \to \infty$ for every $x \ge 0$ if and only if*

$$\int_1^\infty \log(u)n(du) < \infty.$$

Under the above condition, we have

$$\int_{[0,\infty)} e^{-\lambda y}\mu(dy) = \mathbf{P}\bigg[\exp\bigg\{ - \int_{-\infty}^0 \psi(v^\xi_{s,0}(\lambda))ds\bigg\}\bigg]. \tag{5.7}$$

Proof. Under the assumption, we may adjust the parameters in (1.3) so that $b > 0 > a_1$. Then $u^\xi_{-t,0}(\lambda) \le e^{-bt}\lambda$ by Gronwall's inequality, and hence $u^\xi_{-t,0}(\lambda) \to 0$ as $t \to \infty$. In view of (5.3), by applying [23, Theorem 1.20] and dominated convergence we conclude that $\bar{Q}_t(x, \cdot)$ converges to a probability measure μ as $t \to \infty$ for every $x \ge 0$ if and only if a.s.

$$f(\xi, \lambda) := \int_{-\infty}^0 \psi(v^\xi_{s,0}(\lambda))ds = \int_{-\infty}^0 \psi(e^{-\xi(s)}u^\xi_{s,0}(\lambda))ds < \infty, \qquad \lambda \ge 0.$$

Clearly, μ is given by (5.7) if the above condition is satisfied. For any $z > 0$, define $\tau_\lambda(z) = \sup\{r < 0 : u^\xi_{r,0}(\lambda) \le z\}$. By (5.5) and a change of the variable, we have

$$f(\xi, \lambda) = \int_{-\infty}^0 \frac{\psi(e^{-\xi(s)}u^\xi_{s,0}(\lambda))}{\phi(e^{-\xi(s)}u^\xi_{s,0}(\lambda))}e^{-\xi(s)}du^\xi_{s,0}(\lambda) = \int_0^\lambda \frac{\psi(e^{-\xi(\tau_\lambda(z))}z)}{\phi(e^{-\xi(\tau_\lambda(z))}z)}e^{-\xi(\tau_\lambda(z))}dz.$$

Since $\xi(t) \to \infty$ as $t \to -\infty$, we have a.s. $M := \sup_{s\leq 0} e^{-\xi(s)} < \infty$. It is simple to see that $\phi(z) = bz + o(z)$ as $z \to 0$. Then $f(\xi, \lambda) < \infty$ if and only if

$$\begin{aligned}\int_0^\lambda dz \int_0^\infty \frac{1-\exp\{-e^{-\xi(\tau_\lambda(z))}zu\}}{z} n(du) \\ = \int_0^\infty n(du) \int_0^{\lambda u} \frac{1-\exp\{-e^{-\xi(\tau_\lambda(y/u))}y\}}{y} dy < \infty. \end{aligned} \tag{5.8}$$

For all $u > 0$ we have $1 - \exp\{-e^{-\xi(\tau_\lambda(y/u))}y\} \leq My$. It follows that

$$\int_{(0,1]} n(du) \int_0^{\lambda u} \frac{1-\exp\{-e^{-\xi(\tau_\lambda(y/u))}y\}}{y} dy \leq M\lambda \int_{(0,1]} un(du) < \infty.$$

For $u > 1$ we have $1 - \exp\{-e^{-\xi(\tau_\lambda(y/u))}y\} \to 1$ as $y \to \infty$. Then (5.8) holds if and only if

$$\int_{(1,\infty)} n(du) \int_0^{\lambda u} \frac{1}{y} dy = \int_{(1,\infty)} \log(\lambda u) n(du) < \infty.$$

That implies the desired result. □

Corollary 5.7 *Suppose that $a_1 := \mathbf{P}[\xi(1)] < b$. Let $(\tilde{Q}_t)_{t\geq 0}$ be the transition semigroup defined by (3.10). Then there is a probability measure μ on $[0, \infty)$ so that $\tilde{Q}_t(x, \cdot) \to \mu$ weakly as $t \to \infty$ for every $x \geq 0$ if and only if*

$$\int_1^\infty u \log(u) m(du) < \infty.$$

Under the above condition, we have

$$\int_{[0,\infty)} e^{-\lambda y} \mu(dy) = \mathbf{P}\Big[\exp\Big\{-\int_{-\infty}^0 \phi_0'(v_{s,0}^\xi(\lambda)) ds\Big\}\Big],$$

where $\phi_0'(z) = \phi'(z) - b$.

Proof. One can see that (3.10) is the special form of (5.3) with $\psi = \phi_0'$. Then we get the results by Theorem 5.6. □

References

[1] Afanasyev, V. I.; Böinghoff, C.; Kersting, G. and Vatutin, V.A. (2012): Limit theorems for a weakly subcritical branching process in a random environment. *J. Theoret. Probab.* **25**, 703–732.

[2] Afanasy'ev, V.I.; Geiger, J.; Kersting, G. and Vatutin, V.A. (2005): Criticality for branching processes in random environment. *Ann. Probab.* **33**, 645–673.

[3] Aliev, S.A. (1985). A limit theorem for the Galton–Watson branching processes with immigration. *Ukrainian Math. J.* **37**, 535–438.

[4] Bansaye, V.; Millan, J.C.P. and Smadi, C. (2013): On the extinction of continuous state branching processes with catastrophes. *Electron. J. Probab.* **18**, Paper No. 106, 1–31.

[5] Böinghoff, C. and Hutzenthaler, M. (2012): Branching diffusions in random environment. *Markov Process. Related Fields*, **18**, 269–310.

[6] Dawson, D.A. and Li, Z. (2006): Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34**, 1103–1142.

[7] Dawson, D.A. and Li, Z. (2012): Stochastic equations, flows and measure–valued processes. *Ann. Probab.* **40**, 813–857.

[8] El Karoui, N. and Méléard, S. (1990): Martingale measures and stochastic calculus. *Probab. Theory Related Fields* **84**, 83–101.

[9] Ethier, S.N. and Kurtz, T.G. (1986): *Markov Processes: Characterization and Convergence.* Wiley, New York.

[10] Feller, W. (1951): Diffusion processes in genetics. In: *Proceedings 2nd Berkeley Symp. Math. Statist. Probab.*, 1950, 227–246. Univ. California Press, Berkeley and Los Angeles.

[11] Fu, Z. and Li, Z. (2010): Stochastic equations of non–negative processes with jumps. *Stochastic Process. Appl.* **120**, 306–330.

[12] Guivarch, Y. and Liu, Q.S. (2001): Propriétés asymptotiques des processus de branchement en environnement aléatoire. *C.R. Acad. Sci. Paris Sér. I Math.* **332**, 339–344.

[13] Hairer, M. and Mattingly, J. (2016): The strong Feller property for singular stochastic PDEs. *arXiv:1610.03415.*

[14] Helland, I.S. (1981): Minimal conditions for weak convergence to a diffusion process on the line. *Ann. Probab.* **9**, 429–452.

[15] Hutzenthaler, M. (2011): Supercritical branching diffusions in random environment. *Electron. Commun. Probab.* **16**, 781–791.

[16] Jiřina, M. (1958): Stochastic branching processes with continuous state space. *Czech. Math. J.* **8**, 292–313.

[17] Kawazu, K. and Watanabe, S. (1971): Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16**, 36–54.

[18] Keiding, N. (1975): Extinction and exponential growth in random environments. *Theor. Population Biology 8*, 49–63.

[19] Kurtz, T.G. (1978): Diffusion approximations for branching processes. In: *Branching processes (Conf., Saint Hippolyte, Que., 1976)*, **Vol.5**, 269–292.

[20] Lamperti, J. (1967a): The limit of a sequence of branching processes. *Z. Wahrsch. verw. Geb.* **7**, 271–288.

[21] Lamperti, J. (1967b): Continuous state branching processes. *Bull. Amer. Math. Soc.* **73**, 382–386.

[22] Li, Z. (2006): A limit theorem for discrete Galton–Watson branching processes with immigration. *J. Appl. Probab.* **43**, 289–295.

[23] Li, Z. (2011): *Measure-Valued Branching Markov Processes.* Springer, Heidelberg.

[24] Li, Z. and Ma, C. (2015): Asymptotic properties of estimators in a stable Cox–Ingersoll–Ross model. *Stochastic Process. Appl.* **125**, 3196–3233.

[25] Li, Z. and Pu, F. (2012): Strong solutions of jump-type stochastic equations. *Electron. Commun. Probab.* **17**, 33, 1–13.

[26] Li, Z. and Xu, W. (2016): Asymptotic results for exponential functionals of Lévy processes. *arXiv:1601.02363v1.* 11 Jan., 2016.

[27] Palau, S. and Pardo, J.C. (2017): Continuous state branching processes in random environment: The Brownian case. *Stochastic Process. Appl.* **127**, 957–994.

[28] Palau, S. and Pardo, J.C. (2015+): Branching processes in a Lévy random environment. *arXiv:1512.07691v1.* 24 Dec., 2015.

[29] Smith, W.L. (1968): Necessary conditions for almost sure extinction of a branching process with random environment. *Ann. Math. Statist.* **39**, 2136–2140.

[30] Smith, W.L. and Wilkinson, W. E. (1969): On branching processes in random environments. *Ann. Math. Statist.* **40**, 814–827.

[31] Sharpe, M. (1988): *General Theory of Markov Processes.* Academic Press, New York.

[32] Vatutin, V.A. (2004): A limit theorem for an intermediate subcritical branching process in a random environment. *Theory Probab. Appl.* **48**, 481–492.

[33] Wang F. (2013): *Harnack Inequalities for Stochastic Partial Differential Equations.* Springer, Heidelberg.