Distributions of jumps in a continuous-state branching process with immigration

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Abstract. We study the distributional properties of jumps in a continuous-state branching process with immigration. In particular, a representation is given for the distribution of the first jump time of the process with jump size in a given Borel set. From this result we derive a characterization for the distribution of the local maximal jump of the process. The equivalence of this distribution and the total Lévy measure is then studied. For the continuous-state branching process without immigration, we also study similar problems for its global maximal jump.

Keywords and phrases: branching process; continuous-state; immigration; maximal jump; jump time; jump size.

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Abbreviated Title: Distributions of jumps in a CBI-process

1 Introduction

A continuous-state branching process (CB-process) is a nonnegative Markov process describing the random evolution of a population in an isolated environment. The branching property means that, if \( X = (X_t : t \geq 0) \) and \( Y = (Y_t : t \geq 0) \) are two independent CB-processes with the same transition semigroup, then \( X + Y = (X_t + Y_t : t \geq 0) \) is also a CB-process with that transition semigroup. A continuous-state branching process with immigration (CBI-process) is a generalization of the CB-process, which considers the possibility of input of immigrants during the evolution of the population. The transition semigroup of the CBI-process is uniquely determined by its branching mechanism \( \Phi \) and immigration mechanism \( \Psi \), both are functions on the nonnegative half line. The reader may refer to Kawazu and Watanabe (1971), Lamperti (1967a, 1967b) for early works on CB- and CBI-processes as biological models. See also Duquesne and Le Gall (2002), Kyprianou (2014) and Li (2011) for up to date treatments of those processes. We also mention that the CBI-process has been used widely in mathematical finance as models of interest rate, asset price and so on. A special form of the process is known in the financial world as the Cox–Ingersoll–Ross model; see, e.g., Brigo and Mercurio (2006) and Lamberton and Lapeyre (1996).

The CBI-process is a Feller process, so it has a càdlàg realization \( X = (X_t : t \geq 0) \). Let \( \Delta X_s := X_s - X_{s-} (\geq 0) \) denote the size of the jump of \( X \) at time \( s > 0 \). In this work, we are interested in distributional properties of jumps of the CBI-process. In particular, we shall

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give a representation of the distribution of the first occurrence time $\tau_A$ of its jump with jump size in some given Borel set $A \subset (0, \infty)$. From this result we derive a characterization for the distribution of the local maximal jump $\max_{0<s\leq t} \Delta X_s$ for any $t > 0$. Under suitable assumptions, we prove this distribution and the total Lévy measure of the process are equivalent. For the CB-process, we also study similar problems for the global maximal jump $\sup_{0<s<\infty} \Delta X_s$.

The tool of stochastic equations of the CBI-process established in Dawson and Li (2006) and Fu and Li (2010) plays a key role in the proof of our main result. The results obtained in this work are of clear interests in applications of the CB- and CBI-processes as biological and financial models.

The paper is organized as follows. In Section 2, some basic facts on CB- and CBI-processes are reviewed. In Section 3, we give the characterization of the distribution of the jump time $\tau_A$ for $A \subset (0, \infty)$. In Section 4, we establish a number of distributional properties of the local and global maximal jumps of the process.

## 2 CB- and CBI-processes

In this section, we review several basic facts on CB- and CBI-processes for the convenience of the reader. Let us fix a branching mechanism $\Phi$, which is a function on $\mathbb{R}_+ := [0, \infty)$ with the representation

$$
\Phi(z) = \alpha z + \beta z^2 + \int_{(0,\infty)} \pi_0(d\theta)(e^{-z\theta} - 1 + z\theta), \quad z \geq 0,
$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ are two constants, and $\pi_0$ is a $\sigma$-finite measure on $(0, \infty)$ satisfying

$$
\int_{(0,\infty)} \pi_0(d\theta)(\theta \wedge \theta^2) < \infty.
$$

A CB-process with branching mechanism $\Phi$ is a nonnegative Markov process with transition semigroup $(P_t)_{t \geq 0}$ defined by

$$
\int_{\mathbb{R}_+} e^{-\lambda y} P_t(x, dy) = e^{-xv_t(\lambda)}, \quad \lambda \geq 0,
$$

where $t \mapsto v_t(\lambda)$ is the unique nonnegative solution of

$$
v_t(\lambda) = \lambda - \int_0^t \Phi(v_s(\lambda))ds, \quad t \geq 0,
$$

or, in the equivalent differential form,

$$
\frac{d}{dt}v_t(\lambda) = -\Phi(v_t(\lambda)), \quad v_0(\lambda) = \lambda.
$$

Under the integrability condition (2.2), the CB-process started from any deterministic initial value has finite expectation. This in particular allows us to compensate large jumps of the process generated by the branching mechanism; see the stochastic integral equation (3.1).

We say the CB-process is **subcritical** if $\alpha > 0$, **critical** if $\alpha = 0$, and **supercritical** if $\alpha < 0$.

In view of (2.1), we have

$$
\Phi'(z) = \alpha + 2\beta z + \int_0^\infty \pi_0(d\theta)\theta(1 - e^{-z\theta}).
$$
which is increasing in \( z \geq 0 \). Then \( \Phi \) is a convex function. Consequently, the limit \( \Phi(\infty) := \lim_{z \to \infty} \Phi(z) \) exists in \([-\infty, 0] \cup \{\infty\}\). The limit \( \Phi'(\infty) := \lim_{z \to \infty} \Phi'(z) \) exists in \((-\infty, \infty]\). In fact, we have

\[
\Phi'(\infty) = \alpha + 2\beta \cdot \infty + \int_0^\infty \theta \pi_0(d\theta)
\]

with \( 0 \cdot \infty = 0 \) by convention. Observe that \( \Phi(\infty) \in [-\infty, 0] \) if and only if \( \Phi'(\infty) \in (-\infty, 0] \), and \( \Phi(\infty) = \infty \) if and only if \( \Phi'(\infty) \in (0, \infty] \). For \( \lambda \geq 0 \) let

\[
\Phi^{-1}(\lambda) = \inf\{z \geq 0 : \Phi(z) > \lambda\}.
\]

Of course, we have \( \Phi^{-1}(\lambda) = \infty \) for all \( \lambda \geq 0 \) if \( \Phi(\infty) \in [-\infty, 0] \). If \( \Phi(\infty) = \infty \), then \( \Phi^{-1} : [0, \infty) \to [\Phi^{-1}(0), \infty) \) is the inverse of the restriction of \( \Phi \) to \([\Phi^{-1}(0), \infty)\).

The CBI-process generalizes the CB-process given above. Let \( \Psi \) be an immigration mechanism, which is a function on \( \mathbb{R}_+ \) with representation

\[
\Psi(z) = \gamma z + \int_{(0,\infty)} \pi_1(d\theta)(1 - e^{-z\theta}), \quad z \geq 0,
\]

where \( \gamma \in \mathbb{R}_+ \) and \( \pi_1 \) is a \( \sigma \)-finite measure on \((0, \infty)\) satisfying

\[
\int_{(0,\infty)} \pi_1(d\theta)(1 \wedge \theta) < \infty.
\]

A nonnegative Markov process is called a CBI-process with branching mechanism \( \Phi \) and immigration mechanism \( \Psi \) if it has transition semigroup \((Q_t)_{t \geq 0}\) given by

\[
\int_{\mathbb{R}_+} e^{-\lambda y} Q_t(x, dy) = \exp\left\{ -xu_t(\lambda) - \int_0^t \Psi(u_s(\lambda))ds \right\}, \quad \lambda \geq 0.
\]

This reduces to a CB-process when \( \Psi \equiv 0 \). The reader may refer to Kawazu and Watanabe (1971) for discussion of CB- and CBI-processes with more general branching and immigration mechanisms.

From (2.7) we see that \((Q_t)_{t \geq 0}\) is a Feller semigroup, so the CBI-process has a Hunt process realization; see, e.g., Chung (1982, p.75). Let \( X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x) \) be such a realization. Then the sample path \( \{X_t : t \geq 0\} \) is \( P_x \)-a.s. càdàg for every \( x \geq 0 \). Let \( E_x \) denote the expectation with respect to the probability measure \( P_x \).

**Proposition 2.1** For \( t \geq 0, x \geq 0 \) and \( \lambda \geq 0 \) we have

\[
E_x \left[ \exp \left\{ -\lambda \int_0^t X_s ds \right\} \right] = \exp \left\{ -xu_t(\lambda) - \int_0^t \Psi(u_s(\lambda))ds \right\},
\]

where \( t \mapsto u_t(\lambda) \) is the unique nonnegative solution of

\[
u_t(\lambda) = t\lambda - \int_0^t \Phi(u_s(\lambda))ds, \quad t \geq 0.
\]

or, in the equivalent differential form,

\[
\frac{d}{dt} u_t(\lambda) = \lambda - \Phi(u_t(\lambda)), \quad u_0(\lambda) = 0.
\]
Proof. As special cases of Theorem 9.16 in Li (2011), we have (2.8) with \( t \to u_t(\lambda) \) being the unique nonnegative solution of (2.9), which is equivalent to its differential form (2.10). □

**Proposition 2.2** For \( \lambda > 0 \), the mapping \( t \to u_t(\lambda) \) is strictly increasing and \( \lim_{t \to \infty} u_t(\lambda) = \Phi^{-1}(\lambda) \).

**Proof.** Consider a Hunt realization \( X \) of the CB-process with branching mechanism \( \Phi \). By Proposition 2.1, we have

\[
E_x \left[ \exp \left\{ -\lambda \int_0^t X_s ds \right\} \right] = e^{-xu(\lambda)}. \tag{2.11}
\]

As observed in the proof of Proposition 3.1 in Li (2011), we have \( P_x(X_t > 0) > 0 \) for \( x > 0 \) and \( t \geq 0 \). By (2.11) we see that \( t \to u_t(\lambda) \) is strictly increasing, so \( (\partial/\partial t)u_t(\lambda) > 0 \) for all \( \lambda > 0 \). Let \( u_\infty(\lambda) = \lim_{t \to \infty} u_t(\lambda) \in (0, \infty) \). In the case \( \Phi(\infty) \in [-\infty, 0] \), we have \( \Phi(z) \leq 0 \) for all \( z \geq 0 \). Then \( (\partial/\partial t)u_t(\lambda) \geq \lambda \) and \( u_\infty(\lambda) = \infty \). In the case \( \Phi(\infty) = \infty \), we note \( \Phi(u_t(\lambda)) = \lambda - (\partial/\partial t)u_t(\lambda) < \lambda \), and hence \( u_t(\lambda) < \Phi^{-1}(\lambda) \), implying \( u_\infty(\lambda) \leq \Phi^{-1}(\lambda) < \infty \). It follows that

\[
0 = \lim_{t \to \infty} \frac{\partial}{\partial t} u_t(\lambda) = \lambda - \lim_{t \to \infty} \Phi(u_t(\lambda)) = \lambda - \Phi(u_\infty(\lambda)).
\]

Then we have \( u_\infty(\lambda) = \Phi^{-1}(\lambda) \). □

**Corollary 2.3** Let \( X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x) \) be a Hunt realization of the CB-process with branching mechanism \( \Phi \). Then for \( x > 0 \) and \( \lambda > 0 \), we have

\[
E_x \left[ \exp \left\{ -\lambda \int_0^\infty X_s ds \right\} \right] = \exp\{-x\Phi^{-1}(\lambda)\}. \tag{2.12}
\]

Note that (2.12) can also be derived from the theory of Lévy processes; see, e.g., Corollary 12.10 in Kyprianou (2014).

### 3 Distributional properties of jump times

Let \( \Phi \) and \( \Psi \) be the branching and immigration mechanisms with representations (2.1) and (2.6), respectively. Suppose that on a suitable filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) satisfying the usual hypotheses, we have a standard \((\mathcal{F}_t)\)-Brownian motion \((B_t : t \geq 0)\), a \((\mathcal{F}_t)\)-Poisson point process \((p_t : t \geq 0)\) on \((0, \infty)^2\) with characteristic measure \(\pi_0(\cdot) dy\), and a \((\mathcal{F}_t)\)-Poisson point process \((q_t : t \geq 0)\) on \((0, \infty)\) with characteristic measure \(\pi_1(\cdot) dz\). Suppose that \((B_t : t \geq 0), (p_t : t \geq 0), \) and \((q_t : t \geq 0)\) are independent. Let \(N_0(ds, dz, dy)\) denote the Poisson random measure on \((0, \infty)^3\) associated with \((p_t : t \geq 0),\) and \(\tilde{N}_0(ds, dz, dy)\) the compensated measure of \(N_0(ds, dz, dy)\). Let \(N_1(ds, dz)\) denote the Poisson random measure on \((0, \infty)^2\) associated with \((p_t : t \geq 0)\). By the results of Dawson and Li (2006) and Fu and Li (2010), for any \(\mathcal{F}_0\)-measurable nonnegative random variable \(X_0\) there is a unique nonnegative strong solution \(X = (X_t : t \geq 0)\) of the stochastic equation

\[
X_t = X_0 + \int_0^t \sqrt{2\beta X_s} dB_s + \int_{(0,t]} \int_{(0,\infty)} \int_{[0,X_{s-}]} z \tilde{N}_0(ds, dz, dy)
\]
For any \( k \) simultaneously, so we have
\[
(I - \alpha X_s)ds + \int_{(0,t]} \int_{(0,\infty)} zN_1(ds, dz).
\]

It was also proved in Dawson and Li (2006) and Fu and Li (2010) that \( X \) is a CBI-process with branching mechanism \( \Phi \) and immigration mechanism \( \Psi \). For \( x \geq 0 \) let \( P_x \) denote the conditional law of \( X \) given \( X_0 = x \).

In the sequel, we give some results on the distributional properties of the first jump time of the CBI-process with jump size in some given sets. To present the results, let us introduce some notation. For any Borel set \( A \subset (0, \infty) \) with \( \pi_0(A) + \pi_1(A) < \infty \), we define
\[
\Psi_A(z) = \Psi(z) - \int_A \pi_1(d\theta)(1 - e^{-z\theta})
\]
and
\[
\Phi_A(z) = \Phi(z) + \int_A \pi_0(d\theta)(1 - e^{-z\theta}).
\]
Then \( \Phi_A \) is also a branching mechanism and \( \Psi_A \) an immigration mechanism. For example, we have
\[
\Phi_A(z) = \alpha_A z + \beta z^2 + \int_{(0,\infty) \setminus A} \pi_0(d\theta)(e^{-z\theta} - 1 + z\theta),
\]
where
\[
\alpha_A = \alpha + \int_A \theta \pi_0(d\theta).
\]

**Proposition 3.1** Suppose that \( A \subset (0, \infty) \) is a Borel set with \( \pi_0(A) + \pi_1(A) < \infty \). For \( t > 0 \) let \( J_t(A) := \text{Card}\{s \in (0, t]: \Delta X_s = X_s - X_{s-} \in A\} \). Then for any \( x \geq 0 \) we have \( P_x(J_t(A) < \infty) = 1 \).

**Proof.** Let \( N_t^A \) and \( N_t^{A^c} \) be the restrictions of \( N_t \) to \( (0, \infty) \times A \times (0, \infty) \) and \( (0, \infty) \times ((0, \infty) \setminus A) \times (0, \infty) \), respectively. Similarly, let \( N_t^A \) and \( N_t^{A^c} \) be the restrictions of \( N_t \) to \( (0, \infty) \times A \) and \( (0, \infty) \times ((0, \infty) \setminus A) \), respectively. Then we can rewrite (3.1) into
\[
X_t = X_0 + \int_0^t \sqrt{2\beta X_s} dB_s + \int_{(0,t]} \int_{(0,\infty) \setminus A} \int_{(0,X_{s-})} zN_0^A(ds, dz, dy)
+ \int_0^t (\gamma - \alpha_A X_s)ds + \int_{(0,t]} \int_{(0,\infty) \setminus A} \int_{(0,X_{s-})} zN_1^A(ds, dz)
+ \int_{(0,t]} \int_A \int_{(0,X_{s-})} zN_0^A(ds, dz, dy) + \int_{(0,t]} \int_A zN_1^A(ds, dz).
\]

Note that the last two terms on the right hand side of the above equation never jump simultaneously, so we have
\[
J_t(A) = \int_{(0,t]} \int_A \int_{(0,X_{s-})} N_0^A(ds, dz, dy) + \int_{(0,t]} \int_A N_1^A(ds, dz).
\]
For any \( k \geq 1 \) let
\[
J_t(k, A) = \int_{(0,t]} \int_A \int_{(0,k)} N_0^A(ds, dz, dy) + \int_{(0,t]} \int_A N_1^A(ds, dz).
\]
It follows that
\[
E_x[J_t(k, A)] = kt\pi_0(A) + t\pi_1(A) < \infty,
\]
and so \( P_x(J_t(k, A) < \infty) = 1 \). Since \( s \mapsto X_t \) is càdlàg, we have \( \sup_{0<s\leq t} X_s < \infty \). Note also that \( J_t(A) \leq J_t(k, A) \) on the event \( \sup_{0<s\leq t} X_s < k \). It follows that
\[
P_x(J_t(A) = \infty) \leq \sum_{k=1}^{\infty} P_x\left( \{ J_t(A) = \infty \} \cap \left\{ \sup_{0<s\leq t} X_s < k \right\} \right)
\]
\[
= \sum_{k=1}^{\infty} P_x\left( \{ J_t(A) = J_t(k, A) = \infty \} \cap \left\{ \sup_{0<s\leq t} X_s < k \right\} \right)
\]
\[
\leq \sum_{k=1}^{\infty} P_x(J_t(k, A) = \infty) = 0.
\]
Then \( P_x(J_t(A) < \infty) = 1 \). \( \square \)

**Theorem 3.2** Suppose that \( A \subseteq (0, \infty) \) is a Borel set with \( \pi_0(A) + \pi_1(A) < \infty \). Let \( \tau_A = \min\{ s > 0 : \Delta X_s = X_s - X_{s-} \in A \} \), which is well-defined by the result of Proposition 3.1. Then for any \( x \geq 0 \) and \( t \geq 0 \) we have
\[
P_x(\tau_A > t) = \exp\left\{ -t\pi_1(A) - xu_t^A(\pi_0(A)) - \int_0^t \Psi_A(u_t^A(\pi_0(A)))ds \right\},
\]
where \( u_t^A(\lambda) \) is the unique nonnegative solution of
\[
\frac{d}{dt} u_t^A(\lambda) = \lambda - \Phi_A(u_t^A(\lambda)), \quad u_0^A(\lambda) = 0.
\]

**Proof.** We shall use the notation introduced in the proof of Proposition 3.1. Let \( (X_t^A : t \geq 0) \) be the solution of
\[
X_t^A = X_0 + \int_0^t \sqrt{2\beta X_s^A} dB_s + \int_{(0,t]} \int_{(0,\infty)\setminus A} \int_{(0,X_{s-}^A]} z\tilde{N}_0^{A^\tau}(ds, dz, dy) + \int_0^t (\gamma - \alpha_A X_s^A)ds + \int_{(0,t]} \int_{(0,\infty)\setminus A} zN_1^{A^\tau}(ds, dz).
\]
Then \( (X_t^A : t \geq 0) \) is a CBI-process with branching mechanism \( \Phi_A \) and immigration mechanism \( \Psi_A \). By Theorem 2.2 in Dawson and Li (2012) we have \( X_t^A \leq X_t \) for all \( t \geq 0 \). (Intuitively, we can obtain \( (X_t^A : t \geq 0) \) by removing from \( (X_t : t \geq 0) \) all masses produced by jumps of sizes in the set \( A \).) We claim that, up to a null set,
\[
\{ \tau_A > t \} = \left\{ \int_{(0,t]} \int_A \int_{(0,X_{s-}^A]} N_0^A(ds, dz, dy) + \int_{(0,t]} \int_A N_1^A(ds, dz) = 0 \right\}.
\]
Indeed, since \( X_s = X_s^A \) for \( 0 \leq s < \tau_A \), we have
\[
\{ \tau_A > t \} = \{ \tau_A > t \} \cap \left\{ \int_{(0,t]} \int_A \int_{(0,X_{s-}]} N_0^A(ds, dz, dy) \right\}.
\]
Taking any $x > 0$ we have

$$e^{-xu_A^1(\pi_0(A))} = P_x(\tau_A > t) \leq P_x(\tau_B > t) = e^{-xu_B^1(\pi_0(B))}.$$ 

Taking any $x > 0$ we get the result. (2) By Theorem 3.2 we have $P_x(\tau_A > t) = 1$ for every $t \geq 0$. Then $P_x(\tau_A = \infty) = \lim_{t \to \infty} P_x(\tau_A > t) = 1$. (3) By choosing a smaller set if it is necessary, we may assume $0 < \pi_0(A) + \pi_1(A) < \infty$. If $\pi_1(A) > 0$, then $t\pi_1(A) \to \infty$ as $t \to \infty$. Then in the case $\pi_1(A) = 0$, we must have $\pi_0(A) > 0$, so $s \mapsto u_A^1(\pi_0(A))$ is strictly increasing by Proposition 2.2. Since $\Psi \neq 0$, one can see

$$\lim_{t \to \infty} \int_0^t \Psi_A(u_A^1(\pi_0(A)))ds = \infty.$$ 

In view of (3.6), we have $P_x(\tau_A = \infty) = \lim_{t \to \infty} P_x(\tau_A > t) = 0$ in both cases. □

**Corollary 3.4** Suppose that $\Psi \equiv 0$. Then for any $x \geq 0$ and Borel set $A \subset (0, \infty)$ satisfying $0 < \pi_0(A) < \infty$, we have

$$P_x(\tau_A = \infty) = \exp\{ -x\Phi_A^{-1}(\pi_0(A))\}.$$
Proof. By applying Theorem 3.2 to the special case Ψ ≡ 0, we have
\[
P_x(\tau_1 = \infty) = \lim_{t \to \infty} P_x(\tau_1 > t) = \lim_{t \to \infty} \exp \{-x u_r^1(\pi_0(A))\}.
\]
Then the result follows by Proposition 2.2.

4 Local and global maximal jumps

Let \( X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P_x) \) be a Hunt realization of the CBI-process with branching mechanism \( \Phi \) and immigration mechanism \( \Psi \) given by (2.1) and (2.6), respectively. In this section, we shall give some characterizations of the local and global maximal jumps of the process.

Theorem 4.1 Suppose that \( r \geq 0 \) and \( \pi_0(r, \infty) + \pi_1(r, \infty) < \infty \). Then for any \( x \geq 0 \) we have
\[
P_x\left( \max_{s \in [0,t]} \Delta X_s \leq r \right) = \exp \left\{ -t \pi_1(r, \infty) - xu_r^1(\pi_0(r, \infty)) \right\} \int_0^t \Psi(r, \infty)(u_r^1(\pi_0(r, \infty)))ds,
\]
where \( u_r^1(\lambda) \) is the unique nonnegative solution of
\[
\frac{d}{dr} u_r^1(\lambda) = \lambda - \Phi(r, \infty)(u_r^1(\lambda)), \quad u_r^1(0) = 0.
\]

Proof. Since \( P_x(\max_{s \in [0,t]} \Delta X_s \leq r) = P_x(\tau_{(r, \infty)} > t) \), the result follows by Theorem 3.2.

Corollary 4.2 Suppose that \( \Psi \not\equiv 0 \). Then \( P_x(\sup_{s \in (0,\infty)} \Delta X_s = \sup(\pi_0 + \pi_1)) = 1 \) for any \( x \geq 0 \), where \( \sup(\pi_0 + \pi_1) = \sup \sup(\pi_0 + \pi_1) \).

Proof. Since \( \pi_0 + \pi_1 = 0 \), for any \( t > 0 \) we have \( P_x(\sup_{s \in (0,t]} \Delta X_s \leq \sup(\pi_0 + \pi_1)) = 1 \) by Theorem 4.1. Then
\[
P_x\left( \sup_{s \in (0,\infty)} \Delta X_s \leq \sup(\pi_0 + \pi_1) \right) = \lim_{t \to \infty} P_x\left( \sup_{s \in (0,t]} \Delta X_s \leq \sup(\pi_0 + \pi_1) \right) = 1.
\]
For any \( z < \sup(\pi_0 + \pi_1) \) we have \( (\pi_0 + \pi_1)[z, \sup(\pi_0 + \pi_1)] > 0 \). By Corollary 3.3 (3),
\[
P_x\left( \sup_{s \in (0,\infty)} \Delta X_s \in [z, \sup(\pi_0 + \pi_1)] \right) \geq P_x(\tau_{[z, \sup(\pi_0 + \pi_1)]} < \infty) = 1.
\]
Since \( z < \sup(\pi_0 + \pi_1) \) was arbitrary, it follows that \( P_x(\sup \Delta X = \sup(\pi_0 + \pi_1)) = 1 \).

Corollary 4.3 Suppose that \( \Psi \equiv 0 \). Then for any \( x \geq 0 \) and \( r \geq 0 \) satisfying \( 0 < \pi_0(r, \infty) < \infty \) we have
\[
P_x\left( \sup_{s \in (0,\infty)} \Delta X_s \leq r \right) = \exp \{-x \Phi^{-1}(\pi_0(r, \infty))\}.
\]

Proof. This follows by Theorem 4.1 and Proposition 2.2.
Corollary 4.4 Suppose that $\Psi \equiv 0$ and let $\sup(\pi_0) = \sup \text{supp}(\pi_0)$. Then for any $x \geq 0$ we have
\[ P_x \left( \sup_{s \geq 0} \Delta X_s = \sup(\pi_0) \right) = 1 - \exp \{-x\Phi^{-1}(\pi_0(\{\sup(\pi_0)\})\}\}
\]
with $\Phi_{\{0\}} = \Phi_{\{\infty\}} = \Phi$ and $\pi_0(\{0\}) = \pi_0(\{\infty\}) = 0$ by convention.

Proof. By the proof of Corollary 4.2 we have $P_x(\sup_{s \geq 0} \Delta X_s \leq \sup(\pi_0)) = 1$. For any $z < \sup(\pi_0)$ we have $\pi_0(z, \sup(\pi_0)) > 0$. By Corollary 4.3 it follows that
\[ P_x \left( \sup_{s \geq 0} \Delta X_s \in (z, \sup(\pi_0)) \right) = 1 - \exp \{-x\Phi^{-1}(\pi_0(z, \sup(\pi_0))\}\}
\]
and the desired result follows from Corollary 4.3.

Corollary 4.5 Suppose that $\alpha > 0$ and $\Psi \equiv 0$. If the measure $\pi_0$ has unbounded support, then for any $x > 0$, we have, as $r \to \infty$,
\[ P_x \left( \sup_{s \geq 0} \Delta X_s > r \right) = 1 - \exp \{-x\Phi^{-1}(\pi_0(r, \infty))\} \sim \frac{x}{\alpha} \pi_0(r, \infty).
\]

Proof. By (3.4) we see by dominated convergence that $(\partial/\partial z)\Phi_{(r,\infty)}(0) = \alpha_{(r,\infty)}$. It follows that $(\partial/\partial z)\Phi^{-1}_{(r,\infty)}(0) = 1/\alpha_{(r,\infty)}$. Then, as $r \to \infty$,
\[ \Phi^{-1}_{(r,\infty)}(\pi_0(r, \infty)) \sim \pi_0(r, \infty)/\alpha_{(r,\infty)} \sim \pi_0(r, \infty)/\alpha,
\]
and the desired result follows from Corollary 4.3.

We remark that a special form of Corollary 4.3 has been obtained by Bertoin (2011). The next theorem establishes the equivalence of the distribution of the local maximal jump of the CBI-process and the total Lévy measure $\pi_0 + \pi_1$. In view of Theorem 4.1, we may have $P_x(\max_{s \in (0,t]} \Delta X_s = 0) > 0$, so we only discuss the absolute continuity on the set $(0, \infty)$.

Theorem 4.6 Suppose that $x + \gamma > 0$. Then for any $t > 0$, the measure $\pi_0 + \pi_1$ and the distribution $P_x(\max_{s \in (0,t]} \Delta X_s \in \cdot)\big|_{(0,\infty)}$ are equivalent.

Proof. Recall that $\Psi$ and $\Psi_A$ are defined by (2.6) and (3.2), respectively. If $A \subset (0, \infty)$ is a Borel set with $\pi_0(A) + \pi_1(A) = 0$, by Theorem 3.2 we have
\[ P_x \left( \max_{s \in (0,t]} \Delta X_s \in A \right) \leq P_x(\tau_A \leq t) = 1 - P_x(\tau_A > t) = 0.
\]
Then $P_x(\max_{s \in (0,t]} \Delta X \in \cdot)\big|_{(0,\infty)}$ is absolutely continuous with respect to $\pi_0 + \pi_1$. To prove the absolute continuity of $\pi_0 + \pi_1$ with respect to $P_x(\max_{s \in (0,t]} \sup \Delta X \in \cdot)\big|_{(0,\infty)}$, we consider a Borel set $A \subset (0, \infty)$ and a constant $r > 0$. Since
\[ \left\{ \max_{s \in (0,t]} \Delta X_s \in A \right\} \supset \left\{ \max_{s \in (0,t]} \Delta X_s \in A \cap [r, \infty) \right\}
\]
\[ \supset \left\{ \tau_{\Delta X_s \in [r, \infty)} \leq t \right\} \cap \{\pi_{[r, \infty)}, A > t\}
\]
\[ = \{\pi_{[r, \infty)} \leq t\} \setminus \{\pi_{[r, \infty)}, A \leq t\},
\]
we have
\[ P_x \left( \max_{s \in [0,t]} \Delta X_s \in A \right) \geq P_x (\tau_{(r,\infty)} \leq t) - P_x (\tau_{[r,\infty) \setminus A} \leq t). \]

Suppose that \( P_x (\sup_{s \in (0,t]} \Delta X_s \in A) = 0. \) Then \( P_x (\tau_{[r,\infty)} \leq t) = P_x (\tau_{[r,\infty) \setminus A} \leq t), \) so the result of Theorem 3.2 implies
\[
t \pi_1 [r, \infty) + xu_t^{[r,\infty)}(\pi_0 | r, \infty)) + \int_0^t \Psi_{[r,\infty)}(u_s^{[r,\infty)}(\pi_0 | r, \infty))) ds
= t \pi_1 ([r, \infty) \setminus A) + xu_t^{[r,\infty)\setminus A}(\pi_0 | [r, \infty) \setminus A)
+ \int_0^t \Psi_{[r,\infty)\setminus A}(u_s^{[r,\infty)\setminus A}(\pi_0 | [r, \infty) \setminus A))) ds. \tag{4.2}
\]

By Corollary 3.3 (1) we have
\[
u_t^{[r,\infty)}(\pi_0 | r, \infty)) \geq u_t^{[r,\infty)\setminus A}(\pi_0 | [r, \infty) \setminus A), \quad t \geq 0. \tag{4.3}
\]

Then (4.2) implies
\[
t \pi_1 [r, \infty) \leq \int_0^t ds \int_{[r,\infty) \setminus A} (1 - e^{-\theta u_s^{[r,\infty)}(\pi_0 | r, \infty))) \pi_1 (d\theta).
\]

It follows that \( \pi_1 ([r, \infty) \setminus A) = 0. \) Since \( r > 0 \) was arbitrary in the above, we have proved \( \pi_1 (A) = 0. \) Using (4.2) and (4.3) we have
\[
xu_t^{[r,\infty)}(\pi_0 | r, \infty)) + \gamma \int_0^t u_s^{[r,\infty)}(\pi_0 | r, \infty))) ds
\leq xu_t^{[r,\infty)\setminus A}(\pi_0 | [r, \infty) \setminus A)) + \gamma \int_0^t u_s^{[r,\infty)\setminus A}(\pi_0 | [r, \infty) \setminus A) ds,
\]
and so using (4.3) again we get
\[
u_t^{[r,\infty)}(\pi_0 | r, \infty)) = u_t^{[r,\infty)\setminus A}(\pi_0 | [r, \infty) \setminus A)) = a(r, t).
\]

It follows that
\[
\frac{\partial u_t^{[r,\infty)}(\pi_0 | r, \infty))}{\partial t} = \frac{\partial u_t^{[r,\infty)\setminus A}(\pi_0 | [r, \infty) \setminus A))}{\partial t}.
\]

Then we can use (3.7) to see
\[
\pi_0 ([r, \infty) - \Phi_{[r,\infty)}(a(r, t)) = \pi_0 ([r, \infty) \setminus A) - \Phi_{[r,\infty)\setminus A}(a(r, t)),
\]
and hence
\[
\Phi_{[r,\infty)}(a(r, t)) = \Phi_{[r,\infty)\setminus A}(a(r, t)) + \pi_0 ([r, \infty) \setminus A).
\]

But, by (3.3), we should have
\[
\Phi_{[r,\infty)}(a(r, t)) = \Phi_{[r,\infty)\setminus A}(a(r, t)) + \int_{[r,\infty) \setminus A} \pi_0 (d\theta) (1 - e^{-a(r, t)\theta}).
\]
It follows that \( \pi_0([r, \infty) \cap A) = 0 \), implying \( \pi_0(A) = 0 \). That completes the proof. \( \square \)

The conclusion of Theorem 4.6 is not necessarily true in the case \( x = \gamma = 0 \). As a counterexample, consider the case where \( X_0 = 0 \), \( \pi_0 = \delta_1 \) and \( \pi_1 = \delta_2 \). In this case, we have \( \tau_{(2)} \leq t \) when \( \tau_{(1)} \leq t \), for otherwise \( X_s = 0 \) for all \( s \in [0, t] \). It follows that

\[
P_0 \left( \max_{s \in [0,t]} \Delta X_s \geq 1 \right) = 0.
\]

Then \( \pi_0 \) is not absolutely continuous with respect to \( P_0(\max_{s \in [0,t]} \Delta X_s \in \cdot) \).

For critical and subcritical branching CB-processes without immigration, we may also discuss the absolute continuity of the distribution of its global maximal jump. Such a result is presented in the following:

**Theorem 4.7** Suppose that \( \alpha \geq 0 \) and \( \Psi \equiv 0 \). Then for any \( x > 0 \) the Lévy measure \( \pi_0 \) and the distribution \( P_x(\sup_{s \in (0,\infty)} \Delta X_s \in \cdot) | (0,\infty) \) are equivalent.

**Proof.** Since \( \Psi = 0 \) and \( \alpha \geq 0 \), we have \( X_t \to 0 \) almost surely as \( t \to \infty \). If \( A \subset (0,\infty) \) is a Borel set so that \( \pi_0(A) = 0 \), by Corollary 3.3 (2) we have

\[
P_x \left( \sup_{s \in (0,\infty)} \Delta X_s \in A \right) \leq P_x(\tau_A < \infty) = 1 - P_x(\tau_A = \infty) = 0.
\]

Then \( P_x(\sup_{s \in (0,\infty)} \Delta X_s \in \cdot) | (0,\infty) \) is absolutely continuous with respect to \( \pi_0 \). Now suppose that \( A \subset (0,\infty) \) is a Borel set with \( \pi_0(A) > 0 \). For any \( r > 0 \), one can see as in the proof of Theorem 4.6 that

\[
P_x \left( \sup_{s \in (0,\infty)} \Delta X_s \in A \right) \geq P_x(\tau_{[r,\infty)} < \infty) - P_x(\tau_{[r,\infty)} \setminus A < \infty).
\]

If \( P_x(\sup_{s \in (0,\infty)} \Delta X_s \in A) = 0 \), we have \( P_x(\tau_{[r,\infty)} < \infty) = P_x(\tau_{[r,\infty)} \setminus A < \infty) \), so Corollary 3.4 implies

\[
\Phi^{-1}_{[r,\infty)}(\pi_0([r, \infty) \setminus A)) = \Phi^{-1}_{[r,\infty)}(\pi_0([r, \infty) \setminus A)) =: a(r).
\]

It follows that

\[
\Phi_{[r,\infty)}(a(r)) = \pi_0([r, \infty) \setminus A) + \pi_0(A \cap [r, \infty)) = \Phi_{[r,\infty) \setminus A}(\pi_0([r, \infty) \setminus A)) + \pi_0(A \cap [r, \infty)) = \Phi_{[r,\infty) \setminus A}(a(r)) + \pi_0(A \cap [r, \infty)).
\]

Then, as in the proof of Theorem 4.6, we must have \( \pi_0(A \cap [r, \infty)) = 0 \). This contradicts \( \pi_0(A) > 0 \) since \( r > 0 \) was arbitrary. It then follows that \( P_x(\sup_{s \in (0,\infty)} \Delta X_s \in A) > 0 \). \( \square \)

In the above theorem, we only consider the critical and subcritical cases. The supercritical case is more subtle since in that case we may have \( \sup_{s \in (0,\infty)} \Delta X_s = \sup(\pi_0) \) with strictly positive probability by Corollary 4.4. We leave the consideration of the details to the interested reader.

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References


