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Distributions of jumps in a continuous-state branching process with immigration [1]

Xin He and Zenghu Li [2]

Abstract. We study the distributional properties of jumps in a continuous-state branching process with immigration. In particular, a representation is given for the distribution of the first jump time of the process with jump size in a given Borel set. From this result we derive a characterization for the distribution of the local maximal jump of the process. The equivalence of this distribution and the total Lévy measure is then studied. For the continuous-state branching process without immigration, we also study similar problems for its global maximal jump.

Keywords and phrases: branching process; continuous-state; immigration; maximal jump; jump time; jump size.

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Abbreviated Title: Distributions of jumps in a CBI-process

1 Introduction

A continuous-state branching process (CB-process) is a nonnegative Markov process describing the random evolution of a population in an isolated environment. The *branching property* means that, if $X = (X_t : t \geq 0)$ and $Y = (Y_t : t \geq 0)$ are two independent CB-processes with the same transition semigroup, then $X + Y = (X_t + Y_t : t \geq 0)$ is also a CB-process with that transition semigroup. A continuous-state branching process with immigration (CBI-process) is a generalization of the CB-process, which considers the possibility of input of immigrants during the evolution of the population. The transition semigroup of the CBI-process is uniquely determined by its branching mechanism Φ and immigration mechanism Ψ, both are functions on the nonnegative half line. The reader may refer to Kawazu and Watanabe (1971), Lamperti (1967a, 1967b) for early works on CB- and CBI-processes as biological models. See also Duquesne and Le Gall (2002), Kyprianou (2014) and Li (2011) for up to date treatments of those processes. We also mention that the CBI-process has been used widely in mathematical finance as models of interest rate, asset price and so on. A special form of the process is known in the financial world as the Cox–Ingersoll–Ross model; see, e.g., Brigo and Mercurio (2006) and Lamberton and Lapeyre (1996).

The CBI-process is a Feller process, so it has a càdlàg realization $X = (X_t : t \geq 0)$. Let $\Delta X_s := X_s - X_{s-}$ (≥ 0) denote the size of the jump of X at time $s > 0$. In this work, we are interested in distributional properties of jumps of the CBI-process. In particular, we shall

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[2] Postal address: School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P. R. China. E-mails: hexin@bnu.edu.cn and lizh@bnu.edu.cn

give a representation of the distribution of the first occurrence time τ_A of its jump with jump size in some given Borel set $A \subset (0, \infty)$. From this result we derive a characterization for the distribution of the local maximal jump $\max_{0<s\leq t} \Delta X_s$ for any $t > 0$. Under suitable assumptions, we prove this distribution and the total Lévy measure of the process are equivalent. For the CB-process, we also study similar problems for the global maximal jump $\sup_{0<s<\infty} \Delta X_s$. The tool of stochastic equations of the CBI-process established in Dawson and Li (2006) and Fu and Li (2010) plays a key role in the proof of our main result. The results obtained in this work are of clear interests in applications of the CB- and CBI-processes as biological and financial models.

The paper is organized as follows. In Section 2, some basic facts on CB- and CBI-processes are reviewed. In Section 3, we give the characterization of the distribution of the jump time τ_A for $A \subset (0, \infty)$. In Section 4, we establish a number of distributional properties of the local and global maximal jumps of the process.

2 CB- and CBI-processes

In this section, we review several basic facts on CB- and CBI-processes for the convenience of the reader. Let us fix a *branching mechanism* Φ, which is a function on $\mathbb{R}_+ := [0, \infty)$ with the representation

$$\Phi(z) = \alpha z + \beta z^2 + \int_{(0,\infty)} \pi_0(d\theta)(e^{-z\theta} - 1 + z\theta), \qquad z \geq 0, \tag{2.1}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ are two constants, and π_0 is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \pi_0(d\theta)(\theta \wedge \theta^2) < \infty. \tag{2.2}$$

A *CB-process* with branching mechanism Φ is a nonnegative Markov process with transition semigroup $(P_t)_{t\geq 0}$ defined by

$$\int_{\mathbb{R}_+} e^{-\lambda y} P_t(x, dy) = e^{-x v_t(\lambda)}, \qquad \lambda \geq 0, \tag{2.3}$$

where $t \mapsto v_t(\lambda)$ is the unique nonnegative solution of

$$v_t(\lambda) = \lambda - \int_0^t \Phi(v_s(\lambda)) ds, \qquad t \geq 0, \tag{2.4}$$

or, in the equivalent differential form,

$$\frac{d}{dt} v_t(\lambda) = -\Phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda. \tag{2.5}$$

Under the integrability condition (2.2), the CB-process started from any deterministic initial value has finite expectation. This in particular allows us to compensate large jumps of the process generated by the branching mechanism; see the stochastic integral equation (3.1).

We say the CB-process is *subcritical* if $\alpha > 0$, *critical* if $\alpha = 0$, and *supercritical* if $\alpha < 0$. In view of (2.1), we have

$$\Phi'(z) = \alpha + 2\beta z + \int_0^\infty \pi_0(d\theta)\theta(1 - e^{-z\theta})$$

which is increasing in $z \geq 0$. Then Φ is a convex function. Consequently, the limit $\Phi(\infty) := \lim_{z\to\infty} \Phi(z)$ exists in $[-\infty, 0] \cup \{\infty\}$. The limit $\Phi'(\infty) := \lim_{z\to\infty} \Phi'(z)$ exists in $(-\infty, \infty]$. In fact, we have

$$\Phi'(\infty) = \alpha + 2\beta \cdot \infty + \int_0^\infty \theta \pi_0(d\theta)$$

with $0 \cdot \infty = 0$ by convention. Observe that $\Phi(\infty) \in [-\infty, 0]$ if and only if $\Phi'(\infty) \in (-\infty, 0]$, and $\Phi(\infty) = \infty$ if and only if $\Phi'(\infty) \in (0, \infty]$. For $\lambda \geq 0$ let

$$\Phi^{-1}(\lambda) = \inf\{z \geq 0 : \Phi(z) > \lambda\}.$$

Of course, we have $\Phi^{-1}(\lambda) = \infty$ for all $\lambda \geq 0$ if $\Phi(\infty) \in [-\infty, 0]$. If $\Phi(\infty) = \infty$, then $\Phi^{-1} : [0, \infty) \to [\Phi^{-1}(0), \infty)$ is the inverse of the restriction of Φ to $[\Phi^{-1}(0), \infty)$.

The CBI-process generalizes the CB-process given above. Let Ψ be an *immigration mechanism*, which is a function on $\mathbb{R}_+$ with representation

$$\Psi(z) = \gamma z + \int_{(0,\infty)} \pi_1(d\theta)(1 - e^{-z\theta}), \qquad z \geq 0, \tag{2.6}$$

where $\gamma \in \mathbb{R}_+$ and π_1 is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \pi_1(d\theta)(1 \wedge \theta) < \infty.$$

A nonnegative Markov process is called a *CBI-process* with branching mechanism Φ and immigration mechanism Ψ if it has transition semigroup $(Q_t)_{t\geq 0}$ given by

$$\int_{\mathbb{R}_+} e^{-\lambda y} Q_t(x, dy) = \exp\bigg\{ - x v_t(\lambda) - \int_0^t \Psi(v_s(\lambda)) ds \bigg\}, \qquad \lambda \geq 0. \tag{2.7}$$

This reduces to a CB-process when $\Psi \equiv 0$. The reader may refer to Kawazu and Watanabe (1971) for discussion of CB- and CBI-processes with more general branching and immigration mechanisms.

From (2.7) we see that $(Q_t)_{t\geq 0}$ is a Feller semigroup, so the CBI-process has a Hunt process realization; see, e.g., Chung (1982, p.75). Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \mathbf{P}_x)$ be such a realization. Then the sample path $\{X_t : t \geq 0\}$ is $\mathbf{P}_x$-a.s. càdàg for every $x \geq 0$. Let $\mathbf{E}_x$ denote the expectation with respect to the probability measure $\mathbf{P}_x$.

Proposition 2.1 *For $t \geq 0$, $x \geq 0$ and $\lambda \geq 0$ we have*

$$\mathbf{E}_x\bigg[\exp\bigg\{ - \lambda \int_0^t X_s ds\bigg\}\bigg] = \exp\bigg\{ - x u_t(\lambda) - \int_0^t \Psi(u_s(\lambda)) ds \bigg\}, \tag{2.8}$$

where $t \mapsto u_t(\lambda)$ is the unique nonnegative solution of

$$u_t(\lambda) = t\lambda - \int_0^t \Phi(u_s(\lambda)) ds, \qquad t \geq 0. \tag{2.9}$$

or, in the equivalent differential form,

$$\frac{d}{dt} u_t(\lambda) = \lambda - \Phi(u_t(\lambda)), \qquad u_0(\lambda) = 0. \tag{2.10}$$

Proof. As special cases of Theorem 9.16 in Li (2011), we have (2.8) with $t \mapsto u_t(\lambda)$ being the unique nonnegative solution of (2.9), which is equivalent to its differential form (2.10). $\square$

Proposition 2.2 *For $\lambda > 0$, the mapping $t \mapsto u_t(\lambda)$ is strictly increasing and* $\lim_{t\to\infty} u_t(\lambda) = \Phi^{-1}(\lambda)$.

Proof. Consider a Hunt realization X of the CB-process with branching mechanism Φ. By Proposition 2.1, we have

$$\mathbf{E}_x\bigg[\exp\Big\{-\lambda\int_0^t X_s ds\Big\}\bigg] = e^{-xu_t(\lambda)}. \tag{2.11}$$

As observed in the proof of Proposition 3.1 in Li (2011), we have $\mathbf{P}_x(X_t > 0) > 0$ for $x > 0$ and $t \geq 0$. By (2.11) we see that $t \mapsto u_t(\lambda)$ is strictly increasing, so $(\partial/\partial t)u_t(\lambda) > 0$ for all $\lambda > 0$. Let $u_\infty(\lambda) = \lim_{t\to\infty} u_t(\lambda) \in (0,\infty]$. In the case $\Phi(\infty) \in [-\infty, 0]$, we have $\Phi(z) \leq 0$ for all $z \geq 0$. Then $(\partial/\partial t)u_t(\lambda) \geq \lambda$ and $u_\infty(\lambda) = \infty$. In the case $\Phi(\infty) = \infty$, we note $\Phi(u_t(\lambda)) = \lambda - (\partial/\partial t)u_t(\lambda) < \lambda$, and hence $u_t(\lambda) < \Phi^{-1}(\lambda)$, implying $u_\infty(\lambda) \leq \Phi^{-1}(\lambda) < \infty$. It follows that

$$0 = \lim_{t\to\infty}\frac{\partial}{\partial t}u_t(\lambda) = \lambda - \lim_{t\to\infty}\Phi(u_t(\lambda)) = \lambda - \Phi(u_\infty(\lambda)).$$

Then we have $u_\infty(\lambda) = \Phi^{-1}(\lambda)$. $\square$

Corollary 2.3 *Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \mathbf{P}_x)$ be a Hunt realization of the CB-process with branching mechanism Φ. Then for $x > 0$ and $\lambda > 0$, we have*

$$\mathbf{E}_x\bigg[\exp\Big\{-\lambda\int_0^\infty X_s ds\Big\}\bigg] = \exp\{-x\Phi^{-1}(\lambda)\}. \tag{2.12}$$

Note that (2.12) can also be derived from the theory of Lévy processes; see, e.g., Corollary 12.10 in Kyprianou (2014).

3 Distributional properties of jump times

Let Φ and Ψ be the branching and immigration mechanisms with representations (2.1) and (2.6), respectively. Suppose that on a suitable filtered probability space $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{P})$ satisfying the usual hypotheses, we have a standard $(\mathscr{G}_t)$-Brownian motion $(B_t : t \geq 0)$, a $(\mathscr{G}_t)$-Poisson point process $(p_t : t \geq 0)$ on $(0,\infty)^2$ with characteristic measure $\pi_0(dz)dy$, and a $(\mathscr{G}_t)$-Poisson point process $(q_t : t \geq 0)$ on $(0,\infty)$ with characteristic measure $\pi_1(dz)$. Suppose that $(B_t : t \geq 0)$, $(p_t : t \geq 0)$, and $(q_t : t \geq 0)$ are independent. Let $N_0(ds, dz, dy)$ denote the Poisson random measure on $(0,\infty)^3$ associated with $(p_t : t \geq 0)$, and $\tilde{N}_0(ds, dz, dy)$ the compensated measure of $N_0(ds, dz, dy)$. Let $N_1(ds, dz)$ denote the Poisson random measure on $(0,\infty)^2$ associated with $(p_t : t \geq 0)$. By the results of Dawson and Li (2006) and Fu and Li (2010), for any $\mathscr{G}_0$-measurable nonnegative random variable X_0 there is a unique nonnegative strong solution $X = (X_t : t \geq 0)$ of the stochastic equation

$$X_t = X_0 + \int_0^t \sqrt{2\beta X_s}dB_s + \int_{(0,t]}\int_{(0,\infty)}\int_{(0,X_{s-}]} z\tilde{N}_0(ds, dz, dy)$$

$$+\int_0^t(\gamma-\alpha X_s)ds+\int_{(0,t]}\int_{(0,\infty)}zN_1(ds,dz). \tag{3.1}$$

It was also proved in Dawson and Li (2006) and Fu and Li (2010) that X is a CBI-process with branching mechanism Φ and immigration mechanism Ψ. For $x\geq 0$ let $\mathbf{P}_x$ denote the conditional law of X given $X_0=x$.

In the sequel, we give some results on the distributional properties of the first jump time of the CBI-process with jump size in some given sets. To present the results, let us introduce some notation. For any Borel set $A\subset(0,\infty)$ with $\pi_0(A)+\pi_1(A)<\infty$, we define

$$\Psi_A(z)=\Psi(z)-\int_A\pi_1(d\theta)(1-e^{-z\theta}) \tag{3.2}$$

and

$$\Phi_A(z)=\Phi(z)+\int_A\pi_0(d\theta)(1-e^{-z\theta}). \tag{3.3}$$

Then Φ_A is also a branching mechanism and Ψ_A an immigration mechanism. For example, we have

$$\Phi_A(z)=\alpha_Az+\beta z^2+\int_{(0,\infty)\setminus A}\pi_0(d\theta)(e^{-z\theta}-1+z\theta), \tag{3.4}$$

where

$$\alpha_A=\alpha+\int_A\theta\pi_0(d\theta).$$

Proposition 3.1 *Suppose that $A\subset(0,\infty)$ is a Borel set with $\pi_0(A)+\pi_1(A)<\infty$. For $t>0$ let $J_t(A):=\mathrm{Card}\{s\in(0,t]:\Delta X_s=X_s-X_{s-}\in A\}$. Then for any $x\geq 0$ we have $\mathbf{P}_x(J_t(A)<\infty)=1$.*

Proof. Let N_0^A and $N_0^{A^c}$ be the restrictions of N_0 to $(0,\infty)\times A\times(0,\infty)$ and $(0,\infty)\times((0,\infty)\setminus A)\times(0,\infty)$, respectively. Similarly, let N_1^A and $N_1^{A^c}$ be the restrictions of N_1 to $(0,\infty)\times A$ and $(0,\infty)\times((0,\infty)\setminus A)$, respectively. Then we can rewrite (3.1) into

$$\begin{aligned} X_t\; &= X_0+\int_0^t\sqrt{2\beta X_s}dB_s+\int_{(0,t]}\int_{(0,\infty)\setminus A}\int_{(0,X_{s-}]}z\tilde{N}_0^{A^c}(ds,dz,dy)\\ &\quad+\int_0^t(\gamma-\alpha_AX_s)ds+\int_{(0,t]}\int_{(0,\infty)\setminus A}zN_1^{A^c}(ds,dz)\\ &\quad+\int_{(0,t]}\int_A\int_{(0,X_{s-}]}zN_0^A(ds,dz,dy)+\int_{(0,t]}\int_A zN_1^A(ds,dz). \end{aligned}\tag{3.5}$$

Note that the last two terms on the right hand side of the above equation never jump simultaneously, so we have

$$J_t(A)=\int_{(0,t]}\int_A\int_{(0,X_{s-}]}N_0^A(ds,dz,dy)+\int_{(0,t]}\int_A N_1^A(ds,dz).$$

For any $k\geq 1$ let

$$J_t(k,A)=\int_{(0,t]}\int_A\int_{(0,k]}N_0^A(ds,dz,dy)+\int_{(0,t]}\int_A N_1^A(ds,dz).$$

It follows that

$$\mathbf{E}_x[J_t(k,A)] = kt\pi_0(A) + t\pi_1(A) < \infty,$$

and so $\mathbf{P}_x(J_t(k,A) < \infty) = 1$. Since $s \mapsto X_t$ is càdlàg, we have $\sup_{0<s\le t} X_s < \infty$. Note also that $J_t(A) \le J_t(k,A)$ on the event $\sup_{0<s\le t} X_s < k$. It follows that

$$\begin{aligned}
\mathbf{P}_x(J_t(A) = \infty) &\le \sum_{k=1}^{\infty} \mathbf{P}_x\Big(\{J_t(A) = \infty\} \cap \Big\{\sup_{0<s\le t} X_s < k\Big\}\Big) \\
&= \sum_{k=1}^{\infty} \mathbf{P}_x\Big(\{J_t(A) = J_t(k,A) = \infty\} \cap \Big\{\sup_{0<s\le t} X_s < k\Big\}\Big) \\
&\le \sum_{k=1}^{\infty} \mathbf{P}_x(J_t(k,A) = \infty) = 0.
\end{aligned}$$

Then $\mathbf{P}_x(J_t(A) < \infty) = 1$. □

Theorem 3.2 *Suppose that $A \subset (0,\infty)$ is a Borel set with $\pi_0(A) + \pi_1(A) < \infty$. Let $\tau_A = \min\{s > 0 : \Delta X_s = X_s - X_{s-} \in A\}$, which is well-defined by the result of Proposition 3.1. Then for any $x \ge 0$ and $t \ge 0$ we have*

$$\mathbf{P}_x(\tau_A > t) = \exp\Big\{ - t\pi_1(A) - xu_t^A(\pi_0(A)) - \int_0^t \Psi_A(u_s^A(\pi_0(A)))ds\Big\}, \tag{3.6}$$

where $u_t^A(\lambda)$ is the unique nonnegative solution of

$$\frac{d}{dt}u_t^A(\lambda) = \lambda - \Phi_A(u_t^A(\lambda)), \qquad u_0^A(\lambda) = 0. \tag{3.7}$$

Proof. We shall use the notation introduced in the proof of Proposition 3.1. Let $(X_t^A : t \ge 0)$ be the solution of

$$\begin{aligned}
X_t^A \;=\; & X_0 + \int_0^t \sqrt{2\beta X_s^A} dB_s + \int_{(0,t]} \int_{(0,\infty)\setminus A} \int_{(0,X_{s-}^A]} z\tilde{N}_0^{A^c}(ds,dz,dy) \\
& + \int_0^t (\gamma - \alpha_A X_s^A)ds + \int_{(0,t]} \int_{(0,\infty)\setminus A} zN_1^{A^c}(ds,dz).
\end{aligned} \tag{3.8}$$

Then $(X_t^A : t \ge 0)$ is a CBI-process with branching mechanism Φ_A and immigration mechanism Ψ_A. By Theorem 2.2 in Dawson and Li (2012) we have $X_t^A \le X_t$ for all $t \ge 0$. (Intuitively, we can obtain $(X_t^A : t \ge 0)$ by removing from $(X_t : t \ge 0)$ all masses produced by jumps of sizes in the set A.) We claim that, up to a null set,

$$\{\tau_A > t\} \;=\; \Big\{ \int_{(0,t]} \int_A \int_{(0,X_{s-}^A]} N_0^A(ds,dz,dy) + \int_{(0,t]} \int_A N_1^A(ds,dz) = 0\Big\}. \tag{3.9}$$

Indeed, since $X_s = X_s^A$ for $0 \le s < \tau_A$, we have

$$\{\tau_A > t\} \;=\; \{\tau_A > t\} \cap \Big\{ \int_{(0,t]} \int_A \int_{(0,X_{s-}]} N_0^A(ds,dz,dy)$$

$$
\begin{aligned}
&\qquad\qquad + \int_{(0,t]}\int_A N_1^A(ds,dz) = 0\Big\} \\
&\subset \Big\{ \int_{(0,t]}\int_A\int_{(0,X_{s-}^A]} N_0^A(ds,dz,dy) + \int_{(0,t]}\int_A N_1^A(ds,dz) = 0\Big\}.
\end{aligned}
$$

Since $\Delta X_{\tau_A} \in A$ when $\tau_A \le t$, we have

$$
\begin{aligned}
\{\tau_A \le t\} &\subset \{\tau_A \le t\} \cap \Big\{ \int_{\{\tau_A\}}\int_A\int_{(0,X_{s-}]} N_0^A(ds,dz,dy) \\
&\qquad\qquad + \int_{\{\tau_A\}}\int_A N_1^A(ds,dz) > 0\Big\} \\
&= \{\tau_A \le t\} \cap \Big\{ \int_{\{\tau_A\}}\int_A\int_{(0,X_{s-}^A]} N_0^A(ds,dz,dy) \\
&\qquad\qquad + \int_{\{\tau_A\}}\int_A N_1^A(ds,dz) > 0\Big\} \\
&\subset \Big\{ \int_{(0,t]}\int_A\int_{(0,X_{s-}^A]} N_0^A(ds,dz,dy) + \int_{(0,t]}\int_A N_1^A(ds,dz) > 0\Big\}.
\end{aligned}
$$

Then (3.9) holds. Since $(X_t^A : t \ge 0)$ is a strong solution of (3.8), it is progressively measurable with respect to the filtration generated by B, $N_0^{A^c}$ and $N_1^{A^c}$, which is independent of N_0^A and N_1^A. Then we have

$$
\mathbf{P}_x(\tau_A > t) = \exp\{-t\pi_1(A)\}\mathbf{E}_x\Big[\exp\Big\{-\pi_0(A)\int_0^t X_s^A ds\Big\}\Big]. \tag{3.10}
$$

Finally, we get (3.6) by Proposition 2.1. □

Corollary 3.3 *(1) If A and B are Borel subsets of $(0,\infty)$ such that $A \subset B$ and $\pi_0(B) < \infty$, then $u_t^A(\pi_0(A)) \le u_t^B(\pi_0(B))$ for $t \ge 0$. (2) If $A \subset (0,\infty)$ is a Borel set satisfying $\pi_0(A) + \pi_1(A) = 0$, then $\mathbf{P}_x(\tau_A = \infty) = 1$ for $x \ge 0$. (3) If $\Psi \ne 0$ and $A \subset (0,\infty)$ is a Borel set satisfying $\pi_0(A) + \pi_1(A) > 0$, then $\mathbf{P}_x(\tau_A < \infty) = 1$ for $x \ge 0$.*

Proof. (1) By applying Theorem 3.2 to the special case $\Psi \equiv 0$ we have

$$
e^{-xu_t^A(\pi_0(A))} = \mathbf{P}_x(\tau_A > t) \ge \mathbf{P}_x(\tau_B > t) = e^{-xu_t^B(\pi_0(B))}.
$$

Taking any $x > 0$ we get the result. (2) By Theorem 3.2 we have $\mathbf{P}_x(\tau_A > t) = 1$ for every $t \ge 0$. Then $\mathbf{P}_x(\tau_A = \infty) = \lim_{t\to\infty}\mathbf{P}_x(\tau_A > t) = 1$. (3) By choosing a smaller set if it is necessary, we may assume $0 < \pi_0(A) + \pi_1(A) < \infty$. If $\pi_1(A) > 0$, then $t\pi_1(A) \to \infty$ as $t \to \infty$. In the case $\pi_1(A) = 0$, we must have $\pi_0(A) > 0$, so $s \mapsto u_s^A(\pi_0(A))$ is strictly increasing by Proposition 2.2. Since $\Psi \ne 0$, one can see

$$
\lim_{t\to\infty}\int_0^t \Psi_A(u_s^A(\pi_0(A)))ds = \infty.
$$

In view of (3.6), we have $\mathbf{P}_x(\tau_A = \infty) = \lim_{t\to\infty}\mathbf{P}_x(\tau_A > t) = 0$ in both cases. □

Corollary 3.4 *Suppose that $\Psi \equiv 0$. Then for any $x \ge 0$ and Borel set $A \subset (0,\infty)$ satisfying $0 < \pi_0(A) < \infty$, we have*

$$
\mathbf{P}_x(\tau_A = \infty) = \exp\{-x\Phi_A^{-1}(\pi_0(A))\}.
$$

Proof. By applying Theorem 3.2 to the special case $\Psi \equiv 0$, we have

$$\mathbf{P}_x(\tau_A = \infty) = \lim_{t\to\infty} \mathbf{P}_x(\tau_A > t) = \lim_{t\to\infty} \exp\{-xu_t^A(\pi_0(A))\}.$$

Then the result follows by Proposition 2.2. $\square$

4 Local and global maximal jumps

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \mathbf{P}_x)$ be a Hunt realization of the CBI-process with branching mechanism Φ and immigration mechanism Ψ given by (2.1) and (2.6), respectively. In this section, we shall give some characterizations of the local and global maximal jumps of the process.

Theorem 4.1 *Suppose that $r \geq 0$ and $\pi_0(r,\infty) + \pi_1(r,\infty) < \infty$. Then for any $x \geq 0$ and $t > 0$ we have*

$$\mathbf{P}_x\Big(\max_{s\in(0,t]} \Delta X_t \leq r\Big) = \exp\bigg\{ - t\pi_1(r,\infty) - xu_t^r(\pi_0(r,\infty)) - \int_0^t \Psi_{(r,\infty)}(u_s^r(\pi_0(r,\infty)))ds\bigg\},$$

where $u_t^r(\lambda)$ is the unique nonnegative solution of

$$\frac{d}{dt}u_t^r(\lambda) = \lambda - \Phi_{(r,\infty)}(u_t^r(\lambda)), \qquad u_0^r(\lambda) = 0. \tag{4.1}$$

Proof. Since $\mathbf{P}_x(\max_{s\in(0,t]} \Delta X_t \leq r) = \mathbf{P}_x(\tau_{(r,\infty)} > t)$, the result follows by Theorem 3.2. $\square$

Corollary 4.2 *Suppose that $\Psi \neq 0$. Then $\mathbf{P}_x(\sup_{s\in(0,\infty)} \Delta X_s = \sup(\pi_0 + \pi_1)) = 1$ for any $x \geq 0$, where $\sup(\pi_0 + \pi_1) = \sup \mathrm{supp}(\pi_0 + \pi_1)$.*

Proof. Since $(\pi_0 + \pi_1)(\sup(\pi_0 + \pi_1), \infty) = 0$, for any $t > 0$ we have $\mathbf{P}_x(\sup_{s\in(0,t]} \Delta X_s \leq \sup(\pi_0 + \pi_1)) = 1$ by Theorem 4.1. Then

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \leq \sup(\pi_0 + \pi_1)\Big) = \lim_{t\to\infty} \mathbf{P}_x\Big(\sup_{s\in(0,t]} \Delta X_s \leq \sup(\pi_0 + \pi_1)\Big) = 1.$$

For any $z < \sup(\pi_0 + \pi_1)$ we have $(\pi_0 + \pi_1)[z, \sup(\pi_0 + \pi_1)] > 0$. By Corollary 3.3 (3),

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \in [z, \sup(\pi_0 + \pi_1)]\Big) \geq \mathbf{P}_x(\tau_{[z,\sup(\pi_0+\pi_1)]} < \infty) = 1.$$

Since $z < \sup(\pi_0 + \pi_1)$ was arbitrary, it follows that $\mathbf{P}_x(\sup \Delta X = \sup(\pi_0 + \pi_1)) = 1$. $\square$

Corollary 4.3 *Suppose that $\Psi \equiv 0$. Then for any $x \geq 0$ and $r \geq 0$ satisfying $0 < \pi_0(r,\infty) < \infty$ we have*

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \leq r\Big) = \exp\{-x\Phi_{(r,\infty)}^{-1}(\pi_0(r,\infty))\}.$$

Proof. This follows by Theorem 4.1 and Proposition 2.2. $\square$

Corollary 4.4 *Suppose that $\Psi \equiv 0$ and let $\sup(\pi_0) = \sup \mathrm{supp}(\pi_0)$. Then for any $x \geq 0$ we have*

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s = \sup(\pi_0)\Big) = 1 - \exp\{-x\Phi^{-1}_{\{\sup(\pi_0)\}}(\pi_0(\{\sup(\pi_0)\}))\}$$

with $\Phi_{\{0\}} = \Phi_{\{\infty\}} = \Phi$ and $\pi_0(\{0\}) = \pi_0(\{\infty\}) = 0$ by convention.

Proof. By the proof of Corollary 4.2 we have $\mathbf{P}_x(\sup_{s\in(0,\infty)} \Delta X_s \leq \sup(\pi_0)) = 1$. For any $z < \sup(\pi_0)$ we have $\pi_0(z, \sup(\pi_0)] > 0$. By Corollary 4.3 it follows that

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \in (z, \sup(\pi_0)]\Big) = 1 - \exp\{-x\Phi^{-1}_{(z,\sup(\pi_0)]}(\pi_0(z, \sup(\pi_0)])\}$$

Then we get the desired result by letting $z \to \sup(\pi_0)$. □

Corollary 4.5 *Suppose that $\alpha > 0$ and $\Psi \equiv 0$. If the measure π_0 has unbounded support, then for any $x > 0$, we have, as $r \to \infty$,*

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s > r\Big) = 1 - \exp\{-x\Phi^{-1}_{(r,\infty)}(\pi_0(r,\infty))\} \sim \frac{x}{\alpha}\pi_0(r,\infty).$$

Proof. By (3.4) we see by dominated convergence that $(\partial/\partial z)\Phi_{(r,\infty)}(0) = \alpha_{(r,\infty)}$. It follows that $(\partial/\partial z)\Phi^{-1}_{(r,\infty)}(0) = 1/\alpha_{(r,\infty)}$. Then, as $r \to \infty$,

$$\Phi^{-1}_{(r,\infty)}(\pi_0(r,\infty)) \sim \pi_0(r,\infty)/\alpha_{(r,\infty)} \sim \pi_0(r,\infty)/\alpha,$$

and the desired result follows from Corollary 4.3. □

We remark that a special form of Corollary 4.3 has been obtained by Bertoin (2011). The next theorem establishes the equivalence of the distribution of the local maximal jump of the CBI-process and the *total Lévy measure* $\pi_0 + \pi_1$. In view of Theorem 4.1, we may have $\mathbf{P}_x(\max_{s\in(0,t]} \Delta X_s = 0) > 0$, so we only discuss the absolute continuity on the set $(0, \infty)$.

Theorem 4.6 *Suppose that $x + \gamma > 0$. Then for any $t > 0$, the measure $\pi_0 + \pi_1$ and the distribution $\mathbf{P}_x(\max_{s\in(0,t]} \Delta X_s \in \cdot)|_{(0,\infty)}$ are equivalent.*

Proof. Recall that Ψ and Ψ_A are defined by (2.6) and (3.2), respectively. If $A \subset (0, \infty)$ is a Borel set with $\pi_0(A) + \pi_1(A) = 0$, by Theorem 3.2 we have

$$\mathbf{P}_x\Big(\max_{s\in(0,t]} \Delta X_s \in A\Big) \leq \mathbf{P}_x(\tau_A \leq t) = 1 - \mathbf{P}_x(\tau_A > t) = 0.$$

Then $\mathbf{P}_x(\max_{s\in(0,t]} \Delta X \in \cdot)|_{(0,\infty)}$ is absolutely continuous with respect to $\pi_0 + \pi_1$. To prove the absolute continuity of $\pi_0 + \pi_1$ with respect to $\mathbf{P}_x(\max_{s\in(0,t]} \sup \Delta X \in \cdot)|_{(0,\infty)}$, we consider a Borel set $A \subset (0, \infty)$ and a constant $r > 0$. Since

$$\begin{aligned}\Big\{\max_{s\in(0,t]} \Delta X_s \in A\Big\} &\supset \Big\{\max_{s\in(0,t]} \Delta X_s \in A \cap [r, \infty)\Big\} \\ &\supset \{\tau_{A\cap[r,\infty)} \leq t\} \cap \{\tau_{[r,\infty)\setminus A} > t\} \\ &= \{\tau_{[r,\infty)} \leq t\} \setminus \{\tau_{[r,\infty)\setminus A} \leq t\},\end{aligned}$$

we have

$$\mathbf{P}_x\Big(\max_{s\in(0,t]} \Delta X \in A\Big) \geq \mathbf{P}_x(\tau_{[r,\infty)} \leq t) - \mathbf{P}_x(\tau_{[r,\infty)\setminus A} \leq t).$$

Suppose that $\mathbf{P}_x(\sup_{s\in(0,t]} \Delta X \in A) = 0$. Then $\mathbf{P}_x(\tau_{[r,\infty)} \leq t) = \mathbf{P}_x(\tau_{[r,\infty)\setminus A} \leq t)$, so the result of Theorem 3.2 implies

$$\begin{aligned} t\pi_1[r,\infty) &+ xu_t^{[r,\infty)}(\pi_0[r,\infty)) + \int_0^t \Psi_{[r,\infty)}(u_s^{[r,\infty)}(\pi_0[r,\infty)))ds \\ &= t\pi_1([r,\infty)\setminus A) + xu_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A)) \\ &\qquad + \int_0^t \Psi_{[r,\infty)\setminus A}(u_s^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A)))ds. \end{aligned} \tag{4.2}$$

By Corollary 3.3 (1) we have

$$u_t^{[r,\infty)}(\pi_0[r,\infty)) \geq u_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A)), \qquad t \geq 0. \tag{4.3}$$

Then (4.2) implies

$$\begin{aligned} t\pi_1[r,\infty) &+ xu_t^{[r,\infty)}(\pi_0[r,\infty)) + \int_0^t \Psi_{[r,\infty)}(u_s^{[r,\infty)}(\pi_0[r,\infty)))ds \\ &\leq t\pi_1([r,\infty)\setminus A) + xu_t^{[r,\infty)}(\pi_0[r,\infty)) + \int_0^t \Psi_{[r,\infty)\setminus A}(u_s^{[r,\infty)}(\pi_0[r,\infty)))ds. \end{aligned}$$

By reorganizing the terms in the above inequality we obtain

$$t\pi_1([r,\infty)\cap A) \leq \int_0^t ds \int_{[r,\infty)\cap A} (1 - e^{-\theta u_s^{[r,\infty)}(\pi_0[r,\infty))})\pi_1(d\theta).$$

It follows that $\pi_1([r,\infty)\cap A) = 0$. Since $r > 0$ was arbitrary in the above, we have proved $\pi_1(A) = 0$. Using (4.2) and (4.3) we have

$$\begin{aligned} xu_t^{[r,\infty)}&(\pi_0[r,\infty)) + \gamma\int_0^t u_s^{[r,\infty)}(\pi_0[r,\infty))ds \\ &\leq xu_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A)) + \gamma\int_0^t u_s^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))ds, \end{aligned}$$

and so using (4.3) again we get

$$u_t^{[r,\infty)}(\pi_0[r,\infty)) = u_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A)) =: a(r,t).$$

It follows that

$$\frac{\partial u_t^{[r,\infty)}(\pi_0[r,\infty))}{\partial t} = \frac{\partial u_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))}{\partial t}.$$

Then we can use (3.7) to see

$$\pi_0([r,\infty)) - \Phi_{[r,\infty)}(a(r,t)) = \pi_0([r,\infty)\setminus A) - \Phi_{[r,\infty)\setminus A}(a(r,t)),$$

and hence

$$\Phi_{[r,\infty)}(a(r,t)) = \Phi_{[r,\infty)\setminus A}(a(r,t)) + \pi_0([r,\infty)\cap A).$$

But, by (3.3), we should have

$$\Phi_{[r,\infty)}(a(r,t)) = \Phi_{[r,\infty)\setminus A}(a(r,t)) + \int_{[r,\infty)\cap A} \pi_0(d\theta)(1 - e^{-a(r,t)\theta}).$$

It follows that $\pi_0([r,\infty)\cap A)=0$, implying $\pi_0(A)=0$. That completes the proof. $\square$

The conclusion of Theorem 4.6 is not necessarily true in the case $x=\gamma=0$. As a counterexample, consider the case where $X_0=0$, $\pi_0=\delta_1$ and $\pi_1=\delta_2$. In this case, we have $\tau_{\{2\}}\le t$ when $\tau_{\{1\}}\le t$, for otherwise $X_s=0$ for all $s\in[0,t]$. It follows that

$$\mathbf{P}_0\Big(\max_{s\in(0,t]}\Delta X_s=1\Big)=0.$$

Then π_0 is not absolutely continuous with respect to $\mathbf{P}_0(\max_{s\in(0,t]}\Delta X_s\in\cdot)$.

For critical and subcritical branching CB-processes without immigration, we may also discuss the absolute continuity of the distribution of its global maximal jump. Such a result is presented in the following:

Theorem 4.7 *Suppose that $\alpha\ge 0$ and $\Psi\equiv 0$. Then for any $x>0$ the Lévy measure π_0 and the distribution $\mathbf{P}_x(\sup_{s\in(0,\infty)}\Delta X_s\in\cdot)|_{(0,\infty)}$ are equivalent.*

Proof. Since $\Psi=0$ and $\alpha\ge 0$, we have $X_t\to 0$ almost surely as $t\to\infty$. If $A\subset(0,\infty)$ is a Borel set so that $\pi_0(A)=0$, by Corollary 3.3 (2) we have

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)}\Delta X_s\in A\Big)\le\mathbf{P}_x(\tau_A<\infty)=1-\mathbf{P}_x(\tau_A=\infty)=0.$$

Then $\mathbf{P}_x(\sup_{s\in(0,\infty)}\Delta X_s\in\cdot)|_{(0,\infty)}$ is absolutely continuous with respect to π_0. Now suppose that $A\subset(0,\infty)$ is a Borel set with $\pi_0(A)>0$. For any $r>0$, one can see as in the proof of Theorem 4.6 that

$$\mathbf{P}_x\Big(\sup_{s\in(0,\infty)}\Delta X_s\in A\Big)\ge\mathbf{P}_x(\tau_{[r,\infty)}<\infty)-\mathbf{P}_x(\tau_{[r,\infty)\setminus A}<\infty).$$

If $\mathbf{P}_x(\sup_{s\in(0,\infty)}\Delta X_s\in A)=0$, we have $\mathbf{P}_x(\tau_{[r,\infty)}<\infty)=\mathbf{P}_x(\tau_{[r,\infty)\setminus A}<\infty)$, so Corollary 3.4 implies

$$\Phi^{-1}_{[r,\infty)}(\pi_0[r,\infty))=\Phi^{-1}_{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))=:a(r).$$

It follows that

$$\begin{aligned}\Phi_{[r,\infty)}(a(r)) &= \pi_0[r,\infty)=\pi_0([r,\infty)\setminus A)+\pi_0(A\cap[r,\infty))\\ &= \Phi_{[r,\infty)\setminus A}\circ\Phi^{-1}_{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))+\pi_0(A\cap[r,\infty))\\ &= \Phi_{[r,\infty)\setminus A}(a(r))+\pi_0(A\cap[r,\infty)).\end{aligned}$$

Then, as in the proof of Theorem 4.6, we must have $\pi_0(A\cap[r,\infty))=0$. This contradicts $\pi_0(A)>0$ since $r>0$ was arbitrary. It then follows that $\mathbf{P}_x(\sup_{s\in(0,\infty)}\Delta X_s\in A)>0$. $\square$

In the above theorem, we only consider the critical and subcritical cases. The supercritical case is more subtle since in that case we may have $\sup_{s\in(0,\infty)}\Delta X_s=\sup(\pi_0)$ with strictly positive probability by Corollary 4.4. We leave the consideration of the details to the interested reader.

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