Published in: *The Annals of Probability* **42** (2014), 1: 41–79

Path-valued branching processes and nonlocal branching superprocesses [1]

By Zenghu Li

Beijing Normal University

Abstract: A family of continuous-state branching processes with immigration are constructed as the solution flow of a stochastic equation system driven by time–space noises. The family can be regarded as an inhomogeneous increasing path-valued branching process with immigration. Two nonlocal branching immigration superprocesses can be defined from the flow. We identify explicitly the branching and immigration mechanisms of those processes. The results provide new perspectives into the tree-valued Markov processes of Aldous and Pitman [*Ann. Inst. H. Poincaré Probab. Statist.* **34** (1998), 637–686] and Abraham and Delmas [*Ann. Probab.* **40** (2012), 1167–1211].

Mathematics Subject Classification (2010): Primary 60J80, 60J68; secondary 60H20, 92D25

Key words and phrases: Stochastic equation, solution flow, continuous-state branching process, path-valued branching process, immigration, nonlocal branching, superprocess.

Running title: Path-valued processes and superprocesses

1 Introduction

Continuous-state branching processes (CB-processes) are positive Markov processes introduced by Jiřina (1958) to model the evolution of large populations of small particles. Continuous-state branching processes with immigration (CBI-processes) are generalizations of them describing the situation where immigrants may come from other sources of particles; see, for example, Kawazu and Watanabe (1971). The law of a CB-process is determined by its *branching mechanism* ϕ, which is a function with the representation

$$\phi(\lambda) = b\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_0^\infty (e^{-z\lambda} - 1 + z\lambda)m(dz), \tag{1.1}$$

where $\sigma \geq 0$ and b are constants and $(z \wedge z^2)m(dz)$ is a finite measure on $(0, \infty)$. In most cases, we only define the function ϕ on $[0, \infty)$, but it can usually be extended to an

[1] Supported by NSFC (No. 11131003), 973 Program (No. 2011CB808001) and 985 Program.

analytic function on an interval strictly larger than $[0, \infty)$. The branching mechanism is said to be *critical*, *subcritical* or *supercritical* according as $b = 0$, $b > 0$ or $b < 0$.

A CB-process can be obtained as the small particle limit of a sequence of discrete Galton–Watson branching processes; see, for example, Lamperti (1967). A genealogical tree is naturally associated with a Galton–Watson process. The genealogical structures of CB-processes were investigated by introducing *continuum random trees* in the pioneer work of Aldous (1991, 1993), where the *quadratic branching mechanism* $\phi(\lambda) = \lambda^2$ was considered. Continuum random trees corresponding to general branching mechanisms were constructed in Le Gall and Le Jan (1998a, 1998b) and were studied further in Duquesne and Le Gall (2002). By pruning a Galton–Watson tree, Aldous and Pitman (1998) constructed a decreasing tree-valued process. Then they used time-reversal to obtain an increasing tree-valued process starting with the trivial tree. They gave some characterizations of the increasing process up to the *ascension time*, the first time when the increasing tree becomes infinite.

Tree-valued processes associated with general CB-processes were studied in Abraham and Delmas (2010). By shifting a critical branching mechanism, they defined a family of branching mechanisms $\{\psi_\theta : \theta \in \Theta\}$, where $\Theta = [\theta_\infty, \infty)$ or (θ_∞, ∞) for some $\theta_\infty \in [-\infty, 0]$. Abraham and Delmas (2010) constructed a decreasing tree-valued Markov process $\{\mathscr{T}_\theta : \theta \in \Theta\}$ by pruning a continuum tree, where the tree $\mathscr{T}_\theta$ has branching mechanism ψ_θ. The *explosion time* A was defined as the smallest negative time when the tree (or the total mass of the corresponding CB-process) is finite. Abraham and Delmas (2010) gave some characterizations of the evolution of the tree after this time under an excursion law. For the quadratic branching mechanism, they obtained explicit expressions for some interesting distributions. Those extend the results of Aldous and Pitman (1998) on Galton–Watson trees in the time-reversed form. The main tool of Abraham and Delmas (2010) was the exploration process of Le Gall and Le Jan (1998a, 1998b) and Duquesne and Le Gall (2002). Some general ways of pruning random trees in discrete and continuous settings were introduced in Abraham et al. (2010, 2012).

In this paper, we study a class of increasing path-valued Markov processes using the techniques of stochastic equations and measure-valued processes developed in recent years. Those path-valued processes are counterparts of the tree-valued processes of Abraham and Delmas (2010). A special case of the model is described as follows. Let $T = [0, \infty)$ or $[0, a]$ or $[0, a)$ for some $a > 0$. Let $(\theta, \lambda) \mapsto \zeta_\theta(\lambda)$ be a continuous function on $T \times [0, \infty)$ with the representation

$$\zeta_\theta(\lambda) = \beta_\theta \lambda + \int_0^\infty (1 - e^{-z\lambda}) n_\theta(dz), \qquad \theta \in T, \lambda \geq 0,$$

where $\beta_\theta \geq 0$ and $z n_\theta(dz)$ is a finite kernel from T to $(0, \infty)$. Let ϕ be a branching mechanism given by (1.1). Then the function

$$\phi_q(\lambda) := \phi(\lambda) - \int_0^q \zeta_\theta(\lambda) d\theta, \qquad \lambda \geq 0 \tag{1.2}$$

also has the representation (1.1) with the parameters $(b,m) = (b_q, m_q)$ depending on $q \in T$. Let $m(dy,dz)$ be the measure on $T \times (0,\infty)$ defined by

$$m([0,q] \times [c,d]) = m_q[c,d], \qquad q \in T, d > c > 0.$$

Let $W(ds,du)$ be a white noise on $(0,\infty)^2$ based on the Lebesgue measure and let $\tilde{N}_0(ds,dy,dz,du)$ be a compensated Poisson random measure on $(0,\infty) \times T \times (0,\infty)^2$ with intensity $dsm(dy,dz)du$. Let $\mu \geq 0$ be a constant. For $q \in T$ we consider the stochastic equation

$$\begin{aligned} X_t(q) = \mu - b_q \int_0^t X_{s-}(q)ds + \sigma \int_0^t \int_0^{X_{s-}(q)} W(ds,du) \\ + \int_0^t \int_{[0,q]} \int_0^\infty \int_0^{X_{s-}(q)} z\tilde{N}_0(ds,dy,dz,du). \end{aligned} \tag{1.3}$$

We shall see that there is a pathwise unique positive càdlàg solution $\{X_t(q) : t \geq 0\}$ to (1.3). Then we can talk about the solution flow $\{X_t(q) : t \geq 0, q \in T\}$ of the equation system. We prove that each $\{X_t(q) : t \geq 0\}$ is a CB-process with branching mechanism ϕ_q, and $\{(X_t(q))_{t\geq 0} : q \in T\}$ is an inhomogeneous path-valued increasing Markov process with state space $D^+[0,\infty)$, the space of positive càdlàg paths on $[0,\infty)$ endowed with the Skorokhod topology.

The formulation of path-valued processes provides new perspectives into the evolution of the random trees of Aldous and Pitman (1998) and Abraham and Delmas (2010). From this formulation we can derive some structural properties of the model that have not been discovered before. For $q \in T$ let us define the random measure $Z_q(dt) = X_t(q)dt$ on $[0,\infty)$. We shall see that $\{Z_q : q \in T\}$ is an inhomogeneous increasing superprocess involving a nonlocal branching structure, and the total mass process

$$\sigma(q) := \int_0^\infty X_s(q)ds, \qquad q \in T$$

is an inhomogeneous CB-process. Then one can think of $\{X(q) : q \in T\}$ as a path-valued branching process. On the other hand, for each $t \geq 0$ the random increasing function $q \mapsto X_t(q)$ induces a random measure $Y_t(dq)$ on T such that $X_t(q) = Y_t[0,q]$ for $q \in T$. We prove that $\{Y_t : t \geq 0\}$ is a homogeneous superprocess with both local and nonlocal branching structures. We also establish some properties of an excursion law $\mathbf{N}_0$ for the superprocess $\{Y_t : t \geq 0\}$. Given a branching mechanism ϕ of the form (1.1), for a suitable interval T we can define a family of branching mechanisms $\{\phi_q : q \in T\}$ by

$$\phi_q(\lambda) = \phi(\lambda - q) - \phi(-q), \qquad \lambda \geq 0,$$

where the two terms on the right-hand side are defined using (1.1). The family can be represented by (1.2) with $\zeta_\theta(\lambda) = -(\partial/\partial\lambda)\phi_\theta(\lambda)$. In this case, the path-valued process

$q \mapsto (X_t(q))_{t\geq 0}$ under the excursion law $\mathbf{N}_0$ corresponds to the time-reversal of the tree-valued process $\theta \mapsto \mathscr{T}_\theta$ of Abraham and Delmas (2010). In general, we may associate $\{X(q) : q \in T\}$ with a "forest-valued branching process".

To make the exploration self-contained, we shall consider a slightly generalized form of the equation system (1.3) involving some additional immigration structures. In Section 2, we present some preliminary results on inhomogeneous immigration superprocesses and CBI-processes. In Section 3 a class of CBI-processes with predictable immigration rates are constructed as pathwise unique solutions of stochastic integral equations driven by time–space noises. In Section 4 we introduce the path-valued increasing Markov processes and identify them as path-valued branching processes with immigration. A construction of those processes is given in Section 5 using a system of stochastic equations generalizing (1.3). In Section 6 we derive a homogeneous nonlocal branching immigration superprocess from the flow. The properties of the process under an excursion law are studied in Section 7.

We sometimes write $\mathbb{R}_+$ for $[0,\infty)$. Let $F(T)$ denote the set of positive right continuous increasing functions on an interval $T \subset \mathbb{R}$. For a measure μ and a function f on a measurable space we write $\langle \mu, f\rangle = \int f d\mu$ if the integral exists. Throughout this paper, we make the conventions

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}$$

for any $b \geq a \in \mathbb{R}$. Other notations are explained as they first appear.

2 Inhomogeneous immigration superprocesses

In this section, we present some preliminary results on inhomogeneous immigration superprocesses and CBI-processes. Suppose that $T \subset \mathbb{R}$ is an interval, and E is a Lusin topological space. Let $\tilde{E} = T \times E$. A function $(s,x) \mapsto f(s,x)$ on $\tilde{E}$ is said to be *locally bounded* if for each compact interval $S \subset T$ the restriction of $(s,x) \mapsto f(s,x)$ to $S \times E$ is bounded. Let $M(E)$ be the space of finite Borel measures on E endowed with the topology of weak convergence. Let $B^+(E)$ be the set of bounded positive Borel functions on E. Let $\mathscr{I}(E)$ denote the set of all functionals I on $B^+(E)$ with the representation

$$I(f) = \langle \lambda, f\rangle + \int_{M(E)^\circ} (1 - e^{-\langle \nu, f\rangle}) L(d\nu), \qquad f \in B^+(E), \tag{2.1}$$

where $\lambda \in M(E)$ and $(1 \wedge \langle \nu, 1\rangle) L(d\nu)$ is a finite measure on $M(E)^\circ := M(E) \setminus \{0\}$. Let $\mathscr{J}(E)$ denote the set of all functionals on $B^+(E)$ of the form $f \mapsto J(f) := a + I(f)$ with $a \geq 0$ and $I \in \mathscr{I}(E)$. By Theorems 1.35 and 1.37 in Li (2011) one can prove the following:

Theorem 2.1 *There is a one-to-one correspondence between functionals $V \in \mathscr{J}(E)$ and infinitely divisible sub-probability measures Q on $M(E)$, which is determined by*

$$\int_{M(E)} e^{-\langle \nu, f\rangle} Q(d\nu) = \exp\{-V(f)\}, \qquad f \in B^+(E). \tag{2.2}$$

Theorem 2.2 *If $U \in \mathscr{J}(E)$ and if $V : f \mapsto v(\cdot, f)$ is an operator on $B^+(E)$ such that $v(x, \cdot) \in \mathscr{J}(E)$ for all $x \in E$, then $U \circ V \in \mathscr{J}(E)$.*

Suppose that $(P_{r,t} : t \geq r \in T)$ is an inhomogeneous Borel right transition semigroup on E. Let $\xi = (\Omega, \mathscr{F}, \mathscr{F}_{r,t}, \xi_t, \mathbf{P}_{r,x})$ be a right continuous inhomogeneous Markov process realizing $(P_{r,t} : t \geq r \in T)$. Let $(s, x) \mapsto b_s(x)$ be a Borel function on $\tilde{E}$, and let $(s, x) \mapsto c_s(x)$ be a positive Borel function on $\tilde{E}$. Let $\eta_s(x, dy)$ be a kernel from $\tilde{E}$ to E, and let $H_s(x, d\nu)$ be a kernel from $\tilde{E}$ to $M(E)^\circ$. Suppose that the function

$$|b_s(x)| + c_s(x) + \eta_s(x, E) + \int_{M(E)^\circ} (\langle \nu, 1\rangle \wedge \langle \nu, 1\rangle^2 + \langle \nu_x, 1\rangle) H_s(x, d\nu)$$

on $S \times E$ is locally bounded, where $\nu_x(dy)$ denotes the restriction of $\nu(dy)$ to $E \setminus \{x\}$. For $(s, x) \in \tilde{E}$ and $f \in B^+(E)$ define

$$\begin{aligned} \phi_s(x, f) = {} & b_s(x)f(x) + c_s(x)f(x)^2 - \int_E f(y)\eta_s(x, dy) \\ & + \int_{M(E)^\circ} [e^{-\langle \nu, f\rangle} - 1 + \nu(\{x\})f(x)] H_s(x, d\nu). \end{aligned} \tag{2.3}$$

Let $T_t = T \cap (-\infty, t]$ for $t \in T$. By Theorem 6.10 in Li (2011) one can show there is an inhomogeneous Borel right transition semigroup $(Q_{r,t} : t \geq r \in T)$ on the state space $M(E)$ defined by

$$\int_{M(E)} e^{-\langle \nu, f\rangle} Q_{r,t}(\mu, d\nu) = \exp\{-\langle \mu, V_{r,t}f\rangle\}, \quad f \in B^+(E), \tag{2.4}$$

where $(r, x) \mapsto v_{r,t}(x) := V_{r,t}f(x)$ is the unique locally bounded positive solution to the integral equation

$$v_{r,t}(x) = \mathbf{P}_{r,x}[f(\xi_t)] - \int_r^t \mathbf{P}_{r,x}[\phi_s(\xi_s, v_{s,t})] ds, \quad r \in T_t, x \in E. \tag{2.5}$$

Let us consider a right continuous realization $X = (W, \mathscr{G}, \mathscr{G}_{r,t}, X_t, \mathbf{Q}_{r,\mu})$ of the transition semigroup $(Q_{r,t} : t \geq r \in T)$ defined by (2.4). Suppose that $(s, x) \mapsto g_s(x)$ is a locally bounded positive Borel function on $\tilde{E}$. Let $\psi_s(x, f) = -g_s(x) + \phi_s(x, f)$ for $f \in B^+(E)$. Following the proofs of Theorems 5.15 and 5.16 in Li (2011), one can see

$$\mathbf{Q}_{r,\mu} \exp\Big\{-\langle X_t, f\rangle - \int_r^t \langle X_s, g_s\rangle ds\Big\} = \exp\{-\langle \mu, U_{r,t}f\rangle\}, \tag{2.6}$$

where $(r,x) \mapsto u_{r,t}(x) := U_{r,t}f(x)$ is the unique locally bounded positive solution to

$$u_{r,t}(x) = \mathbf{P}_{r,x}[f(\xi_t)] - \int_r^t \mathbf{P}_{r,x}[\psi_s(\xi_s, u_{s,t})]ds, \quad r \in T_t, x \in E. \tag{2.7}$$

Then there is an inhomogeneous Borel right sub-Markov transition semigroup $(Q^g_{r,t} : t \geq r \in T)$ on $M(E)$ given by

$$\int_{M(E)} e^{-\langle \nu, f\rangle} Q^g_{r,t}(\mu, d\nu) = \exp\{-\langle \mu, U_{r,t}f\rangle\}. \tag{2.8}$$

A Markov process with transition semigroup given by (2.8) is called an *inhomogeneous superprocess* with *branching mechanisms* $\{\psi_s : s \in T\}$. The family of operators $(U_{r,t} : t \geq r \in T)$ is called the *cumulant semigroup* of the superprocess. From (2.8) one can derive the following *branching property*:

$$Q^g_{r,t}(\mu_1 + \mu_2, \cdot) = Q^g_{r,t}(\mu_1, \cdot) * Q^g_{r,t}(\mu_2, \cdot) \tag{2.9}$$

for $t \geq r \in T$ and $\mu_1, \mu_2 \in M(E)$, where "$*$" denotes the convolution operation. Some special branching mechanisms are given in Dawson et al. (2002), Dynkin (1993) and Li (1992, 2011). Clearly, the semigroup $(Q_{r,t} : t \geq r \in T)$ given by (2.4) corresponds to a conservative inhomogeneous superprocess. In general, the inhomogeneous superprocess is not necessarily conservative.

We can append an additional immigration structure to the inhomogeneous superprocess. Suppose that $\rho(ds)$ is a Radon measure on T and $\{J_s : s \in T\} \subset \mathscr{I}(E)$ is a family of functionals such that $s \mapsto J_s(f)$ is a locally bounded Borel function on T for each $f \in B^+(E)$.

Theorem 2.3 *There is an inhomogeneous transition semigroup $(Q^{\rho,J}_{r,t} : t \geq r \in T)$ on $M(E)$ given by*

$$\int_{M(E)} e^{-\langle \nu, f\rangle} Q^{\rho,J}_{r,t}(\mu, d\nu) = \exp\Big\{-\langle \mu, U_{r,t}f\rangle - \int_r^t J_s(U_{s,t}f)\rho(ds)\Big\}, \tag{2.10}$$

where $(r,x) \mapsto u_{r,t}(x) := U_{r,t}f(x)$ is the unique locally bounded positive solution to (2.7).

Proof. By Theorems 2.1 and 2.2, for any $t \geq r \in T$ we can define an infinitely divisible sub-probability measure $N_{r,t}$ on $M(E)$ by

$$\int_{M(E)} e^{-\langle \nu, f\rangle} N_{r,t}(d\nu) = \exp\Big\{-\int_r^t J_s(U_{s,t}f)\rho(ds)\Big\}.$$

It is easy to check that

$$N_{r,t} = (N_{r,s}Q^g_{s,t}) * N_{s,t}, \qquad t \geq s \geq r \in T,$$

where

$$N_{r,s}Q^g_{s,t} = \int_{M(E)} N_{r,s}(d\mu)Q^g_{s,t}(\mu, \cdot).$$

Following the arguments in Li (2002, 2011) one can show

$$Q^{\rho,J}_{r,t}(\mu, \cdot) = Q^g_{r,t}(\mu, \cdot) * N_{r,t}, \qquad t \geq r \in T \tag{2.11}$$

defines an inhomogeneous sub-Markov transition semigroup on $M(E)$. Clearly, the Laplace functional of this transition semigroup is given by (2.10). □

If a Markov process with state space $M(E)$ has transition semigroup $(Q^{\rho,J}_{r,t} : t \geq r \in T)$ given by (2.10), we call it an *inhomogeneous immigration superprocess* with *immigration mechanisms* $\{J_s : s \in T\}$ and *immigration measure* ρ. The intuitive meaning of the model is clear in view of (2.11). That is, the population at any time $t \geq 0$ is made up of two parts, the native part generated by the mass $\mu \in M(E)$ at time $r \geq 0$ has distribution $Q^g_{r,t}(\mu, \cdot)$ and the immigration in the time interval $(r, t]$ gives the distribution $N_{r,t}$. When E shrinks to a singleton, we can identify $M(E)$ with the positive half line $\mathbb{R}_+ = [0, \infty)$. In this case, the transition semigroups given by (2.8) and (2.10) determine one-dimensional CB- and CBI-processes, respectively.

Now let us consider a branching mechanism ϕ of the form (1.1). We can define the transition semigroup $(P_t)_{t\geq 0}$ of a homogeneous CB-process by

$$\int_{\mathbb{R}_+} e^{-\lambda y} P_t(x, dy) = e^{-xv_t(\lambda)}, \qquad t, \lambda \geq 0, \tag{2.12}$$

where $t \mapsto v_t(\lambda)$ is the unique locally bounded positive solution of

$$v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda))ds,$$

which is essentially a special form of (2.5). We can write the above integral equation into its differential form

$$\frac{d}{dt}v_t(\lambda) = -\phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda. \tag{2.13}$$

The Chapman–Kolmogorov equation of $(P_t)_{t\geq 0}$ implies $v_r(v_t(\lambda)) = v_{r+t}(\lambda)$ for all $r, t, \lambda \geq 0$. The set of functions $(v_t)_{t\geq 0}$ is the *cumulant semigroup*. Observe that $\lambda \mapsto \phi(\lambda)$ is continuously differentiable with

$$\phi'(\lambda) = b + \sigma^2\lambda + \int_0^\infty z(1 - e^{-z\lambda})m(dz), \qquad \lambda \geq 0.$$

By differentiating (2.12) and (2.13) in $\lambda \geq 0$ one can show

$$\int_{\mathbb{R}_+} yP_t(x, dy) = x\frac{d}{d\lambda}v_t(\lambda)\Big|_{\lambda=0+} = xe^{-bt}. \tag{2.14}$$

It is easy to see that $(P_t)_{t\geq 0}$ is a Feller semigroup. Let us consider a càdlàg realization $X = (\Omega, \mathscr{F}, \mathscr{F}_{r,t}, X_t, \mathbf{P}_{r,x})$ of the corresponding CB-process with an arbitrary initial time $r \geq 0$. Let $\eta(ds)$ be a Radon measure on $[0,\infty)$. By Theorem 5.15 in Li (2011), for $t \geq r \geq 0$ and $f \in B^+[0,t]$, we have

$$\mathbf{P}_{r,x}\Big[\exp\Big\{-\int_{[r,t]} f(s)X_s\eta(ds)\Big\}\Big] = \exp\{-xu^t(r,f)\}, \tag{2.15}$$

where $r \mapsto u^t(r,f)$ is the unique bounded positive solution to

$$u^t(r,f) + \int_r^t \phi(u^t(s,f))ds = \int_{[r,t]} f(s)\eta(ds), \quad 0 \leq r \leq t. \tag{2.16}$$

In particular, for $r \geq 0$ and $f \in B^+[0,\infty)$ with compact support, we have

$$\mathbf{P}_{r,x}\Big[\exp\Big\{-\int_r^\infty f(s)X_s ds\Big\}\Big] = \exp\{-xu(r,f)\}, \tag{2.17}$$

where $r \mapsto u(r,f)$ is the unique compactly supported bounded positive function on $[0,\infty)$ solving

$$u(r,f) + \int_r^\infty \phi(u(s,f))ds = \int_r^\infty f(s)ds, \qquad r \geq 0. \tag{2.18}$$

It is not hard to see that $u(r,f) = 0$ for $r > l_f := \sup\{t \geq 0 : f(t) > 0\}$. For any $r \geq 0$ let

$$\sigma_r(X) = \int_r^\infty X_s ds.$$

Theorem 2.4 *Suppose that $\phi(\lambda) \to \infty$ as $\lambda \to \infty$. Then for any $\lambda \geq 0$ we have*

$$\mathbf{P}_{r,x}\Big[e^{-\lambda\sigma_r(X)}1_{\{\sigma_r(X)<\infty\}}\Big] = \exp\{-x\phi^{-1}(\lambda)\}, \tag{2.19}$$

where ϕ^{-1} is the right inverse of ϕ defined by

$$\phi^{-1}(\lambda) = \inf\{z \geq 0 : \phi(z) > \lambda\}. \tag{2.20}$$

Proof. A proof of (2.19) was already given in Abraham and Delmas (2010). We here give a simple derivation of the result since the argument is also useful to prove the next theorem. By (2.15) and (2.16), for any $t \geq r$ and $z, \theta \geq 0$ we have

$$\mathbf{P}_{r,x}\Big[\exp\Big\{-zX_t - \theta\int_r^t X_s ds\Big\}\Big] = \exp\{-xu^t(r,z,\theta)\},$$

where $r \mapsto u^t(r,z,\theta)$ is the unique bounded positive solution to

$$u^t(r,z,\theta) + \int_r^t \phi(u^t(s,z,\theta))ds = z + \theta(t-r), \quad 0 \le r \le t.$$

Then one can see $u^t(r,z,\phi(z)) = z$. It follows that

$$\mathbf{P}_{r,x}\Big[\exp\Big\{-zX_t - \phi(z)\int_r^t X_s ds\Big\}\Big] = e^{-zx}.$$

Since $\sigma_r(X) < \infty$ implies $\lim_{t\to\infty} X_t = 0$, if $\phi(z) > 0$, we get

$$\mathbf{P}_{r,x}\Big[e^{-\phi(z)\sigma_r(X)} 1_{\{\sigma_r(X)<\infty\}}\Big] = e^{-zx}.$$

That gives (2.19) first for $\lambda = \phi(z) > 0$ and then for all $\lambda \ge 0$. □

Let $t \mapsto \rho(t)$ be a locally bounded positive Borel function on $[0,\infty)$. Suppose that $h \ge 0$ is a constant and $zn(dz)$ is a finite measure on $(0,\infty)$. Let ψ be an *immigration mechanism* given by

$$\psi(\lambda) = h\lambda + \int_0^\infty (1 - e^{-z\lambda}) n(dz), \qquad \lambda \ge 0. \tag{2.21}$$

By Theorem 2.3 we can define an inhomogeneous transition semigroup $\{P^\rho_{r,t} : t \ge r \ge 0\}$ on $\mathbb{R}_+$ by

$$\int_{\mathbb{R}_+} e^{-\lambda y} P^\rho_{r,t}(x,dy) = \exp\Big\{-xv_{t-r}(\lambda) - \int_r^t \psi(v_{t-s}(\lambda))\rho(s)ds\Big\}. \tag{2.22}$$

A positive Markov process with transition semigroup $(P^\rho_{r,t})_{t\ge r\ge 0}$ is called an inhomogeneous CBI-process with *immigration rate* $\rho = \{\rho(t) : t \ge 0\}$. It is easy to see that the homogeneous time–space semigroup associated with $(P^\rho_{r,t})_{t\ge r\ge 0}$ is a Feller transition semigroup. Then $(P^\rho_{r,t})_{t\ge r\ge 0}$ has a càdlàg realization $Y = (\Omega, \mathscr{F}, \mathscr{F}_{r,t}, Y_t, \mathbf{P}^\rho_{r,x})$. A modification of the proof of Theorem 5.15 in Li (2011) shows that, for $t \ge r \ge 0$ and $f \in B^+[0,t]$,

$$\begin{aligned}\mathbf{P}^\rho_{r,x}&\Big[\exp\Big\{-\int_{[r,t]} f(s)Y_s\eta(ds)\Big\}\Big]\\ &= \exp\Big\{-xu^t(r,f) - \int_r^t \psi(u^t(s,f))\rho(s)ds\Big\},\end{aligned} \tag{2.23}$$

where $r \mapsto u^t(r,f)$ is the unique bounded positive solution to (2.16). In particular, for $r \ge 0$ and $f \in B^+[0,\infty)$ with compact support, we have

$$\mathbf{P}^\rho_{r,x}\Big[\exp\Big\{-\int_r^\infty f(s)Y_s ds\Big\}\Big]$$

$$= \exp\Big\{ - xu(r,f) - \int_r^\infty \psi(u(s,f))\rho(s)ds\Big\}, \tag{2.24}$$

where $r \mapsto u(r,f)$ is the unique compactly supported bounded positive solution to (2.18). For any $r \geq 0$ let

$$\sigma_r(Y) = \int_r^\infty Y_s ds.$$

By a modification of the proof of Theorem 2.4, we get the following:

Theorem 2.5 *Suppose that $\phi(\lambda) \to \infty$ as $\lambda \to \infty$. Then for any $r, \lambda \geq 0$ we have*

$$\mathbf{P}^\rho_{r,x}\Big[e^{-\lambda\sigma_r(Y)}1_{\{\sigma_r(Y)<\infty\}}\Big] = \exp\Big\{ - x\phi^{-1}(\lambda) - \psi(\phi^{-1}(\lambda))\int_r^\infty \rho(s)ds\Big\},$$

where $\phi^{-1}(\lambda)$ is defined by (2.20).

3 The predictable immigration rate

The main purpose of this section is to give a construction of the CBI-process with transition semigroup $(P^\rho_{r,t})_{t\geq r\geq 0}$ defined by (2.22) as the pathwise unique solution of a stochastic integral equation driven by time–space noises. For the convenience of applications, we shall generalize the model slightly by considering a random immigration rate. This is essential for our study of the path-valued Markov processes. The reader is referred to Bertoin and Le Gall (2006), Dawson and Li (2006, 2011), Fu and Li (2010) and Li and Mytnik (2011) for some related results.

Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Let $\{W(t,\cdot) : t \geq 0\}$ be an $(\mathscr{F}_t)$-white noise on $(0,\infty)$ based on the Lebesgue measure and let $\{p_0(t) : t \geq 0\}$ and $\{p_1(t) : t \geq 0\}$ be $(\mathscr{F}_t)$-Poisson point processes on $(0,\infty)^2$ with characteristic measures $m(dz)du$ and $n(dz)du$, respectively. We assume that the white noise and the Poisson processes are independent of each other. Let $W(ds, du)$ denote the stochastic integral on $(0,\infty)^2$ with respect to the white noise. Let $N_0(ds, dz, du)$ and $N_1(ds, dz, du)$ denote the Poisson random measures on $(0,\infty)^3$ associated with $\{p_0(t)\}$ and $\{p_1(t)\}$, respectively. Let $\tilde{N}_0(ds, dz, du)$ denote the compensated random measure associated with $\{p_0(t)\}$. Suppose that $\rho = \{\rho(t) : t \geq 0\}$ is a positive $(\mathscr{F}_t)$-predictable process such that $t \mapsto \mathbf{P}[\rho(t)]$ is locally bounded. We are interested in positive càdlàg solutions of the stochastic equation

$$\begin{aligned} Y_t = Y_0 &+ \sigma\int_0^t\int_0^{Y_{s-}} W(ds,du) + \int_0^t\int_0^\infty\int_0^{Y_{s-}} z\tilde{N}_0(ds,dz,du) \\ &+ \int_0^t (h\rho(s) - bY_{s-})ds + \int_0^t\int_0^\infty\int_0^{\rho(s)} zN_1(ds,dz,du). \end{aligned} \tag{3.1}$$

For any positive càdlàg solution $\{Y_t : t \geq 0\}$ of (3.1) satisfying $\mathbf{P}[Y_0] < \infty$, one can use a standard stopping time argument to show that $t \mapsto \mathbf{P}[Y_t]$ is locally bounded and

$$\mathbf{P}[Y_t] = \mathbf{P}[Y_0] + \psi'(0) \int_0^t \mathbf{P}[\rho(s)]ds - b \int_0^t \mathbf{P}[Y_s]ds, \tag{3.2}$$

where

$$\psi'(0) = h + \int_0^\infty zn(dz).$$

By Itô's formula, it is easy to see that $\{Y_t : t \geq 0\}$ solves the following martingale problem: For every $f \in C^2(\mathbb{R}_+)$,

$$\begin{aligned} f(Y_t) = f(Y_0) &+ \text{local mart.} - b \int_0^t Y_s f'(Y_s)ds + \frac{1}{2}\sigma^2 \int_0^t Y_s f''(Y_s)ds \\ &+ \int_0^t Y_s ds \int_0^\infty [f(Y_s + z) - f(Y_s) - zf'(Y_s)]m(dz) \\ &+ \int_0^t \rho(s)\Big\{hf'(Y_s) + \int_0^\infty [f(Y_s + z) - f(Y_s)]n(dz)\Big\}ds. \end{aligned} \tag{3.3}$$

Proposition 3.1 *Suppose that $\{Y_t : t \geq 0\}$ is a positive càdlàg solution of* (3.1) *and $\{Z_t : t \geq 0\}$ is a positive càdlàg solution of the equation with (b, ρ) replaced by (c, η). Then we have*

$$\begin{aligned} \mathbf{P}[|Z_t - Y_t|] \leq \mathbf{P}[|Z_0 - Y_0|] &+ \psi'(0) \int_0^t \mathbf{P}[|\eta(s) - \rho(s)|]ds \\ &+ |c| \int_0^t \mathbf{P}[|Z_s - Y_s|]ds + |b - c| \int_0^t \mathbf{P}[Y_s]ds. \end{aligned}$$

Proof. For each integer $n \geq 0$ define $a_n = \exp\{-n(n + 1)/2\}$. Then $a_n \to 0$ decreasingly as $n \to \infty$ and

$$\int_{a_n}^{a_{n-1}} z^{-1}dz = n, \qquad n \geq 1.$$

Let $x \mapsto g_n(x)$ be a positive continuous function supported by (a_n, a_{n-1}), so that

$$\int_{a_n}^{a_{n-1}} g_n(x)dx = 1$$

and $g_n(x) \leq 2(nx)^{-1}$ for every $x > 0$. Let

$$f_n(z) = \int_0^{|z|} dy \int_0^y g_n(x)dx, \qquad z \in \mathbb{R}.$$

It is easy to see that $|f_n'(z)| \leq 1$ and

$$0 \leq |z| f_n''(z) = |z| g_n(|z|) \leq 2n^{-1}, \qquad z \in \mathbb{R}.$$

Moreover, we have $f_n(z) \to |z|$ increasingly as $n \to \infty$. Let $\alpha_t = Z_t - Y_t$ for $t \geq 0$. From (3.1) we have

$$\begin{aligned}\alpha_t &= \alpha_0 + h \int_0^t [\eta(s) - \rho(s)] ds - c \int_0^t \alpha_{s-} ds + (b - c) \int_0^t Y_{s-} ds \\ &\quad + \sigma \int_0^t \int_{Y_{s-}}^{Z_{s-}} W(ds, du) + \int_0^t \int_0^\infty \int_{Y_{s-}}^{Z_{s-}} z \tilde{N}_0(ds, dz, du) \\ &\quad + \int_0^t \int_0^\infty \int_{\rho(s)}^{\eta(s)} z N_1(ds, dz, du). \end{aligned} \tag{3.4}$$

By this and Itô's formula,

$$\begin{aligned} f_n(\alpha_t) &= f_n(\alpha_0) + h \int_0^t f_n'(\alpha_s)[\eta(s) - \rho(s)] ds - c \int_0^t f_n'(\alpha_s) \alpha_s ds \\ &\quad + (b - c) \int_0^t f_n'(\alpha_s) Y_s ds + \frac{1}{2}\sigma^2 \int_0^t f_n''(\alpha_s) |\alpha_s| ds \\ &\quad + \int_0^t \alpha_s 1_{\{\alpha_s > 0\}} ds \int_0^\infty [f_n(\alpha_s + z) - f_n(\alpha_s) - z f_n'(\alpha_s)] m(dz) \\ &\quad - \int_0^t \alpha_s 1_{\{\alpha_s < 0\}} ds \int_0^\infty [f_n(\alpha_s - z) - f_n(\alpha_s) + z f_n'(\alpha_s)] m(dz) \\ &\quad + \int_0^t [\eta(s) - \rho(s)] 1_{\{\eta(s) > \rho(s)\}} ds \int_0^\infty [f_n(\alpha_s + z) - f_n(\alpha_s)] n(dz) \\ &\quad - \int_0^t [\rho(s) - \eta(s)] 1_{\{\rho(s) > \eta(s)\}} ds \int_0^\infty [f_n(\alpha_s - z) - f_n(\alpha_s)] n(dz) \\ &\quad + \text{mart.} \end{aligned} \tag{3.5}$$

It is easy to see that $|f_n(a + x) - f_n(a)| \leq |x|$ for any $a, x \in \mathbb{R}$. If $ax \geq 0$, we have

$$|f_n(a + x) - f_n(a) - x f_n'(a)| \leq (2|ax|) \wedge (n^{-1}|x|^2).$$

Taking the expectation in both sides of (3.5) gives

$$\begin{aligned} \mathbf{P}[f_n(\alpha_t)] &\leq \mathbf{P}[f_n(\alpha_0)] + h \int_0^t \mathbf{P}[|\eta(s) - \rho(s)|] ds + |c| \int_0^t \mathbf{P}[|\alpha_s|] ds \\ &\quad + |b - c| \int_0^t \mathbf{P}[Y_s] ds + \int_0^t \mathbf{P}[|\eta(s) - \rho(s)|] ds \int_0^\infty z n(dz) \\ &\quad + n^{-1}\sigma^2 t + \int_0^t ds \int_0^\infty \{(2z\mathbf{P}[|\alpha_s|]) \wedge (n^{-1}z^2)\} m(dz). \end{aligned}$$

Then we get the desired estimate by letting $n \to \infty$. $\square$

Proposition 3.2 *Suppose that* $\{Y_t : t \geq 0\}$ *is a positive càdlàg solution of* (3.1), *and* $\{Z_t : t \geq 0\}$ *is a positive càdlàg solution of the equation with* (b, ρ) *replaced by* (c, η). *Then we have*

$$\begin{aligned}\mathbf{P}\Big[\sup_{0\leq s\leq t}|Z_s - Y_s|\Big] \leq \mathbf{P}[|Z_0 - Y_0|] &+ \psi'(0)\int_0^t \mathbf{P}[|\eta(s) - \rho(s)|]ds \\ &+\Big(|c| + 2\int_1^\infty zm(dz)\Big)\int_0^t \mathbf{P}[|Z_s - Y_s|]ds \\ &+ |b-c|\int_0^t \mathbf{P}[Y_s]ds + 2\sigma\Big(\int_0^t \mathbf{P}[|Z_s - Y_s|]ds\Big)^{\frac{1}{2}} \\ &+ 2\Big(\int_0^t \mathbf{P}[|Z_s - Y_s|]ds\int_0^1 z^2 m(dz)\Big)^{\frac{1}{2}}.\end{aligned}$$

Proof. This follows by applying Doob's martingale inequality to (3.4). □

Theorem 3.3 *For any* $Y_0 \geq 0$ *there is a pathwise unique positive càdlàg solution* $\{Y_t : t \geq 0\}$ *of* (3.1).

Proof. The pathwise uniqueness of the solution follows by Proposition 3.1 and Gronwall's inequality. Without loss of generality, we may assume $Y_0 \geq 0$ is deterministic in proving the existence of the solution. We give the proof in three steps.

Step 1. Let $B(t) = W((0, t] \times (0, 1])$. Then $\{B(t) : t \geq 0\}$ is a standard Brownian motion. By Theorems 5.1 and 5.2 in Dawson and Li (2006), for any constant $\rho \geq 0$ there is a pathwise unique positive solution to

$$\begin{aligned}Y_t = Y_0 + \sigma\int_0^t \sqrt{Y_{s-}}dB(s) &+ \int_0^t\int_0^\infty\int_0^{Y_{s-}} z\tilde{N}_0(ds, dz, du) \\ + \int_0^t (h\rho - bY_{s-})ds &+ \int_0^t\int_0^\infty\int_0^\rho zN_1(ds, dz, du).\end{aligned}$$

It is simple to see that $\{Y_t : t \geq 0\}$ is a weak solution to

$$\begin{aligned}Y_t = Y_0 + \sigma\int_0^t\int_0^{Y_{s-}} W(ds, du) &+ \int_0^t\int_0^\infty\int_0^{Y_{s-}} z\tilde{N}_0(ds, dz, du) \\ + \int_0^t (h\rho - bY_{s-})ds &+ \int_0^t\int_0^\infty\int_0^\rho zN_1(ds, dz, du).\end{aligned} \tag{3.6}$$

As pointed out at the beginning of this proof, the pathwise uniqueness holds for (3.6).

Step 2. Let $0 = r_0 < r_1 < r_2 < \cdots$ be an increasing sequence. For each $i \geq 1$ let η_i be a positive integrable random variable measurable with respect to $\mathscr{F}_{r_{i-1}}$. Let $\rho = \{\rho(t) : t \geq 0\}$ be the positive $(\mathscr{F}_t)$-predictable step process given by

$$\rho(t) = \sum_{i=1}^\infty \eta_i 1_{(r_{i-1}, r_i]}(t), \qquad t \geq 0.$$

By the result in the first step, on each interval $(r_{i-1}, r_i]$ there is a pathwise unique solution $\{Y_t : r_{i-1} < t \leq r_i\}$ to

$$\begin{aligned}Y_t = Y_{r_{i-1}} + \sigma \int_{r_{i-1}}^t \int_0^{Y_{s-}} W(ds, du) + \int_{r_{i-1}}^t \int_0^\infty \int_0^{Y_{s-}} z\tilde{N}_0(ds, dz, du) \\ + \int_{r_{i-1}}^t (h\eta_i - bY_{s-})ds + \int_{r_{i-1}}^t \int_0^\infty \int_0^{\eta_i} zN_1(ds, dz, du).\end{aligned}$$

Then $\{Y_t : t \geq 0\}$ is a solution to (3.1).

Step 3. Suppose that $\rho = \{\rho(t) : t \geq 0\}$ is general positive $(\mathscr{F}_t)$-predictable process such that $t \mapsto \mathbf{P}[\rho(t)]$ is locally bounded. Take a sequence of positive predictable step processes $\rho_k = \{\rho_k(t) : t \geq 0\}$ so that

$$\mathbf{P}\Big[\int_0^t |\rho_k(s) - \rho(s)|ds\Big] \to 0 \tag{3.7}$$

for every $t \geq 0$ as $k \to \infty$. Let $\{Y_k(t) : t \geq 0\}$ be the solution to (3.1) with $\rho = \rho_k$. By Proposition 3.1, Gronwall's inequality and (3.7), one sees

$$\sup_{0\leq s\leq t} \mathbf{P}[|Y_k(s) - Y_i(s)|] \to 0$$

for every $t \geq 0$ as $i, k \to \infty$. Then Proposition 3.2 implies

$$\mathbf{P}\Big[\sup_{0\leq s\leq t} |Y_k(s) - Y_i(s)|\Big] \to 0$$

for every $t \geq 0$ as $i, k \to \infty$. Thus there is a subsequence $\{k_i\} \subset \{k\}$ and a càdlàg process $\{Y_t : t \geq 0\}$ so that

$$\sup_{0\leq s\leq t} |Y_{k_i}(s) - Y_s| \to 0$$

almost surely for every $t \geq 0$ as $i \to \infty$. It is routine to show that $\{Y_t : t \geq 0\}$ is a solution to (3.1). □

Theorem 3.4 *If $\rho = \{\rho(t) : t \geq 0\}$ is a deterministic locally bounded positive Borel function, the solution $\{Y_t : t \geq 0\}$ of* (3.1) *is an inhomogeneous CBI-process with transition semigroup $\{P^\rho_{r,t} : t \geq r \geq 0\}$ defined by* (2.22).

Proof. By the martingale problem (3.3), when $\rho(t) = \rho$ is a deterministic constant function, the process $\{Y_t : t \geq 0\}$ is a Markov process with transition semigroup $\{P^\rho_{r,t} : t \geq r \geq 0\}$; see, for example, Theorem 9.30 in Li (2011). If $\rho = \{\rho(t) : t \geq 0\}$ is a general deterministic locally bounded positive Borel function, we can take each step function $\rho_k = \{\rho_k(t) : t \geq 0\}$ in the last proof to be deterministic. Then the solution

$\{Y_k(t) : t \geq 0\}$ of (3.1) with $\rho = \rho_k$ is an inhomogeneous CBI-process with transition semigroup $\{P^{\rho_k}_{r,t} : t \geq r \geq 0\}$. In other words, for any $\lambda \geq 0$, $t \geq r \geq 0$ and $G \in \mathscr{F}_r$ we have

$$\mathbf{P}[1_G e^{-\lambda Y_k(t)}] = \mathbf{P}\Big[1_G \exp\Big\{ - Y_k(r) v_{t-r}(\lambda) - \int_r^t \rho_k(s)\psi(v_{t-s}(\lambda))ds\Big\}\Big].$$

Letting $k \to \infty$ along the sequence $\{k_i\}$ mentioned in the last proof gives

$$\mathbf{P}[1_G e^{-\lambda Y_t}] = \mathbf{P}\Big[1_G \exp\Big\{ - Y_r v_{t-r}(\lambda) - \int_r^t \rho(s)\psi(v_{t-s}(\lambda))ds\Big\}\Big].$$

Then $\{Y_t : t \geq 0\}$ is a CBI-process with immigration rate $\rho = \{\rho(t) : t \geq 0\}$. □

In view of the result of Theorem 3.4, the solution $\{Y_t : t \geq 0\}$ to (3.1) can be called an inhomogeneous CBI-process with branching mechanism ϕ, immigration mechanism ψ and *predictable immigration rate* $\rho = \{\rho(t) : t \geq 0\}$.

4 Path-valued branching processes

In this section, we introduce some path-valued Markov processes, which are essentially special forms of the immigration superprocesses defined by (2.7) and (2.10). Suppose that $T \subset \mathbb{R}$ is an interval, and $\{\phi_q : q \in T\}$ is a family of branching mechanisms, where ϕ_q is given by (1.1) with the parameters $(b, m) = (b_q, m_q)$ depending on $q \in T$. We call $\{\phi_q : q \in T\}$ an *admissible family* if for each $\lambda \geq 0$, the function $q \mapsto \phi_q(\lambda)$ is decreasing and continuously differentiable with the derivative $\zeta_q(\lambda) := -(\partial/\partial q)\phi_q(\lambda)$ of the form

$$\zeta_q(\lambda) = \beta_q \lambda + \int_0^\infty (1 - e^{-z\lambda}) n_q(dz), \qquad q \in T, \lambda \geq 0, \tag{4.1}$$

where $\beta_q \geq 0$ and $n_q(dz)$ is a σ-finite kernel from T to $(0, \infty)$ satisfying

$$\sup_{p \leq \theta \leq q} \Big[\beta_\theta + \int_0^\infty z n_\theta(dz)\Big] < \infty, \qquad q \geq p \in T.$$

For an admissible family $\{\phi_q : q \in T\}$, we clearly have

$$\phi_{p,q}(\lambda) := \phi_p(\lambda) - \phi_q(\lambda) = \int_p^q \zeta_\theta(\lambda) d\theta. \tag{4.2}$$

It follows that

$$b_q = b_p - \int_p^q \beta_\theta d\theta - \int_p^q d\theta \int_0^\infty z n_\theta(dz) \tag{4.3}$$

and

$$m_q(dz) = m_p(dz) + \int_{\{p<\theta\leq q\}} n_\theta(dz)d\theta. \tag{4.4}$$

We say $q_0 \in T$ is a *critical point* of the admissible family $\{\phi_q : q \in T\}$ if $b_{q_0} = 0$, which means ϕ_{q_0} is a critical branching mechanism. By (4.3) one can see $q \mapsto b_q$ is a continuous decreasing function on T, so the set of critical points $T_0 \subset T$ can only be an interval.

Let us consider a function $\mu \in F(T)$ and an admissible family of branching mechanisms $\{\phi_q : q \in T\}$. Write $\mu(p, q] = \mu(q) - \mu(p)$ for $q \geq p \in T$. Recall that (2.22) defines the transition semigroup $\{P^\rho_{r,t} : t \geq r \geq 0\}$ of an inhomogeneous CBI-process $\{Y_t : t \geq 0\}$. Let $\mathbf{P}^\rho_x(\phi, \psi, dw)$ denote the law on $D^+[0,\infty)$ of such a process with initial value $Y_0 = x \geq 0$. Given any $\rho \in D^+[0,\infty)$, we define the probability measure $\mathbf{P}_{p,q}(\rho, dw)$ on $D^+[0,\infty)$ by

$$\mathbf{P}_{p,q}(\rho, B) = \int_{D^+[0,\infty)} 1_B(\rho + w)\mathbf{P}^\rho_{\mu(p,q]}(\phi_q, \phi_{p,q}, dw) \tag{4.5}$$

for Borel sets $B \subset D^+[0,\infty)$. In view of (2.24), for any $f \in B^+[0,\infty)$ with compact support, we have

$$\begin{aligned}\int_{D^+[0,\infty)} &\exp\Big\{ - \int_0^\infty f(s)w(s)ds\Big\}\mathbf{P}_{p,q}(\rho, dw) \\ &= \exp\Big\{ - \mu(p, q]u_q(0, f) - \int_0^\infty u_{p,q}(s, f)\rho(s)ds\Big\},\end{aligned} \tag{4.6}$$

where $s \mapsto u_q(s) := u_q(s, f)$ is the unique compactly supported bounded positive solution to

$$u_q(s) + \int_s^\infty \phi_q(u_q(t))dt = \int_s^\infty f(t)dt, \tag{4.7}$$

and

$$u_{p,q}(s, f) = f(s) + \phi_{p,q}(u_q(s, f)), \quad s \geq 0. \tag{4.8}$$

We remark that $u_q(s, f) = u_{p,q}(s, f) = 0$ for $s > l_f := \sup\{t \geq 0 : f(t) > 0\}$.

Proposition 4.1 *For any $f \in B^+[0,\infty)$ with compact support, we have*

$$u_p(s, u_{p,q}(\cdot, f)) = u_q(s, f), \qquad s \geq 0, p \leq q \in T \tag{4.9}$$

and

$$u_{p,\theta}(s, u_{\theta,q}(\cdot, f)) = u_{p,q}(s, f), \quad s \geq 0, p \leq \theta \leq q \in T. \tag{4.10}$$

Proof. From (4.2) and (4.7) we can see that $s \mapsto v(s) := u_q(s, f)$ is a solution of

$$v(s) = \int_s^\infty [f(t) + \phi_{p,q}(u_q(t, f))]dt - \int_s^\infty \phi_p(v(t))dt. \tag{4.11}$$

On the other hand, by (4.7) and (4.8) we have

$$\begin{aligned} u_p(s, u_{p,q}(\cdot, f)) &= \int_s^\infty u_{p,q}(t, f)dt - \int_s^\infty \phi_p(u_p(t, u_{p,q}(\cdot, f)))dt \\ &= \int_s^\infty [f(t) + \phi_{p,q}(u_q(t, f))]dt \\ &\qquad - \int_s^\infty \phi_p(u_p(t, u_{p,q}(\cdot, f)))dt. \end{aligned}$$

Then $s \mapsto u_p(s, u_{p,q}(\cdot, f))$ is also a solution to (4.11). By the uniqueness of the solution to the equation, we get (4.9). It follows that

$$\begin{aligned} u_{p,\theta}(s, u_{\theta,q}(\cdot, f)) &= u_{\theta,q}(s, f) + \phi_{p,\theta}(u_\theta(s, u_{\theta,q}(\cdot, f))) \\ &= f(s) + \phi_{\theta,q}(u_q(s, f)) + \phi_{p,\theta}(u_q(s, f)) \\ &= f(s) + \phi_{p,q}(u_q(s, f)). \end{aligned}$$

Then we have (4.10). □

Proposition 4.2 *For any $f \in B^+[0, \infty)$ with compact support we have*

$$u_{p,q}(s, f) = f(s) + \int_p^q \psi_\theta(s, u_{\theta,q}(\cdot, f))d\theta, \quad s \geq 0, q \geq p \in T, \tag{4.12}$$

where $\psi_\theta(s, f) = \zeta_\theta(u_\theta(s, f))$.

Proof. By (4.2) and (4.8) one can see $p \mapsto u_{p,q}(s, f)$ is a decreasing function. In view of (4.9) and (4.10), for $q > \theta > p \in T$, we get

$$u_{p,q}(s, f) = u_{p,\theta}(s, u_{\theta,q}(\cdot, f)) = u_{\theta,q}(s, f) + \phi_{p,\theta}(u_\theta(s, u_{\theta,q}(\cdot, f))).$$

Then we differentiate both sides to see

$$\frac{d}{dp} u_{p,q}(s, f)\Big|_{p=\theta-} = \frac{d}{dp}\phi_{p,\theta}(u_\theta(s, u_{\theta,q}(\cdot, f)))\Big|_{p=\theta-} = -\zeta_\theta(u_\theta(s, u_{\theta,q}(\cdot, f))),$$

which implies (4.12). □

From (4.6) one can see that $\mathbf{P}_{p,q}(\rho, dw)$ is a probability kernel on $D^+[0, \infty)$. By (4.9) and (4.10) it is easy to check that the family of kernels $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ satisfies the Chapman–Kolmogorov equation. Then $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ form an inhomogeneous Markov transition semigroup on $D^+[0, \infty)$. This semigroup is closely related to some nonlocal branching superprocesses. For $\alpha \geq 0$ let $M[0, \alpha]$ be the space of finite Borel measures on $[0, \alpha]$ furnished with the topology of weak convergence.

Theorem 4.3 *There is a Markov transition semigroup $\{\mathbf{Q}_{p,q}^\alpha : q \geq p \in T\}$ on $M[0,\alpha]$ such that, for $f \in B^+[0,\alpha]$,*

$$\int_{M[0,\alpha]} e^{-\langle \nu, f\rangle} \mathbf{Q}_{p,q}^\alpha(\eta, d\nu) = \exp\Big\{ -\mu(p,q] u_q^\alpha(0,f) - \langle \eta, u_{p,q}^\alpha(\cdot, f)\rangle \Big\}, \tag{4.13}$$

where

$$u_q^\alpha(s,f) = u_q(s, f1_{[0,\alpha]}), \quad u_{p,q}^\alpha(s,f) = u_{p,q}(s, f1_{[0,\alpha]}). \tag{4.14}$$

Proof. We first consider an absolutely continuous measure $\eta \in M[0,\alpha]$ with a density $\rho \in D^+[0,\alpha]$. Suppose that $\{X_t : t \geq 0\}$ is a random path with distribution $\mathbf{P}_{p,q}(\rho 1_{[0,\alpha]}, \cdot)$ on $D^+[0,\infty)$. Let $\mathbf{Q}_{p,q}^\alpha(\eta, \cdot)$ be the distribution on $M[0,\alpha]$ of the random measure X such that $X(dt) = X_t dt$ for $0 \leq t \leq \alpha$. The Laplace function of $\mathbf{Q}_{p,q}^\alpha(\eta,\cdot)$ is clearly given by (4.13) and (4.14). In particular, we can use those two formulas to define a probability measure on $M[0,\alpha]$. For an arbitrary $\eta \in M[0,\alpha]$, choose a sequence of absolutely continuous measures $\{\eta_n\} \subset M[0,\alpha]$ with densities in $D^+[0,\alpha]$ so that $\eta_n \to \eta$ weakly. Let $\mathbf{Q}_{p,q}^\alpha(\eta_n, \cdot)$ be the probability measure on $M[0,\alpha]$ defined by

$$\int_{M[0,\alpha]} e^{-\langle \nu, f\rangle} \mathbf{Q}_{p,q}^\alpha(\eta_n, d\nu) = \exp\Big\{ -\mu(p,q] u_q^\alpha(0,f) - \langle \eta_n, u_{p,q}^\alpha(\cdot, f)\rangle \Big\}.$$

For $f \in C^+[0,\alpha]$ one can see from (4.7) and (4.8) that $u_{p,q}^\alpha(\cdot, f) \in C^+[0,\alpha]$, and hence

$$\lim_{n\to\infty} \int_{M[0,\alpha]} e^{-\langle \nu, f\rangle} \mathbf{Q}_{p,q}^\alpha(\eta_n, d\nu) = \exp\Big\{ -\mu(p,q] u_q^\alpha(0,f) - \langle \eta, u_{p,q}^\alpha(\cdot, f)\rangle \Big\}.$$

Then (4.13) really gives the Laplace functional of a probability measure $\mathbf{Q}_{p,q}^\alpha(\eta,\cdot)$ on $M[0,\alpha]$ which is the weak limit of $\mathbf{Q}_{p,q}^\alpha(\eta_n,\cdot)$ as $n \to \infty$. It is easy to see that $\mathbf{Q}_{p,q}^\alpha(\eta, d\nu)$ is a kernel on $M[0,\alpha]$. The semigroup property of the family $\{\mathbf{Q}_{p,q}^\alpha : q \geq p \in T\}$ follows from (4.9) and (4.10). $\square$

Theorem 4.4 *Let $q \in T$ and $f \in B^+[0,\alpha]$. Then $(p,s) \mapsto u_{p,q}^\alpha(s) := u_{p,q}^\alpha(s,f)$ is the unique locally bounded positive solution to*

$$u_{p,q}^\alpha(s) = f(s) + \int_p^q \psi_\theta^\alpha(s, u_{\theta,q}^\alpha) d\theta, \quad s \in [0,\alpha], q \geq p \in T, \tag{4.15}$$

where $\psi_\theta^\alpha(s,f) = \zeta_\theta(u_\theta^\alpha(s,f))$. Moreover, the transition semigroup $\{\mathbf{Q}_{p,q}^\alpha : q \geq p \in T\}$ defines an immigration superprocess in $M[0,\alpha]$ with branching mechanisms $\{-\psi_\theta^\alpha : \theta \in T\}$, immigration mechanisms $\{u_\theta^\alpha(0,\cdot) : \theta \in T\}$ and immigration measure μ.

Proof. From (4.12) one can see that $u^\alpha_{p,q}(s) = u^\alpha_{p,q}(s,f)$ satisfies (4.15). By letting $t = \alpha$ and $\eta(ds) = ds$ in (2.15), we infer that the functional $f \mapsto u^\alpha_\theta(s,f)$ on $B^+[0,\alpha]$ is the Laplace exponent of an infinitely divisible probability measure carried by $M[s,\alpha]$. It is easy to see that $\psi^\alpha_\theta(s,0) = 0$. By Theorem 2.2 the composed functional $f \mapsto \psi^\alpha_\theta(s,f)$ is also the Laplace exponent of an infinitely divisible probability measure on $M[s,\alpha]$. Then it has the representation

$$\psi^\alpha_\theta(s,f) = \langle \eta^\alpha_\theta(s), f\rangle + \int_{M[s,\alpha]^\circ} (1 - e^{-\langle \nu, f\rangle}) H^\alpha_\theta(s,d\nu), \tag{4.16}$$

where $\eta^\alpha_\theta(s) \in M[s,\alpha]$ and $(1 \wedge \langle \nu, 1\rangle) H^\alpha_\theta(s,d\nu)$ is a finite measure on $M[s,\alpha]^\circ$. By letting $f(t) = \lambda$ and taking the derivatives in both sides of (4.16), we have

$$\frac{d}{d\lambda}\psi^\alpha_\theta(s,\lambda)\Big|_{\lambda=0+} = \langle \eta^\alpha_\theta(s), 1\rangle + \int_{M[s,\alpha]^\circ} \langle \nu, 1\rangle H^\alpha_\theta(s,d\nu).$$

On the other hand, using (2.14) and (2.15),

$$\frac{d}{d\lambda}u^\alpha_\theta(s,\lambda)\Big|_{\lambda=0+} = \int_s^\alpha e^{-b_\theta(t-s)}dt.$$

From (4.1) we have

$$\frac{d}{d\lambda}\zeta_\theta(\lambda)\Big|_{\lambda=0+} = \beta_\theta + \int_0^\infty z n_\theta(dz).$$

It follows that

$$\frac{d}{d\lambda}\psi^\alpha_\theta(s,\lambda)\Big|_{\lambda=0+} = \Big[\beta_\theta + \int_0^\infty z n_\theta(dz)\Big] \int_s^\alpha e^{-b_\theta(t-s)}dt.$$

As a function of (θ, s), the above quantity is bounded on $S \times [0,\alpha]$ for each bounded closed interval $S \subset T$. By Example 2.5 of Li (2011) one sees that $f \mapsto -\psi^\alpha_\theta(\cdot, f)$ is a special form of the operator given by (2.3), and so (4.15) is a special case of (2.7). Thus $(p,s) \mapsto u^\alpha_{p,q}(s,f)$ is the unique locally bounded positive solution to (4.15). By (4.9) we have

$$\mu(p,q]u^\alpha_q(0,f) = \int_p^q u^\alpha_q(0,f)\mu(d\theta) = \int_p^q u^\alpha_\theta(0, u^\alpha_{\theta,q}(\cdot,f))\mu(d\theta). \tag{4.17}$$

Then $\{\mathbf{Q}^\alpha_{p,q} : q \geq p \in T\}$ defines an immigration superprocess in $M[0,\alpha]$ with branching mechanisms $\{-\psi^\alpha_\theta : \theta \in T\}$, immigration mechanisms $\{u^\alpha_\theta(0,\cdot) : \theta \in T\}$ and immigration measure μ. $\square$

Let $\mathscr{M}[0,\infty)$ denote the space of Radon measures on $[0,\infty)$ endowed with the topology of vague convergence. For any $\alpha \geq 0$ we regard $M[0,\alpha]$ as the subset of $\mathscr{M}[0,\infty)$ consisting of the measures supported by $[0,\alpha]$. We can also embed $D^+[0,\infty)$ continuously into $\mathscr{M}[0,\infty)$ by identifying the path $w \in D^+[0,\infty)$ and the measure $\nu \in \mathscr{M}[0,\infty)$ such that $\nu(ds) = w(s)ds$ for $s \geq 0$.

Theorem 4.5 *There is an extension $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$ of $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ on $\mathscr{M}[0,\infty)$, which is given by*

$$\int_{\mathscr{M}[0,\infty)} e^{-\langle \nu, f\rangle} \mathbf{Q}_{p,q}(\eta, d\nu) = \exp\Big\{ -\mu(p,q]u_q(0,f) - \langle \eta, u_{p,q}(\cdot, f)\rangle\Big\} \qquad (4.18)$$

for $f \in B^+[0,\infty)$ with compact support.

Proof. Given $\eta \in \mathscr{M}[0,\infty)$, we define $\pi_\alpha\eta \in M[0,\alpha]$ by $\pi_\alpha\eta(ds) = 1_{[0,\alpha]}\eta(ds)$. It is easy to check that $\pi_\alpha\pi_\beta\eta = \pi_\alpha\eta$ for $\beta \geq \alpha \geq 0$. Then the sequence of probability measures $\{\mathbf{Q}_{p,q}(\pi_k\eta, \cdot) : k = 1, 2, \cdots\}$ induce a consistent family of finite-dimensional distributions on the product space $M_\infty := \prod_{k=1}^\infty M[0,k]$. Let $\mathbf{Q}$ be the unique probability measure on M_∞ determined by the family. Then under $\mathbf{Q}$ the canonical sequence $(X_1, X_2, \cdots)$ of M_∞ converges almost surely to a random Radon measure X on $[0,\infty)$, which has distribution $\mathbf{Q}_{p,q}(\eta, \cdot)$ on $\mathscr{M}[0,\infty)$ given by (4.18). It is easy to show that $\mathbf{Q}_{p,q}(\eta, d\nu)$ is a probability kernel on the space $\mathscr{M}[0,\infty)$. The semigroup property of $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$ follows from (4.9) and (4.10). $\square$

Since the state space $\mathscr{M}[0,\infty)$ contains infinite measures, the transition semigroup $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$ defined by (4.18) does not fit exactly into the setup of the second section. However, if $\{Z_q : q \in T\}$ is a Markov process in $\mathscr{M}[0,\infty)$ with transition semigroup $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$, for each $\alpha \geq 0$, the restriction of $\{Z_q : q \in T\}$ to $[0,\alpha]$ is an inhomogeneous immigration superprocess with transition semigroup $\{\mathbf{Q}^\alpha_{p,q} : q \geq p \in T\}$. Then we can think of the original process $\{Z_q : q \in T\}$ as an inhomogeneous immigration superprocess with the *extended state space* $\mathscr{M}[0,\infty)$. The model can be described intuitively as follows. The offspring born by a "particle" at site $s \geq 0$ at time $\theta \in T$ are spread over the interval $[s,\infty)$ according to the law determined by $\psi_\theta(s,\cdot)$. Thus the superprocess only involves a nonlocal branching structure. The immigration rate is given by $\mu(d\theta)$ and the immigrants coming at time $\theta \in T$ are distributed in $[0,\infty)$ according to the law given by $u_\theta(0,\cdot)$. The spatial motion of the immigration superprocess is trivial.

Suppose that $\{(X_t(q))_{t\geq 0} : q \in T\}$ is a Markov process with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ defined by (4.6). We can identify the random path $(X_t(q))_{t\geq 0}$ with the absolutely continuous random measure Z_q on $[0,\infty)$ with $(X_t(q))_{t\geq 0}$ as a density. By Theorem 4.5, the measure-valued process $\{Z_q : q \in T\}$ is an immigration superprocess with transition semigroup $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$ defined by (4.18). Therefore we can naturally call $\{(X_t(q))_{t\geq 0} : q \in T\}$ a *path-valued branching process with immigration*. By (4.5) we have $X_q \geq X_p$ almost surely for $q \geq p \in T$. If $\mu(q) = \mu$ independent of $q \in T$, we simply call $\{X_q : q \geq 0\}$ a *path-valued branching process*.

By (4.8) or (4.12) we have $u_{p,q}(s,f) \geq f(s)$ for any $s \geq 0$ and $f \in B^+[0,\infty)$ with compact support. Then (4.18) implies that the set of infinite measures on $[0,\infty)$ is absorbing for $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$. Let $\{\mathbf{Q}^\infty_{p,q} : q \geq p \in T\}$ denote the sub-Markov restriction of $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$ to the space $M[0,\infty)$ of finite measures on $[0,\infty)$. If $f \in B^+[0,\infty)$ is bounded away from zero, we define

$$u_q^\infty(s,f) = \lim_{\alpha\to\infty} u_q(s, f1_{[0,\alpha]}), \quad u_{p,q}^\infty(s,f) = \lim_{\alpha\to\infty} u_{p,q}(s, f1_{[0,\alpha]}).$$

For an arbitrary $f \in B^+[0,\infty)$, define

$$u_q^\infty(s,f) = \lim_{n\to\infty} u_q^\infty(s, f+1/n), \quad u_{p,q}^\infty(s,f) = \lim_{n\to\infty} u_{p,q}^\infty(s, f+1/n).$$

By (4.15) one can see $u_{p,q}^\infty(s) := u_{p,q}^\infty(s,f)$ solves

$$u_{p,q}^\infty(s) = f(s) + \int_p^q \psi_\theta^\infty(s, u_{\theta,q}^\infty) d\theta, \qquad s \geq 0, q \geq p \in T, \tag{4.19}$$

where $\psi_\theta^\infty(s,f) = \zeta_\theta(u_\theta^\infty(s,f))$. From (4.8) we obtain

$$u_{p,q}^\infty(s,f) = f(s) + \phi_{p,q}(u_q^\infty(s,f)), \qquad s \geq 0. \tag{4.20}$$

It is easy to show that, for $f \in B^+[0,\infty)$,

$$\int_{M[0,\infty)} e^{-\langle \nu,f\rangle} \mathbf{Q}_{p,q}^\infty(\eta, d\nu) = \exp\Big\{ -\mu(p,q] u_q^\infty(0,f) - \langle \eta, u_{p,q}^\infty(\cdot,f)\rangle \Big\}. \tag{4.21}$$

To avoid the triviality of $\{\mathbf{Q}_{p,q}^\infty : q \geq p \in T\}$, we need to assume $\phi_q(\lambda) \to \infty$ as $\lambda \to \infty$ for every $q \in T$. In this case, we can define the right inverse ϕ_q^{-1} of ϕ_q as in (2.20). By (2.17), (2.19) and (4.20), we have

$$u_q^\infty(s,\lambda) = \phi_q^{-1}(\lambda), \; u_{p,q}^\infty(s,\lambda) = \phi_p(\phi_q^{-1}(\lambda)), \quad s \geq 0, \lambda \geq 0. \tag{4.22}$$

Theorem 4.6 *Suppose that $\phi_q(\lambda) \to \infty$ as $\lambda \to \infty$ for every $q \in T$. Let $S \subset T$ be an interval not containing critical points of $\{\phi_q : q \in T\}$. Then for any $q \in S$ and $f \in B^+[0,\infty)$ there is a unique locally bounded positive solution $(p,s) \mapsto u_{p,q}^\infty(s) := u_{p,q}^\infty(s,f)$ to (4.19) on $S \times [0,\infty)$. Moreover, the sub-Markov transition semigroup $\{\mathbf{Q}_{p,q}^\infty : q \geq p \in S\}$ defines an inhomogeneous immigration superprocess in $M[0,\infty)$ with branching mechanisms $\{-\psi_\theta^\infty : \theta \in S\}$, immigration mechanisms $\{u_\theta^\infty(0,\cdot) : \theta \in S\}$ and immigration measure μ.*

Proof. For any $s \geq 0$ and $\theta \in S$ one can see by (2.17) that the functional $f \mapsto u_\theta^\infty(s,f)$ on $B^+[0,\infty)$ is the exponent of an infinitely divisible sub-probability measure carried by $M[s,\infty)$. Then we have the representation

$$u_\theta^\infty(s,f) = a_\theta^\infty(s) + \langle \eta_\theta^\infty(s), f\rangle + \int_{M[s,\infty)^\circ} (1 - e^{-\langle \nu,f\rangle}) H_\theta^\infty(s,d\nu),$$

where $a_\theta^\infty(s) \geq 0$, $\eta_\theta^\infty(s) \in M[s,\infty)$ and $(1 \wedge \langle \nu, 1\rangle)H_\theta^\infty(s,d\nu)$ is a finite measure on $M[s,\infty)^\circ$. By the first equality in (4.22) we get $a_\theta^\infty(s) = u_\theta^\infty(s,0) = \phi_\theta^{-1}(0)$. It follows that

$$\langle \eta_\theta^\infty(s), 1\rangle + \int_{M[s,\infty)^\circ} \langle \nu, 1\rangle H_\theta^\infty(s,d\nu) = 1/\phi_\theta'(\phi_\theta^{-1}(0)).$$

The right-hand side is bounded on each compact subinterval of S. By Theorem 2.2, the composed functional $f \mapsto \psi_\theta^\infty(s,f) = \zeta_\theta(u_\theta^\infty(s,f))$ is the exponent of an infinitely divisible sub-probability measure carried by $M[s,\infty)$. Then $f \mapsto \psi_\theta^\infty(s,0) - \psi_\theta^\infty(s,f)$ can be represented by a special form of (2.3). That shows (4.19) is a special case of (2.7). The desired result now follows in view of (4.21) and (4.17) with $\alpha = \infty$. $\square$

If $\phi_q(\lambda) \to \infty$ as $\lambda \to \infty$ for every $q \in T$, we can restrict $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ to the space $D^+_{\rm in}[0,\infty)$ of integrable paths in $D^+[0,\infty)$ to get a sub-Markov transition semigroup $\{\mathbf{P}^\infty_{p,q} : q \geq p \in T\}$. This semigroup can also be regarded as a restriction of $\{\mathbf{Q}^\infty_{p,q} : q \geq p \in T\}$. For $f \in B^+[0,\infty)$, we have

$$\begin{aligned}\int_{D^+_{\rm in}[0,\infty)} & \exp\Big\{ - \int_0^\infty f(s)w(s)ds\Big\}\mathbf{P}^\infty_{p,q}(\eta, dw) \\ &= \exp\Big\{ - \mu(p,q]u_q^\infty(0,f) - \int_0^\infty u_{p,q}^\infty(s,f)\eta(s)ds\Big\}. \end{aligned} \tag{4.23}$$

For an inhomogeneous immigration superprocess $\{Z_q : q \in T\}$ with transition semigroup $\{\mathbf{Q}_{p,q} : q \geq p \in T\}$ or $\{\mathbf{Q}^\infty_{p,q} : q \geq p \in T\}$, we define its *total mass process* $\{\sigma(q) : q \in T\}$ by $\sigma(q) = Z_q[0,\infty)$. For a path-valued branching process with immigration $\{(X_t(q))_{t\geq 0} : q \in T\}$ with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ or $\{\mathbf{P}^\infty_{p,q} : q \geq p \in T\}$, its *total mass process* is defined as

$$\sigma(q) = \int_0^\infty X_s(q)ds, \qquad q \in T.$$

We here think of $\{\sigma(q) : q \in T\}$ as a process with state space $\mathbb{R}_+$ and cemetery ∞. In view of (4.21), (4.22) and (4.23), we have

Theorem 4.7 *Suppose that $\phi_q(\lambda) \to \infty$ as $\lambda \to \infty$ for every $q \in T$. Then $\{\sigma(q) : q \in T\}$ is an inhomogeneous Markov process with transition semigroup $\{R_{p,q} : q \geq p \in T\}$ such that, for $\lambda \geq 0$,*

$$\int_{\mathbb{R}_+} e^{-\lambda y} R_{p,q}(x,dy) = \exp\Big\{ - x\phi_p(\phi_q^{-1}(\lambda)) - \mu(p,q]\phi_q^{-1}(\lambda)\Big\}. \tag{4.24}$$

Before concluding this section, let us consider the admissible family of branching mechanisms $\{\phi_q : q \in \mathbb{R}\}$ defined by $\phi_q(\lambda) = \lambda^2 - 2q\lambda$ for $\lambda \geq 0$. In this special case, zero is the only critical point of the family $\{\phi_q : q \in \mathbb{R}\}$. Let $\{(X_t(q))_{t\geq 0} : q \in \mathbb{R}\}$ be a corresponding path-valued branching process. Let $\{\sigma(q) : q \in \mathbb{R}\}$ be the process of total mass. By Theorem 4.7 one can see that $\{\sigma(q) : q \in \mathbb{R}\}$ is an inhomogeneous Markov process with transition semigroup $\{R_{p,q} : q \geq p \in \mathbb{R}\}$ defined by

$$\int_{\mathbb{R}_+} e^{-\lambda y} R_{p,q}(x, dy) = \exp\{-x v_{p,q}(\lambda)\}, \qquad \lambda \geq 0, \tag{4.25}$$

where

$$v_{p,q}(\lambda) = \lambda + 2(q-p)(\sqrt{q^2+\lambda} + q).$$

This process can be obtained from two homogeneous CB-processes by simple transformations. For $t, \lambda \geq 0$ let

$$u_t^-(\lambda) = e^{-2t}\lambda + 2e^{-t}(1-e^{-t})(\sqrt{1+\lambda} - 1).$$

It is easy to check that

$$u_{t-s}^-(\lambda) = e^{2s} v_{-e^{-s}, -e^{-t}}(e^{-2t}\lambda), \quad \lambda \geq 0, t \geq s \in \mathbb{R}.$$

From this and (4.25) one can see that $\{e^{-2t}\sigma(-e^{-t}) : t \in \mathbb{R}\}$ is a homogeneous Markov process with transition semigroup $(R_t^-)_{t\geq 0}$ defined by

$$\int_{\mathbb{R}_+} e^{-\lambda y} R_t^-(x, dy) = e^{-x u_t^-(\lambda)}, \qquad \lambda \geq 0.$$

Moreover, we have

$$\frac{d}{dt} u_t^-(\lambda) = -\phi_-(u_t^-(\lambda)),$$

where

$$\phi_-(z) = 2z - 2(\sqrt{1+z} - 1).$$

Then $\{e^{-2t}\sigma(-e^{-t}) : t \in \mathbb{R}\}$ is actually a conservative homogeneous CB-process in $[0, \infty)$ with branching mechanism ϕ_-. Similarly, one sees $\{e^{2t}\sigma(e^t) : t \in \mathbb{R}\}$ is a homogeneous Markov process with transition semigroup $(R_t^+)_{t\geq 0}$ defined by

$$\int_{\mathbb{R}_+} e^{-\lambda y} R_t^+(x, dy) = e^{-x u_t^+(\lambda)}, \qquad \lambda \geq 0,$$

where

$$u_t^+(\lambda) = e^{2t}\lambda + 2e^t(e^t - 1)(\sqrt{1+\lambda} + 1).$$

One can easily see that

$$\frac{d}{dt} u_t^+(\lambda) = -\phi_+(u_t^+(\lambda)),$$

where

$$\phi_+(z) = -2z - 2(\sqrt{1+z} + 1).$$

Then $\{e^{2t}\sigma(e^t) : t \in \mathbb{R}\}$ is a CB-process with branching mechanism ϕ_+.

5 Construction by stochastic equations

In this section, we give a construction of the path-valued process $\{(X_t(q))_{t\geq 0} : q \in T\}$ with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ defined by (4.6) as the solution flow of a system of stochastic equations driven by time–space noises. We shall assume $T = [0, \infty)$ or $[0, a]$ or $[0, a)$ for some $a > 0$. This specification of the index set is clearly not essential for the applications. Let $\mu \in F(T)$, and let $\{\phi_q : q \in T\}$ be an admissible family of branching mechanisms, where ϕ_q is given by (1.1) with the parameters $(b, m) = (b_q, m_q)$ depending on $q \in T$. Let $\mu(p, q] = \mu(q) - \mu(p)$ for $q \geq p \in T$, and let $m(dy, dz)$ be the measure on $T \times (0, \infty)$ defined by

$$m([0, q] \times [c, d]) = m_q[c, d], \qquad q \in T, d > c > 0. \tag{5.1}$$

Let $\rho = \rho(s)$ be a locally bounded positive Borel function on $[0, \infty)$ and let ψ be an immigration mechanism given by (2.21).

Suppose that $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbf{P})$ is a filtered probability space satisfying the usual hypotheses. Let $W(ds, du)$ be an $(\mathscr{F}_t)$-white noise on $(0, \infty)^2$ based on the Lebesgue measure, let $\tilde{N}_0(ds, dy, dz, du)$ be a compensated $(\mathscr{F}_t)$-Poisson random measure on $(0, \infty) \times T \times (0, \infty)^2$ with intensity $dsm(dy, dz)du$ and let $N_1(ds, dz, du)$ be an $(\mathscr{F}_t)$-Poisson random measure on $(0, \infty)^3$ with intensity $dsn(dz)du$. Suppose that $W(ds, du)$, $\tilde{N}_0(ds, dy, dz, du)$ and $N_1(ds, dz, du)$ are independent of each other. For $q \in T$ it is easy to see that

$$\tilde{N}(ds, dz, du) := \int_{\{0 \leq y \leq q\}} \tilde{N}_0(ds, dy, dz, du)$$

is a compensated Poisson random measure with intensity $dsm_q(dz)du$. By Theorem 3.3 for every $q \in T$ there is a pathwise unique solution to the stochastic equation

$$\begin{aligned} X_t(q) = \mu(q) &- b_q \int_0^t X_{s-}(q)ds + \sigma \int_0^t \int_0^{X_{s-}(q)} W(ds, du) \\ &+ \int_0^t \int_{[0,q]} \int_0^\infty \int_0^{X_{s-}(q)} z\tilde{N}_0(ds, dy, dz, du) \\ &+ h \int_0^t \rho(s)ds + \int_0^t \int_0^\infty \int_0^{\rho(s)} zN_1(ds, dz, du). \end{aligned} \tag{5.2}$$

By Theorem 3.4 the solution $\{X_t(q) : t \geq 0\}$ is a CBI-process with branching mechanism ϕ_q, immigration mechanism ψ and immigration rate ρ.

Theorem 5.1 *The process $\{(X_t(q))_{t\geq 0} : q \in T\}$ is a path-valued branching process with immigration in $D^+[0, \infty)$ having transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ defined by (4.6).*

Proof. We can rewrite equation (5.2) into

$$X_t(q) = \mu(q) - h_q \int_0^t X_{s-}(q)ds + \sigma \int_0^t \int_0^{X_{s-}(q)} W(ds, du)$$

$$
\begin{aligned}
&+ h\int_0^t \rho(s)ds + \int_0^t\int_0^q\int_0^\infty\int_0^{X_{s-}(q)} zN_0(ds,dy,dz,du)\\
&+ \int_0^t\int_{\{0\}}\int_0^\infty\int_0^{X_{s-}(q)} z\tilde{N}_0(ds,dy,dz,du)\\
&+ \int_0^t\int_0^\infty\int_0^{\rho(s)} zN_1(ds,dz,du),
\end{aligned}
\tag{5.3}
$$

where

$$
q \mapsto h_q := b_q + \int_0^q d\theta\int_0^\infty zn_\theta(dz) = b_0 - \int_0^q \beta_\theta d\theta
$$

is a decreasing function. Then, for $q \geq p \in T$, one can see by a simple modification of Theorem 2.2 in Dawson and Li (2011) that $X_t(q) \geq X_t(p)$ for every $t \geq 0$ with probability one. Let $\xi_t(p,q) = X_t(q) - X_t(p)$ for $t \geq 0$. From (5.3) we have

$$
\begin{aligned}
\xi_t(p,q) = \mu(p,q] &- b_q\int_0^t \xi_{s-}(p,q)ds + \int_p^q \beta_\theta d\theta\int_0^t X_{s-}(p)ds\\
&+ \sigma\int_0^t\int_0^{\xi_{s-}(p,q)} W(ds, X_{s-}(p)+du)\\
&+ \int_0^t\int_{[0,q]}\int_0^\infty\int_0^{\xi_{s-}(p,q)} z\tilde{N}_0(ds,dy,dz,X_{s-}(p)+du)\\
&+ \int_0^t\int_p^q\int_0^\infty\int_0^{X_{s-}(p)} zN_0(ds,dy,dz,du).
\end{aligned}
\tag{5.4}
$$

Here $W(ds, X_{s-}(p)+du)$ is a white noise based on the Lebesgue measure. Note also that

$$
\int_{\{0\leq y\leq q\}} N_0(ds,dy,dz,X_{s-}(p)+du)
$$

is a Poisson random measure with intensity $dsm_q(dz)du$, and

$$
\int_{\{p<y\leq q\}} N_0(ds,dy,dz,du)
$$

is a Poisson random measure with intensity

$$
\int_{\{p<\theta\leq q\}} dsn_\theta(dz)dud\theta.
$$

Clearly, the white noise and the two random measures are independent. By Theorem 3.4, conditioned upon $\{X_t(p) : t \geq 0\}$ the process $\{\xi_t(p,q) : t \geq 0\}$ is a CBI-process with branching mechanism ϕ_q, immigration mechanism $\phi_{p,q}$, and immigration rate $\{X_{t-}(p) : t \geq 0\}$. Conditioned upon $\{X_t(p) : t \geq 0\}$, the process $\{\xi_t(p,q) : t \geq 0\}$ is clearly independent of the σ-algebra generated by $\{X_t(v) : t \geq 0, v \in [0,p]\}$. Then $\{(X_t(q))_{t\geq 0} : q \in T\}$ is a path-valued Markov process with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$. $\square$

Theorem 5.2 *There is a positive function $(t, u) \mapsto C(t, u)$ on $[0, \infty) \times T$ bounded on compact sets so that, for any $t \geq 0$ and $p \leq q \leq u \in T$,*

$$\mathbf{P}\Big\{\sup_{0\leq s\leq t}[X_s(q) - X_s(p)]\Big\} \leq C(t, u)\Big\{\mu(p, q] + b_p - b_q \\ + \sqrt{\mu(p, q]} + \sqrt{b_p - b_q}\Big\}. \tag{5.5}$$

Proof. Since $\{X_t(p) : t \geq 0\}$ is a CBI-process, we see from (3.2) that $t \mapsto \mathbf{P}[X_t(p)]$ is locally bounded. Let $\{\xi_t(p, q) : t \geq 0\}$ be defined as in the last proof. By (5.4) we have

$$\mathbf{P}[\xi_t(p, q)] = \mu(p, q] - b_q \int_0^t \mathbf{P}[\xi_s(p, q)]ds + (b_p - b_q) \int_0^t \mathbf{P}[X_s(p)]ds.$$

By Gronwall's inequality one can find a positive function $(t, u) \mapsto C_0(t, u)$ on $[0, \infty) \times T$ bounded on compact sets so that, for any $t \geq 0$ and $p \leq q \leq u \in T$,

$$\mathbf{P}[\xi_t(p, q)] \leq C_0(t, u)\{\mu(p, q] + b_p - b_q\}. \tag{5.6}$$

Applying Doob's inequality to the martingales in (5.4), we obtain

$$\begin{aligned}\mathbf{P}\Big\{\sup_{0\leq s\leq t}\xi_s(p, q)\Big\} \leq\ & \mu(p, q] + 2\sigma\Big(\int_0^t \mathbf{P}[\xi_s(p, q)]ds\Big)^{\frac{1}{2}} \\ & + |b_q|\int_0^t \mathbf{P}[\xi_s(p, q)]ds + (b_p - b_q)\int_0^t \mathbf{P}[X_s(p)]ds \\ & + \int_1^\infty z m_q(dz)\int_0^t \mathbf{P}[\xi_s(p, q)]ds \\ & + 2\Big(\int_0^t \mathbf{P}[\xi_s(p, q)]ds\int_0^1 z^2 m_q(dz)\Big)^{\frac{1}{2}}.\end{aligned}$$

Then the desired estimate follows from (5.6). □

Now let us consider a special admissible family of branching mechanisms. Suppose that ϕ is a critical or supercritical branching mechanism given by (1.1) with $b \leq 0$. Let $T = T(\phi)$ be the set of $q \geq 0$ so that

$$\int_1^\infty z e^{qz} m(dz) < \infty.$$

Then $T = [0, a]$ or $[0, a)$, where $a = \sup(T)$. We can define an admissible family of branching mechanisms $\{\phi_q : q \in T\}$ by

$$\phi_q(\lambda) = \phi(\lambda - q) - \phi(-q), \qquad \lambda \geq 0, \tag{5.7}$$

where the two terms on the right-hand side are defined using formula (1.1). Let $\{X_t(q) : t \geq 0, q \in T\}$ be the solution flow of stochastic equation system (1.3). By Theorem 5.1

we see that $\{(X_t(q))_{t\geq 0} : q \in T\}$ is an inhomogeneous path-valued branching process with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ given by

$$\int_{D^+[0,\infty)} e^{-\int_0^\infty f(s)w(s)ds}\mathbf{P}_{p,q}(\eta, dw) = \exp\Big\{ - \int_0^\infty u_{p,q}(s,f)\eta(s)ds\Big\}, \tag{5.8}$$

where $f \in B^+[0,\infty)$ has compact support, and $u_{p,q}(s,f)$ is given by (4.8). If $\phi(\lambda) \to \infty$ as $\lambda \to \infty$, by Theorem 4.7 the corresponding total mass process $\{\sigma(q) : q \in T\}$ is an inhomogeneous CB-process with transition semigroup $\{R_{p,q} : q \geq p \in T\}$ given by

$$\int_{\mathbb{R}_+} e^{-\lambda y}R_{p,q}(x, dy) = \exp\{ - x\phi_p(\phi_q^{-1}(\lambda))\}, \qquad \lambda \geq 0. \tag{5.9}$$

By Theorem 2.5 we have

$$\mathbf{P}[e^{-\lambda\sigma(q)}1_{\{\sigma(q)<\infty\}}] = e^{-\mu\phi_q^{-1}(\lambda)}, \qquad \lambda \geq 0, q \in T. \tag{5.10}$$

It is simple to see that

$$q \mapsto \phi_q^{-1}(0) = q + \phi^{-1}(\phi(-q))$$

is continuous on T. Let $A = \inf\{q \in T : \sigma(q) = \infty\}$ be the explosion time of $\{\sigma(q) : q \in T\}$. For any $q \in T$ we can let $\lambda = 0$ in (5.10) to obtain

$$\mathbf{P}\{A > q\} = \mathbf{P}\{\sigma(q) < \infty\} = e^{-\mu\phi_q^{-1}(0)}. \tag{5.11}$$

This gives a characterization of the distribution of A.

Theorem 5.3 *Suppose that $\phi(\lambda) \to \infty$ as $\lambda \to \infty$. Then for any $\theta \in T$, we have*

$$\mathbf{P}[\sigma(\theta)1_{\{\sigma(\theta)<\infty\}}] = \frac{\mu e^{-\mu[\theta+\phi^{-1}(\phi(-\theta))]}}{\phi'(\phi^{-1}(\phi(-\theta)))}. \tag{5.12}$$

Proof. Let $\lambda \geq 0$ and $u = \phi_\theta^{-1}(\lambda)$. By (5.10) we have

$$\mathbf{P}[\sigma(\theta)e^{-\lambda\sigma(\theta)}1_{\{\sigma(\theta)<\infty\}}] = -\frac{d}{d\lambda}e^{-\mu\phi_\theta^{-1}(\lambda)} = \mu e^{-\mu\phi_\theta^{-1}(\lambda)}\frac{d}{d\lambda}\phi_\theta^{-1}(\lambda).$$

From the relation $\phi_\theta(u) = \phi(u-\theta) - \phi(-\theta)$, one can see

$$\phi_\theta^{-1}(\lambda) = \theta + \phi^{-1}(\lambda + \phi(-\theta)).$$

It follows that

$$\mathbf{P}[\sigma(\theta)e^{-\lambda\sigma(\theta)}1_{\{\sigma(\theta)<\infty\}}] = \frac{\mu e^{-\mu[\theta+\phi^{-1}(\lambda+\phi(-\theta))]}}{\phi'(\phi^{-1}(\lambda + \phi(-\theta)))}.$$

Then we get (5.12) by letting $\lambda = 0$. $\square$

Theorem 5.4 *Suppose that $\phi(\lambda) \to \infty$ as $\lambda \to \infty$. Let $\theta \in [0, a)$ and let $G(\theta)$ be a positive random variable measurable with respect to the σ-algebra generated by $\{X_t(v) : t \geq 0, 0 < v \leq \theta\}$. Then we have*

$$\mathbf{P}[G(\theta)|A = \theta] = \frac{\phi'(\phi^{-1}(\phi(-\theta)))}{\mu e^{-\mu[\theta+\phi^{-1}(\phi(-\theta))]}}\mathbf{P}[G(\theta)\sigma(\theta)1_{\{\sigma(\theta)<\infty\}}]. \tag{5.13}$$

Proof. Since $q \mapsto \phi_q^{-1}(0)$ is continuous on T, for any $q \in (\theta, a)$ we can see by (5.9) that

$$\mathbf{P}[G(\theta)1_{\{A>q\}}] = \mathbf{P}[G(\theta)1_{\{\sigma(q)<\infty\}}] = \mathbf{P}[G(\theta)\exp\{-\sigma(\theta)\phi_\theta(\phi_q^{-1}(0))\}].$$

It is easy to see that

$$\phi_\theta(\phi_q^{-1}(0)) = \phi_\theta(\bar{q} + q) = \phi(\bar{q} + q - \theta) - \phi(-\theta),$$

where $\bar{q} = \phi^{-1}(\phi(-q))$. By elementary calculations,

$$\frac{d}{dq}\phi_\theta(\phi_q^{-1}(0)) = \phi'(\bar{q} + q - \theta)\Big(1 - \frac{\phi'(-q)}{\phi'(\bar{q})}\Big).$$

It follows that

$$-\frac{d}{dq}\mathbf{P}[G(\theta)1_{\{A>q\}}]\Big|_{q=\theta+} = [\phi'(\bar{\theta}) - \phi'(-\theta)]\mathbf{P}[G(\theta)\sigma(\theta)1_{\{\sigma(\theta)<\infty\}}],$$

and hence

$$\mathbf{P}[G(\theta)|A = \theta] = \frac{\mathbf{P}[G(\theta)\sigma(\theta)1_{\{\sigma(\theta)<\infty\}}]}{\mathbf{P}[\sigma(\theta)1_{\{\sigma(\theta)<\infty\}}]}. \tag{5.14}$$

Then we get (5.13) from (5.12) and (5.14). □

6 A nonlocal branching superprocess

In this section, we consider a nonlocal branching superprocess defined from the solution flow of (5.2). We first assume $T = [0, a]$ for some $a > 0$. Let $\mu \in F(T)$, and let $\{\phi_q : q \in T\}$ be an admissible family of branching mechanisms, where ϕ_q is given by (1.1) with the parameters $(b, m) = (b_q, m_q)$ depending on $q \in T$. Let $m(dy, dz)$ be the measure on $T \times (0, \infty)$ defined by (5.1). Let $\rho = \{\rho(t) : t \geq 0\}$ be a locally bounded positive Borel function on $[0, \infty)$. Let ψ be an immigration mechanism given by (2.21). Let $X(q) = \{X_t(q) : t \geq 0\}$ be the solution of (5.2) for $q \in T$. Then the path-valued Markov process $\{X(q) : q \in T\}$ has transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ defined by (4.6). Let Q_T denote the set of rationals in T. For any $t \geq 0$ we define the random function $Y_t \in F(T)$ by $Y_t(a) = X_t(a)$ and

$$Y_t(q) = \inf\{X_t(u) : u \in Q_T \cap (q, a]\}, \qquad 0 \leq q < a. \tag{6.1}$$

Similarly, for any $t > 0$, define $Z_t \in F(T)$ by $Z_t(a) = X_{t-}(a)$ and

$$Z_t(q) = \inf\{X_{t-}(u) : \; u \in Q_T \cap (q, a]\}, \qquad 0 \leq q < a. \tag{6.2}$$

By Theorem 5.2, for each $q \in T$ we have

$$\mathbf{P}\{Y_t(q) = X_t(q) \text{ and } Z_t(q) = X_{t-}(q) \text{ for all } t \geq 0\} = 1. \tag{6.3}$$

Consequently, for every $q \in T$ the process $\{Y_t(q) : t \geq 0\}$ is almost surely càdlàg and solves (5.2), so it is a CBI-process with branching mechanism ϕ_q, immigration mechanism ψ and immigration rate ρ. In view of (4.3) and (4.4), for every $q \in T$ we almost surely have

$$\begin{aligned} Y_t(q) = {} & \mu(q) + A_t + \sigma \int_0^t \int_0^{Y_{s-}(q)} W(ds, du) \\ & - b_0 \int_0^t Y_{s-}(q) ds + \int_0^q \beta_\theta d\theta \int_0^t Y_{s-}(q) ds \\ & + \int_0^t \int_{\{0\}} \int_0^\infty \int_0^{Y_{s-}(q)} z \tilde{N}_0(ds, dy, dz, du) \\ & + \int_0^t \int_0^q \int_0^\infty \int_0^{Y_{s-}(q)} z N_0(ds, dy, dz, du), \end{aligned} \tag{6.4}$$

where

$$A_t = h \int_0^t \rho(s) ds + \int_0^t \int_0^\infty \int_0^{\rho(s)} z N_1(ds, dz, du).$$

For $t \geq 0$ let $Y_t(dx)$ and $Z_t(dx)$ denote the random measures on T induced by the random functions Y_t and $Z_t \in F(T)$, respectively. For any $f \in C^1(T)$ one can use Fubini's theorem to see

$$\langle Y_t, f \rangle = f(a) Y_t(a) - \int_0^a f'(q) Y_t(q) dq. \tag{6.5}$$

Fix an integer $n \geq 1$ and let $q_i = ia/2^n$ for $i = 0, 1, \cdots, 2^n$. By (6.3) and (6.4) it holds almost surely that

$$\begin{aligned} \sum_{i=1}^{2^n} f'(q_i) Y_t(q_i) = {} & \sum_{i=1}^{2^n} f'(q_i) \mu(q_i) + \sigma \sum_{i=1}^{2^n} f'(q_i) \int_0^t \int_0^{Z_s(q_i)} W(ds, du) \\ & + A_t \sum_{i=1}^{2^n} f'(q_i) - b_0 \sum_{i=1}^{2^n} f'(q_i) \int_0^t Z_s(q_i) ds \\ & + \sum_{i=1}^{2^n} f'(q_i) \int_0^{q_i} \beta_\theta d\theta \int_0^t ds \int_{[0, q_i]} Z_s(dx) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{2^n} f'(q_i)\int_0^t\int_{\{0\}}\int_0^\infty\int_0^{Z_s(q_i)} z\tilde{N}_0(ds,dy,dz,du)\\
&+\sum_{i=1}^{2^n} f'(q_i)\int_0^t\int_0^{q_i}\int_0^\infty\int_0^{Z_s(q_i)} zN_0(ds,dy,dz,du)\\
=&\sum_{i=1}^{2^n} f'(q_i)\mu(q_i)+\sigma\int_0^t\int_0^{Z_s(a)} F_n(s,0,u)W(ds,du)\\
&+A_t\sum_{i=1}^{2^n} f'(q_i)-b_0\int_0^t\Big[\sum_{i=1}^{2^n} f'(q_i)Z_s(q_i)\Big]ds\\
&+\int_0^t ds\int_T Z_s(dx)\int_0^a F_n(s,x\vee\theta,0)\beta_\theta d\theta\\
&+\int_0^t\int_{\{0\}}\int_0^\infty\int_0^{Z_s(a)} zF_n(s,0,u)\tilde{N}_0(ds,dy,dz,du)\\
&+\int_0^t\int_0^a\int_0^\infty\int_0^{Z_s(a)} zF_n(s,y,u)N_0(ds,dy,dz,du), \qquad (6.6)
\end{aligned}
$$

where

$$
F_n(s,y,u)=\sum_{i=1}^{2^n} f'(q_i)1_{\{y\le q_i\}}1_{\{u\le Z_s(q_i)\}}.
$$

By the right continuity of $q\mapsto Z_s(q)$ it is not hard to see that, as $n\to\infty$,

$$
2^{-n}F_n(s,y,u)\to F(s,y,u):=\int_y^a 1_{\{u\le Z_s(q)\}}f'(q)dq. \qquad (6.7)
$$

Then we can multiply (6.6) by 2^{-n} and let $n\to\infty$ to see, almost surely,

$$
\begin{aligned}
\int_0^a f'(q)Y_t(q)dq =& \int_0^a f'(q)\mu(q)dq+\sigma\int_0^t\int_0^{Z_s(a)} F(s,0,u)W(ds,du)\\
&+A_t\int_0^a f'(q)dq-b_0\int_0^t ds\int_0^a f'(q)Z_s(q)dq\\
&+\int_0^t ds\int_T Z_s(dx)\int_0^a F(s,x\vee\theta,0)\beta_\theta d\theta\\
&+\int_0^t\int_{\{0\}}\int_0^\infty\int_0^{Z_s(a)} zF(s,0,u)\tilde{N}_0(ds,dy,dz,du)\\
&+\int_0^t\int_0^a\int_0^\infty\int_0^{Z_s(a)} zF(s,y,u)N_0(ds,dy,dz,du). \qquad (6.8)
\end{aligned}
$$

From (6.4), (6.5) and (6.8) it follows that, almost surely,

$$
\langle Y_t,f\rangle=\langle\mu,f\rangle+f(0)A_t+\sigma\int_0^t\int_0^{Z_s(a)}[f(a)-F(s,0,u)]W(ds,du)
$$

$$
\begin{aligned}
&- b_0 \int_0^t \langle Z_s, f\rangle ds + \int_0^t ds \int_T Z_s(dx) \int_0^a f(x \vee \theta)\beta_\theta d\theta \\
&+ \int_0^t \int_{\{0\}} \int_0^\infty \int_0^{Z_s(a)} z[f(a) - F(s,0,u)]\tilde{N}_0(ds, dy, dz, du) \\
&+ \int_0^t \int_0^a \int_0^\infty \int_0^{Z_s(a)} z[f(a) - F(s,y,u)]N_0(ds, dy, dz, du).
\end{aligned} \tag{6.9}
$$

Theorem 6.1 *The measure-valued process $\{Y_t : t \geq 0\}$ has a càdlàg modification.*

Proof. By (6.9) one can see $\{\langle Y_t, f\rangle : t \geq 0\}$ has a càdlàg modification for every $f \in C^1(T)$. Let $\mathscr{U}$ be the countable set of polynomials having rational coefficients. Then $\mathscr{U}$ is uniformly dense in both $C^1(T)$ and $C(T)$. For $f \in \mathscr{U}$ let $\{Y_t^*(f) : t \geq 0\}$ be a càdlàg modification of $\{\langle Y_t, f\rangle : t \geq 0\}$. By removing a null set from Ω if it is necessary, we obtain a càdlàg process $\{Y_t^* : t \geq 0\}$ of rational linear functionals on $\mathscr{U}$, which can immediately be extended to a càdlàg process of real linear functionals on $C(T)$. By Riesz's representation, the latter determines a measure-valued process, which is clearly a càdlàg modification of $\{Y_t : t \geq 0\}$. □

Theorem 6.2 *The càdlàg modification of $\{Y_t : t \geq 0\}$ is the unique solution of the following martingale problem: For every $G \in C^2(\mathbb{R})$ and $f \in C(T)$,*

$$
\begin{aligned}
G(\langle Y_t, f\rangle) = G(\langle \mu, f\rangle) &+ \int_0^t G'(\langle Y_s, f\rangle) ds \int_T Y_s(dx) \int_T f(x \vee \theta)\beta_\theta d\theta \\
&- b_0 \int_0^t G'(\langle Y_s, f\rangle)\langle Y_s, f\rangle ds + \frac{1}{2}\sigma^2 \int_0^t G''(\langle Y_s, f\rangle)\langle Y_s, f^2\rangle ds \\
&+ \int_0^t ds \int_T Y_s(dx) \int_0^\infty \Big[G(\langle Y_s, f\rangle + zf(x)) \\
&- G(\langle Y_s, f\rangle) - zf(x)G'(\langle Y_s, f\rangle)\Big] m_0(dz) \\
&+ \int_0^t ds \int_T Y_s(dx) \int_T d\theta \int_0^\infty \Big[G(\langle Y_s, f\rangle + zf(x \vee \theta)) \\
&- G(\langle Y_s, f\rangle)\Big] n_\theta(dz) + hf(0) \int_0^t G'(\langle Y_s, f\rangle)\rho(s) ds \\
&+ \int_0^t \rho(s) ds \int_0^\infty \Big[G(\langle Y_s, f\rangle + zf(0)) - G(\langle Y_s, f\rangle)\Big] n(dz) \\
&+ \textit{local mart.}
\end{aligned} \tag{6.10}
$$

Proof. We first assume $f \in C^1(T)$. By (6.9) and Itô's formula, we get

$$
G(\langle Y_t, f\rangle) = G(\langle \mu, f\rangle) - b_0 \int_0^t G'(\langle Z_s, f\rangle)\langle Z_s, f\rangle ds
$$

$$
\begin{aligned}
&+\frac{1}{2}\sigma^2\int_0^t ds\int_0^{Z_s(a)} G''(\langle Z_s,f\rangle)[f(a)-F(s,0,u)]^2 du\\
&+\int_0^t G'(\langle Z_s,f\rangle)ds\int_T Z_s(dx)\int_T f(x\vee\theta)\beta_\theta d\theta\\
&+\int_0^t ds\int_0^{Z_s(a)} du\int_0^\infty \Big[G(\langle Z_s,f\rangle+z[f(a)-F(s,0,u)])\\
&-G(\langle Z_s,f\rangle)-z[f(a)-F(s,0,u)]G'(\langle Z_s,f\rangle)\Big]m_0(dz)\\
&+\int_0^t ds\int_0^{Z_s(a)} du\int_T d\theta\int_0^\infty \Big[G(\langle Z_s,f\rangle+z[f(a)-F(s,\theta,u)])\\
&-G(\langle Z_s,f\rangle)\Big]n_\theta(dz)+hf(0)\int_0^t G'(\langle Z_s,f\rangle)\rho(s)ds\\
&+\int_0^t \rho(s)ds\int_0^\infty \Big[G(\langle Z_s,f\rangle+zf(0))-G(\langle Z_s,f\rangle)\Big]n(dz)\\
&+\text{local mart.}
\end{aligned}
$$

For $s,u>0$ let $Z_s^{-1}(u)=\inf\{q\geq 0: Z_s(q)>u\}$. It is easy to see that $\{q\geq 0: u\leq Z_s(q)\}=[Z_s^{-1}(u),\infty)$, except for at most countably many $u>0$. Then in the above we can replace $f(a)-F(s,\theta,u)$ by

$$
f(a)-\int_\theta^a 1_{\{Z_s^{-1}(u)\leq q\}}f'(q)dq = f(Z_s^{-1}(u)\vee\theta).
$$

It follows that

$$
\begin{aligned}
G(\langle Y_t,f\rangle) &= G(\langle\mu,f\rangle)-b_0\int_0^t G'(\langle Z_s,f\rangle)\langle Z_s,f\rangle ds\\
&\quad+\frac{1}{2}\sigma^2\int_0^t ds\int_0^{Z_s(a)} G''(\langle Z_s,f\rangle)f(Z_s^{-1}(u))^2 du\\
&\quad+\int_0^t G'(\langle Z_s,f\rangle)ds\int_T Z_s(dx)\int_T f(x\vee\theta)\beta_\theta d\theta\\
&\quad+\int_0^t ds\int_0^{Z_s(a)} du\int_0^\infty \Big[G(\langle Z_s,f\rangle+zf(Z_s^{-1}(u)))\\
&\quad-G(\langle Z_s,f\rangle)-zf(Z_s^{-1}(u))G'(\langle Z_s,f\rangle)\Big]m_0(dz)\\
&\quad+\int_0^t ds\int_0^{Z_s(a)} du\int_T d\theta\int_0^\infty \Big[G(\langle Z_s,f\rangle+zf(Z_s^{-1}(u)\vee\theta))\\
&\quad-G(\langle Z_s,f\rangle)\Big]n_\theta(dz)+hf(0)\int_0^t G'(\langle Z_s,f\rangle)\rho(s)ds\\
&\quad+\int_0^t \rho(s)ds\int_0^\infty \Big[G(\langle Z_s,f\rangle+zf(0))-G(\langle Z_s,f\rangle)\Big]n(dz)\\
&\quad+\text{local mart.}\\
&= G(\langle\mu,f\rangle)+\int_0^t G'(\langle Z_s,f\rangle)ds\int_T Z_s(dx)\int_T f(x\vee\theta)\beta_\theta d\theta
\end{aligned}
$$

$$
\begin{aligned}
&-b_0\int_0^t G'(\langle Z_s,f\rangle)\langle Z_s,f\rangle ds+\frac{1}{2}\sigma^2\int_0^t G''(\langle Z_s,f\rangle)\langle Z_s,f^2\rangle ds\\
&+\int_0^t ds\int_T Z_s(dx)\int_0^\infty\Big[G(\langle Z_s,f\rangle+zf(x))\\
&-G(\langle Z_s,f\rangle)-zf(x)G'(\langle Z_s,f\rangle)\Big]m_0(dz)\\
&+\int_0^t ds\int_T Z_s(dx)\int_T d\theta\int_0^\infty\Big[G(\langle Z_s,f\rangle+zf(x\vee\theta))\\
&-G(\langle Z_s,f\rangle)\Big]n_\theta(dz)+hf(0)\int_0^t G'(\langle Z_s,f\rangle)\rho(s)ds\\
&+\int_0^t\rho(s)ds\int_0^\infty\Big[G(\langle Z_s,f\rangle+zf(0))-G(\langle Z_s,f\rangle)\Big]n(dz)\\
&+\text{local mart.}
\end{aligned}
$$

For each $q\in T$ the càdlàg process $\{X_t(q):t\geq 0\}$ has at most countably many discontinuity points $A_q:=\{t>0:Y_{t-}(q)\neq Y_t(q)\}$. In view of (6.1) and (6.2), we have $Z_t(q)=Y_t(q)$ for all $q\in T$ and $t\in B:=A_a^c\cap(\cap_{u\in Q_T}A_u^c)$. Here $B\subset[0,\infty)$ is a set with full Lebesgue measure. Then we have (6.10) for $f\in C^1(T)$. For an arbitrary $f\in C(T)$, we get (6.10) by an approximation argument. The uniqueness (in distribution) of the solution to the martingale problem follows by a modification of the proof of Theorem 7.13 in Li (2011). $\square$

The martingale problem (6.10) is essentially a special case of the one given in Theorem 10.18 of Li (2011); see also Theorem 9.18 of Li (2011). Let $f\mapsto\Psi(\cdot,f)$ be the operator on $C^+(T)$ defined by

$$
\Psi(x,f)=\int_T f(x\vee\theta)\beta_\theta d\theta+\int_T d\theta\int_0^\infty(1-e^{-zf(x\vee\theta)})n_\theta(dz). \tag{6.11}
$$

By modifying the proof of Theorem 3.4 one can show the following:

Theorem 6.3 *The solution $\{Y_t:t\geq 0\}$ of the martingale problem* (6.10) *is an immigration superprocess with transition semigroup $(Q_t)_{t\geq 0}$ defined by*

$$
\int_{M(T)}e^{-\langle\nu,f\rangle}Q_t(\mu,d\nu)=\exp\Big\{-\langle\mu,V_tf\rangle-\int_0^t\psi(V_sf(0))\rho(s)ds\Big\}, \tag{6.12}
$$

where $f\in C^+(T)$, and $t\mapsto V_tf$ is the unique locally bounded positive solution of

$$
V_tf(x)=f(x)-\int_0^t[\phi_0(V_sf(x))-\Psi(x,V_sf)]ds,\quad t\geq 0,x\in T. \tag{6.13}
$$

The branching mechanism of the immigration superprocess $\{Y_t:t\geq 0\}$ has *local part* $(x,f)\mapsto\phi_0(f(x))$ and *nonlocal part* $(x,f)\mapsto\Psi(x,f)$; see Example 2.5 in Li (2011). The process has immigration mechanism $f \mapsto \psi(f(0))$ and immigration rate $\rho = \{\rho(s) : s \geq 0\}$. Then the immigrants only come at the origin. The spatial motion in this model is trivial. Heuristically, when an infinitesimal particle dies at site $x \in T$, some offspring are born at this site according to the local branching mechanism and some are born in the interval $(x, a]$ according to the nonlocal branching mechanism. Therefore the branching of an infinitesimal particle located at $x \in T$ does not make any influence on the population in the interval $[0, x)$. This explains the Markov property of the path-valued process $\{(Y_t(q))_{t\geq 0} : q \in T\}$.

The cumulant semigroup $(V_t)_{t\geq 0}$ can also be defined by a differential evolution equation. In fact, by Theorem 7.11 of Li (2011), for any $f \in C^+(T)$, the integral equation (6.13) is equivalent to

$$\begin{cases} \dfrac{dV_tf}{dt}(x) = -\,\phi_0(V_tf(x)) + \Psi(x, V_tf), & t \geq 0, x \in T, \\ V_0f(x) = f(x), & x \in T. \end{cases} \tag{6.14}$$

Then the transition semigroup $(Q_t)_{t\geq 0}$ can also be defined by (6.12) for $f \in C^+(T)$ with $t \mapsto V_tf$ being the unique locally bounded positive solution of (6.14).

Theorem 6.4 *Let $Y = (\Omega, \mathscr{G}, \mathscr{G}_t, Y_t, \mathbf{Q}_\mu)$ be any càdlàg immigration superprocess with transition semigroup $(Q_t)_{t\geq 0}$ defined by* (6.12) *and* (6.13)*. Then under $\mathbf{Q}_\mu$ for every $q \in T$ the process $\{Y_t[0, q] : t \geq 0\}$ has a càdlàg version and $\{(Y_t[0, q])_{t\geq 0} : q \in T\}$ is a path-valued branching process with immigration with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ defined by* (4.6)*.*

Proof. By Theorem 6.2, one can see that for each $q \in T$ the restriction of $\{Y_t : t \geq 0\}$ to $[0, q]$ is also an immigration superprocess with state space $M[0, q]$. In particular, the process $\{Y_t[0, q] : t \geq 0\}$ has a càdlàg version. Clearly, the finite-dimensional distributions of the path-valued process $\{(Y_t[0, q])_{t\geq 0} : q \in T\}$ are uniquely determined by the initial state $\mu \in M(T)$ and transition semigroup $(Q_t)_{t\geq 0}$. Then $\{(Y_t[0, q])_{t\geq 0} : q \in T\}$ has identical finite-dimensional distributions with the process $\{(Y_t(q))_{t\geq 0} : q \in T\}$ defined by (6.4). Since $\{(Y_t(q))_{t\geq 0} : q \in T\}$ is a Markov process with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$, so is $\{(Y_t[0, q])_{t\geq 0} : q \in T\}$. $\square$

If $T = [0, \infty)$ or $[0, a)$ for some $a > 0$, we may apply the above results to the interval $[0, q] \subset T$ for $q \in T$. Then for each $q \in T$, there is a immigration superprocess $\{Y_t^q : t \geq 0\}$ in $M[0, q]$. Those processes determine a nonlocal branching immigration superprocess $\{Y_t : t \geq 0\}$ in $\mathscr{M}(T)$, the space of Radon measures on T furnished with the topology of vague convergence. The results established in this section hold for this process with obvious modifications.

7 The excursion law

In this section we assume $T = [0, a]$ for some $a > 0$. However, the results obtained here can be modified to the case $T = [0, a)$ or $[0, \infty)$ obviously. Let $\{\phi_q : q \in T\}$ be an admissible family of branching mechanisms, where ϕ_q is given by (1.1) with the parameters $(b, m) = (b_q, m_q)$ depending on $q \in T$. In addition, we assume $\phi_0'(\lambda) \to \infty$ as $\lambda \to \infty$. By Theorem 6.3, we can define the transition semigroup $(Q_t)_{t\geq 0}$ of a non-local branching superprocess by

$$\int_{M(T)} e^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = \exp\{-\langle \mu, V_t f\rangle\}, \qquad f \in C^+(T), \tag{7.1}$$

where $t \mapsto V_t f$ is the unique locally bounded positive solution of (6.13). Let $(Q_t^\circ)_{t\geq 0}$ denote the restriction of the semigroup to $M(T)^\circ$.

Theorem 7.1 *The cumulant semigroup of $(V_t)_{t\geq 0}$ in (7.1) admits the representation*

$$V_t f(x) = \int_{M(T)^\circ} (1 - e^{-\langle \nu, f\rangle}) L_t(x, d\nu), \qquad t > 0, x \in T, \tag{7.2}$$

where $(L_t(x, \cdot))_{t>0}$ is a σ-finite entrance law for $(Q_t^\circ)_{t\geq 0}$.

Proof. We need a modification of the characterization (6.14) of the cumulant semigroup. Let us consider a jump process ξ in T with generator A defined by

$$Af(x) = \int_0^a (f(q) - f(x))\gamma(dq), \qquad x \in T, f \in C(T),$$

where

$$\gamma(dq) = \beta_q dq + \int_{\{0<z<\infty\}} z n_q(dz) dq.$$

Let $\phi_*(\lambda) = \gamma[0, a]\lambda + \phi_0(\lambda)$, and let $f \mapsto \Psi_*(\cdot, f)$ be the operator on $C^+(T)$ defined by

$$\Psi_*(x, f) = \int_0^a \int_0^\infty [e^{-zf(x\vee y)} - 1 + zf(x \vee y)] m(dy, dz).$$

Now the first equation in (6.14) can be rewritten as

$$\frac{dV_t f}{dt}(x) = AV_t f(x) - \phi_*(V_t f(x)) - \Psi_*(x, V_t f).$$

Then we may think of $(V_t)_{t\geq 0}$ as the cumulant semigroup of a superprocess with underlying spatial motion ξ and branching mechanism $(x, f) \mapsto \phi_*(f(x)) + \Psi_*(x, f)$. Since clearly $\phi_*'(\lambda) \to \infty$ as $\lambda \to \infty$, the result follows by Theorem 8.6 of Li (2011). □

Let us consider a canonical càdlàg realization $Y = (\Omega, \mathscr{G}, \mathscr{G}_t, Y_t, \mathbf{Q}_\mu)$ of the non-local branching superprocess with transition semigroup $(Q_t)_{t\geq 0}$ defined by (6.13) and (7.1), where $\Omega = D([0,\infty), M(T))$. Let $Y_t(q) = Y_t[0,q]$ for $t \geq 0$ and $q \in T$. By Theorem 6.4, we have

$$\mathbf{Q}_\mu \exp\Big\{ - \int_0^\infty Y_s(q) f(s) ds \Big\} = \exp\{ - \mu[0,q] u_q(0,f)\}, \tag{7.3}$$

where $s \mapsto u_q(s,f)$ is the unique compactly supported bounded positive solution to (4.7). By Theorems 8.22 and 8.23 of Li (2011), for each $x \in T$ there is an excursion law $\mathbf{N}_x$ on $D([0,\infty), M(T))$ of the superprocess such that $\mathbf{N}_x\{Y_0 \neq 0\} = 0$ and

$$\mathbf{N}_x\Big[1 - e^{-\int_0^\infty \langle Y_s, f_s\rangle ds}\Big] = -\log \mathbf{Q}_{\delta_x}\Big[e^{-\int_0^\infty \langle Y_s, f_s\rangle ds}\Big] \tag{7.4}$$

for any bounded positive Borel function $(s,y) \mapsto f_s(y)$ on $[0,\infty)\times T$ with compact support. In view of (7.3) and (7.4), for any $f \in B^+[0,\infty)$ with compact support, we have

$$\mathbf{N}_0\Big[1 - e^{-\int_0^\infty Y_s(q) f(s) ds}\Big] = u_q(0,f), \tag{7.5}$$

where $s \mapsto u_q(s,f)$ is the unique compactly supported bounded positive solution to (4.7).

Theorem 7.2 *Under $\mathbf{N}_0$ the path-valued process $\{(Y_t(q))_{t\geq 0} : q \in T\}$ satisfies the Markov property with transition semigroup $\{\mathbf{P}_{p,q} : q \geq p \in T\}$ such that*

$$\int_{D^+[0,\infty)} e^{-\int_0^\infty f(s) w(s) ds} \mathbf{P}_{p,q}(\eta, dw) = \exp\Big\{ - \int_0^\infty u_{p,q}(s,f)\eta(s) ds\Big\}, \tag{7.6}$$

where $f \in B^+[0,\infty)$ has compact support, and $u_{p,q}(s,f)$ is defined by (4.8).

Proof. By Theorem 6.4, under $\mathbf{Q}_{\delta_0}$ the process $\{(Y_t(q))_{t\geq 0} : q \in T\}$ satisfies the Markov property with transition semigroup defined by (7.6). Suppose that $(s,x) \mapsto f_s(x)$ is a bounded positive Borel function on $[0,\infty) \times T$, and $s \mapsto g_s$ is a bounded positive Borel function on $[0,\infty)$, both with compact supports. Then we have

$$\begin{aligned} &\mathbf{Q}_{\delta_0}\Big[\exp\Big\{ - \int_0^\infty [\langle Y_s, f_s 1_{[0,p]}\rangle + Y_s(q) g_s] ds\Big\}\Big] \\ &\quad = \mathbf{Q}_{\delta_0}\Big[\exp\Big\{ - \int_0^\infty [\langle Y_s, f_s 1_{[0,p]}\rangle + Y_s(p) u_{p,q}(s,g)] ds\Big\}\Big]. \end{aligned}$$

From this and (7.4) it follows that

$$\begin{aligned} &\mathbf{N}_0\Big[1 - \exp\Big\{ - \int_0^\infty [\langle Y_s, f_s 1_{[0,p]}\rangle + Y_s(q) g_s] ds\Big\}\Big] \\ &\quad = \mathbf{N}_0\Big[1 - \exp\Big\{ - \int_0^\infty [\langle Y_s, f_s 1_{[0,p]}\rangle + Y_s(p) u_{p,q}(s,g)] ds\Big\}\Big]. \end{aligned}$$

Then subtracting the quantity

$$\mathbf{N}_0\Big[1-\exp\Big\{-\int_0^\infty \langle Y_s, f_s 1_{[0,p]}\rangle ds\Big\}\Big]$$

from both sides, we get

$$\begin{aligned}\mathbf{N}_0\Big[&\exp\Big\{-\int_0^\infty \langle Y_s, f_s 1_{[0,p]}\rangle ds\Big\}\\ &\qquad\cdot\Big(1-\exp\Big\{-\int_0^\infty Y_s(q)g_s ds\Big\}\Big)\Big]\\ =\mathbf{N}_0\Big[&\exp\Big\{-\int_0^\infty \langle Y_s, f_s 1_{[0,p]}\rangle ds\Big\}\\ &\qquad\cdot\Big(1-\exp\Big\{-\int_0^\infty Y_s(p)u_{p,q}(s,g) ds\Big\}\Big)\Big].\end{aligned}$$

A monotone class argument shows that

$$\begin{aligned}&\mathbf{N}_0\Big[F\Big(1-\exp\Big\{-\int_0^\infty Y_s(q)g_s ds\Big\}\Big)\Big]\\ &\quad=\mathbf{N}_0\Big[F\Big(1-\exp\Big\{-\int_0^\infty Y_s(p)u_{p,q}(s,g) ds\Big\}\Big)\Big]\end{aligned}$$

for any positive Borel function F on $D([0,\infty), M(T))$ measurable with respect to the σ-algebra generated by $\{Y_t[0,v] : t \geq 0, 0 \leq v \leq p\}$. That implies the desired Markov property of the process $\{(Y_t(q))_{t\geq 0} : q \in T\}$. □

A characterization of the finite-dimensional distributions of the path-valued process $\{(Y_t(q))_{t\geq 0} : q \in T\}$ under the excursion law $\mathbf{N}_0$ can be given by combining (7.5) and (7.6). Similarly, one can obtain characterizations of the finite-dimensional distributions of the path-valued process $\{(Y_t(q))_{0\leq t\leq \alpha} : q \in T\}$ for $\alpha > 0$ and the total mass process

$$\sigma(q) := \int_0^\infty Y_t(q)dt, \qquad q \in T.$$

The following result should be compared with Theorem 6.7 of Abraham and Delmas (2010).

Theorem 7.3 *Suppose that ϕ is a branching mechanism such that $\phi(\lambda) \to \infty$ as $\lambda \to \infty$. Let $\{\phi_q : q \in T\}$ be the admissible family defined by (5.7). Let $\theta \in T$ be a strictly positive constant. Then for any positive random variable $G(\theta)$ measurable with respect to the σ-algebra generated by $\{Y_t(v) : t \geq 0, v \in [0,\theta]\}$, we have*

$$\mathbf{N}_0[G(\theta)|A=\theta] = \phi'(\phi^{-1}(\phi(-\theta)))\mathbf{N}_0[G(\theta)\sigma(\theta)1_{\{\sigma(\theta)<\infty\}}].$$

Proof. Based on Theorem 7.2 and the Markov property of the path-valued process $\{(Y_t(q))_{t\geq 0} : q \in T\}$, this follows as in the proofs of Theorems 5.3 and 5.4. □

Acknowledgments

I would like to acknowledge the Laboratory of Mathematics and Complex Systems (Ministry of Education) for providing me the research facilities. I am grateful to Dr. Leif Döring for helpful comments on the presentation of the paper.

References

Abraham, R. and Delmas, J.-F. (2012): A continuum tree-valued Markov process. *Ann. Probab.* **40**, 1167–1211

Abraham, R., Delmas, J.-F. and He, H. (2012): Pruning Galton–Watson trees and tree-valued Markov processes. *Ann. Inst. H. Poincaré Probab. Statist.* **48**, 688–705.

Abraham, R., Delmas, J.-F. and Voisin, G. (2010): Pruning a Lévy continuum random tree. *Electr. J. Probab.* **15**, 1429–1473.

Aldous, D. (1991): The continuum random tree I. *Ann. Probab.* **19**, 1–28.

Aldous, D. (1993): The continuum random tree III. *Ann. Probab.* **21**, 248–289.

Aldous, D. and Pitman, J. (1998): Tree-valued Markov chains derived from Galton–Watson processes. *Ann. Inst. H. Poincaré Probab. Statist.* **34**, 637–686.

Bertoin, J. and Le Gall, J.-F. (2006): Stochastic flows associated to coalescent processes III: Limit theorems. *Illinois J. Math.* **50**, 147–181.

Dawson, D.A., Gorostiza, L.G. and Li, Z. (2002): Non-local branching superprocesses and some related models. *Acta Appl. Math.* **74**, 93–112.

Dawson, D.A. and Li, Z. (2006): Skew convolution semigroups and affine Markov processes. *Ann. Probab.* **34**, 1103–1142.

Dawson, D.A. and Li, Z. (2012): Stochastic equations, flows and measure-valued processes. *Ann. Probab.* **40**, 813–857.

Duquesne, T. and Le Gall, J.-F. (2002): *Random Trees, Lévy Processes and Spatial Branching Processes*. Astérisque **281**.

Dynkin, E.B. (1993): Superprocesses and partial differential equations. *Ann. Probab.* **21**, 1185–1262.

Fu, Z. and Li, Z. (2010): Stochastic equations of non-negative processes with jumps. *Stochastic Process. Appl.* **120**, 306–330.

Jiřina, M. (1958): Stochastic branching processes with continuous state space. *Czech. Math. J.* **8**, 292–313.

Kawazu, K. and Watanabe, S. (1971): Branching processes with immigration and related limit theorems. *Theory Probab. Appl.* **16**, 36–54.

Lamperti, J. (1967): The limit of a sequence of branching processes. *Z. Wahrsch. verw. Geb.* **7**, 271–288.

Le Gall, J.-F. and Le Jan, Y. (1998a): Branching processes in Lévy processes: The exploration process. *Ann. Probab.* **26**, 213–252.

Le Gall, J.-F. and Le Jan, Y. (1998b): Branching processes in Lévy processes: Laplace functionals of snakes and superprocesses. *Ann. Probab.* **26**, 1407–1432.

Li, Z. (1992): Branching particle systems in random environments. *Chinese Science Bulletin (Chinese Edition)* **37**, 1541–1543.

Li, Z. (2002): Skew convolution semigroups and related immigration processes. *Theory Probab. Appl.* **46**, 274–296.

Li, Z. (2011): *Measure-Valued Branching Markov Processes*. Springer, Heidelberg.

Li, Z. and Mytnik, L. (2011): Strong solutions for stochastic differential equations with jumps. *Ann. Inst. H. Poincaré Probab. Statist.* **47**, 1055–1067.

School of Mathematical Sciences
Beijing Normal University
Beijing 100875, P. R. China
E-mail: `lizh@bnu.edu.cn`
URL: `http://math.bnu.edu.cn/~lizh/`