Published in: *Journal of Theoretical Probability* **26** (2013), 1: 31–45

An integral test on time dependent local extinction for super-coalescing Brownian motion with Lebesgue initial measure[1]

Hui He, Zenghu Li and Xiaowen Zhou

Abstract. This paper concerns the almost sure time dependent local extinction behavior for super-coalescing Brownian motion X with $(1+\beta)$-stable branching and Lebesgue initial measure on $\mathbb{R}$. We first give a representation of X using excursions of a continuous state branching process and Arratia's coalescing Brownian flow. For any nonnegative, nondecreasing and right continuous function g, put

$$\tau := \sup\{t \geq 0 : X_t([-g(t), g(t)]) > 0\}.$$

We prove that $\mathbb{P}\{\tau = \infty\} = 0$ or 1 according as whether the integral $\int_1^\infty g(t)t^{-1-1/\beta}dt$ is finite or infinite.

AMS 2000 subject classifications: Primary 60G57, 60J80; Secondary 60F20

Keywords: super-coalescing Brownian motion, almost sure local extinction, excursion representation, integral test.

1 Introduction

By a *super-coalescing Brownian motion* (SCBM in short) we mean a measure-valued stochastic process describing the time-space-mass evolution of a particle system in $\mathbb{R}$. In such a system the particles move according to (instantaneous) coalescing Brownian motions and the masses of those particles evolve according to independent continuous state branching processes (CSBPs in short) with $(1+\beta)$-stable branching law. Whenever two particles are in the same location their masses are added up with total mass continuing with the independent $(1+\beta)$-stable branching. Note that this scheme is well defined due to the additivity of the CSBP. For coalescence to happen with a positive probability we only consider SCBM on $\mathbb{R}$.

The SCBM has been studied in [4, 10, 11]. With arbitrary finite initial measure it can be obtained by taking a high-density/small-particle limit of the empirical measure process of coalescing-branching particle system with Poisson initial measure. Its probability law can be specified by the duality on coalescing Brownian motions.

Formally, the SCBM with Radon initial measure μ can be constructed by taking a monotone limit of SCBMs with initial measures μ truncated over increasing finite intervals.

[1]Supported by NSFC(10721091,11071021) and NSERC grants.

In this paper we will present a direct construction of the SCBM using excursions of the CSBP and Arratia's coalescing Brownian flow following Dawson et al. [4]. A similar construction was first proposed in Dawson and Li [3] for superprocess with dependent spatial motion. This procedure allows us to construct the SCBM simultaneously for all times $t \geq 0$. It turns out to be handy for the later coupling arguments in proving our main results.

Almost sure *local extinction* for super-Brownian motion on $\mathbb{R}^n$ says that, given any bounded Borel set in $\mathbb{R}^n$, almost surely the measure-valued process does not charge on it after a time long enough. It often occurs in low dimensions. For super-Brownian motion with Lebesgue initial measure it was first studied in Iscoe [8] via analyzing the super-Brownian occupation time using nonlinear PDE arising from the Laplace functional. By time dependent local extinction we mean the local extinction behavior where the size of the above-mentioned set also depends on t. Almost sure time dependent local extinction was discussed in Fleischmann et al [5]. An integral test was found in Zhou [12] on the almost sure time dependent local extinction for super-Brownian motion with $(1+\beta)$-stable branching and Lebesgue initial measure. Its proof is a Borel-Cantelli argument based on estimates of extinction probabilities. The additivity for super-Brownian motion plays a crucial role in the proof there since it allows us to decompose one super-Brownian motion into independent super-Brownian motions for different purposes and then treat them separately.

The SCBM often shares similar asymptotic properties as super-Brownian motion. In this paper we are going to show that the same integral test in Zhou [12] is also valid for the SCBM on $\mathbb{R}$. More precisely, for any nonnegative, nondecreasing and right continuous function g on $\mathbb{R}^+$, we are going to show that the probability of seeing any mass over interval $[-g(t), g(t)]$ for time t large enough is either 0 or 1 depending on whether the integral $\int_1^\infty g(t)t^{-1-1/\beta}dt$ is finite or not.

It is evident that the integral test remains the same for the superprocess with trivial spatial motion. This should not come as a surprise in view of the law of iterated logarithm of the Brownian motion. Unfortunately, we were unable to reduce the proof of the result for the SCBM to the trivial spatial motion case using the law of iterated logarithm. Instead we give a direct proof.

The main difficulty of the direct proof is that the SCBM is no longer additive due to the dependence of coalescing spatial motion. As a result we have to adopt strategies that are quite different from Zhou [12] to tackle this problem. The excursion representation plays an important role in our proof. By the excursion representation, one SCBM can be represented as sum of several SCBMs starting from disjoint intervals. The key is that those SCBMs are not independent. So in one direction of our proof, we will use a coupling argument by introducing two coalescing Brownian systems and comparing the asymptotical behaviors of the two systems since the Borel-Cantelli lemma requires that the events are independent. More details will be given in the following.

For the case $\int_1^\infty g(t)t^{-1-1/\beta}dt < \infty$ we first choose a sequence of times (t_n) increasing geometrically. Then for each n we decompose the SCBM with Lebesgue initial measure into two (dependent) SCBMs J^n and K^n starting from Lebesgue measures restricted to interval $[-2g(t_n), 2g(t_n)]$ and its complement, respectively. We can show that, for large n, both the probability for J^n_t to survive up to time t_n and the probability for K^n_t to ever charge interval $[-g(t_n), g(t_n)]$ before time t_n are small enough. Consequently, the almost sure local extinction with respect to g occurs following a Borel-Cantelli argument.

In the other direction of our proof, when g increases fast enough we first choose a sequence of times (t_n) strictly increasing to ∞ and the associated disjoint intervals $[l_n, r_n], n = 1, 2, \dots$ in $\mathbb{R}^+$. We then consider an SCBM $\bar{X}$ starting from Lebesgue measure restricted to the region $(\cup_{i=1}^{\infty}[-r_i, -l_i]) \cup (\cup_{j=1}^{\infty}[l_j, r_j])$. We are able to choose the spacings between intervals properly to satisfy the following constrains. On one hand, the spacing is not too small so that for each n, up to time t_n the mass started from interval $[-r_n, -l_n] \cup [l_n, r_n]$ at time 0 is very unlikely to interact with masses initiated from the other intervals. On the other hand, the spacing is also not too large so that the process $\bar{X}$ still has enough initial mass to start with and by time t_n the probability $\mathbb{P}\{\bar{X}_{t_n}([-g(t_n), g(t_n)]) > 0\}$ is not too small. Then the proof can be carried out by coupling arguments together with several Borel-Cantelli arguments.

The approaches developed in this paper can be modified to study the almost sure time dependent local extinction for SCBM with Lévy branching mechanism other than stable branching. But we do not expect the result to be as clean.

The rest of the paper is arranged as follows. In Section 2 we present the construction of SCBM using Arratia's flow and the branching excursion law. Our main results of integral tests on almost sure local extinction, Theorem 3.1 and Theorem 3.2 and their proofs are presented in Section 3.

2 A construction of SCBM with branching excursions and Arratia's flow

Let $\gamma \geq 0$ and $0 < \beta \leq 1$ be fixed constants. A *continuous state critical branching process* (CSBP) with $(1+\beta)$-stable branching is a right continuous strong Markov process taking values in $[0, \infty)$ whose transition semigroup $(Q_t)_{t\geq 0}$ is determined by

$$\int_0^\infty e^{-zy} Q_t(x, dy) = \exp\{-x\psi_t(z)\}, \quad t, x, z \geq 0, \tag{2.1}$$

where $\psi_t(z)$ is the unique solution of

$$\frac{\partial}{\partial t}\psi_t(z) = \frac{1}{1+\beta}\gamma\psi_t(z)^{1+\beta}, \qquad \psi_0(z) = z.$$

It is easy to find that

$$\psi_t(z) = z\left(\frac{1+\beta}{1+\beta+\gamma\beta t z^\beta}\right)^{1/\beta}.$$

In the sequel of the paper, we shall always assume $\gamma > 0$ unless otherwise specified. Then for any $t > 0$ we have

$$\lim_{z\to\infty} \psi_t(z) = \left(\frac{1+\beta}{\gamma\beta t}\right)^{1/\beta} =: \psi_t(\infty) < \infty. \tag{2.2}$$

Letting $z \to \infty$ in (2.1) yields

$$Q_t(x, \{0\}) = \exp\{-x\psi_t(\infty)\}, \quad t > 0, \, x \geq 0.$$

From (2.1) it follows that

$$Q_t(x_1 + x_2, \cdot) = Q_t(x_1, \cdot) * Q_t(x_2, \cdot), \quad t, x_1, x_2 \geq 0.$$

In view of this infinite divisibility and (2.2), we have

$$\psi_t(z) = \int_0^\infty (1 - e^{-zy})\kappa_t(dy), \quad t > 0,\ z \geq 0, \tag{2.3}$$

for a family of finite diffuse measures $(\kappa_t)_{t>0}$ on $(0, \infty)$; see, e.g., Bertoin and Le Gall [2].

A *coalescing Brownian flow* $\{\phi(a,t) : a \in \mathbb{R}, t \geq 0\}$ is by definition an $\mathbb{R}$-valued two-parameter process such that for every $a \in \mathbb{R}$, $t \mapsto \phi(a,t)$ is continuous; for every $t \geq 0$, $a \mapsto \phi(a,t)$ is nondecreasing and right continuous; and for any $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{R}^n$ the probability law of $(\phi(a_1,t), \ldots, \phi(a_n,t))$ follows that of the *coalescing Brownian motion* starting at $(a_1, \ldots, a_n)$; see [1] and [6] for more details.

Let $\mathcal{M}$ be the space of Radon measures on $\mathbb{R}$ endowed with the topology of vague convergence. Let $\mathcal{M}_a$ be the subset of $\mathcal{M}$ consisting of purely atomic Radon measures. Let $\mathcal{B}_0$ be the space of bounded Borel functions on $\mathbb{R}$ with bounded supports. Suppose that $\{(\phi_1(t), \phi_2(t), \ldots) : t \geq 0\}$ is a countable system of coalescing Brownian motions and $\{(\xi_1(t), \xi_2(t), \ldots) : t \geq 0\}$ is a countable system of independent CSBP's with $(1+\beta)$-stable branching law. We assume that the two systems are defined on a complete probability space and are independent of each other. In addition, we assume that $\{\phi_1(0), \phi_2(0), \ldots\} \cap [-l, l]$ is a finite set for every finite $l \geq 1$. Then we define the $\mathcal{M}_a$-valued process

$$X_t = \sum_{i=1}^\infty \xi_i(t)\delta_{\phi_i(t)}, \qquad t \geq 0. \tag{2.4}$$

For any $t \geq 0$ let $\mathcal{G}_t = \sigma(\mathcal{F}_t^\phi \cup \mathcal{F}_t^\xi)$, where

$$\mathcal{F}_t^\phi := \sigma(\{\phi_i(s) : 0 \leq s \leq t; i = 1, 2, \ldots\})$$

and

$$\mathcal{F}_t^\xi := \sigma(\{\xi_i(s) : 0 \leq s \leq t; i = 1, 2, \ldots\}).$$

Theorem 2.1 *The process $\{X_t : t \geq 0\}$ defined by (2.4) is a right continuous $(\mathcal{G}_t)$-Markov process with transition semigroup $(P_t)_{t\geq 0}$ given by*

$$\int_{\mathcal{M}_a} e^{-\langle \nu, f\rangle} P_t(\mu, d\nu) = \mathbb{E}\bigg[\exp\bigg\{ -\int_{\mathbb{R}} \psi_t(f(\phi(a,t)))\mu(da)\bigg\}\bigg] \tag{2.5}$$

for $\mu \in \mathcal{M}_a$ and $f \in \mathcal{B}_0$, where $\phi(a,t)$ is a coalescing Brownian flow.

Proof. By the additivity of the CSBP's it is easy to see $\{X_t : t \geq 0\}$ a right continuous $(\mathcal{G}_t)$-Markov process. By the independence of the two systems $(\phi_1(t), \phi_2(t), \ldots)$ and $(\xi_1(t), \xi_2(t), \ldots)$ we have

$$\mathbb{E}[\exp\{-\langle X_t, f\rangle\}] = \mathbb{E}\bigg[\mathbb{E}\bigg[\exp\bigg\{ -\sum_{i=1}^\infty \xi_i(t) f(\phi_i(t))\bigg\}\bigg|\mathcal{F}_t^\phi\bigg]\bigg]$$

$$
\begin{aligned}
=&\mathbb{E}\Big[\exp\Big\{-\sum_{i=1}^{\infty}\xi_i(0)\psi_t(f(\phi_i(t)))\Big\}\Big]\\
=&\mathbb{E}\Big[\exp\Big\{-\sum_{i=1}^{\infty}\xi_i(0)\psi_t(f(\phi(a_i,t)))\Big\}\Big]\\
=&\mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{R}}X_0(da)\psi_t(f(\phi(a,t)))\Big\}\Big],
\end{aligned}
$$

where $a_i = \phi_i(0)$ for $i \geq 1$ and $\phi(a,t)$ is a coalescing Brownian flow independent of $(\xi_1(t), \xi_2(t), \ldots)$. A similar calculation shows that $\{X_t : t \geq 0\}$ has the transition semigroup $(P_t)_{t\geq 0}$ given by (2.5). □

Definition 2.1 *By a super-coalescing Brownian motion (SCBM) we mean a Markov process whose transition semigroup $(P_t)_{t\geq 0}$ is given by (2.5) with $\mu \in \mathcal{M}$.*

From (2.5), it is easy to see that $P_t(\mu, \mathcal{M}_a) = 1$ for $t > 0$ and $\mu \in \mathcal{M}$. Then the process constructed by (2.4) is a special case. We will give a formulation of the SCBM with an arbitrary initial state $\mu \in \mathcal{M}$. To this end, let us review some basic facts on CSBP's. Let $Q_t^0(x,\cdot)$ denote the restriction of the measure $Q_t(x,\cdot)$ to $(0,\infty)$. Since the origin 0 is a trap for the CSBP, the family of kernels $(Q_t^0)_{t\geq 0}$ also constitutes a semigroup. Based on (2.1) and (2.3) one can check that

$$
\int_0^\infty (1-e^{-zy})\kappa_{s+t}(dy) = \int_0^\infty \kappa_s(dx)\int_0^\infty (1-e^{-zy})Q_t^0(x,dy), \quad s,t>0,\ z\geq 0.
$$

Then $\kappa_s Q_t^0 = \kappa_{s+t}$. Therefore, $(\kappa_t)_{t>0}$ is an entrance law for $(Q_t^0)_{t\geq 0}$.

Let $\mathbf{W}$ be the set of right continuous nonnegative functions on $(0,\infty)$ satisfying $w(t) = 0$ for $t \geq \tau_0(w)$, where $\tau_0(w) := \inf\{s > 0 : w(s) = 0\}$. Let 0 denote the path that is constantly zero. Let $\mathfrak{B}(\mathbf{W})$ be the natural σ-algebra on $\mathbf{W}$ generated by the coordinate process $\{w(s) : s > 0\}$. By the general theory of Markov processes, there exits a unique σ-finite measure $\mathbf{Q}_\kappa$ on $(\mathbf{W}, \mathfrak{B}(\mathbf{W}))$ such that $\mathbf{Q}_\kappa(\{0\}) = 0$ and

$$
\begin{aligned}
&\mathbf{Q}_\kappa(w(t_1)\in dy_1, w(t_2)\in dy_2, \ldots, w(t_n)\in dy_n)\\
&\quad = \kappa_{t_1}(dy_1)Q^0_{t_2-t_1}(y_1,dy_2)\ldots Q^0_{t_n-t_{n-1}}(y_{n-1},dy_n)
\end{aligned}
\tag{2.6}
$$

for $0 < t_1 < t_2 < \ldots < t_n$ and $y_1, y_2, \ldots, y_n \in (0,\infty)$; see Proposition 3.5 of Getoor and Glover [7]. By Theorem 8.22 of Li [9] we have $w(t) \to 0$ as $t \to 0$ for $\mathbf{Q}_\kappa$-almost every $w \in \mathbf{W}$. Then we can think of $\mathbf{Q}_\kappa$ as a σ-finite measure on the set $\mathbf{W}_0$ of right continuous nonnegative functions on $[0,\infty)$ satisfying $w(0) = w(t) = 0$ for $t \geq \tau_0(w)$, where $\tau_0(w) := \inf\{s > 0 : w(s) = 0\}$. The measure $\mathbf{Q}_\kappa$ is known as the *excursion law* of the CSBP. Let $\mathfrak{B}(\mathbf{W}_0)$ and $\mathfrak{B}_t = \mathfrak{B}_t(\mathbf{W}_0)$ denote the natural σ-algebras on $\mathbf{W}_0$ generated by $\{w(s) : s \geq 0\}$ and $\{w(s) : 0 \leq s \leq t\}$, respectively. For $r > 0$, let $\mathbf{Q}_{\kappa,r}$ denote the restriction of $\mathbf{Q}_\kappa$ to $\mathbf{W}_r := \{w \in \mathbf{W}_0 : \tau_0(w) > r\}$. Note that

$$
\mathbf{Q}_\kappa(\mathbf{W}_r) = \mathbf{Q}_{\kappa,r}(\mathbf{W}_r) = \kappa_r(0,\infty) = \psi_r(\infty) < \infty.
$$

Lemma 2.2 *The coordinate process $\{w(t+r) : t \geq 0\}$ under $\mathbf{Q}_{\kappa,r}\{\cdot|\mathfrak{B}_r\}$ is a CSBP with transition semigroup $(Q_t)_{t\geq 0}$.*

Proof. This follows from (2.6) by standard arguments. We here give a detailed proof for the convenience of the reader. For any $A \in \mathfrak{B}((0,\infty))$ we can use (2.6) to see

$$\begin{aligned}\mathbf{Q}_{\kappa,r}(w(r)\in A, w(r+t)=0) &= \mathbf{Q}_{\kappa,r}(w(r)\in A) - \mathbf{Q}_{\kappa,r}(w(r)\in A, w(r+t)>0)\\ &= \kappa_r(A) - \int_A \kappa_r(dx) Q_t^0(x,(0,\infty))\\ &= \kappa_r(A) - \int_A \kappa_r(dx)(1 - Q_t(x,\{0\}))\\ &= \int_A \kappa_r(dx) Q_t(x,\{0\})\end{aligned}$$

for $t > 0$. Since 0 is a trap, we have

$$\begin{aligned}&\mathbf{Q}_{\kappa,r}(w(r)\in dy, w(t_1)\in dy_1,\ldots,w(t_n)\in dy_n)\\ &\quad = \kappa_r(dy) Q_{t_1-r}(y,dy_1)\ldots Q_{t_n-t_{n-1}}(y_{n-1},dy_n),\end{aligned}$$

for $0 < r < t_1 < t_2 < \ldots < t_n$ and $y > 0$, $y_1, y_2, \ldots, y_n \in [0,\infty)$. Thus for $A \in \mathfrak{B}((0,\infty))$ and $A_1,\ldots,A_n \in \mathfrak{B}([0,\infty))$,

$$\begin{aligned}&\int_{\{w(r)\in A\}} \mathbf{Q}_{\kappa,r}\{w(t_1)\in A_1,\ldots,w(t_n)\in A_n|\mathfrak{B}_r\}\mathbf{Q}_{\kappa,r}(dw)\\ &= \int_{\{w(r)\in A\}} 1\{w(t_1)\in A_1 \ldots w(t_n)\in A_n\}\mathbf{Q}_{\kappa,r}(dw)\\ &= \int_A\int_{A_1}\cdots\int_{A_n} \kappa_r(dy) Q_{t_1-r}(y,dy_1)\ldots Q_{t_n-t_{n-1}}(y_{n-1},dy_n)\\ &= \int_{\{w(r)\in A\}}\int_{A_1}\cdots\int_{A_n} Q_{t_1-r}(w_r,dy_1)\ldots Q_{t_n-t_{n-1}}(y_{n-1},dy_n)\mathbf{Q}_{\kappa,r}(dw).\end{aligned}$$

Then an application of monotone class theorem yields the desired result. □

We now consider an arbitrary initial measure $\mu \in \mathcal{M}$. Suppose that on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ we have a coalescing Brownian flow $\{\phi(a,t) : a \in \mathbb{R}, t \geq 0\}$ and a Poisson random measure $N(da,dw)$ on $\mathbb{R}\times\mathbf{W}_0$ with intensity measure $\mu(da)\mathbf{Q}_\kappa(dw)$. We assume that $\{\phi(a,t)\}$ and $\{N(da,dw)\}$ are independent of each other. Denote the support of N by $\mathrm{supp}(N) = \{(a_i,w_i) : i \geq 1\}$. For $t \geq 0$ let $\mathcal{G}_t = \sigma(\mathcal{F}_t^N \cup \mathcal{F}_t^\phi)$, where

$$\mathcal{F}_t^N := \sigma(\{w_i(s) : 0 \leq s \leq t; i \geq 1\})$$

and

$$\mathcal{F}_t^\phi := \sigma(\{\phi(a,s) : 0 \leq s \leq t; a \in \mathbb{R}\}).$$

Then we define the $\mathcal{M}$-valued process

$$X_t = \int_{\mathbb{R}}\int_{\mathbf{W}_0} w(t)\delta_{\phi(a,t)}N(da,dw), \quad t > 0, \quad \text{and} \quad X_0 = \mu. \tag{2.7}$$

Theorem 2.3 *The process $\{X_t : t \geq 0\}$ defined by (2.7) is an SCBM starting from μ and*

$$\mathbb{E}[\exp\{-\langle X_t,f\rangle\}] = \mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{R}}\psi_t(f(\phi(a,t)))\mu(da)\Big\}\Big] \tag{2.8}$$

for $t > 0$ and $f \in \mathcal{B}_0$.

Proof. Note that for any $l \geq 1$ and $r > 0$ we have a.s. $m(l,r) := N([-l,l] \times \mathbf{W}_r) < \infty$. In fact, we have

$$\mathbb{E}[m(l,r)] = \mu([-l,l])\mathbf{Q}_\kappa(\mathbf{W}_r) = \mu([-l,l])\kappa_r(0,\infty) < \infty.$$

Then, since $\kappa_r(dx)$ is a diffuse measure, given $\mathcal{G}_r$ we can re-enumerate the set $\mathrm{supp}(N)$ into $\{(a_{k_i}, w_{k_i}) : i \geq 1\}$ so that: (i) $|a_{k_1}| \leq |a_{k_2}| \leq \ldots$; and (ii) $|a_{k_i}| = |a_{k_{i+1}}|$ implies $w_{k_i}(r) < w_{k_{i+1}}(r)$. Note that this enumeration only uses information from $\mathcal{G}_r$. As in the proof of Lemma 3.4 of Dawson and Li [3] one can see that $\{w_{k_i}(r+t) : t \geq 0; i \geq 1\}$ under $\mathbb{P}\{\cdot|\mathcal{G}_r\}$ are independent CSBP's which are independent of $\{\phi(a, r+t) : t \geq 0; a \in \mathbb{R}\}$. Observe that

$$X_{r+t} = \sum_{i=1}^{\infty} w_{k_i}(r+t)\delta_{\phi(a_{k_i}, r+t)}, \qquad t \geq 0.$$

Then Theorem 2.1 implies that $\{X_{r+t} : t \geq 0\}$ under $\mathbb{P}\{\cdot|\mathcal{G}_r\}$ is a right continuous $(\mathcal{G}_{r+t})$-Markov process with transition semigroup $(P_t)_{t\geq 0}$. Thus $\{X_t : t > 0\}$ under the non-conditioned probability $\mathbb{P}$ is a right continuous $(\mathcal{G}_t)$-Markov process with transition semigroup $(P_t)_{t\geq 0}$. On the other hand, we have

$$\begin{aligned}
\mathbb{E}[\exp\{-\langle X_t, f\rangle\}] &= \mathbb{E}\bigg[\exp\bigg\{-\int_{\mathbb{R}}\int_{\mathbf{W}_0} w(t)f(\phi(a,t))N(da,dw)\bigg\}\bigg] \\
&= \mathbb{E}\bigg[\exp\bigg\{-\int_{\mathbb{R}}\mu(da)\int_0^\infty \Big(1 - e^{-uf(\phi(a,t))}\Big)\kappa_t(du)\bigg\}\bigg] \\
&= \mathbb{E}\bigg[\exp\bigg\{-\int_{\mathbb{R}}\psi_t(f(\phi(a,t)))\mu(da)\bigg\}\bigg].
\end{aligned}$$

That proves (2.8) and also gives that for each $t > 0$, the random measure X_t has distribution $P_t(\mu, \cdot)$. We have thus completed the proof. □

Remark 2.4 *Based on (2.8) one can show $X_t \to \mu$ in probability (in fact almost surely with a little more work) as $t \to 0$. From the above construction we see that starting from an arbitrary initial state in $\mathcal{M}$, the SCBM collapses immediately into a purely atomic random measure with a countable support. Then the masses located at different points evolve according to independent CSBP's with the supporting points evolving according to coalescing Brownian motions. The construction (2.7) of the SCBM generalizes Theorem 3.5 of Dawson et al. [4], where it was assumed that the SCBM starts from a finite measure on $\mathbb{R}$.*

We can also give a useful alternate characterization of the SCBM following Zhou [11]. For $y_1 \leq \ldots \leq y_{2n} \in \mathbb{R}^{2n}$, we write $(Y_1(t), \ldots, Y_{2n}(t))$ for a system of coalescing Brownian motion starting at $(y_1, \ldots, y_{2n})$. Given $\{a_1, \ldots, a_n\} \subset \mathbb{R}^n$, throughout this paper we always put

$$h_t(x) := \sum_{j=1}^{n} a_j 1_{]Y_{2j-1}(t), Y_{2j}(t)]}(x), \qquad t \geq 0, x \in \mathbb{R}. \tag{2.9}$$

By applying Theorem 2.3 to $f = h_0$ we obtain

Theorem 2.5 *Let $\{X_t : t \geq 0\}$ be the SCBM defined by (2.7). Then for any $t > 0$ we have*

$$\mathbb{E}[\exp\{-\langle X_t, h_0\rangle\}] = \mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{R}} \psi_t(h_t(a))\mu(da)\Big\}\Big]. \tag{2.10}$$

The above theorem shows that, the SCBM constructed in Zhou [11] using approximation is actually the special case with $\beta = 1$ of the SCBM defined by (2.7).

3 An integral test on almost sure local extinction for SCBM

Throughout this paper let $g(t), t > 0$, be any nonnegative, nondecreasing and right continuous function on $[0, \infty)$. Let X be an SCBM. For such a function g we define the *extinction time* as

$$\tau := \sup\{t \geq 0 : X_t([-g(t), g(t)]) \neq 0\}$$

with the convention $\sup \emptyset = 0$. Recall a standard result for Brownian motion: If $\{B_t : t \geq 0\}$ is a Brownian motion, then

$$\mathbf{P}\left\{\sup_{0\leq s\leq T} B_s \geq C\right\} \leq \exp\left\{-\frac{C^2}{2T}\right\}, \quad C \geq 0, T > 0. \tag{3.1}$$

Theorem 3.1 *Assume that $X_0(dx) = dx$ and $\gamma > 0$. If*

$$\int_1^\infty g(y)y^{-1-1/\beta}dy < \infty, \tag{3.2}$$

then

$$\mathbb{P}\{\tau < \infty\} = 1. \tag{3.3}$$

Proof. Without loss of generality we may assume that X is defined as (2.7). We first show that (3.3) holds given $g(t) \geq t^\delta$ for some constant $1/2 < \delta < 1$ and for t large enough. Put $t_n := e^n$. Set

$$I_g^n := [-2g(t_{n+1}), 2g(t_{n+1})].$$

By the excursion representaion, we have that

$$X_t = \int_{I_g^n}\int_{\mathbf{W}_0} w(t)\delta_{\phi(a,t)}N(da, dw) + \int_{\mathbb{R}\setminus I_g^n}\int_{\mathbf{W}_0} w(t)\delta_{\phi(a,t)}N(da, dw) =: J_t^n + K_t^n.$$

Note that on the event

$$\Omega_{t_{n+1}} := \left\{\sup_{0\leq t\leq t_{n+1}} \phi(-2g(t_{n+1}), t) < -g(t_{n+1})\right\} \bigcap \left\{\inf_{0\leq t\leq t_{n+1}} \phi(2g(t_{n+1}), t) > g(t_{n+1})\right\},$$

we have

$$X_t([-g(t_{n+1}), g(t_{n+1})]) = J_t^n([-g(t_{n+1}), g(t_{n+1})]), \quad \forall\, t \leq t_{n+1}.$$

Thus

$$
\begin{aligned}
&\mathbf{P}\{\exists t_n < t \le t_{n+1}, X_t([-g(t_{n+1}), g(t_{n+1})]) \neq 0\} \\
&\quad \le \mathbf{P}\{\exists t_n < t \le t_{n+1}, J_t^n([-g(t_{n+1}), g(t_{n+1})]) \neq 0\} + \mathbf{P}\{\Omega_{t_{n+1}}^c\} \\
&\quad = 1 - \mathbf{P}\{J_t^n([-g(t_{n+1}), g(t_{n+1})]) = 0, \forall\, t_n < t \le t_{n+1}\} + \mathbf{P}\{\Omega_{t_{n+1}}^c\} \\
&\quad \le 1 - \mathbf{P}\{J_t^n(\mathbb{R}) = 0, \forall\, t_n < t \le t_{n+1}\} + \mathbf{P}\{\Omega_{t_{n+1}}^c\},
\end{aligned}
$$

where the first inequality comes from the fact that $a \mapsto \phi(a,t)$ is non-decreasing. By (3.1), we have

$$
\mathbf{P}\{\Omega_{t_{n+1}}^c\} \le 2\exp\left\{-\frac{g^2(t_{n+1})}{2t_{n+1}}\right\} \le 2\exp\left\{-\frac{t_{n+1}^{2\delta-1}}{2}\right\} \le 2e^{-n}
$$

for sufficiently large $n \ge 1$. It follows that

$$
\sum_{n=1}^{\infty} \mathbf{P}\{\Omega_{t_{n+1}}^c\} < \infty.
$$

On the other hand, the fact that $J_t^n(\mathbb{R})$ is a CSBP starting from $4g(t_{n+1})$ yields

$$
\mathbf{P}\{J_t^n(\mathbb{R}) = 0, \forall\, t_n < t \le t_{n+1}\} = \mathbf{P}\{J_{t_n}^n(\mathbb{R}) = 0\} = \exp\left\{-4g(t_{n+1})\psi_{t_n}(\infty)\right\}.
$$

We have

$$
1 - \mathbf{P}\{J_t^n(\mathbb{R}) = 0, \forall\, t_n < t \le t_{n+1}\} \le 4g(t_{n+1})\psi_{t_n}(\infty).
$$

Moreover,

$$
\begin{aligned}
\sum_{n=m}^{\infty} g(t_{n+1})\psi_{t_n}(\infty) &= \sum_{n=m}^{\infty} g(t_{n+1}) \left(\frac{1+\beta}{\gamma\beta t_n}\right)^{1/\beta} \\
&\le c(\gamma,\beta) \int_{m+1}^{\infty} \frac{g(e^x)}{e^{(x-2)/\beta}} dx \\
&\le e^{2/\beta} c(\gamma,\beta) \int_{t_{m+1}}^{\infty} \frac{g(y)}{y^{1+1/\beta}} dy \\
&< \infty,
\end{aligned}
\tag{3.4}
$$

where $c(\gamma,\beta) = \frac{1}{\beta}\left(\frac{1+\beta}{\gamma\beta}\right)^{1/\beta}$. Therefore, the desired result follows from Borel-Cantelli lemma.

To show the desired result for any g satisfying (3.2), we can consider function $g(t) + t^\delta$ instead. It follows from the previous result that (3.3) holds for function $g(t) + t^\delta$. Then plainly, it also holds for $g(t)$. □

Theorem 3.2 *Assume $X_0(dx) = dx$ and $\gamma > 0$. If*

$$
\int_1^{\infty} g(y) y^{-1-1/\beta} dy = \infty,
$$

then

$$
\mathbb{P}\{\tau = \infty\} = 1.
$$

Proof. Delayed.

Before proceeding with the proof for Theorem 3.2 we first define the nonnegative and strictly increasing sequences $(t_n), (l_n)$ and (r_n) as follows. Take $t_0 \geq 0$ so that $g(t_0) \geq 1$ and define inductively

$$t_{n+1} := \inf\{t \geq t_n : g(t) \geq 3g(t_n)\}, \quad n \geq 1.$$

Then

$$g(t_n) \leq g(t_{n+1}-) \leq 3g(t_n) \leq g(t_{n+1}). \tag{3.5}$$

Define

$$r_n := \frac{9}{10}g(t_n) \text{ and } l_n := \frac{31}{30}g(t_{n-1}).$$

By (3.5) we have

$$\begin{aligned} r_n - l_n &= \frac{3}{10}g(t_n) - \frac{3}{30}g(t_{n-1}) + \frac{6}{10}g(t_n) - \frac{28}{30}g(t_{n-1}) \\ &\geq \frac{1}{10}\left(g(t_{n+1}-) - g(t_n-)\right) + \frac{6}{10}g(t_n) - \frac{28}{30}g(t_{n-1}) \\ &\geq \frac{1}{10}\left(g(t_{n+1}-) - g(t_n-)\right). \end{aligned} \tag{3.6}$$

Let $I_n := [-r_n, -l_n] \cup [l_n, r_n]$ and $I := \cup_{n=0}^{\infty} I_n$. We need the following coalescing Brownian systems:

$$C_n = \{C_n(x,t); x \in I_n, t \geq 0\}$$

and

$$C = \{C(x,t); x \in I, t \geq 0\}$$

such that, for any finite set $A \subset \mathbb{R}$, both $\{C_n(x,\cdot); x \in A \cap I_n\}$ and $\{C(x,\cdot); x \in A \cap I\}$ are coalescing Brownian motions starting from $A \cap I_n$ and $A \cap I$, respectively.

Define

$$\tau(x,y) := \inf\{t \geq 0 : C(x,t) = C(y,t)\}.$$

According to the construction of coalescing Brownian motions on $\mathbb{R}$ in [1] (see also [6] for a more general model), we may construct $\{C_n; n \geq 1\}$ and C from a countable family of independent Brownian motions starting from $Q := \{i/2^k; i, k \in \mathbb{Z}\}$ such that C_n and C_m are independent for $n \neq m$ and

$$C_n(x,t) = C(x,t) \quad \text{for } x \in I_n \text{ and } t \leq \tau_n,$$

where $\tau_n := \tau(-r_n, -l_{n+1}) \wedge \tau(-l_n, -r_{n-1}) \wedge \tau(r_{n-1}, l_n) \wedge \tau(r_n, l_{n+1})$.

In the sequel of this paper, by $N(da, dw)$ we always denote a Poisson random measure on $\mathbb{R} \times \mathbf{W}_0$ with intensity measure $da\mathbf{Q}_\kappa(dw)$. For Lebesgue measure L let $\{X_t^n : t \geq 0\}$ and $\{\bar{X}_t : t \geq 0\}$ be defined by $X_0^n = 1_{I_n}(x)dx$, $\bar{X}_0 = 1_I(x)dx$ and for $t > 0$,

$$X_t^n = \int_{I_n}\int_{\mathbf{W}_0} w(t)\delta_{C_n(a,t)}N(da,dw), \quad \bar{X}_t = \int_I\int_{\mathbf{W}_0} w(t)\delta_{C(a,t)}N(da,dw).$$

Then by Theorem 2.3, $\{X_t^n : t \geq 0\}$ and $\{\bar{X}_t : t \geq 0\}$ are SCBMs starting from $1_{I_n}(x)dx$ and $1_I(x)dx$, respectively. We first prove the following lemma of key estimates. Theorem 3.2 will be easily deduced from the lemma.

Lemma 3.3 *Set* $B_n := [-g(t_n), g(t_n)]$. *Assume that* $t^{1/2+\epsilon} \leq g(t) \leq 3^t$ *for* $t \geq 1$ *and some* $0 < \epsilon < \frac{1}{2}$. *Then we have*

$$\sum_{n=0}^{\infty} \mathbb{P}\left\{X_{t_n}^n(B_n^c) > 0\right\} < \infty; \tag{3.7}$$

$$\sum_{n=0}^{\infty} \mathbb{P}\left\{X_{t_n}^n(\mathbb{R}) > 0\right\} = \infty; \tag{3.8}$$

$$\sum_{n=0}^{\infty} \mathbb{P}\left\{X_{t_n}^n(B_n) > \bar{X}_{t_n}(B_n)\right\} < \infty. \tag{3.9}$$

Proof. Proof for (3.7). Set

$$\Gamma_n := \left\{\inf_{0\leq s\leq t_n} C_n(-r_n, s) \geq -g(t_n)\right\} \bigcap \left\{\sup_{0\leq s\leq t_n} C_n(r_n, s) \leq g(t_n)\right\}.$$

Then by (3.1)

$$\begin{aligned}\mathbb{P}\{\Gamma_n^c\} &\leq \mathbb{P}\left\{\inf_{0\leq s\leq t_n} C_n(-r_n, s) \leq -g(t_n)\right\} + \mathbb{P}\left\{\sup_{0\leq s\leq t_n} C_n(r_n, s) \geq g(t_n)\right\} \\ &\leq 2\exp\left\{-\frac{g^2(t_n)}{200t_n}\right\} \leq 2\exp\left\{-\frac{t_n^{2\epsilon}}{200}\right\}.\end{aligned}$$

By (3.5) and the assumption that $g(t) \leq 3^t$ for $t > 1$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(\Gamma_n^c) < \infty.$$

Then (3.7) follows from

$$\begin{aligned}\mathbb{P}\left\{X_{t_n}^n(B_n^c) > 0\right\} &= \mathbb{P}\left\{X_{t_n}^n(B_n^c) > 0; \Gamma_n\right\} + \mathbb{P}\left\{X_{t_n}^n(B_n^c) > 0; \Gamma_n^c\right\} \\ &\leq \mathbb{P}\{\Gamma_n^c\}.\end{aligned}$$

Proof for (3.8). Since $X_\cdot^n(\mathbb{R})$ is a branching process starting from $2(r_n - l_n)$,

$$\begin{aligned}\mathbb{P}\{X_{t_n}^n(\mathbb{R}) > 0\} &= 1 - \exp\{-2(r_n - l_n)\psi_{t_n}(\infty)\} \\ &\geq \min\left\{\frac{\psi_{t_n}(\infty)}{10}\left(g(t_{n+1}-) - g(t_n-)\right), 1 - e^{-1}\right\},\end{aligned}$$

where the inequality is deduced from (3.6) and the elementary inequality $1 - e^{-x} \geq x/2$ for $0 \leq x \leq 1$. Recall $c(\gamma, \beta)$ in (3.4). Then (3.8) follows from

$$\begin{aligned}
&\sum_{n=1}^{\infty} \psi_{t_n}(\infty)\left(g(t_{n+1}-) - g(t_n-)\right) \\
&\quad = -\psi_{t_1}(\infty) g(t_1-) + \sum_{n=2}^{\infty} g(t_{n+1}-)(\psi_{t_n}(\infty) - \psi_{t_{n+1}}(\infty)) \\
&\quad = -\frac{1}{t_1} g(t_1-) - \sum_{n=2}^{\infty} g(t_{n+1}-) \int_{t_n}^{t_{n+1}} \frac{\partial \psi_t(\infty)}{\partial t} dt \\
&\quad \geq -\frac{1}{t_1} g(t_1-) + c(\gamma, \beta) \sum_{n=2}^{\infty} \int_{t_n}^{t_{n+1}} g(t) t^{-1-1/\beta} dt \\
&\quad = \infty.
\end{aligned}$$

Proof for (3.9). Set

$$\Gamma_1(x, t, y) := \left\{ \inf_{0 \leq s \leq t} C(x, t) \geq y \right\} \quad \text{and} \quad \Gamma_2(x, t, y) := \left\{ \sup_{0 \leq s \leq t} C(x, t) \leq y \right\}.$$

Define

$$\begin{aligned}
\Omega_1 &: = \Gamma_1(-r_n, t_n, -g(t_n)) \cap \Gamma_2(-l_n, t_n, -g(t_{n-1})); \\
\Omega_2 &: = \Gamma_1(-r_{n-1}, t_n, -g(t_{n-1})) \cap \Gamma_2(r_{n-1}, t_n, g(t_{n-1})); \\
\Omega_3 &: = \Gamma_1(l_n, t_n, g(t_{n-1})) \cap \Gamma_2(r_n, t_n, g(t_n)); \\
\Omega_4 &: = \Gamma_1(l_{n+1}, t_n, g(t_n)) \cap \Gamma_2(-l_{n+1}, t_n, -g(t_n)).
\end{aligned}$$

Then again by (3.1)

$$\begin{aligned}
\mathbb{P}\left\{\cup \Omega_i^c\right\} &\leq 4 \exp\left\{ -\frac{g^2(t_n)}{1800 t_n} \right\} + 4 \exp\left\{ -\frac{g^2(t_{n-1})}{1800 t_n} \right\} \\
&\leq 4 \exp\left\{ -\frac{t_n^{2\epsilon}}{1800} \right\} + 4 \exp\left\{ -\frac{t_n^{2\epsilon}}{200} \right\},
\end{aligned}$$

where the last inequality follows from (3.5). Now, we are ready to deduce (3.9). Note that on $\cap \Omega_i$,

$$C_n(x, t) = C(x, t) \quad \text{for } t \leq t_n,\ x \in I_n.$$

It follows that

$$\begin{aligned}
&\mathbb{P}\left\{ X_{t_n}^n(B_n) > \bar{X}_{t_n}(B_n) \right\} \\
&\quad \leq \mathbb{P}\left\{ X_{t_n}^n(B_n) > \bar{X}_{t_n}(B_n); \cap \Omega_i \right\} + \mathbb{P}\{\cup \Omega_i^c\} \\
&\quad \leq 4 \exp\left\{ -\frac{t_n^{2\epsilon}}{1800} \right\} + 4 \exp\left\{ -\frac{t_n^{2\epsilon}}{200} \right\}.
\end{aligned}$$

We thus obtain (3.9) by applying (3.5) again and the assumption that $g(t) \leq 3^t$ for $t > 1$.
□

Proof for Theorem 3.2. Firstly, suppose that $t^{1/2+\epsilon} \le g(t) \le 3^t$ for $t \ge 1$ and some $0 < \epsilon < \frac{1}{2}$. Define

$$\tau_0 := \sup\{t \ge 0 : \bar{X}_t\left([-g(t), g(t)]\right) > 0\}.$$

Note that $\{X^n_\cdot; n = 1, 2, \ldots\}$ are independent SCBMs. In addition, the sequence (t_n) defined after Theorem 3.2 satisfies $t_n \to \infty$ since g is increasing and $g(t) \le e^t$ for all $t > 0$. By (3.7), (3.8) and the Borel-Cantelli lemma, a.s. all but a finite number of the events $\{X^n_{t_n}(B^c_n) = 0\}$ occur and an infinite number of the events $\{X^n_{t_n}(\mathbb{R}) > 0\}$ occur. So a.s. an infinite number of the events $\{X^n_{t_n}(B_n) > 0\}$ occur. But by (3.9) and the Borel-Cantelli lemma again, a.s. all but a finite number of the events $\{X^n_{t_n}(B_n) \le \bar{X}_{t_n}(B_n)\}$ occur. Thus an infinite number of the events $\{\bar{X}_{t_n}(B_n) > 0\}$ occur. This gives

$$\mathbb{P}\{\tau_0 = \infty\} = 1.$$

Let X_t be defined as (2.7) with $X_0 = dx$. Define $\{\tilde{X}_t : t \ge 0\}$ with $\tilde{X}_0 = 1_I(x)dx$ and

$$\tilde{X}_t := \int_I \int_{\mathbf{W}_0} w(t)\delta_{\phi(a,t)} N(da, dw), \quad t > 0.$$

Then $\tilde{X}$ is an SCBM starting from $1_I(x)dx$. Obviously,

$$X_t\left([-g(t), g(t)]\right) > \tilde{X}_t\left([-g(t), g(t)]\right).$$

Then the fact that $\tilde{X}$ has the same distribution with $\bar{X}$ yields

$$\mathbb{P}\{\tau = \infty\} = 1.$$

For more general g satisfying $\int_1^\infty g(y)y^{-1-1/\beta}dy = \infty$, we can consider function

$$g_0(y) := (g(y) \wedge 3^y) \vee y^{\frac{1}{2}+\epsilon}.$$

First, one can check that

$$\int_1^\infty (g(y) \wedge 3^y)y^{-1-1/\beta}dy = \infty.$$

So,

$$\sup\{t \ge 0 : X_t([-g_0(t), g_0(t)]) > 0\} = \infty, \text{ a.s.}$$

Meanwhile, according to Theorem 3.1,

$$\sup\{t \ge 0 : X_t([-t^{\frac{1}{2}+\epsilon}, t^{\frac{1}{2}+\epsilon}]) > 0\} < \infty \quad \text{a.s.}$$

Then

$$\sup\{t \ge 0 : X_t([-g(t) \wedge 3^t, g(t) \wedge 3^t]) > 0\} = \infty \quad \text{a.s.}$$

Then the desired result follows from $g(t) \ge g(t) \wedge 3^t$. We have thus finished the proof. □

Acknowlegments: The authors would like to thank an anonymous referee for his/her remarks that improved significantly the presentation of the paper.

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Hui He and Zenghu Li: Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China.
E-mail: hehui@bnu.edu.cn and lizh@bnu.edu.cn

Xiaowen Zhou: Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec, H3G 1M8, Canada.
E-mail: xzhou@mathstat.concordia.ca