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Stochastic equations, flows and measure-valued processes

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Abstract: We first prove some general results on pathwise uniqueness, comparison property and existence of non-negative strong solutions of stochastic equations driven by white noises and Poisson random measures. The results are then used to prove the strong existence of two classes of stochastic flows associated with coalescents with multiple collisions, that is, generalized Fleming–Viot flows and flows of continuous-state branching processes with immigration. One of them unifies the different treatments of three kinds of flows in Bertoin and Le Gall [*Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005), 307–333]. Two scaling limit theorems for the generalized Fleming–Viot flows are proved, which lead to sub-critical branching immigration superprocesses. From those theorems we derive easily a generalization of the limit theorem for finite point motions of the flows in Bertoin and Le Gall [*Illinois J. Math.* **50** (2006), 147–181].

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Running title: Stochastic equations, flows and processes

1 Introduction

A class of stochastic flows of bridges were introduced by Bertoin and Le Gall (2003) to study the coalescent processes with multiple collisions of Pitman (1999); see also Sagitov (1999). The law of such a coalescent process is determined by a finite measure $\Lambda(dz)$ on $[0,1]$. The Kingman coalescent corresponds to $\Lambda = \delta_0$ and the Bolthausen–Sznitman coalescent corresponds to $\Lambda =$ Lebesgue measure on $[0,1]$; see Bolthausen and Sznitman (1998) and

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Kingman (1982). In fact, Bertoin and Le Gall (2003) established a remarkable connection between the coalescents with multiple collisions and the stochastic flows of bridges. Based on this connection, they have developed a theory of the coalescents and the flows in the series of papers; see Bertoin and Le Gall (2003, 2005, 2006). We refer the reader to Le Jan and Raimond (2004), Ma and Xiang (2001) and Xiang (2009) for the study of stochastic flows of mappings and measures in abstract settings.

Let $\{B_{s,t} : -\infty < s \leq t < \infty\}$ be the stochastic flow of bridges associated to a Λ-coalescent in the sense of Bertoin and Le Gall (2003). A number of precise characterizations of the flow $\{B_{-t,0}(v) : t \geq 0, v \in [0,1]\}$ were given in Bertoin and Le Gall (2003). For any $t \geq 0$, the function $v \mapsto B_{-t,0}(v)$ induces a random probability measure $\rho_t(dv)$ on $[0,1]$. The process $\{\rho_t : t \geq 0\}$ was characterized in Bertoin and Le Gall (2003) as the unique solution of a martingale problem. In fact, this process is a measure-valued dual to the Λ-coalescent process. It was also pointed out in Bertoin and Le Gall (2003) that $\{\rho_t : t \geq 0\}$ can be regarded as a *generalized Fleming–Viot process*; see also Donnelly and Kurtz (1999a, 1999b).

Let $\Lambda(dz)$ be a finite measure on $[0,1]$ such that $\Lambda(\{0\}) = 0$, and let $\{M(ds,dz,du)\}$ be a Poisson random measure on $(0,\infty)\times(0,1]^2$ with intensity $z^{-2}ds\Lambda(dz)du$. It was proved in Bertoin and Le Gall (2005) that there is weak solution flow $\{X_t(v) : t \geq 0, v \in [0,1]\}$ to the stochastic equation

$$X_t(v) = v + \int_0^t \int_0^1 \int_0^1 z[1_{\{u \leq X_{s-}(v)\}} - X_{s-}(v)]M(ds,dz,du). \tag{1.1}$$

Moreover, Bertoin and Le Gall (2005) showed that for any $0 \leq r_1 < \cdots < r_p \leq 1$ the p-point motion $\{(B_{-t,0}(r_1), \cdots, B_{-t,0}(r_p)) : t \geq 0\}$ is equivalent to $\{(X_t(r_1), \cdots, X_t(r_p)) : t \geq 0\}$. Therefore, the solutions of (1.1) give a realization of the flow of bridges associated with the Λ-coalescent process. A separate treatment for the Kingman coalescent flow was also given in Bertoin and Le Gall (2005). In that case they showed the p-point motion $\{(B_{-t,0}(r_1), \cdots, B_{-t,0}(r_p)) : t \geq 0\}$ is a diffusion process in

$$D_p := \{x = (x_1, \cdots, x_p) \in \mathbb{R}^p : 0 \leq x_1 \leq \cdots \leq x_p \leq 1\}$$

with generator A_0 defined by

$$A_0 f(x) = \frac{1}{2} \sum_{i,j=1}^{p} x_{i\wedge j}(1 - x_{i\vee j}) \frac{\partial^2 f}{\partial x_i \partial x_j}(x). \tag{1.2}$$

Given a Λ-coalescent flow $\{B_{s,t} : -\infty < s \leq t < \infty\}$, we define the *flow of inverses* by

$$B_{s,t}^{-1}(v) = \inf\{u \in [0,1] : B_{s,t}(u) > v\}, \qquad v \in [0,1)$$

and $B_{s,t}^{-1}(1) = B_{s,t}^{-1}(1-)$. In the Kingman coalescent case, it was proved in Bertoin and Le Gall (2005) that the p-point motion $\{(B_{0,t}^{-1}(r_1), \cdots, B_{0,t}^{-1}(r_p)) : t \geq 0\}$ is a diffusion process in D_p with generator A_1 given by

$$A_1 f(x) = A_0 f(x) + \sum_{i=1}^{p} \Big(\frac{1}{2} - x_i\Big) \frac{\partial f}{\partial x_i}(x), \tag{1.3}$$

where A_0 is given by (1.2). The analogous characterization for the Λ-coalescent flow with $\Lambda(\{0\}) = 0$ was also provided in Bertoin and Le Gall (2005). Those results give deep insights into the structures of the stochastic flows associated with the Λ-coalescents.

The asymptotic properties of Λ-coalescent flows were studied in Bertoin and Le Gall (2006). For each integer $k \geq 1$ let $\Lambda_k(dx)$ be a finite measure on $[0,1]$ with $\Lambda_k(\{0\}) = 0$ and let $\{X_k(t,v) : t \geq 0, v \in [0,1]\}$ be defined by (1.1) from a Poisson random measure $\{M_k(ds,dz,du)\}$ on $(0,\infty) \times (0,1]^2$ with intensity $z^{-2} ds \Lambda_k(dz) du$. Suppose that $z^{-2}(z \wedge z^2)\Lambda_k(k^{-1}dz)$ converges weakly as $k \to \infty$ to a finite measure on $(0,\infty)$ denoted by $z^{-2}(z \wedge z^2)\Lambda(dz)$. By a limit theorem of Bertoin and Le Gall (2006) the rescaled p-point motion $\{(kX_k(kt, r_1/k), \cdots, kX_k(kt, r_p/k)) : t \geq 0\}$ converges in distribution to those of the weak solution flow of the stochastic equation

$$Y_t(v) = v + \int_0^t \int_0^\infty \int_0^\infty x 1_{\{u \leq Y_{s-}(v)\}} \tilde{N}(ds, dx, du), \tag{1.4}$$

where $\tilde{N}(ds,dx,du)$ is a compensated Poisson random measure on $[0,\infty) \times (0,\infty)^2$ with intensity $z^{-2} ds \Lambda(dz) du$. It was pointed out in Bertoin and Le Gall (2006) that the solution of (1.4) is a special critical *continuous-state branching process* (CB-process).

In this paper we study two classes of stochastic flows defined by stochastic equations that generalize (1.1) and (1.4). We shall first treat the generalization of (1.4) since it involves simpler structures. Suppose that $\sigma \geq 0$ and b are constants, $v \mapsto \gamma(v)$ is a non-negative and non-decreasing continuous function on $[0,\infty)$, and $(z \wedge z^2)m(dz)$ is a finite measures on $(0,\infty)$. Let $\{W(ds,du)\}$ be a white noise on $(0,\infty)^2$ based on the Lebesgue measure $dsdu$. Let $\{N(ds,dz,du)\}$ be a Poisson random measure on $(0,\infty)^3$ with intensity $dsm(dz)du$. Let $\{\tilde{N}(ds,dz,du)\}$ be the compensated measure of $N(ds,dz,du)$. We shall see that for any $v \geq 0$ there is a pathwise unique non-negative solution of the stochastic equation

$$\begin{aligned} Y_t(v) &= v + \sigma \int_0^t \int_0^{Y_{s-}(v)} W(ds,du) + \int_0^t [\gamma(v) - bY_{s-}(v)] ds \\ &\quad + \int_0^t \int_0^\infty \int_0^{Y_{s-}(v)} z \tilde{N}(ds,dz,du). \end{aligned} \tag{1.5}$$

It is not hard to show each solution $Y(v) = \{Y_t(v) : t \geq 0\}$ is a *continuous-state branching process with immigration* (CBI-process). Then it is natural to call the two-parameter process $\{Y_t(v) : t \geq 0, v \geq 0\}$ a *flow of CBI-processes*. We prove that the flow has a version with the following properties:

(i) for each $v \geq 0$, $t \mapsto Y_t(v)$ is a càdlàg process on $[0, \infty)$ and solves (1.5);

(ii) for each $t \geq 0$, $v \mapsto Y_t(v)$ is a non-negative and non-decreasing càdlàg process on $[0, \infty)$.

The proof of those properties is based on the observation that $\{Y(v) : v \geq 0\}$ is a path-valued process with independent increments. For any $t \geq 0$, the random function $v \mapsto Y_t(v)$ induces a random Radon measure $Y_t(dv)$ on $[0, \infty)$. We shall see that $\{Y_t : t \geq 0\}$ is actually an *immigration superprocess* in the sense of Li (2011) with trivial underlying spatial motion. One could replace the diffusion term in (1.5) by the stochastic integral $\sigma \int_0^t \sqrt{Y_{s-}(v)} dW(s)$ using a one-dimensional Brownian motion $\{W(t) : t \geq 0\}$ as in Dawson and Li (2006). The resulted equation defines an equivalent CBI-process for any fixed $v \geq 0$, but it does not give an equivalent flow.

To describe our generalization of (1.1), let us assume that $\sigma \geq 0$ and $b \geq 0$ are constants, $v \mapsto \gamma(v)$ is a non-decreasing continuous function on $[0, 1]$ such that $0 \leq \gamma(v) \leq 1$ for all $0 \leq v \leq 1$ and $z^2\nu(dz)$ is a finite measure on $(0, 1]$. Let $\{B(ds, du)\}$ be a white noise on $(0, \infty) \times (0, 1]$ based on $dsdu$ and let $\{M(ds, dz, du)\}$ be a Poisson random measure on $(0, \infty) \times (0, 1]^2$ with intensity $ds\nu(dz)du$. We show that for any $v \in [0, 1]$ there is a pathwise unique solution $X(v) = \{X_t(v) : t \geq 0\}$ to the equation

$$\begin{aligned} X_t(v) &= v + \sigma \int_0^t \int_0^1 [1_{\{u \leq X_{s-}(v)\}} - X_{s-}(v)] B(ds, du) \\ &\quad + b \int_0^t [\gamma(v) - X_{s-}(v)] ds \\ &\quad + \int_0^t \int_0^1 \int_0^1 z[1_{\{u \leq X_{s-}(v)\}} - X_{s-}(v)] M(ds, dz, du). \end{aligned} \tag{1.6}$$

Clearly, the above equation unifies and generalizes the flows described by (1.1), (1.2) and (1.3). Here it is essential to use the white noise as the diffusion driving force. We show there is a version of the random field $\{X_t(v) : t \geq 0, 0 \leq v \leq 1\}$ with the following properties:

(i) for each $v \in [0, 1]$, $t \mapsto X_t(v)$ is càdlàg on $[0, \infty)$ and solves (1.6);

(ii) for each $t \geq 0$, $v \mapsto X_t(v)$ is non-decreasing and càdlàg on $[0, 1]$ with $X_t(0) \geq 0$ and $X_t(1) \leq 1$.

We refer to $\{X_t(v) : t \geq 0, 0 \leq v \leq 1\}$ as a *generalized Fleming–Viot flow* following Bertoin and Le Gall (2003, 2005, 2006). In particular, our result gives the strong existence of the flows associated with the coalescents with multiple collisions. The study of this flow is more involved than the one defined by (1.5) as the path-valued process $\{X(v) : 0 \leq v \leq 1\}$ does not have independent increments. However, we shall see it is still an inhomogeneous Markov process. From the random field $\{X_t(v) : t \geq 0, 0 \leq v \leq 1\}$ we can define a càdlàg sub-probability-valued process $\{X_t : t \geq 0\}$ on $[0, 1]$, which is a counterpart of the generalized Fleming–Viot process of Bertoin and Le Gall (2003). We prove two scaling limit theorems for the generalized Fleming–Viot processes, which lead to a special form of the immigration superprocess defined from (1.5). From the theorems we derive easily a generalization of the limit theorem for the finite point motions in Bertoin and Le Gall (2006).

The techniques of this paper are mainly based on the strong solutions of (1.5) and (1.6), which are different from those of Bertoin and Le Gall (2005, 2006). In Section 2 we give some general results for the pathwise uniqueness, comparison property and existence of non-negative strong solutions of stochastic equations driven by white noises and Poisson random measures. Those extend the results in Fu and Li (2010) and provide the basis for the investigation of the strong solution flows of (1.5) and (1.6). They should also be of interest on their own right. In Section 3 we study the flows of CBI-processes and their associated immigration superprocesses. The generalized Fleming–Viot flows are discussed in Section 4. Finally, we prove the scaling limit theorems in Section 5.

Notation. For a measure μ and a function f on a measurable space $(E, \mathscr{E})$ write $\langle \mu, f \rangle = \int_E f d\mu$ if the integral exists. For any $a \geq 0$ let $M[0, a]$ be the set of finite measures on $[0, a]$ endowed with the topology of weak convergence. Let $M_1[0, a]$ be the subspace of $M[0, a]$ consisting of sub-probability measures. Let $B[0, a]$ be the Banach space of bounded Borel functions on $[0, a]$ endowed with the supremum norm $\| \cdot \|$ and let $C[0, a]$ denote its subspace of continuous functions. We use $B[0, a]^+$ and $C[0, a]^+$ to denote the subclasses of non-negative elements. Throughout this paper, we make the conventions

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}$$

for any $b \geq a \geq 0$. Given a function f defined on a subset of $\mathbb{R}$, we write

$$\varDelta_z f(x) = f(x + z) - f(x) \quad \text{and} \quad D_z f(x) = \varDelta_z f(x) - f'(x)z$$

for $x, z \in \mathbb{R}$ if the right-hand side is meaningful. Let λ denote the Lebesgue measure on $[0, \infty)$.

2 Strong solutions of stochastic equations

In this section, we prove some results on stochastic equations of one-dimensional processes driven by white noises and Poisson random measures. The results extend those of Fu and Li (2010). Since our aim is to apply the results to the generalized Fleming–Viot flows and the flows of CBI-processes, we only discuss equations of non-negative processes. However, the arguments can be modified to deal with general one-dimensional equations.

Let E, U_0 and U_1 be separable topological spaces whose topologies can be defined by complete metrics. Suppose that $\pi(dz)$, $\mu_0(du)$ and $\mu_1(du)$ are σ-finite Borel measures on E, U_0 and U_1, respectively. We say the parameters (σ, b, g_0, g_1) are *admissible* if:

- $x \mapsto b(x)$ is a continuous function on $\mathbb{R}_+$ satisfying $b(0) \geq 0$;
- $(x, u) \mapsto \sigma(x, u)$ is a Borel function on $\mathbb{R}_+ \times E$ satisfying $\sigma(0, u) = 0$ for $u \in E$;
- $(x, u) \mapsto g_0(x, u)$ is a Borel function on $\mathbb{R}_+ \times U_0$ satisfying $g_0(0, u) = 0$ and $g_0(x, u) + x \geq 0$ for $x > 0$ and $u \in U_0$;
- $(x, u) \mapsto g_1(x, u)$ is a Borel function on $\mathbb{R}_+ \times U_1$ satisfying $g_1(x, u) + x \geq 0$ for $x \geq 0$ and $u \in U_1$.

Let $\{W(ds, du)\}$ be a white noise on $(0, \infty) \times E$ with intensity $ds\pi(dz)$. Let $\{N_0(ds, du)\}$ and $\{N_1(ds, du)\}$ be Poisson random measures on $(0, \infty) \times U_0$ and $(0, \infty) \times U_1$ with intensities $ds\mu_0(du)$ and $ds\mu_1(du)$, respectively. Suppose that $\{W(ds, du)\}$, $\{N_0(ds, du)\}$ and $\{N_1(ds, du)\}$ are defined on some complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and are independent of each other. Let $\{\tilde{N}_0(ds, du)\}$ denote the compensated measure of $\{N_0(ds, du)\}$. A non-negative càdlàg process $\{x(t) : t \geq 0\}$ is called a *solution* of

$$\begin{aligned} x(t) &= x(0) + \int_0^t \int_E \sigma(x(s-), u) W(ds, du) \\ &\quad + \int_0^t b(x(s-)) ds + \int_0^t \int_{U_0} g_0(x(s-), u) \tilde{N}_0(ds, du) \\ &\quad + \int_0^t \int_{U_1} g_1(x(s-), u) N_1(ds, du) \end{aligned} \tag{2.1}$$

if it satisfies the stochastic equation almost surely for every $t \geq 0$. We say $\{x(t) : t \geq 0\}$ is a *strong solution* if, in addition, it is adapted to the augmented natural filtration generated by $\{W(ds, du)\}$, $\{N_0(ds, du)\}$ and $\{N_1(ds, du)\}$; see, e.g., Situ (2005, page 76). Since $x(s-) \neq x(s)$ for at most countably many $s \geq 0$, we can also use $x(s)$ instead of $x(s-)$ in the integrals with respect to $W(ds, du)$ and ds on the right-hand side of (2.1). For the convenience of the statements of the results, we write $b(x) = b_1(x) - b_2(x)$, where $x \mapsto b_1(x)$ is continuous and $x \mapsto b_2(x)$ is continuous and non-decreasing. Let us formulate the following conditions:

(2.a) there is a constant $K \geq 0$ so that

$$b(x) + \int_{U_1} |g_1(x,u)|\mu_1(du) \leq K(1+x)$$

for every $x \geq 0$;

(2.b) there is a non-decreasing function $x \mapsto L(x)$ on $\mathbb{R}_+$ and a Borel function $(x,u) \mapsto \bar{g}_0(x,u)$ on $\mathbb{R}_+ \times U_0$ so that $\sup_{0 \leq y \leq x} |g_0(y,u)| \leq \bar{g}_0(x,u)$ and

$$\int_E \sigma(x,u)^2 \pi(du) + \int_{U_0} [\bar{g}_0(x,u) \wedge \bar{g}_0(x,u)^2]\mu_0(du) \leq L(x)$$

for every $x \geq 0$;

(2.c) for each $m \geq 1$ there is a non-decreasing concave function $z \mapsto r_m(z)$ on $\mathbb{R}_+$ such that $\int_{0+} r_m(z)^{-1}dz = \infty$ and

$$|b_1(x) - b_1(y)| + \int_{U_1} |g_1(x,u) - g_1(y,u)|\mu_1(du) \leq r_m(|x-y|)$$

for every $0 \leq x, y \leq m$;

(2.d) for each $m \geq 1$ there is a non-negative non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \rho_m(z)^{-2}dz = \infty$,

$$\int_E |\sigma(x,u) - \sigma(y,u)|^2 \pi(du) \leq \rho_m(|x-y|)^2$$

and

$$\int_{U_0} \mu_0(du) \int_0^1 \frac{l_0(x,y,u)^2(1-t)1_{\{|l_0(x,y,u)| \leq n\}}}{\rho_m(|(x-y) + tl_0(x,y,u)|)^2} dt \leq c(m,n)$$

for every $n \geq 1$ and $0 \leq x, y \leq m$, where $l_0(x,y,u) = g_0(x,u) - g_0(y,u)$ and $c(m,n) \geq 0$ is a constant.

Theorem 2.1 *Suppose that (σ, b, g_0, g_1) are admissible parameters satisfying conditions* (2.a,b,c,d). *Then the pathwise uniqueness of solutions holds for (2.1).*

Proof. We first fix the integer $m \geq 1$. Let $a_0 = 1$ and choose $a_k \to 0$ decreasingly so that $\int_{a_k}^{a_{k-1}} \rho_m(z)^{-2}dz = k$ for $k \geq 1$. Let $x \mapsto \psi_k(x)$ be a non-negative continuous function on $\mathbb{R}$ which has support in (a_k, a_{k-1}) and satisfies $\int_{a_k}^{a_{k-1}} \psi_k(x)dx = 1$ and $0 \leq \psi_k(x) \leq 2k^{-1}\rho_m(x)^{-2}$ for $a_k < x < a_{k-1}$. For each $k \geq 1$ we define the non-negative and twice continuously differentiable function

$$\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x)dx, \qquad z \in \mathbb{R}. \tag{2.2}$$

It is easy to see that $\phi_k(z) \to |z|$ non-decreasingly as $k \to \infty$ and $0 \le \phi_k'(z) \le 1$ for $z \ge 0$ and $-1 \le \phi_k'(z) \le 0$ for $z \le 0$. By condition (2.d) and the choice of $x \mapsto \psi_k(x)$,

$$\begin{aligned}\phi_k''(x-y)\int_E |\sigma(x,u)-\sigma(y,u)|^2\pi(du)& \\ \le \psi_k(|x-y|)\rho_m(|x-y|)^2 \le \frac{2}{k}&\end{aligned} \tag{2.3}$$

for $0 \le x, y \le m$. Then the left-hand side tends to zero uniformly in $0 \le x, y \le m$ as $k \to \infty$. For $h, \zeta \in \mathbb{R}$, by Taylor's expansion we have

$$D_h\phi_k(\zeta) = \int_0^1 h^2\phi_k''(\zeta + th)(1-t)dt = \int_0^1 h^2\psi_k(|\zeta + th|)(1-t)dt.$$

It follows that

$$D_h\phi_k(\zeta) \le \frac{2}{k}\int_0^1 h^2\rho_m(|\zeta + th|)^{-2}(1-t)dt. \tag{2.4}$$

Observe also that

$$D_h\phi_k(\zeta) = \Delta_h\phi_k(\zeta) - \phi_k'(\zeta)h \le 2|h|. \tag{2.5}$$

For $0 \le x, y \le m$ and $n \ge 1$ we can use (2.4) and (2.5) to get

$$\begin{aligned}&\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) \\ &\quad\le \frac{2}{k}\int_{U_0}\mu_0(du)\int_0^1 \frac{l_0(x,y,u)^2(1-t)1_{\{|l_0(x,y,u)|\le n\}}}{\rho_m(|(x-y)+tl_0(x,y,u)|)^2}dt \\ &\quad\quad + 2\int_{U_0}|l_0(x,y,u)|1_{\{|l_0(x,y,u)|>n\}}\mu_0(du) \\ &\quad\le \frac{2}{k}c(m,n) + 4\int_{U_0}\bar{g}_0(m,u)1_{\{\bar{g}_0(m,u)>n/2\}}\mu_0(du).\end{aligned} \tag{2.6}$$

By conditions (2.b,d) one sees the right-hand side tends to zero uniformly in $0 \le x, y \le m$ as $k \to \infty$. Then the pathwise uniqueness for (2.1) follows by a trivial modification of Theorem 3.1 in Fu and Li (2010). □

The key difference between the above theorem and Theorems 3.2 and 3.3 of Fu and Li (2010) is that here we do not assume $x \mapsto g_0(x,u)$ is non-decreasing. This is essential for the applications to stochastic equations like (1.6).

Theorem 2.2 *Let (σ, b', g_0, g_1') and $(\sigma, b'', g_0, g_1'')$ be two sets of admissible parameters satisfying conditions* (2.a,b,c,d). *In addition, assume that*

(i) *for every $u \in U_1$, $x \mapsto x + g_1'(x,u)$ or $x \mapsto x + g_1''(x,u)$ is non-decreasing;*

(ii) $b'(x) \le b''(x)$ *and* $g_1'(x,u) \le g_1''(x,u)$ *for every* $x \ge 0$ *and* $u \in U_1$.

Suppose that $\{x'(t) : t \ge 0\}$ *is a solution of (2.1) with* $(b, g_1) = (b', g_1')$ *and* $\{x''(t) : t \ge 0\}$ *is a solution of the equation with* $(b, g_1) = (b'', g_1'')$. *If* $x'(0) \le x''(0)$, *then* $\mathbf{P}\{x'(t) \le x''(t)$ *for all* $t \ge 0\} = 1$.

Proof. Let $\zeta(t) = x'(t) - x''(t)$ for $t \ge 0$. Let $x \mapsto \psi_k(x)$ be defined as in the proof of Theorem 2.1. Instead of (2.2), for each $k \ge 1$ we now define

$$\phi_k(z) = \int_0^z dy \int_0^y \psi_k(x)dx, \qquad z \in \mathbb{R}. \tag{2.7}$$

Then $\phi_k(z) \to z^+ := 0 \vee z$ non-decreasingly as $k \to \infty$. Let

$$l_0(t,u) = g_0(x'(t),u) - g_0(x''(t),u), \qquad t \ge 0, u \in U_0,$$

and

$$l_1(t,u) = g_1'(x'(t),u) - g_1''(x''(t),u), \qquad t \ge 0, u \in U_1.$$

For $\zeta(s-) \le 0$ we have $\phi_k(\zeta(s-)) = \phi_k'(\zeta(s-)) = 0$. Since $x \mapsto x + f(x,u)$ is non-decreasing for $f = g_1'$ or g_1'', for $\zeta(s-) = x'(s-) - x''(s-) \le 0$ we also have

$$\begin{aligned} \zeta(s-) + l_1(s-,u) &= x'(s-) - x''(s-) + g_1'(x'(s-),u) - g_1''(x''(s-),u) \\ &\le x'(s-) - x''(s-) + f(x'(s-),u) - f(x''(s-),u) \le 0. \end{aligned}$$

The latter implies

$$\varDelta_{l_1(s-,u)}\phi_k(\zeta(s-)) = \phi_k(\zeta(s-) + l_1(s-,u)) - \phi_k(\zeta(s-)) = 0.$$

By Itô's formula we have

$$\begin{aligned} \phi_k(\zeta(t)) &= \phi_k(\zeta(0)) + \frac{1}{2}\int_0^t ds \int_E \phi_k''(\zeta(s-))[\sigma(x'(s-),u) \\ &\qquad\qquad - \sigma(x''(s-),u)]^2 \pi(du) \\ &\quad + \int_0^t \phi_k'(\zeta(s-))[b'(x'(s-)) \\ &\qquad\qquad - b''(x''(s-))]1_{\{\zeta(s-)>0\}} ds \\ &\quad + \int_0^t ds \int_{U_1} \varDelta_{l_1(s-,u)}\phi_k(\zeta(s-))1_{\{\zeta(s-)>0\}}\mu_1(du) \\ &\quad + \int_0^t ds \int_{U_0} D_{l_0(s-,u)}\phi_k(\zeta(s-))\mu_0(du) + M_m(t), \end{aligned} \tag{2.8}$$

where

$$M_m(t) = \int_0^t \int_E \phi_k'(\zeta(s-))[\sigma(x'(s-),u)$$

$$
\begin{aligned}
&\qquad\qquad - \sigma(x''(s-),u)]W(ds,du) \\
&+ \int_0^t \int_{U_1} \Delta_{l_1(s-,u)}\phi_k(\zeta(s-))\tilde{N}_1(ds,du) \\
&+ \int_0^t \int_{U_0} \Delta_{l_0(s-,u)}\phi_k(\zeta(s-))\tilde{N}_0(ds,du).
\end{aligned}
$$

Let $\tau_m = \inf\{t \geq 0 : x'(t) \geq m \text{ or } x''(t) \geq m\}$ for $m \geq 1$. Under conditions (2.b,c) it is easy to show that $\{M_m(t \wedge \tau_m)\}$ is a martingale. Recall that $b'(x) \leq b''(x)$ and $b'(x) = b_1'(x) - b_2'(x)$ for a non-decreasing function $x \mapsto b_2'(x)$. Then under the restriction $\zeta(s-) > 0$ we have

$$
\begin{aligned}
&\phi_k'(\zeta(s-))[b'(x'(s-)) - b''(x''(s-))] \\
&\qquad \leq \phi_k'(\zeta(s-))[b'(x'(s-)) - b'(x''(s-))] \\
&\qquad \leq \phi_k'(\zeta(s-))[b_1'(x'(s-)) - b_1'(x''(s-))] \\
&\qquad \leq |b_1'(x'(s-)) - b_1'(x''(s-))|
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathit{\Delta}_{l_1(s-,u)}\phi_k(\zeta(s-)) \\
&\quad = \phi_k(\zeta(s-) + g_1'(x'(s-),u) - g_1''(x''(s-),u)) - \phi_k(\zeta(s-)) \\
&\quad \leq \phi_k(\zeta(s-) + g_1'(x'(s-),u) - g_1'(x''(s-),u)) - \phi_k(\zeta(s-)) \\
&\quad \leq |g_1'(x'(s-),u) - g_1'(x''(s-),u)|.
\end{aligned}
$$

The estimates (2.3) and (2.6) are still valid. If $x'(0) \leq x''(0)$, we can take the expectation in (2.8) and let $k \to \infty$ to get

$$
\begin{aligned}
\mathbf{E}[\zeta(t \wedge \tau_m)^+] \;&\leq\; \mathbf{E}\bigg[\int_0^{t\wedge\tau_m} r_m(|\zeta(s-)|)1_{\{\zeta(s-)>0\}} ds\bigg] \\
&\leq\; \int_0^t r_m(\mathbf{E}[\zeta(s \wedge \tau_m)^+])ds,
\end{aligned}
$$

where the second inequality holds by the concaveness of $z \mapsto r_m(z)$. Then $\mathbf{E}[\zeta(t \wedge \tau_m)^+] = 0$ for all $t \geq 0$. Since $\tau_m \to \infty$ as $m \to \infty$, we get the desired comparison property. $\square$

We say the *comparison property* of solutions holds for (2.1) if for any two solutions $\{x_1(t) : t \geq 0\}$ and $\{x_2(t) : t \geq 0\}$ satisfying $x_1(0) \leq x_2(0)$ we have $\mathbf{P}\{x_1(t) \leq x_2(t) \text{ for all } t \geq 0\} = 1$. From Theorem 2.2 we get the following:

Theorem 2.3 *Let (σ, b, g_0, g_1) be admissible parameters satisfying conditions* (2.a,b,c,d). *In addition, assume that for every $u \in U_1$ the function $x \mapsto x + g_1(x,u)$ is non-decreasing. Then the comparison property holds for the solutions of (2.1).*

The monotonicity assumption on the function $x \mapsto x + g_1(x,u)$ in Theorem 2.3 is natural. To see this, suppose that $\{x_1(t)\}$ and $\{x_2(t)\}$ are two solutions of (2.1) and $\{(s_i, u_i) : i \geq 1\}$ is the set of atoms of $\{N_1(ds, du)\}$. The assumption guarantees that $x_1(s_i-) \leq x_2(s_i-)$ implies

$$\begin{aligned} x_1(s_i) &= x_1(s_i-) + g_1(x_1(s_i-), u_i) \\ &\leq x_2(s_i-) + g_1(x_2(s_i-), u_i) = x_2(s_i). \end{aligned}$$

A similar explanation can be given to Theorem 2.2. In some applications the kernel $x \mapsto g_0(x, u)$ may be non-decreasing. When this is true, we can replace (2.d) by the following simpler condition:

(2.e) For each $u \in U_0$ the function $x \mapsto g_0(x, u)$ is non-decreasing, and for each $m \geq 1$ there is a non-negative and non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ so that $\int_{0+} \rho_m(z)^{-2} dz = \infty$ and

$$\begin{aligned} \int_E |\sigma(x, u) - \sigma(y, u)|^2 \pi(du) + \int_{U_0} |l_0(x, y, u)| \wedge |l_0(x, y, u)|^2 \mu_0(du) \\ \leq \rho_m(|x - y|)^2 \end{aligned}$$

for all $0 \leq x, y \leq m$, where $l_0(x, y, u) = g_0(x, u) - g_0(y, u)$.

Proposition 2.4 *Let* (σ, b, g_0, g_1) *be admissible parameters. If* (2.e) *holds, then* (2.d) *holds.*

Proof. Since $x \mapsto g_0(x, u)$ is non-decreasing, it is not hard to see $|(x - y) + tl_0(x, y, u)| \geq |x - y|$. By condition (2.e) and the monotonicity of $z \mapsto \rho(z)$ we have

$$\begin{aligned} \int_0^1 dt \int_{U_0} \frac{(1-t) l_0(x, y, u)^2 1_{\{|l_0(x,y,u)| \leq n\}}}{\rho_m(|(x - y) + tl_0(x, y, u)|)^2} \mu_0(du) \\ \leq n \int_0^1 dt \int_{U_0} \frac{[|l_0(x, y, u)| \wedge l_0(x, y, u)^2]}{\rho_m(|x - y|)^2} \mu_0(du) \leq n. \end{aligned}$$

Then condition (2.d) is satisfied. □

Theorem 2.5 *Suppose that* (σ, b, g_0, g_1) *are admissible parameters satisfying conditions* (2.a,c,e)*. Then there is a unique strong solution to (2.1).*

Proof. We first note that (2.b) follows from (2.e). By Proposition 2.4, we also have (2.d) from (2.e). Let $\{V_n\}$ be a non-decreasing sequence of Borel subsets of U_0 so that $\cup_{n=1}^{\infty} V_n = U_0$ and $\mu_0(V_n) < \infty$ for every $n \geq 1$. For $m, n \geq 1$ one can use (2.e) to see

$$x \mapsto \beta_m(x) := \int_{U_0} [g_0(x, u) - g_0(x, u) \wedge m] \mu_0(du)$$

and

$$x \mapsto \gamma_{m,n}(x) := \int_{V_n} [g_0(x,u) \wedge m] \mu_0(du)$$

are continuous non-decreasing functions. By the results for continuous-type stochastic equations as in Ikeda and Watanabe (1989, page 169), one can show there is a non-negative weak solution to

$$\begin{aligned} x(t) &= x(0) + \int_0^t \int_E \sigma(x(s) \wedge m, u) W(ds, du) \\ &\quad + \int_0^t b_m(x(s) \wedge m) ds - \int_0^t \gamma_{m,n}(x(s) \wedge m) ds, \end{aligned}$$

where $b_m(x) = b(x) - \beta_m(x)$. The pathwise uniqueness holds for the above equation by Theorem 2.1. Then it has a unique strong solution. Let $\{W_n\}$ be a non-decreasing sequence of Borel subsets of U_1 so that $\cup_{n=1}^{\infty} W_n = U_1$ and $\mu_1(W_n) < \infty$ for every $n \geq 1$. Following the proof of Proposition 2.2 of Fu and Li (2010) one can show there is a unique strong solution $\{x_{m,n}(t) : t \geq 0\}$ to

$$\begin{aligned} x(t) &= x(0) + \int_0^t \int_E \sigma(x(s-) \wedge m, u) W(ds, du) \\ &\quad + \int_0^t b_m(x(s-) \wedge m) ds - \int_0^t \gamma_{m,n}(x(s) \wedge m) ds \\ &\quad + \int_0^t \int_{V_n} [g_0(x(s-) \wedge m, u) \wedge m] N_0(ds, du) \\ &\quad + \int_0^t \int_{W_n} [g_1(x(s-) \wedge m, u) \wedge m] N_1(ds, du). \end{aligned}$$

We can rewrite the above equation into

$$\begin{aligned} x(t) &= x(0) + \int_0^t \int_E \sigma(x(s-) \wedge m, u) W(ds, du) \\ &\quad + \int_0^t b_m(x(s-) \wedge m) ds \\ &\quad + \int_0^t \int_{V_n} [g_0(x(s-) \wedge m, u) \wedge m] \tilde{N}_0(ds, du) \\ &\quad + \int_0^t \int_{W_n} [g_1(x(s-) \wedge m, u) \wedge m] N_1(ds, du). \end{aligned}$$

As in the proof of Lemma 4.3 of Fu and Li (2010) one can see the sequence $\{x_{m,n}(t) : t \geq 0\}$, $n = 1, 2, \cdots$ is tight in $D([0, \infty), \mathbb{R}_+)$. Following the proof of Theorem 4.4 of Fu and Li (2010) it is easy to show that any weak limit point $\{x_m(t) : t \geq 0\}$ of the sequence is a non-negative weak solution to

$$x(t) = x(0) + \int_0^t \int_E \sigma(x(s-) \wedge m, u) W(ds, du)$$

$$
\begin{aligned}
&+\int_0^t b_m(x(s-)\wedge m)ds\\
&+\int_0^t\int_{U_0}[g_0(x(s-)\wedge m,u)\wedge m]\tilde{N}_0(ds,du)\\
&+\int_0^t\int_{U_1}[g_1(x(s-)\wedge m,u)\wedge m]N_1(ds,du).
\end{aligned}
\tag{2.9}
$$

By Theorem 2.1 the pathwise uniqueness holds for (2.9), so the equation has a unique strong solution; see, e.g., Situ (2005, page 104). Then the result follows by a simple modification of the proof of Proposition 2.4 of Fu and Li (2010). See El Karoui and Méléard (1990) and Kurtz (2007, 2010) for the general theory of stochastic equations driven by white noises and Poisson random measures. □

3 Stochastic flows of CBI-processes

In this section, we give the constructions and characterizations of the flow of CBI-processes and the associated immigration superprocess. Suppose that $\sigma\geq 0$ and b are constants and $(u\wedge u^2)m(du)$ is a finite measures on $(0,\infty)$. Let ϕ be a function given by

$$
\phi(z)=bz+\frac{1}{2}\sigma^2z^2+\int_0^\infty(e^{-zu}-1+zu)m(du),\quad z\geq 0. \tag{3.1}
$$

A Markov process with state space $\mathbb{R}_+:=[0,\infty)$ is called a *CB-process* with branching mechanism ϕ if it has transition semigroup $(p_t)_{t\geq 0}$ given by

$$
\int_{\mathbb{R}_+}e^{-\lambda y}p_t(x,dy)=e^{-xv_t(\lambda)},\qquad \lambda\geq 0, \tag{3.2}
$$

where $(t,\lambda)\mapsto v_t(\lambda)$ is the unique locally bounded non-negative solution of

$$
\frac{d}{dt}v_t(\lambda)=-\phi(v_t(\lambda)),\quad v_0(\lambda)=\lambda,\qquad t\geq 0.
$$

Given any $\beta\geq 0$ we can also define a transition semigroup $(q_t)_{t\geq 0}$ on $\mathbb{R}_+$ by

$$
\int_{\mathbb{R}_+}e^{-\lambda y}q_t(x,dy)=\exp\bigg\{-xv_t(\lambda)-\int_0^t\beta v_s(\lambda)ds\bigg\}. \tag{3.3}
$$

A non-negative real-valued Markov process with transition semigroup $(q_t)_{t\geq 0}$ is called a *CBI-process* with branching mechanism ϕ and immigration rate β. It is easy to see that both $(p_t)_{t\geq 0}$ and $(q_t)_{t\geq 0}$ are Feller semigroups. See, for example, Kawazu and Watanabe (1971) and Li (2011, Chapter 3).

Let $\{W(ds,du)\}$ be a white noise on $(0,\infty)^2$ based on the Lebesgue measure $dsdu$ and let $\{N(ds,dz,du)\}$ be Poisson random measure on $(0,\infty)^3$ with intensity $dsm(dz)du$. Let $\{\tilde{N}(ds,dz,du)\}$ be the compensated measure of $\{N(ds,dz,du)\}$.

Theorem 3.1 *There is a unique non-negative strong solution of the stochastic equation*

$$\begin{aligned} Y_t &= Y_0 + \sigma \int_0^t \int_0^{Y_{s-}} W(ds, du) + \int_0^t (\beta - bY_{s-}) ds \\ &\quad + \int_0^t \int_0^\infty \int_0^{Y_{s-}} z\tilde{N}(ds, dz, du). \end{aligned}$$

Moreover, the solution $\{Y_t : t \geq 0\}$ *is a CBI-process with branching mechanism* ϕ *and immigration rate* β*.*

Proof. The existence and uniqueness of the strong solution follows by an application of Theorem 2.5; see also Dawson and Li (2006). Using Itô's formula one can see that $\{Y_t(v) : t \geq 0\}$ solves the martingale problem associated with the generator L defined by

$$Lf(x) = \frac{1}{2}\sigma^2 x f''(x) + (\beta - bx)f'(x) + x\int_0^\infty D_z f(x) m(dz). \tag{3.4}$$

Then it is a CBI-process with branching mechanism ϕ and immigration rate β; see Kawazu and Watanabe (1971) and Li (2011, Section 9.5). □

Let $v \mapsto \gamma(v)$ be a non-negative and non-decreasing continuous function on $[0, \infty)$. We denote by $\gamma(dv)$ the Radon measure on $[0, \infty)$ so that $\gamma([0, v]) = \gamma(v)$ for $v \geq 0$. By Theorem 3.1 for each $v \geq 0$ there is a pathwise unique non-negative solution $Y(v) = \{Y_t(v) : t \geq 0\}$ to the stochastic equation

$$\begin{aligned} Y_t(v) &= v + \sigma \int_0^t \int_0^{Y_{s-}(v)} W(ds, du) + \int_0^t [\gamma(v) - bY_{s-}(v)] ds \\ &\quad + \int_0^t \int_0^\infty \int_0^{Y_{s-}(v)} z\tilde{N}(ds, dz, du). \end{aligned} \tag{3.5}$$

Theorem 3.2 *For any* $v_2 \geq v_1 \geq 0$ *we have* $\mathbf{P}\{Y_t(v_2) \geq Y_t(v_1)$ *for all* $t \geq 0\} = 1$ *and* $\{Y_t(v_2) - Y_t(v_1) : t \geq 0\}$ *is a CBI-process with branching mechanism* ϕ *and immigration rate* $\beta := \gamma(v_2) - \gamma(v_1) \geq 0$*.*

Proof. The comparison property follows by applying Theorem 2.2 and Proposition 2.4 to (3.5). Let $Z_t = Y_t(v_2) - Y_t(v_1)$ for $t \geq 0$. From (3.5) we have

$$\begin{aligned} Z_t &= v_2 - v_1 + \sigma \int_0^t \int_{Y_{s-}(v_1)}^{Y_{s-}(v_2)} W(ds, du) + \int_0^t (\beta - bZ_{s-}) ds \\ &\quad + \int_0^t \int_0^\infty \int_{Y_{s-}(v_1)}^{Y_{s-}(v_2)} z\tilde{N}(ds, dz, du) \\ &= v_2 - v_1 + \sigma \int_0^t \int_0^{Z_{s-}} W_1(ds, du) + \int_0^t (\beta - bZ_{s-}) ds \end{aligned}$$

$$+ \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\tilde{N}_1(ds, dz, du), \tag{3.6}$$

where

$$W_1(ds, du) = W(ds, Y_{s-}(v_1) + du)$$

is a white noise with intensity $dsdu$ and

$$N_1(ds, dz, du) = N(ds, dz, Y_{s-}(v_1) + du)$$

is a Poisson random measure with intensity $dsm(dz)du$. That shows $\{Z_t : t \geq 0\}$ is a weak solution of (3.5). Then it a CBI-process with branching mechanism ϕ and immigration rate β. $\square$

Theorem 3.3 *Let $v_2 \geq v_1 \geq u_2 \geq u_1 \geq 0$. Then $\{Y_t(u_2) - Y_t(u_1) : t \geq 0\}$ and $\{Y_t(v_2) - Y_t(v_1) : t \geq 0\}$ are independent CBI-processes with immigration rates $\alpha := \gamma(u_2) - \gamma(u_1)$ and $\beta := \gamma(v_2) - \gamma(v_1)$, respectively.*

Proof. Let L_α and L_β denote the generators of the CBI-processes with immigration rates α and β, respectively. Let $X_t = Y_t(u_2) - Y_t(u_1)$ and $Z_t = Y_t(v_2) - Y_t(v_1)$. For any $G \in C^2(\mathbb{R}_+^2)$ one can use Itô's formula to show

$$\begin{aligned} G(X_t, Z_t) &= G(X_0, Z_0) + \int_0^t L_\alpha G(X_s, Z_s)ds \\ &\quad + \int_0^t L_\beta G(X_s, Z_s)ds + \text{local mart.}, \end{aligned} \tag{3.7}$$

where L_α and L_β act on the first and second coordinates of G, respectively. Then $\{X_t : t \geq 0\}$ and $\{Z_t : t \geq 0\}$ are independent CBI-processes with immigration rates α and β, respectively. $\square$

Proposition 3.4 *There is a locally bounded non-negative function $t \mapsto C(t)$ on $[0, \infty)$ so that*

$$\begin{aligned} \mathbf{E}\Big\{\sup_{0\leq s\leq t} [Y_s(v_2) - Y_s(v_1)]\Big\} &\leq C(t)\Big\{(v_2 - v_1) + [\gamma(v_2) - \gamma(v_1)] \\ &\quad + \sqrt{v_2 - v_1} + \sqrt{\gamma(v_2) - \gamma(v_1)}\Big\} \end{aligned} \tag{3.8}$$

for $t \geq 0$ and $v_2 \geq v_1 \geq 0$.

Proof. Let $Z_t = Y_t(v_2) - Y_t(v_1)$ for $t \geq 0$. Taking the expectation in (3.6) we have

$$\mathbf{E}(Z_t) = (v_2 - v_1) + t[\gamma(v_2) - \gamma(v_1)] - b\int_0^t \mathbf{E}(Z_s)ds.$$

Solving the above integral equation gives

$$\mathbf{E}(Z_t) = (v_2 - v_1)e^{-bt} + [\gamma(v_2) - \gamma(v_1)]b^{-1}(1 - e^{-bt}) \tag{3.9}$$

with $b^{-1}(1-e^{-bt}) = t$ for $b = 0$ by convention. By (3.6) and Doob's martingale inequality,

$$\begin{aligned}
\mathbf{E}\Big\{\sup_{0\le s\le t} Z_s\Big\} \;\le\;& (v_2 - v_1) + 2\sigma \mathbf{E}^{\frac{1}{2}}\bigg\{\bigg(\int_0^t \int_{Y_{s-}(v_1)}^{Y_{s-}(v_2)} W(ds,du)\bigg)^2\bigg\} \\
&+ \int_0^t \{[\gamma(v_2) - \gamma(v_1)] + |b|\mathbf{E}(Z_s)\}ds \\
&+ 2\mathbf{E}^{\frac{1}{2}}\bigg\{\bigg(\int_0^t \int_0^1 \int_{Y_{s-}(v_1)}^{Y_{s-}(v_2)} z\tilde{N}(ds,dz,du)\bigg)^2\bigg\} \\
&+ \mathbf{E}\bigg[\int_0^t \int_1^\infty \int_{Y_{s-}(v_1)}^{Y_{s-}(v_2)} zN(ds,dz,du)\bigg] \\
\le\;& (v_2 - v_1) + t[\gamma(v_2) - \gamma(v_1)] + 2\sigma\bigg[\int_0^t \mathbf{E}(Z_s)ds\bigg]^{\frac{1}{2}} \\
&+ 2\bigg[\int_0^1 z^2\nu(dz)\bigg]^{\frac{1}{2}}\bigg[\int_0^t \mathbf{E}(Z_s)ds\bigg]^{\frac{1}{2}} \\
&+ \bigg[|b| + \int_1^\infty z\nu(dz)\bigg]\int_0^t \mathbf{E}(Z_s)ds.
\end{aligned}$$

Then (3.8) follows by (3.9). □

Suppose that (E, ρ) is a complete metric space. Let F be a subset of $[0, \infty)$ such that $0 \in F$ and let $t \mapsto x(t)$ be a path from F to E. For any $\epsilon > 0$ the number of ϵ-oscillations of this path on F is defined as

$$\begin{aligned}
\mu(\epsilon) \;:=\; \sup\{n \ge 0 :\ & \text{there are } 0 = t_0 < t_1 < \cdots < t_n \in F \\
& \text{so that } \rho(x(t_{i-1}), x(t_i)) \ge \epsilon \text{ for all } 1 \le i \le n\}.
\end{aligned}$$

If F is dense in $[0, \infty)$, it is simple to show the limits $y(t) := \lim_{F\ni s\to t+} x(s)$ exist for all $t \ge 0$ and constitute a càdlàg path $t \mapsto y(t)$ on $[0, \infty)$ if and only if $t \mapsto x(t)$ has at most a finite number of ϵ-oscillations on $F \cap [0, T]$ for every $\epsilon > 0$ and $T \ge 0$.

Lemma 3.5 *Suppose that $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{P})$ is a filtered probability space and $\{X_t : t \ge 0\}$ is a $(\mathscr{G}_t)$-Markov process with state space $(E, \mathscr{E})$ and transition semigroup $(P_{s,t})_{t\ge s}$. Suppose that ρ is a complete metric on E so that:*

(i) *for $\epsilon > 0$ and $0 \le s, t \le u$ we have $\{\omega \in \Omega : \rho(X_s(\omega), X_t(\omega)) < \epsilon\} \in \mathscr{G}_u$;*

(ii) *for $\epsilon > 0$ and $x \in E$ we have $U_\epsilon(x) := \{y \in E : \rho(x, y) < \epsilon\} \in \mathscr{E}$ and*

$$\alpha_\epsilon(h) := \sup_{0\le t-s\le h} \sup_{x\in E} P_{s,t}(x, U_\epsilon(x)^c) \to 0 \quad (h \to 0). \tag{3.10}$$

Then $\{X_t : t \geq 0\}$ has a ρ-càdlàg modification.

Proof. Let $F = \{0, r_1, r_2, \cdots\}$ be a countable dense subset of $[0, \infty)$ and let $F_n = \{0, r_1, \cdots, r_n\}$. For $\epsilon > 0$ and $a > 0$ let $\nu^a(\epsilon)$ and $\nu_n^a(\epsilon)$ denote respectively the numbers of ϵ-oscillations of $t \mapsto X_t$ on $F \cap [0, a]$ and $F_n \cap [0, a]$. Then $\nu_n^a(\epsilon) \to \nu^a(\epsilon)$ increasingly as $n \to \infty$. Let $\tau_n^\epsilon(0) = 0$ and for $k \geq 0$ define

$$\tau_n^\epsilon(k+1) = \min\{t \in F_n \cap (\tau_n^\epsilon(k), \infty) : \rho(X_{\tau_n^\epsilon(k)}, X_t) \geq \epsilon\}$$

if $\tau_n^\epsilon(k) < \infty$ and $\tau_n^\epsilon(k+1) = \infty$ if $\tau_n^\epsilon(k) = \infty$. Since F_n is discrete, for any $a \geq 0$ we have

$$\{\tau_n^\epsilon(k+1) \leq a\} = \bigcup_{s<t \in F_n \cap [0,a]} (\{\tau_n^\epsilon(k) = s\} \cap \{\rho(X_s, X_t) \geq \epsilon\}).$$

Using property (i) and the above relation it is easy to see successively that each $\tau_n^\epsilon(k)$ is a stopping time. As in the proof of Lemma 9.1 of Wentzell (1981, page 168) one can prove $\mathbf{P}\{\tau_n^\epsilon(1) \leq h\} \leq 2\alpha_{\epsilon/2}(h)$ for $\epsilon > 0$ and $h > 0$. Since the strong Markov property of $\{X_t : t \geq 0\}$ holds at the discrete stopping times $\tau_n^\epsilon(k)$, $k = 1, 2, \cdots$, one can inductively show

$$\mathbf{P}\{\nu_n^h(2\epsilon) \geq k\} \leq \mathbf{P}\{\tau_n^\epsilon(k) \leq h\} \leq [2\alpha_{\epsilon/2}(h)]^k.$$

It follows that

$$\mathbf{P}\{\nu^h(2\epsilon) \geq k\} = \lim_{n\to\infty} \mathbf{P}\{\nu_n^h(2\epsilon) \geq k\} \leq [2\alpha_{\epsilon/2}(h)]^k.$$

Choosing sufficiently small $h = h(\epsilon) \in F \cap (0, \infty)$ so that $\alpha_{\epsilon/2}(h) < 1/2$ and letting $k \to \infty$ we get $\mathbf{P}\{\nu^h(2\epsilon) < \infty\} = 1$. By repeating the above procedure successively on the intervals $[h, 2h]$, $[2h, 3h]$, $\cdots$ we get $\mathbf{P}\{\nu^a(2\epsilon) < \infty\} = 1$ for every $a > 0$. Let $\Omega_1 = \cap_{m=1}^\infty \{\nu^m(1/m) < \infty\}$. Then $\Omega_1 \in \mathscr{G}$ and $\mathbf{P}(\Omega_1) = 1$. Moreover, for $\omega \in \Omega_1$ we can define a ρ-càdlàg path $t \mapsto Y_t(\omega)$ on $[0, \infty)$ by $Y_t(\omega) := \lim_{F \ni s \to t+} X_s(\omega)$. Take $x_0 \in E$ and define $Y_t(\omega) = x_0$ for $t \geq 0$ and $\omega \in \Omega \setminus \Omega_1$. By (3.10) one can see $t \mapsto X_t$ is right continuous in probability, so $Y_t = X_t$ a.s. for every $t \geq 0$. Then $\{Y_t : t \geq 0\}$ is a ρ-càdlàg modification of $\{X_t : t \geq 0\}$. $\square$

Let $D[0, \infty)$ be the space of non-negative càdlàg functions on $[0, \infty)$ and let $\mathscr{B}(D[0, \infty))$ be its Borel σ-algebra generated by the Skorokhod topology. Theorems 3.2 and 3.3 imply that $\{Y(v) : v \geq 0\}$ is a non-decreasing process in $(D[0, \infty), \mathscr{B}(D[0, \infty)))$ with independent increments. Let ρ be the metric on $D[0, \infty)$ defined by

$$\rho(\xi, \zeta) = \int_0^\infty e^{-t} \sup_{0 \leq s \leq t} (|\xi(s) - \zeta(s)| \wedge 1) dt. \tag{3.11}$$

This metric corresponds to the topology of local uniform convergence, which is strictly stronger than the Skorokhod topology.

Theorem 3.6 *The path-valued process $\{Y(v) : v \geq 0\}$ has a ρ-càdlàg modification. Consequently, there is a version of the solution flow $\{Y_t(v) : t \geq 0, v \geq 0\}$ of (3.5) with the following properties:*

(i) for each $v \geq 0$, $t \mapsto Y_t(v)$ is a càdlàg process on $[0, \infty)$ and solves (3.5);

(ii) for each $t \geq 0$, $v \mapsto Y_t(v)$ is a non-negative and non-decreasing càdlàg process on $[0, \infty)$.

Proof. Step 1. For any $T \geq 0$ let $D[0, T]$ be the space of non-negative càdlàg functions on $[0, T]$ and let $\mathscr{B}(D[0, T])$ be its σ-algebra generated by the Skorokhod topology. For $v \geq 0$ let $Y^T(v) = \{Y_t(v) : 0 \leq t \leq T\}$. Theorem 3.3 implies that $\{Y^T(v) : v \geq 0\}$ is a process in $(D[0, T], \mathscr{B}(D[0, T]))$ with independent increments.

Step 2. Let $F_T = \{T, r_1, r_2, \cdots\}$ be a countable dense subset of $[0, T]$. We consider the metric ρ_T on $D[0, T]$ defined by

$$\rho_T(\xi, \zeta) = \sup_{0 \leq s \leq T} |\xi(s) - \zeta(s)| = \sup_{r \in F_T} |\xi(s) - \zeta(s)|.$$

For any $\epsilon > 0$ and $\xi \in D[0, T]$ we have

$$\begin{aligned} \bar{U}_\epsilon(\xi) &:= \{\zeta \in D[0, T] : \rho_T(\xi, \zeta) \leq \epsilon\} \\ &= \bigcap_{r \in F_T} \{\zeta \in D[0, T] : |\xi_r - \zeta_r| \leq \epsilon\}. \end{aligned}$$

Then the above set belongs to $\mathscr{B}(D[0, T])$; see, e.g., Ethier and Kurtz (1986, page 127). It follows that

$$U_\epsilon(\xi) := \{\zeta \in D[0, T] : \rho_T(\xi, \zeta) < \epsilon\} = \bigcup_{n=1}^{\infty} \bar{U}_{\epsilon - 1/n}(\xi)$$

also belongs to $\mathscr{B}(D[0, T])$.

Step 3. Let $(\mathscr{F}_v^T)_{v \geq 0}$ be the natural filtration of $\{Y^T(v) : v \geq 0\}$. For any $\epsilon > 0$ and $0 \leq s, t \leq v$ we have

$$\rho_T(Y^T(s), Y^T(t)) = \sup_{r \in F_T} |Y_r(s) - Y_r(t)|.$$

Then one can show $\{\omega \in \Omega : \rho_T(Y^T(\omega, s), Y^T(\omega, t)) < \epsilon\} \in \mathscr{F}_v^T$.

Step 4. Let $(P_{u,v}^T)_{v \geq u}$ denote the transition semigroup of $\{Y^T(v) : v \geq 0\}$. By Proposition 3.4 for $\epsilon > 0$ and $\xi \in D[0, \infty)$ we have

$$\begin{aligned} P_{u,v}(\xi, U_\epsilon(\xi)^c) &= \mathbf{P}\Big\{\sup_{0 \leq s \leq T} [Y_s(v) - Y_s(u)] \geq \epsilon\Big\} \\ &\leq \epsilon^{-1} \mathbf{E}\Big\{\sup_{0 \leq s \leq T} [Y_s(v) - Y_s(u)]\Big\} \end{aligned}$$

$$\leq \epsilon^{-1}C(t)\Big\{(v-u)+[\gamma(v)-\gamma(u)] + \sqrt{v-u}+\sqrt{\gamma(v)-\gamma(u)}\Big\}.$$

Since $v\mapsto\gamma(v)$ is uniformly continuous on each bounded interval, Lemma 3.5 implies that $\{Y^T(v): v\geq 0\}$ has a ρ_T-càdlàg modification. That implies the existence of a ρ-càdlàg modification of $\{Y(v): v\geq 0\}$. □

In the situation of Theorem 3.6 we call the solution $\{Y_t(v): t\geq 0, v\geq 0\}$ of (3.5) a *flow of CBI-processes*. Let $F[0,\infty)$ be the set of non-negative and non-decreasing càdlàg functions on $[0,\infty)$. Given a finite stopping time τ and a function $\mu\in F[0,\infty)$ let $\{Y^\mu_{\tau,t}(v): t\geq 0\}$ be the solution of

$$\begin{aligned} Y^\mu_{\tau,t}(v) &= \mu(v)+\sigma\int_\tau^{\tau+t}\int_0^{Y^\mu_{\tau,s-}(v)} W(ds,du) \\ &\quad +\int_\tau^{\tau+t}[\gamma(v)-bY^\mu_{\tau,s-}(v)]ds \\ &\quad +\int_\tau^{\tau+t}\int_0^\infty\int_0^{Y^\mu_{\tau,s-}(v)} z\tilde{N}(ds,dz,du), \end{aligned} \tag{3.12}$$

and write simply $\{Y^\mu_t(v): t\geq 0\}$ instead of $\{Y^\mu_{0,t}(v): t\geq 0\}$. The pathwise uniqueness for the above equation follows from that of (3.5) since $\{W(\tau+ds,du)\}$ is a white noise based on $dsdz$ and $\{N(\tau+ds,dz,du)\}$ is a Poisson random measure with intensity $dsm(dz)du$. Let $G_{\tau,t}$ be the random operator on $F[0,\infty)$ that maps μ to $Y^\mu_{\tau,t}$.

Theorem 3.7 *For any finite stopping time τ we have $\mathbf{P}\{Y^\mu_{\tau+t}=G_{\tau,t}Y^\mu_\tau$ for all $t\geq 0\}=1$.*

Proof. By the sample path regularity of $(t,v)\mapsto Y_t(v)$ we only need to show $\mathbf{P}\{Y^\mu_{\tau+t}(v)=G_{\tau,t}Y^\mu_\tau(v)\}=1$ for every $t\geq 0$ and $v\geq 0$. By (3.5) we have

$$\begin{aligned} Y^\mu_{\tau+t}(v) &= Y^\mu_\tau(v)+\sigma\int_\tau^{\tau+t}\int_0^{Y^\mu_{s-}(v)} W(ds,du) \\ &\quad +\int_\tau^{\tau+t}[\gamma(v)-bY^\mu_{s-}(v)]ds \\ &\quad +\int_\tau^{\tau+t}\int_0^\infty\int_0^{Y^\mu_{s-}(v)} z\tilde{N}(ds,dz,du). \end{aligned}$$

By the pathwise uniqueness for (3.12) we get the desired result. □

For any Radon measure $\mu(dv)$ on $[0,\infty)$ with distribution function $v\mapsto\mu(v)$, the random function $v\mapsto Y^\mu_t(v)$ induces a random Radon measure $Y^\mu_t(dv)$ on $[0,\infty)$ so that $Y^\mu_t([0,v])=Y^\mu_t(v)$ for $v\geq 0$. We shall give some characterizations of the measure-valued process $\{Y^\mu_t: t\geq 0\}$.

For simplicity, we fix a constant $a \geq 0$ and consider the restrictions of $\mu(dv)$, $\gamma(dv)$ and $\{Y_t^\mu : t \geq 0\}$ to $[0, a]$ without changing the notation. Let us consider the step function

$$f(x) = c_0 1_{\{0\}}(x) + \sum_{i=1}^n c_i 1_{(a_{i-1}, a_i]}(x), \qquad x \in [0, a], \tag{3.13}$$

where $\{c_0, c_1, \cdots, c_n\} \subset \mathbb{R}$ and $\{0 = a_0 < a_1 < \cdots < a_n = a\}$ is a partition of $[0, a]$. For this function we have

$$\langle Y_t^\mu, f\rangle = c_0 Y_t^\mu(0) + \sum_{i=1}^n c_i [Y_t^\mu(a_i) - Y_t^\mu(a_{i-1})]. \tag{3.14}$$

From (3.12) and (3.14) it is simple to see

$$\begin{aligned} \langle Y_t^\mu, f\rangle &= \langle \mu, f\rangle + \sigma \int_0^t \int_0^\infty g_{s-}^\mu(u) W(ds, du) \\ &\quad + \int_0^t [\langle \gamma, f\rangle - b\langle Y_{s-}^\mu, f\rangle] ds \\ &\quad + \int_0^t \int_0^\infty \int_0^\infty z g_{s-}^\mu(u) \tilde{N}(ds, dz, du), \end{aligned} \tag{3.15}$$

where

$$g_s^\mu(u) = c_0 1_{\{u \leq Y_s^\mu(0)\}} + \sum_{i=1}^n c_i 1_{\{Y_s^\mu(a_{i-1}) < u \leq Y_s^\mu(a_i)\}}. \tag{3.16}$$

Proposition 3.8 *For any $t \geq 0$ and $f \in B[0, a]$ we have*

$$\mathbf{E}[\langle Y_t^\mu, f\rangle] = \langle \mu, f\rangle e^{-bt} + \langle \gamma, f\rangle b^{-1}(1 - e^{-bt}) \tag{3.17}$$

with $b^{-1}(1 - e^{-bt}) = t$ for $b = 0$ by convention.

Proof. We first consider the step function (3.13). By taking the expectation in (3.15) we obtain

$$\mathbf{E}[\langle Y_t^\mu, f\rangle] = \langle \mu, f\rangle + t\langle \gamma, f\rangle - b \int_0^t \mathbf{E}[\langle Y_s^\mu, f\rangle] ds.$$

The above integral equation has the unique solution given by (3.17). For a general function $f \in B[0, a]$ we get (3.17) by a monotone class argument. $\square$

Theorem 3.9 *The measure-valued process $\{Y_t^\mu : t \geq 0\}$ is a càdlàg strong Markov process in $M[0, a]$ with $Y_0^\mu = \mu$.*

Proof. In view of (3.14), the process $t \mapsto \langle Y_t^\mu, f\rangle$ is càdlàg for the step function (3.13). Since any function in $C[0,a]$ can be approximated by a sequence of step functions in the supremum norm, it is easy to conclude $t \mapsto \langle Y_t^\mu, f\rangle$ is càdlàg for all $f \in C[0,a]$. By Theorem 3.7, for any finite stopping time τ we have $Y_{\tau+t}^\mu = G_{\tau,t} Y_\tau^\mu$ almost surely. That clearly implies the strong Markov property of $\{Y_t^\mu : t \geq 0\}$. $\square$

Theorem 3.10 *For any $f \in B[0,a]$ the process $\{\langle Y_t^\mu, f\rangle : t \geq 0\}$ has a càdlàg modification. Moreover, there is a locally bounded function $t \mapsto C(t)$ so that*

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\leq s\leq t}\langle Y_s^\mu, f\rangle\Big] \;\leq\; & C(t)\Big[\langle \mu, f\rangle + \langle \gamma, f\rangle \\ & + \langle \mu, f^2\rangle^{1/2} + \langle \gamma, f^2\rangle^{1/2}\Big] \end{aligned} \tag{3.18}$$

for every $t \geq 0$ and $f \in B[0,a]^+$.

Proof. We first consider a non-negative step function given by (3.13) with constants $\{c_0, c_1, \cdots, c_n\} \subset \mathbb{R}_+$. By (3.15) and Doob's martingale inequality,

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\leq s\leq t}\langle Y_s^\mu, f\rangle\Big] \;\leq\; & \langle \mu, f\rangle + 2\sigma \mathbf{E}^{\frac{1}{2}}\bigg\{\bigg[\int_0^t\int_0^\infty g_{s-}^\mu(u)W(ds,du)\bigg]^2\bigg\} \\ & + t\langle\gamma, f\rangle + |b|\int_0^t \mathbf{E}[\langle Y_s^\mu, f\rangle]ds \\ & + 2\mathbf{E}^{\frac{1}{2}}\bigg\{\bigg[\int_0^t\int_0^1\int_0^\infty z g_{s-}^\mu(u)\tilde{N}(ds,dz,du)\bigg]^2\bigg\} \\ & + \mathbf{E}\bigg[\int_0^t\int_1^\infty\int_0^\infty z g_{s-}^\mu(u)N(ds,dz,du)\bigg] \\ =\; & \langle \mu, f\rangle + 2\sigma\mathbf{E}^{\frac{1}{2}}\bigg[\int_0^t ds\int_0^\infty g_s^\mu(u)^2 du\bigg] \\ & + t\langle\gamma, f\rangle + |b|\int_0^t \mathbf{E}[\langle Y_s^\mu, f\rangle]ds \\ & + 2\mathbf{E}^{\frac{1}{2}}\bigg[\int_0^t ds\int_0^1 z^2 m(dz)\int_0^\infty g_s^\mu(u)^2 du\bigg] \\ & + \mathbf{E}\bigg[\int_0^t ds\int_1^\infty z m(dz)\int_0^\infty g_s^\mu(u)du\bigg] \\ \leq\; & \langle \mu, f\rangle + 2\Big(\int_0^t \mathbf{E}[\langle Y_s^\mu, f^2\rangle]ds\Big)^{\frac{1}{2}}\bigg[\sigma + \Big(\int_0^1 z^2 m(dz)\Big)^{\frac{1}{2}}\bigg] \\ & + t\langle\gamma, f\rangle + \int_0^t \mathbf{E}[\langle Y_s^\mu, f\rangle]ds\bigg[|b| + \int_1^\infty z m(dz)\bigg].\end{aligned}$$

In view of (3.17) we get (3.18) for the step function. Now let $\eta(dv) = \mu(dv) + \gamma(dv)$ and choose a bounded sequence of step functions $\{f_n\}$ so that $f_n \to f$

in $L^2(\eta)$ as $n \to \infty$. By applying (3.18) to the non-negative step function $|f_n - f_m|$ we get

$$\mathbf{E}\Big[\sup_{0\le s\le t} \langle Y_s^\mu, |f_n - f_m|\rangle\Big] \ \le\ C(t)\Big[\langle \eta, |f_n - f_m|\rangle + 2\langle \eta, |f_n - f_m|^2\rangle^{1/2}\Big].$$

The right-hand side tends to zero as $m, n \to \infty$. Then there is a càdlàg process $\{Y_t^\mu(f) : t \ge 0\}$ so that

$$\mathbf{E}\Big[\sup_{0\le s\le t} |\langle Y_s^\mu, f_n\rangle - Y_s^\mu(f)|\Big] \to 0, \qquad n \to \infty. \tag{3.19}$$

On the other hand, from (3.17) we have

$$\mathbf{E}[\langle Y_t^\mu, |f_n - f|\rangle] \ =\ \langle \mu, |f_n - f|\rangle e^{-bt} + b^{-1}(1 - e^{-bt})\langle \gamma, |f_n - f|\rangle,$$

which tends to zero as $n \to \infty$. Then $\{Y_t^\mu(f) : t \ge 0\}$ is a modification of $\{\langle Y_t^\mu, f\rangle : t \ge 0\}$. Finally, we get (3.18) for $f \in B[0,a]^+$ by using (3.19) and the result for step functions. $\square$

Theorem 3.11 *The process $\{Y_t^\mu : t \ge 0\}$ is the unique solution of the following martingale problem: For every $G \in C^2(\mathbb{R})$ and $f \in B[0,a]$,*

$$\begin{aligned} G(\langle Y_t^\mu, f\rangle) \ &=\ G(\langle \mu, f\rangle) + \frac{1}{2}\sigma^2 \int_0^t G''(\langle Y_s^\mu, f\rangle)\langle Y_s^\mu, f^2\rangle ds \\ &\quad + \int_0^t G'(\langle Y_s^\mu, f\rangle)[\langle \gamma, f\rangle - b\langle Y_s^\mu, f\rangle] ds \\ &\quad + \int_0^t ds \int_{[0,a]} Y_s^\mu(dx) \int_0^\infty \Big[G(\langle Y_s^\mu, f\rangle + z f(x)) \\ &\qquad\qquad - G(\langle Y_s^\mu, f\rangle) - z f(x) G'(\langle Y_s^\mu, f\rangle)\Big] m(dz) \\ &\quad + \textit{local mart.} \end{aligned} \tag{3.20}$$

Proof. Again we start with the step function (3.13). Using (3.15) and Itô's formula,

$$\begin{aligned} G(\langle Y_t^\mu, f\rangle) \ &=\ G(\langle \mu, f\rangle) + \frac{1}{2}\sigma^2 \int_0^t ds \int_0^\infty G''(\langle Y_{s-}^\mu, f\rangle) g_{s-}^\mu(u)^2 du \\ &\quad + \int_0^t G'(\langle Y_{s-}^\mu, f\rangle)[\langle \gamma, f\rangle - b\langle Y_{s-}^\mu, f\rangle] ds \\ &\quad + \int_0^t ds \int_0^\infty m(dz) \int_0^\infty \Big[G(\langle Y_s^\mu, f\rangle + z g_s^\mu(u)) \\ &\quad - G(\langle Y_s^\mu, f\rangle) - G'(\langle Y_s^\mu, f\rangle) z g_s^\mu(u)\Big] du + \text{local mart.} \\ &=\ G(\langle \mu, f\rangle) + \frac{1}{2}\sigma^2 \int_0^t G''(\langle Y_s^\mu, f\rangle)\langle Y_s^\mu, f^2\rangle ds \end{aligned}$$

$$+\int_0^t G'(\langle Y_s^\mu, f\rangle)[\langle\gamma, f\rangle - b\langle Y_s^\mu, f\rangle]ds$$
$$+\int_0^t ds\int_0^\infty Y_s^\mu(dx)\int_0^\infty \Big[G(\langle Y_s^\mu, f\rangle + zf(x))$$
$$-G(\langle Y_s^\mu, f\rangle) - G'(\langle Y_s^\mu, f\rangle)zf(x)\Big]m(dz) + \text{local mart.}$$

That proves (3.20) for step functions. For $f \in B[0,a]$ we get the martingale problem using (3.19). The uniqueness of the solution follows from a result in Li (2011, Section 9.3). □

The solution of the martingale problem (3.20) is the special case of the *immigration superprocess* studied in Li (2011) with trivial spatial motion. More precisely, the infinitesimal particles propagate in $[0,a]$ without migration. Then for any disjoint bounded Borel subsets B_1 and B_2 of $[0,a]$, the non-negative real-valued processes $\{Y_t^\mu(B_1) : t \geq 0\}$ and $\{Y_t^\mu(B_2) : t \geq 0\}$ are independent. That explains why the restriction of $\{Y_t^\mu : t \geq 0\}$ to the interval $[0,a]$ is still a Markov process. To consider the process on the half line $[0,\infty)$ we need to introduce a weight function as follows.

Let h be a strictly positive continuous function on $[0,\infty)$ vanishing at infinity. Let $M_h[0,\infty)$ be the space of Radon measures μ on $[0,\infty)$ so that $\langle\mu, h\rangle < \infty$. Let $B_h[0,\infty)$ be the set of Borel functions on $[0,\infty)$ bounded by const$\cdot h$ and let $C_h[0,\infty)$ denote its subset of continuous functions. A topology on $M_h[0,\infty)$ can be defined by the convention: $\mu_n \to \mu$ in $M_h[0,\infty)$ if and only if $\langle\mu_n, f\rangle \to \langle\mu, f\rangle$ for every $f \in C_h[0,\infty)$. Suppose that $\mu \in M_h[0,\infty)$ and $\gamma \in M_h[0,\infty)$. It is easy to show that $\{Y_t^\mu : t \geq 0\}$ is a càdlàg strong Markov process in $M_h[0,\infty)$ and the results of Theorem 3.10 and Theorem 3.11 are also true for $B_h[0,\infty)$.

4 Generalized Fleming–Viot flows

In this section we give a construction of the generalized Fleming–Viot flow as the strong solution of a stochastic integral equation. Let $\sigma \geq 0$, $b \geq 0$ and $0 \leq \beta \leq 1$ be constants, and let $z^2\nu(dz)$ be a finite measure on $(0,1]$. Suppose that $\{B(ds,du)\}$ is a white noise on $(0,\infty)^2$ with intensity $dsdu$ and $\{M(ds,dz,du)\}$ is a Poisson random measure on $(0,\infty)\times(0,1]\times(0,\infty)$ with intensity $ds\nu(dz)du$. Let

$$q(x,u) = 1_{\{u\leq 1\wedge x\}} - (1\wedge x), \qquad x \geq 0, u \in (0,1].$$

We first consider the stochastic integral equation

$$X_t \;=\; X_0 + \int_0^t\int_0^1 \sigma q(X_{s-},u)B(ds,du) + \int_0^t b(\beta - X_{s-})ds$$

$$+\int_0^t\int_0^1\int_0^1 zq(X_{s-},u)\tilde{M}(ds,dz,du), \tag{4.1}$$

where $\tilde{M}(ds,dz,du)$ denotes the compensated measure of $M(ds,dz,du)$. In fact, the compensation in (4.1) can be disregarded as

$$\int_0^1 q(X_{s-},u)du = \int_0^1 [1_{\{u\leq X_{s-}\wedge 1\}} - (X_{s-}\wedge 1)]du = 0.$$

Theorem 4.1 *There is a unique non-negative strong solution to (4.1).*

Proof. We first show the pathwise uniqueness for (4.1). Set $l(x,y,u) = q(x,u)-q(y,u)$. For $x,y\geq 0$ and $0\leq z,t\leq 1$ we have

$$\begin{aligned}&(x-y)+ztl(x,y,u)\\&\quad=[(x-1\wedge x)-(y-1\wedge y)]+(1-zt)(1\wedge x-1\wedge y)\\&\qquad+zt(1_{\{u\leq x\wedge 1\}}-1_{\{u\leq y\wedge 1\}}).\end{aligned}$$

It is then easy to see

$$|(x-y)+ztl(x,y,u)|\geq (1-zt)|1\wedge x-1\wedge y|.$$

Moreover, we have

$$\begin{aligned}\int_0^1 l(x,y,u)^2du &= |1\wedge x-1\wedge y|-(1\wedge x-1\wedge y)^2\\&\leq |1\wedge x-1\wedge y|.\end{aligned}$$

Using the above two inequalities,

$$\begin{aligned}&\int_0^1(1-t)dt\int_0^1\nu(dz)\int_0^1\frac{z^2l(x,y,u)^2}{|(x-y)+ztl(x,y,u)|}du\\&\quad\leq\int_0^1 z^2\nu(dz)\int_0^1\frac{1-t}{1-zt}dt\int_0^1\frac{l(x,y,u)^2}{|1\wedge x-1\wedge y|}du\\&\quad\leq\int_0^1 z^2\nu(dz)\int_0^1\frac{1-t}{1-zt}dt\leq\int_0^1 z^2\nu(dz).\end{aligned}$$

Then condition (2.d) is satisfied with $\rho(z)=\sqrt{z}$. Other conditions of Theorem 2.1 can be checked easily. Then we have the pathwise uniqueness for (4.1). To show the existence of the solution, we may assume $X_0 = v\geq 0$ is a deterministic constant. By Theorem 2.5 there a unique non-negative strong solution of (4.1) if the Poisson integral term is removed. Then for each $k\geq 1$ there is a unique non-negative strong solution to

$$Z_t = Z_0+\int_0^t\int_0^1\sigma q(Z_{s-},u)B(ds,du)+\int_0^t b(\beta-Z_{s-})ds$$

$$+\int_0^t\int_{1/k}^1\int_0^1 zq(Z_{s-},u)M(ds,dz,du) \tag{4.2}$$

because the last term on the right-hand side gives at most a finite number of jumps on each bounded time interval. Let $\{Z_k(t): t\geq 0\}$ be the solution of (4.2) with $Z_k(0)=v$. Let $T_1=\inf\{t\geq 0: Z_k(t)\leq 1\}$. On the time interval $[0,T_1]$, the stochastic integral terms in (4.2) vanish. Then $t\mapsto Z_k(t)$ is non-increasing on $[0,T_1]$. By modifying the proof of Proposition 2.1 in Fu and Li (2010) one can see $Z_k(t)\leq 1$ for $t\geq T_1$. Thus $Z_k(t)\leq (Z_k(0)\vee 1)=(v\vee 1)$ for all $t\geq 0$. Let $\{\tau_k\}$ be a bounded sequence of stopping times. Note that the last term on the right-hand side of (4.2) can be considered as a stochastic integral with respect to the compensated Poisson random measure. Then for any $t\geq 0$ we have

$$\begin{aligned}
&\mathbf{E}\Big\{[Z_k(\tau_k+t)-Z_k(\tau_k)]^2\Big\}\\
&\quad\leq 3\sigma^2\mathbf{E}\bigg[\int_0^t ds\int_0^1 q(Z_k(\tau_k+s),u)^2du\bigg]+3b^2t^2(v\vee 1)^2\\
&\qquad\quad+3\mathbf{E}\bigg[\int_0^t ds\int_0^1 z^2\nu(dz)\int_0^1 q(Z_k(\tau_k+s),u)^2du\bigg]\\
&\quad\leq 3t\bigg[\sigma^2+tb^2(v\vee 1)^2+\int_0^1 z^2\nu(dz)\bigg].
\end{aligned}$$

The right-hand side tends to zero as $t\to 0$. By a criterion of Aldous (1978), the sequence $\{Z_k(t): t\geq 0\}$ is tight in $D([0,\infty),\mathbb{R}_+)$; see also Ethier and Kurtz (1986, pages 137-138). By a modification of the proof of Theorem 4.4 in Fu and Li (2010) one sees that any limit point of this sequence is a weak solution of (4.1). □

Now let $v\mapsto\gamma(v)$ be a non-decreasing continuous function on $[0,1]$ so that $0\leq\gamma(v)\leq 1$ for all $0\leq v\leq 1$. We denote by $\gamma(dv)$ the sub-probability measure on $[0,1]$ so that $\gamma([0,v])=\gamma(v)$ for $0\leq v\leq 1$. By Theorem 4.1 for each $v\geq 0$ there is a pathwise unique non-negative solution $\{X_t(v): t\geq 0\}$ to the equation

$$\begin{aligned}
X_t(v) \;=\; & v+\int_0^t\int_0^1 \sigma[1_{\{u\leq X_{s-}(v)\}}-X_{s-}(v)]B(ds,du)\\
&+\int_0^t b[\gamma(v)-X_{s-}(v)]ds\\
&+\int_0^t\int_0^1\int_0^1 z[1_{\{u\leq X_{s-}(v)\}}-X_{s-}(v)]\tilde{M}(ds,dz,du). \qquad (4.3)
\end{aligned}$$

It is not hard to see that $0\leq v\leq 1$ implies $\mathbf{P}\{0\leq X_t(v)\leq 1 \text{ for all } t\geq 0\}=1$. The compensation for the Poisson random measure can be disregarded, so this equation just coincides with (1.6). By Theorem 2.2 for any $0\leq v_1\leq v_2\leq 1$ we have

$$\mathbf{P}\{X_t(v_1)\leq X_t(v_2) \text{ for all } t\geq 0\}=1.$$

Therefore $\{X(v) : 0 \leq v \leq 1\}$ is a non-decreasing path-valued process in $D[0,\infty)$.

Proposition 4.2 *There is a locally bounded non-negative function $t \mapsto C(t)$ on $[0,\infty)$ so that*

$$\begin{aligned}\mathbf{E}\Big\{\sup_{0\leq s\leq t}[X_s(v_2)-X_s(v_1)]\Big\} \;\leq\; & C(t)\Big\{(v_2-v_1)+[\gamma(v_2)-\gamma(v_1)] \\ & +\sqrt{v_2-v_1}+\sqrt{\gamma(v_2)-\gamma(v_1)}\Big\} \qquad (4.4)\end{aligned}$$

for $t \geq 0$ and $0 \leq v_1 \leq v_2 \leq 1$.

Proof. Let $Z_t = X_t(v_2) - X_t(v_1)$ for $t \geq 0$. From (4.3) we have

$$\begin{aligned} Z_t \;=\; & (v_2-v_1)+\int_0^t\int_0^1 \sigma[Y_{s-}(u)-Z_{s-}]B(ds,du) \\ & +\int_0^t b\{[\gamma(v_2)-\gamma(v_1)]-Z_{s-}\}ds \\ & +\int_0^t\int_0^1\int_0^1 z[Y_{s-}(u)-Z_{s-}]\tilde{M}(ds,dz,du), \qquad (4.5)\end{aligned}$$

where $Y_s(u) = 1_{\{X_s(v_1)<u\leq X_s(v_2)\}}$. Taking the expectation in (4.5) and solving a deterministic integral equation one can show

$$\mathbf{E}[Z_t] = (v_2-v_1)e^{-bt} + [\gamma(v_2)-\gamma(v_1)](1-e^{-bt}). \qquad (4.6)$$

By (4.5) and Doob's martingale inequality,

$$\begin{aligned}\mathbf{E}\Big\{\sup_{0\leq s\leq t} Z_s\Big\} \;\leq\; & (v_2-v_1)+2\sigma\mathbf{E}^{\frac{1}{2}}\Big\{\Big(\int_0^t\int_0^1[Y_{s-}(u)-Z_{s-}]B(ds,du)\Big)^2\Big\} \\ & +\int_0^t b\{[\gamma(v_2)-\gamma(v_1)]+\mathbf{E}[Z_s]\}ds \\ & +2\mathbf{E}^{\frac{1}{2}}\Big\{\Big(\int_0^t\int_0^1\int_0^1 z[Y_{s-}(u)-Z_{s-}]\tilde{M}(ds,dz,du)\Big)^2\Big\} \\ =\; & (v_2-v_1)+2\sigma\mathbf{E}^{\frac{1}{2}}\Big\{\int_0^t ds\int_0^1[Y_s(u)-Z_s]^2du\Big\} \\ & +\int_0^t b\{[\gamma(v_2)-\gamma(v_1)]+\mathbf{E}[Z_s]\}ds \\ & +2\mathbf{E}^{\frac{1}{2}}\Big\{\int_0^t ds\int_0^1 z^2\nu(dz)\int_0^1[Y_s(u)-Z_s]^2du\Big\},\end{aligned}$$

where

$$\int_0^1[Y_s(u)-Z_s]^2du = Z_s(1-Z_s) \leq Z_s.$$

Then we have (4.4) by (4.6). $\square$

Recall that $D[0,\infty)$ is the space of non-negative càdlàg functions on $[0,\infty)$ endowed with the Borel σ-algebra generated by the Skorokhod topology. Let ρ be the metric on $D[0,\infty)$ defined by (3.11).

Theorem 4.3 *The path-valued process $\{X(v) : 0 \leq v \leq 1\}$ is a Markov process in $D[0,\infty)$.*

Proof. Let $0 < v < 1$ and let $\tau_n = \inf\{t \geq 0 : X_t(v) \leq 1/n\}$ for $n \geq 1$. In view of (4.3), we have $X_t(v) = 0$ if $X_{t-}(v) = 0$. Then $\tau_n \to \tau_\infty := \inf\{t \geq 0 : X_t(v) = 0\}$ as $n \to \infty$. For any $p \in [0, v)$ the comparison property and pathwise uniqueness for (4.3) imply $X_t(p) = X_t(v)$ for $t \geq \tau_\infty$. Let $Z_n(t) = X_{t\wedge\tau_n}(v)^{-1}X_{t\wedge\tau_n}(p)$ for $t \geq 0$. By (4.3) and Itô's formula,

$$\begin{aligned}
Z_n(t) &= \frac{p}{v} + \int_0^{t\wedge\tau_n}\int_0^1 \frac{\sigma}{X_{s-}(v)}\Big[1_{\{u\leq X_{s-}(p)\}} - X_{s-}(p)\Big]B(ds,du) \\
&\quad - \int_0^{t\wedge\tau_n}\int_0^1 \frac{\sigma X_{s-}(p)}{X_{s-}(v)^2}\Big[1_{\{u\leq X_{s-}(v)\}} - X_{s-}(v)\Big]B(ds,du) \\
&\quad + \int_0^{t\wedge\tau_n} bX_{s-}(v)^{-1}\Big[\gamma(p) - \gamma(v)X_{s-}(v)^{-1}X_{s-}(p)\Big]ds \\
&\quad + \int_0^{t\wedge\tau_n} ds\int_0^1 \frac{\sigma^2 X_{s-}(p)}{X_{s-}(v)^3}\Big[1_{\{u\leq X_{s-}(v)\}} - X_{s-}(v)\Big]^2 du \\
&\quad - \int_0^{t\wedge\tau_n} ds\int_0^1 \frac{\sigma^2}{X_{s-}(v)^2}\Big[1_{\{u\leq X_{s-}(p)\}} - X_{s-}(p)\Big] \\
&\qquad\qquad \cdot\Big[1_{\{u\leq X_{s-}(v)\}} - X_{s-}(v)\Big]du \\
&\quad + \int_0^{t\wedge\tau_n}\int_0^1\int_0^1 \bigg\{\frac{X_{s-}(p)(1-z) + z1_{\{u\leq X_{s-}(p)\}}}{X_{s-}(v)(1-z) + z1_{\{u\leq X_{s-}(v)\}}} \\
&\qquad\qquad - \frac{X_{s-}(p)}{X_{s-}(v)}\bigg\}M(ds,dz,du) \\
&= \frac{p}{v} + \int_0^{t\wedge\tau_n}\int_0^{X_{s-}(v)} \sigma X_{s-}(v)^{-1}\Big[1_{\{u\leq X_{s-}(p)\}} \\
&\qquad\qquad - X_{s-}(v)^{-1}X_{s-}(p)\Big]B(ds,du) \\
&\quad + \int_0^{t\wedge\tau_n} bX_{s-}(v)^{-1}\Big[\gamma(p) - \gamma(v)X_{s-}(v)^{-1}X_{s-}(p)\Big]ds \\
&\quad + \int_0^{t\wedge\tau_n}\int_0^1\int_0^{X_{s-}(v)} \bigg[\frac{X_{s-}(p)(1-z) + z1_{\{u\leq X_{s-}(p)\}}}{X_{s-}(v)(1-z) + z} \\
&\qquad\qquad - \frac{X_{s-}(p)}{X_{s-}(v)}\bigg]M(ds,dz,du),
\end{aligned}$$

where the two terms involving σ^2 counteract each other. Observe also that the last integral does not change if we replace $M(ds,dz,du)$ by the compensated measure $\tilde{M}(ds, dz, du)$. Then we get the equation

$$\begin{aligned} Z_n(t) \;=\; & \frac{p}{v} + \int_0^{t\wedge\tau_n}\int_0^{X_{s-}(v)} \sigma X_{s-}(v)^{-1}\Big[1_{\{u\le X_{s-}(v)Z_n(s-)\}} \\ & \qquad\qquad - Z_n(s-)\Big]B(ds, du) \\ & + \int_0^{t\wedge\tau_n}\int_0^1\int_0^{X_{s-}(v)} z\bigg[\frac{1_{\{u\le X_{s-}(v)Z_n(s-)\}}}{z+(1-z)X_{s-}(v)} \\ & \qquad\qquad - \frac{Z_n(s-)}{z+(1-z)X_{s-}(v)}\bigg]\tilde{M}(ds, dz, du) \\ & + \int_0^{t\wedge\tau_n} bX_{s-}(v)^{-1}[\gamma(p) - \gamma(v)Z_n(s-)]ds. \end{aligned} \tag{4.7}$$

Since $X_{s-}(v) \ge 1/n$ for $0 < s \le \tau_n$, by a simple generalization of Theorem 2.1 one can show the pathwise uniqueness holds for (4.7). Then, setting $Z_t = \lim_{n\to\infty} Z_n(t)$ we have

$$X_t(p) = Z_t X_t(v) 1_{\{t<\tau_\infty\}} + X_t(v)1_{\{t\ge\tau_\infty\}}, \qquad t \ge 0. \tag{4.8}$$

Now from (4.7) and (4.8) we infer that $\{X_t(p) : t \ge 0\}$ is measurable with respect to the σ-algebra $\mathscr{F}_v$ generated by the process $\{X_t(v) : t \ge 0\}$ and the restricted martingale measures

$$1_{\{u\le X_{s-}(v)\}}B(ds, du),\ 1_{\{u\le X_{s-}(v)\}}\tilde{M}(ds, dz, du).$$

By similar arguments, for any $q \in (v, 1]$ one can see $\{1 - X_t(q) : t \ge 0\}$ is measurable with respect to the σ-algebra $\mathscr{G}_v$ generated by the process $\{1 - X_t(v) : t \ge 0\}$ and the restricted martingale measures

$$1_{\{X_{s-}(v)<u\le 1\}}B(ds, du),\ 1_{\{X_{s-}(v)<u\le 1\}}\tilde{M}(ds, dz, du).$$

Observe that $\{B(ds, X_{s-}(v) + du)\}$ is a white noise with intensity $dsdu$ and $\{M(ds, dz, X_{s-}(v)+du)\}$ is a Poisson random measure with intensity $ds\nu(dz)du$. Then, given $\{X_t(v) : t \ge 0\}$ the σ-algebras $\mathscr{F}_v$ and $\mathscr{G}_v$ are conditionally independent. That implies the Markov property of $\{(X(v), \mathscr{F}_v) : 0 \le v \le 1\}$. □

Theorem 4.4 *The path-valued Markov process $\{X(v) : 0 \le v \le 1\}$ has a ρ-càdlàg modification. Consequently, there is a version of the solution flow $\{X_t(v) : t \ge 0, 0 \le v \le 1\}$ of (4.3) with the following properties:*

(i) for each $v \in [0, 1]$, $t \mapsto X_t(v)$ is càdlàg on $[0,\infty)$ and solves (4.3);

(ii) for each $t \ge 0$, $v \mapsto X_t(v)$ is non-decreasing and càdlàg on $[0, 1]$ with $X_t(0) \ge 0$ and $X_t(1) \le 1$.

Proof. This follows from Lemma 3.5 and Proposition 4.2 by arguments as in the proof of Theorem 3.6. $\square$

We call the solution flow $\{X_t(v) : t \geq 0, v \in [0,1]\}$ of (4.3) specified in Theorem 4.4 a *generalized Fleming–Viot flow* following Bertoin and Le Gall (2003, 2005, 2006). The law of the flow is determined by the parameters (σ, b, γ, ν).

Let $F[0,1]$ be the set of non-decreasing càdlàg functions f on $[0,1]$ such that $0 \leq f(0) \leq f(1) \leq 1$. Given a finite stopping time τ and a function $\mu \in F[0,1]$, let $\{X^\mu_{\tau,t}(v) : t \geq 0\}$ be the solution of

$$\begin{aligned} X^\mu_{\tau,t}(v) \;&=\; \mu(v) + \int_\tau^{\tau+t} \int_0^1 \sigma[1_{\{u \leq X^\mu_{\tau,s-}(v)\}} - X^\mu_{\tau,s-}(v)] B(ds, du) \\ &\quad + \int_\tau^{\tau+t} b[\gamma(v) - X^\mu_{\tau,s-}(v)] ds \\ &\quad + \int_\tau^{\tau+t} \int_0^1 \int_0^1 z[1_{\{u \leq X^\mu_{\tau,s-}(v)\}} - X^\mu_{\tau,s-}(v)] \tilde{M}(ds, dz, du) \end{aligned} \tag{4.9}$$

and write simply $\{X^\mu_t(v) : t \geq 0\}$ instead of $\{X^\mu_{0,t}(v) : t \geq 0\}$. The pathwise uniqueness for the above equation follows from that of (4.3). Let $F_{\tau,t}$ be the random operator on $F[0,1]$ that maps μ to $X^\mu_{\tau,t}$. As for the flow of CBI-processes we have

Theorem 4.5 *For any finite stopping time τ we have $\mathbf{P}\{X^\mu_{\tau+t} = F_{\tau,t} X^\mu_t$ for all $t \geq 0\} = 1$.*

For any sub-probability measure $\mu(dv)$ on $[0,1]$ with distribution function $v \mapsto \mu(v)$ we write $X^\mu_t(dv)$ for the random sub-probability measure on $[0,1]$ determined by the random function $v \mapsto X^\mu_t(v)$. We call $\{X^\mu_t : t \geq 0\}$ the *generalized Fleming–Viot process* associated with the flow $\{X^\mu_t(v) : t \geq 0, v \in [0,1]\}$. The reader may refer to Dawson (1993) and Ethier and Kurtz (1993) for the theory of classical Fleming–Viot processes. To give some characterizations of the generalized Fleming–Viot process, let us consider the step function

$$f(u) = c_0 1_{\{0\}}(u) + \sum_{i=1}^n c_i 1_{(a_{i-1}, a_i]}(u), \qquad u \in [0,1], \tag{4.10}$$

where $\{c_0, c_1, \cdots, c_n\} \subset \mathbb{R}$ and $\{0 = a_0 < a_1 < \cdots < a_n = 1\}$ is a partition of $[0,1]$. For this function we have

$$\langle X^\mu_t, f\rangle = c_0 X^\mu_t(0) + \sum_{i=1}^n c_i [X^\mu_t(a_i) - X^\mu_t(a_{i-1})]. \tag{4.11}$$

By (4.9) and (4.11) we have

$$\langle X^\mu_t, f\rangle \;=\; \langle \mu, f\rangle + \int_0^t \int_0^1 \sigma[g^\mu_{s-}(u) - \langle X^\mu_{s-}, f\rangle] B(ds, du)$$

$$
\begin{aligned}
&+\int_0^t b[\langle\gamma, f\rangle - \langle X_{s-}^\mu, f\rangle]ds \\
&+\int_0^t\int_0^1\int_0^1 z[g_{s-}^\mu(u) - \langle X_{s-}^\mu, f\rangle]\tilde{M}(ds,dz,du), \qquad (4.12)
\end{aligned}
$$

where

$$
g_s^\mu(u) = c_0 1_{\{u\le X_s^\mu(0)\}} + \sum_{i=1}^n c_i 1_{\{X_s^\mu(a_{i-1})<u\le X_s^\mu(a_i)\}}. \tag{4.13}
$$

The proofs of the following three results are similar to those for CBI-processes.

Theorem 4.6 *The generalized Fleming–Viot process $\{X_t^\mu : t\ge 0\}$ defined above is an almost surely càdlàg strong Markov process with $X_0^\mu = \mu$.*

Proposition 4.7 *For any $t\ge 0$ and $f\in B[0,1]$ we have*

$$
\mathbf{E}[\langle X_t^\mu, f\rangle] = \langle\mu, f\rangle e^{-bt} + \langle\gamma, f\rangle(1-e^{-bt}). \tag{4.14}
$$

Theorem 4.8 *For any $f\in B[0,1]$ the process $\{\langle X_t^\mu, f\rangle : t\ge 0\}$ has a càdlàg modification. Moreover, there is a locally bounded function $t\mapsto C(t)$ so that*

$$
\begin{aligned}
\mathbf{E}\Big[\sup_{0\le s\le t}\langle X_s^\mu, f\rangle\Big] \;\le\; C(t)\Big[&\langle\mu, f\rangle + \langle\gamma, f\rangle \\
&+\langle\mu, f^2\rangle^{1/2} + \langle\gamma, f^2\rangle^{1/2}\Big] \qquad (4.15)
\end{aligned}
$$

for any $t\ge 0$ and $f\in B[0,1]^+$.

The generalized Fleming–Viot process can be characterized in terms of a martingale problem. Given any finite family $\{f_1,\cdots,f_p\}\subset B[0,1]$, write

$$
G_{p,\{f_i\}}(\eta) = \prod_{i=1}^p\langle\eta, f_i\rangle, \qquad \eta\in M_1[0,1]. \tag{4.16}
$$

Let $\mathscr{D}_1(L)$ be the linear span of the functions on $M_1[0,1]$ of the form (4.16) and let L be the linear operator on $\mathscr{D}_1(L)$ defined by

$$
\begin{aligned}
LG_{p,\{f_i\}}(\eta) \;=\;& \sigma^2\sum_{i<j}\Big[\langle\eta, f_if_j\rangle\prod_{k\ne i,j}\langle\eta, f_k\rangle - \prod_{k=1}^p\langle\eta, f_k\rangle\Big] \\
&+\sum_{I\subset\{1,\cdots,p\},|I|\ge 2}\beta_{p,|I|}\Big[\Big\langle\eta, \prod_{i\in I} f_i\Big\rangle\prod_{j\notin I}\langle\eta, f_j\rangle - \prod_{k=1}^p\langle\eta, f_k\rangle\Big] \\
&+b\sum_{i=1}^p\Big[\langle\gamma, f_i\rangle\prod_{k\ne i}\langle\eta, f_k\rangle - \prod_{k=1}^p\langle\eta, f_k\rangle\Big], \qquad (4.17)
\end{aligned}
$$

where $|I|$ denotes the cardinality of $I\subset\{1,\cdots,p\}$ and

$$
\beta_{p,|I|} = \int_0^1 z^{|I|}(1-z)^{p-|I|}\nu(dz).
$$

Theorem 4.9 *The generalized Fleming–Viot process $\{X_t^\mu : t \geq 0\}$ is the unique solution of the following martingale problem: For any $p \geq 1$ and $\{f_1, \cdots, f_p\} \subset B[0,1]$,*

$$G_{p,\{f_i\}}(X_t^\mu) = G_{p,\{f_i\}}(\mu) + \int_0^t LG_{p,\{f_i\}}(X_s^\mu)ds + mart. \tag{4.18}$$

Proof. We first consider a collection of step functions $\{f_1, \cdots, f_p\}$. Let $g_i^\mu(s,u)$ be defined by (4.13) with $f = f_i$. Since the compensation of the Poisson random measure in (4.12) can be disregarded, by Itô's formula we get

$$\begin{aligned}
&G_{p,\{f_i\}}(X_t^\mu)\\
&\quad = G_{p,\{f_i\}}(\mu) + \sigma^2 \int_0^t ds \int_0^1 \Big[\sum_{i<j} h_i^\mu(s,u)h_j^\mu(s,u) \prod_{k\neq i,j} \langle X_s^\mu, f_k\rangle\Big] du\\
&\qquad + \int_0^t ds \int_0^1 \nu(dz) \int_0^1 \Big\{ \prod_{k=1}^p [\langle X_s^\mu, f_k\rangle + z h_k^\mu(s,u)] - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle \Big\} du\\
&\qquad + b\int_0^t \sum_{i=1}^p [\langle \gamma, f_i\rangle - \langle X_s^\mu, f_i\rangle] \prod_{k\neq i} \langle X_s^\mu, f_k\rangle ds + \mathrm{mart.}\\
&\quad = G_{p,\{f_i\}}(\mu) + \sigma^2 \int_0^t ds \int_0^1 \Big[\sum_{i<j} l_i^\mu(u)l_j^\mu(u) \prod_{k\neq i,j} \langle X_s^\mu, f_k\rangle\Big] X_s^\mu(du)\\
&\qquad + \int_0^t ds \int_0^1 \nu(dz) \int_0^1 \Big\{ \prod_{k=1}^p [\langle X_s^\mu, f_k\rangle + z l_k^\mu(u)] - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle \Big\} X_s^\mu(du)\\
&\qquad + b\int_0^t \sum_{i=1}^p \Big[\langle \gamma, f_i\rangle \prod_{k\neq i} \langle X_s^\mu, f_k\rangle - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle\Big] ds + \mathrm{mart.},
\end{aligned}$$

where $h_i^\mu(s,u) = g_i^\mu(s,u) - \langle X_s^\mu, f_i\rangle$ and $l_i^\mu(u) = f_i(u) - \langle X_s^\mu, f_i\rangle$. It is simple to show

$$\int_0^1 l_i^\mu(u) l_j^\mu(u) X_s^\mu(du) = \langle X_s^\mu, f_i f_j\rangle - \langle X_s^\mu, f_i\rangle\langle X_s^\mu, f_j\rangle.$$

Then we continue with

$$\begin{aligned}
&G_{p,\{f_i\}}(X_t^\mu)\\
&\quad = G_{p,\{f_i\}}(\mu) + \sigma^2 \int_0^t \sum_{i<j} \Big[\langle X_s^\mu, f_i f_j\rangle \prod_{k\neq i,j} \langle X_s^\mu, f_k\rangle - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle\Big] ds\\
&\qquad + \int_0^t ds \int_0^1 \nu(dz) \int_0^1 \Big\{ \prod_{k=1}^p [(1-z)\langle X_s^\mu, f_k\rangle + z f_k(u)]\\
&\qquad\qquad - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle \Big\} X_s^\mu(du)
\end{aligned}$$

$$
\begin{aligned}
&+ b\int_0^t \sum_{i=1}^p \Big[\langle\gamma, f_i\rangle \prod_{k\neq i}\langle X_s^\mu, f_k\rangle - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle\Big] ds + \text{mart.}\\
&= G_{p,\{f_i\}}(\mu) + \sigma^2\int_0^t \sum_{i<j}\Big[\langle X_s^\mu, f_if_j\rangle \prod_{k\neq i,j}\langle X_s^\mu, f_k\rangle - \prod_{k=1}^p\langle X_s^\mu, f_k\rangle\Big] ds\\
&\quad + \int_0^t ds\int_0^1 \nu(dz)\int_0^1 \Big\{\sum_{I\subset\{1,\cdots,p\}} z^{|I|}(1-z)^{p-|I|}\prod_{i\in I} f_i(u)\prod_{j\notin I}\langle X_s^\mu, f_j\rangle\\
&\qquad\qquad - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle\Big\} X_s^\mu(du)\\
&\quad + b\int_0^t \sum_{i=1}^p \Big[\langle\gamma, f_i\rangle \prod_{k\neq i}\langle X_s^\mu, f_k\rangle - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle\Big] ds + \text{mart.}\\
&= G_{p,\{f_i\}}(\mu) + \sigma^2\int_0^t \sum_{i<j}\Big[\langle X_s^\mu, f_if_j\rangle \prod_{k\neq i,j}\langle X_s^\mu, f_k\rangle - \prod_{k=1}^p\langle X_s^\mu, f_k\rangle\Big] ds\\
&\quad + \int_0^t ds\int_0^1 \nu(dz)\int_0^1 \Big\{\sum_{I\subset\{1,\cdots,p\}} z^{|I|}(1-z)^{p-|I|}\Big[\prod_{i\in I} f_i(u)\prod_{j\notin I}\langle X_s^\mu, f_j\rangle\\
&\qquad\qquad - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle\Big]\Big\} X_s^\mu(du)\\
&\quad + b\int_0^t \sum_{i=1}^p \Big[\langle\gamma, f_i\rangle \prod_{k\neq i}\langle X_s^\mu, f_k\rangle - \prod_{k=1}^p \langle X_s^\mu, f_k\rangle\Big] ds + \text{mart.}
\end{aligned}
$$

That gives (4.18) for step functions $\{f_1,\cdots,f_p\}$. For $\{f_1,\cdots,f_p\}\subset B[0,1]$ one can show (4.18) by approximating the functions in the space $L^2(\mu+\gamma)$ using bounded sequences of step functions. Since $\{X_t^\mu : t\geq 0\}$ is a Markov process and $\mathscr{D}_1(L)$ separates probability measures on $M[0,1]$, the uniqueness for the martingale problem holds; see Ethier and Kurtz (1986, page 182). $\square$

In particular, if $\mu(1)=\gamma(1)=1$, we have $X_t^\mu(1)=1$ for all $t\geq 0$ and the corresponding generalized Fleming–Viot process $\{X_t^\mu : t\geq 0\}$ is a probability-valued Markov process with generator L defined by

$$
\begin{aligned}
LG_{p,\{f_i\}}(\eta) \;=\; &\sigma^2\sum_{i<j}\Big[\langle\eta, f_if_j\rangle\prod_{k\neq i,j}\langle\eta, f_k\rangle - \prod_{k=1}^p\langle\eta, f_k\rangle\Big]\\
&+ \sum_{I\subset\{1,\cdots,p\},|I|\geq 2}\beta_{p,|I|}\Big[\Big\langle\eta, \prod_{i\in I} f_i\Big\rangle\prod_{j\notin I}\langle\eta, f_j\rangle - \prod_{k=1}^p\langle\eta, f_k\rangle\Big]\\
&+ \sum_{i=1}^p\langle\eta, Af_i\rangle\prod_{k\neq i}\langle\eta, f_k\rangle, \qquad (4.19)
\end{aligned}
$$

where

$$
Af(x) = b\int_{[0,1]}[f(y)-f(x)]\gamma(dy), \qquad x\in[0,1].
$$

That is a generalization of a classical Fleming–Viot process; see, e.g., Ethier and Kurtz (1993, page 351). On the other hand, for $b = 0$ the solution flow $\{X_t^\mu(v) : t \geq 0, 0 \leq v \leq 1\}$ of (4.3) corresponds to the Λ-coalescent process with $\Lambda(dz) = \sigma^2\delta_0 + z^2\nu(dz)$, which is clear from (4.18) and the martingale problem given by Theorem 1 in Bertoin and Le Gall (2005). For $b > 0$ it seems the flow determines a coalescent process with a spatial structure. A serious exploration in the subject would be of interest to the understanding of the related dynamic systems.

5 Scaling limit theorems

In this section, we prove some limit theorems for the generalized Fleming–Viot flows. We shall present the results in the setting of measure-valued processes and through the use of Markov process arguments. These are different from the approach of Bertoin and Le Gall (2006), who used the analysis of characteristics of semimartingales. For each $k \geq 1$ let $\sigma_k \geq 0$ and $b_k \geq 0$ be two constants, let $z^2\nu_k(dz)$ be a finite measure on $(0, 1]$, and let $v \mapsto \gamma_k(v)$ be a non-decreasing continuous function on $[0, 1]$ so that $0 \leq \gamma_k(v) \leq 1$ for all $0 \leq v \leq 1$. We denote by $\gamma_k(dv)$ the sub-probability measure on $[0, 1]$ so that $\gamma_k([0, v]) = \gamma_k(v)$ for $0 \leq v \leq 1$. Let $\{X_t^k(v) : t \geq 0, v \in [0, 1]\}$ be a generalized Fleming–Viot flow with parameters $(\sigma_k, b_k, \gamma_k, \nu_k)$ and with $X_0^k(v) = v$ for $v \in [0, 1]$. Let $Y_k(t, v) = kX_{kt}^k(k^{-1}v)$ for $t \geq 0$ and $v \in [0, k]$. Let $\eta_k(z) = k\gamma_k(k^{-1}z)$ and $m_k(dz) = \nu_k(k^{-1}dz)$ for $z \in (0, k]$. In view of (4.3), we can also define $\{Y_k(t, v) : t \geq 0, v \in [0, k]\}$ directly by

$$\begin{aligned} Y_k(t, v) \;=\;& v + k\sigma_k \int_0^t \int_0^k [1_{\{u \leq Y_k(s-,v)\}} - k^{-1}Y_k(s-, v)]W_k(ds, du) \\ & + kb_k \int_0^t [\eta_k(v) - Y_k(s-, v)]ds \\ & + \int_0^t \int_0^k \int_0^k z[1_{\{u \leq Y_k(s-,v)\}} - k^{-1}Y_k(s-, v)]\tilde{N}_k(ds, dz, du), \end{aligned} \tag{5.1}$$

where $\{W_k(ds, du)\}$ is a white noise on $(0, \infty) \times (0, k]$ with intensity $dsdu$ and $\{N_k(ds, dz, du)\}$ is a Poisson random measure on $(0, \infty) \times (0, k]^2$ with intensity $dsm_k(dz)du$. In the sequel, we assume $k \geq a$ for fixed a constant $a \geq 0$. Then the rescaled flow $\{Y_k(t, v) : t \geq 0, v \in [0, k]\}$ induces an $M[0, a]$-valued process $\{Y_k^a(t) : t \geq 0\}$. We are interested in the asymptotic behavior of $\{Y_k^a(t) : t \geq 0\}$ as $k \to \infty$. Recall that λ denotes the Lebesgue measure on $[0, \infty)$.

Lemma 5.1 *For any $G \in C^2(\mathbb{R})$ and $f \in C[0, a]$ we have*

$$G(\langle Y_k^a(t), f\rangle) \;=\; G(\langle \lambda, f\rangle) + kb_k \int_0^t G'(\langle Y_k^a(s), f\rangle)\langle \eta_k, f\rangle ds$$

$$
\begin{aligned}
& - kb_k \int_0^t G'(\langle Y_k^a(s), f\rangle)\langle Y_k^a(s), f\rangle ds \\
& + \frac{1}{2}k^2\sigma_k^2 \int_0^t G''(\langle Y_k^a(s), f\rangle)\langle Y_k^a(s), f^2\rangle ds \\
& - \frac{1}{2}k\sigma_k^2 \int_0^t G''(\langle Y_k^a(s), f\rangle)\langle Y_k^a(s), f\rangle^2 ds \\
& + \int_0^t ds \int_0^k m_k(dz) \int_{[0,a]} \Big\{ G(\langle Y_k^a(s), f\rangle + zf(x)) \\
& \qquad - G(\langle Y_k^a(s), f\rangle) - G'(\langle Y_k^a(s), f\rangle) z f(x) \Big\} Y_k^a(s, dx) \\
& + \int_0^t ds \int_0^k [\epsilon_k(s,z) + \xi_k(s,z)] m_k(dz) + \textit{local mart.},
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon_k(s,z) \;=\; & \int_0^k \Big\{ G(\langle Y_k^a(s), f\rangle + z[f(x) - k^{-1}\langle Y_k^a(s), f\rangle]) \\
& - G(\langle Y_k^a(s), f\rangle + zf(x)) \\
& - k^{-1} G'(\langle Y_k^a(s), f\rangle) z \langle Y_k^a(s), f\rangle \Big\} Y_k^a(s, dx)
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_k(s,z) \;=\; & [k - Y_k(s,a)] \Big[G(\langle Y_k^a(s), f\rangle - k^{-1} z \langle Y_k^a(s), f\rangle) \\
& - G(\langle Y_k^a(s), f\rangle) + k^{-1} G'(\langle Y_k^a(s), f\rangle) z \langle Y_k^a(s), f\rangle \Big].
\end{aligned}
$$

Proof. For the step function defined by (3.13) we get from (5.1) that

$$
\begin{aligned}
\langle Y_k^a(t), f\rangle \;=\; & \langle \lambda, f\rangle + k\sigma_k \int_0^t \int_0^k h_k(s-, u) W_k(ds, du) \\
& + kb_k \int_0^t [\langle \eta_k, f\rangle - \langle Y_k^a(s-), f\rangle] ds \\
& + \int_0^t \int_0^k \int_0^k z h_k(s-, u) \tilde{N}_k(ds, dz, du), \qquad (5.2)
\end{aligned}
$$

where $h_k(s,u) = g_k(s,u) - k^{-1}\langle Y_k^a(s), f\rangle$ and

$$
g_k(s,u) = c_0 1_{\{u \le Y_k(s,0)\}} + \sum_{i=1}^n c_i 1_{\{Y_k(s,a_{i-1}) < u \le Y_k(s,a_i)\}}. \qquad (5.3)
$$

Let $l_k(s,x) = f(x) - k^{-1}\langle Y_k^a(s), f\rangle$. By (5.2) and Itô's formula,

$$
\begin{aligned}
G(\langle Y_k^a(t), f\rangle) \;=\; & G(\langle \lambda, f\rangle) + kb_k \int_0^t G'(\langle Y_k^a(s), f\rangle)[\langle \eta_k, f\rangle - \langle Y_k^a(s), f\rangle] ds \\
& + \frac{1}{2}k^2\sigma_k^2 \int_0^t G''(\langle Y_k^a(s), f\rangle) ds \int_0^k h_k(s,u)^2 du
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t ds\int_0^k m_k(dz)\int_0^k \Big\{G(\langle Y_k^a(s),f\rangle + zh_k(s,u)) \\
&\qquad - G(\langle Y_k^a(s),f\rangle) - G'(\langle Y_k^a(s),f\rangle)zh_k(s,u)\Big\}du \\
&+\text{local mart.} \\
= \;& G(\langle\lambda,f\rangle) + kb_k\int_0^t G'(\langle Y_k^a(s),f\rangle)[\langle\eta_k,f\rangle - \langle Y_k^a(s),f\rangle]ds \\
&+\frac{1}{2}k^2\sigma_k^2\int_0^t G''(\langle Y_k^a(s),f\rangle)[\langle Y_k^a(s),f^2\rangle - k^{-1}\langle Y_k^a(s),f\rangle]ds \\
&+\int_0^t ds\int_0^k m_k(dz)\int_{[0,a]}\Big\{G(\langle Y_k^a(s),f\rangle + zl_k(s,x)) \\
&\qquad - G(\langle Y_k^a(s),f\rangle) - G'(\langle Y_k^a(s),f\rangle)zl_k(s,x)\Big\}Y_k^a(s,dx) \\
&+\int_0^t [k - Y_k(s,a)]ds\int_0^k \Big\{G(\langle Y_k^a(s),f\rangle - k^{-1}z\langle Y_k^a(s),f\rangle) \\
&\qquad - G(\langle Y_k^a(s),f\rangle) + k^{-1}G'(\langle Y_k^a(s),f\rangle)z\langle Y_k^a(s),f\rangle\Big\}m_k(dz) \\
&+\text{local mart.}
\end{aligned}
$$

That gives the desired result for the step function. For $f\in C[0,a]$ it follows by approximating the function by a sequence of step functions. □

Lemma 5.2 *For $t\geq 0$ and $f\in C[0,a]^+$ we have*

$$
\begin{aligned}
&\mathbf{E}\Big[\sup_{0\leq s\leq t}\langle Y_k^a(s),f\rangle\Big] \\
&\quad\leq \langle\lambda,f\rangle + kb_k\langle\eta_k,f\rangle t + 4t\Big[\langle\lambda,f\rangle + \langle\eta_k,f\rangle\Big]\int_1^k zm_k(dz) \\
&\qquad + 2\sqrt{t}\Big[\langle\lambda,f^2\rangle + \langle\eta_k,f^2\rangle\Big]^{\frac{1}{2}}\left[\sigma + \left(\int_0^1 z^2m_k(dz)\right)^{\frac{1}{2}}\right].
\end{aligned}
$$

Proof. We first consider a non-negative step function given by (3.13) with $\{c_0,c_1,\cdots,c_n\}\subset\mathbb{R}_+$. Let $g_k(s,u)$ and $h_k(s,u)$ be defined as in the proof of Lemma 5.1. By (5.2) and Doob's martingale inequality we get

$$
\begin{aligned}
&\mathbf{E}\Big[\sup_{0\leq s\leq t}\langle Y_k^a(s),f\rangle\Big] \\
&\quad\leq \langle\lambda,f\rangle + 2k\sigma_k\mathbf{E}^{\frac{1}{2}}\left\{\left[\int_0^t\int_0^k h_k(s-,u)W(ds,du)\right]^2\right\} \\
&\qquad + kb_k\langle\eta_k,f\rangle t + \mathbf{E}\left[\int_0^t ds\int_1^k zm_k(dz)\int_0^k |h_k(s-,u)|du\right] \\
&\qquad + \mathbf{E}\left[\int_0^t\int_1^k\int_0^k z|h_k(s-,u)|N_k(ds,dz,du)\right] \\
&\qquad + 2\mathbf{E}^{\frac{1}{2}}\left\{\left[\int_0^t\int_0^1\int_0^k zh_k(s-,u)\tilde{N}_k(ds,dz,du)\right]^2\right\}.
\end{aligned}
$$

It then follows that

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{0\le s\le t}\langle Y_k^a(s),f\rangle\Big] \\
&\quad\le \langle\lambda,f\rangle + 2k\sigma_k\mathbf{E}^{\frac{1}{2}}\Big\{\int_0^t ds\int_0^k h_k(s,u)^2du\Big\} \\
&\qquad + kb_k\langle\eta_k,f\rangle t + 2\mathbf{E}\Big\{\int_0^t ds\int_1^k zm_k(dz)\int_0^k |h_k(s,u)|du\Big\} \\
&\qquad + 2\mathbf{E}^{\frac{1}{2}}\Big\{\int_0^t ds\int_0^1 z^2m_k(dz)\int_0^k h_k(s,u)^2du\Big\} \\
&\quad\le \langle\lambda,f\rangle + kb_k\langle\eta_k,f\rangle t + 4\mathbf{E}\Big[\int_0^t\langle Y_k^a(s),f\rangle ds\int_1^k zm_k(dz)\Big] \\
&\qquad + 2\mathbf{E}^{\frac{1}{2}}\Big[\int_0^t\langle Y_k^a(s),f^2\rangle ds\Big]\Big[k\sigma_k + \Big(\int_0^1 z^2m_k(dz)\Big)^{\frac{1}{2}}\Big].
\end{aligned}$$

By Proposition 4.7 one can see

$$\mathbf{E}[\langle Y_k^a(t),f\rangle] = \langle\lambda,f\rangle e^{-kb_kt} + \langle\eta_k,f\rangle(1-e^{-kb_kt}) \le \langle\lambda,f\rangle + \langle\eta_k,f\rangle.$$

Then we have the desired inequality for the step function. The inequality for $f\in C[0,a]^+$ follows by approximating this function with a bounded sequence of positive step functions. $\square$

Lemma 5.3 *Let τ_k be a bounded stopping time for $\{Y_k^a(t): t\ge 0\}$. Then for any $t\ge 0$ and $f\in C[0,a]$ we have*

$$\begin{aligned}
&\mathbf{E}\Big\{|\langle Y_k^a(\tau_k+t),f\rangle - \langle Y_k^a(\tau_k),f\rangle|\Big\} \\
&\quad\le \mathbf{E}^{\frac{1}{2}}\Big[\int_0^t\langle Y_k^a(\tau_k+s),f^2\rangle ds\Big]\Big[k\sigma_k + \Big(\int_0^1 z^2m_k(dz)\Big)^{\frac{1}{2}}\Big] \\
&\qquad + kb_k\mathbf{E}\Big[\int_0^t(\langle\eta_k,|f|\rangle + \langle Y_k^a(\tau_k+s),|f|\rangle)ds\Big] \\
&\qquad + 4\mathbf{E}\Big[\int_0^t\langle Y_k^a(\tau_k+s),|f|\rangle ds\int_1^k zm_k(dz)\Big].
\end{aligned}\tag{5.4}$$

Proof. We first consider the step function given by (3.13). Let $g_k(s,u)$ and $h_k(s,u)$ be defined as in the proof of Lemma 5.1. From (5.2) we have

$$\begin{aligned}
&\mathbf{E}\Big\{|\langle Y_k^a(\tau_k+t),f\rangle - \langle Y_k^a(\tau_k),f\rangle|\Big\} \\
&\quad\le k\sigma_k\mathbf{E}^{\frac{1}{2}}\Big\{\Big[\int_0^t\int_0^k h_k(\tau_k+s-,u)W(\tau_k+ds,du)\Big]^2\Big\} \\
&\qquad + kb_k\mathbf{E}\Big[\int_0^t|\langle\eta_k,f\rangle - \langle Y_k^a(\tau_k+s-),f\rangle|ds\Big] \\
&\qquad + \mathbf{E}^{\frac{1}{2}}\Big\{\Big[\int_0^t\int_0^1\int_0^k zh_k(\tau_k+s-,u)\tilde{N}_k(\tau_k+ds,dz,du)\Big]^2\Big\}
\end{aligned}$$

$$+\mathbf{E}\Big[\int_0^t\int_1^k\int_0^k z|h_k(\tau_k+s-,u)|N_k(\tau_k+ds,dz,du)\Big]$$
$$+\mathbf{E}\Big[\int_0^t ds\int_1^k zm_k(dz)\int_0^k|h_k(\tau_k+s-,u)|du\Big].$$

By the property of independent increments of the white noise and the Poisson random measure,

$$\begin{aligned}
&\mathbf{E}\Big\{|\langle Y_k^a(\tau_k+t),f\rangle-\langle Y_k^a(\tau_k),f\rangle|\Big\}\\
&\quad\le k\sigma_k\mathbf{E}^{\frac12}\Big\{\int_0^t ds\int_0^k h_k(\tau_k+s,u)^2du\Big\}\\
&\qquad+kb_k\mathbf{E}\Big[\int_0^t(\langle\eta_k,|f|\rangle+\langle Y_k^a(\tau_k+s),|f|\rangle)ds\Big]\\
&\qquad+\mathbf{E}^{\frac12}\Big\{\int_0^t ds\int_0^1 z^2m_k(dz)\int_0^k h_k(\tau_k+s,u)^2du\Big\}\\
&\qquad+2\mathbf{E}\Big[\int_0^t ds\int_1^k zm_k(dz)\int_0^k|h_k(\tau_k+s,u)|du\Big]\\
&\quad\le\mathbf{E}^{\frac12}\Big[\int_0^t\langle Y_k^a(\tau_k+s),f^2\rangle ds\Big]\Big[k\sigma_k+\Big(\int_0^1 z^2m_k(dz)\Big)^{\frac12}\Big]\\
&\qquad+kb_k\mathbf{E}\Big[\int_0^t(\langle\eta_k,|f|\rangle+\langle Y_k^a(\tau_k+s),|f|\rangle)ds\Big]\\
&\qquad+4\mathbf{E}\Big[\int_0^t\langle Y_k^a(\tau_k+s),|f|\rangle ds\int_1^k zm_k(dz)\Big].
\end{aligned}$$

Then (5.4) holds for the step function. For $f\in C[0,a]$ the inequality follows by an approximation argument. □

Lemma 5.4 *Suppose that $kb_k\to b$, $\eta_k\to\eta$ weakly on $[0,a]$ and $k^2\sigma_k^2\delta_0(dz)+(z\wedge z^2)m_k(dz)$ converges weakly on $[0,\infty)$ to a finite measure $\sigma^2\delta_0(dz)+(z\wedge z^2)m(dz)$ as $k\to\infty$. Let $\{0\le a_1<\cdots<a_n\}$ be an ordered set of constants. Then $\{(Y_k^{a_1}(t),\cdots,Y_k^{a_n}(t)):t\ge0\}$, $k=1,2,\cdots$ is a tight sequence in $D([0,\infty),M[0,a_1]\times\cdots\times M[0,a_n])$.*

Proof. Let τ_k be a bounded stopping time for $\{Y_k^a(t):t\ge0\}$ and assume the sequence $\{\tau_k:k=1,2,\cdots\}$ is uniformly bounded. Let $f_i\in C[0,a_i]$ for $i=1,\cdots,n$. By (5.4) we see

$$\begin{aligned}
&\mathbf{E}\Big\{\sum_{i=1}^n|\langle Y_k^{a_i}(\tau_k+t),f_i\rangle-\langle Y_k^{a_i}(\tau_k),f_i\rangle|\Big\}\\
&\quad\le\sum_{i=1}^n\mathbf{E}^{\frac12}\Big[\int_0^t\langle Y_k^{a_i}(\tau_k+s),f_i^2\rangle ds\Big]\Big[k\sigma_k+\Big(\int_0^1 z^2m_k(dz)\Big)^{\frac12}\Big]\\
&\qquad+kb_k\sum_{i=1}^n\mathbf{E}\Big[\int_0^t(\langle\eta_k,|f_i|\rangle+\langle Y_k^{a_i}(\tau_k+s),|f_i|\rangle)ds\Big]
\end{aligned}$$

$$+4\sum_{i=1}^n \mathbf{E}\Big[\int_0^t \langle Y_k^{a_i}(\tau_k+s),|f_i|\rangle ds\int_1^k zm_k(dz)\Big]. \tag{5.5}$$

Then the inequality in Lemma 5.2 implies

$$\lim_{t\to 0}\sup_{k\geq 1}\mathbf{E}\Big\{\sum_{i=1}^n |\langle Y_k^{a_i}(\tau_k+t),f_i\rangle - \langle Y_k^{a_i}(\tau_k),f_i\rangle|\Big\}=0.$$

By a criterion of Aldous (1978), the sequence $\{(\langle Y_k^{a_1}(t),f_1\rangle,\cdots,\langle Y_k^{a_n}(t),f_n\rangle):t\geq 0\}$ is tight in $D([0,\infty),\mathbb{R}^n)$; see also Ethier and Kurtz (1986, pages 137-138). Then a simple extension of the tightness criterion of Roelly (1986) implies $\{(Y_k^{a_1}(t),\cdots,Y_k^{a_n}(t)):t\geq 0\}$ is tight in $D([0,\infty),M[0,a_1]\times\cdots\times M[0,a_n])$. □

Suppose that $\sigma\geq 0$ and $b\geq 0$ are two constants, $v\mapsto\eta(v)$ is a non-negative and non-decreasing continuous function on $[0,\infty)$, and $(z\wedge z^2)m(dz)$ is a finite measure on $(0,\infty)$. Let $\eta(dv)$ be the Radon measure on $[0,\infty)$ so that $\eta([0,v])=\eta(v)$ for $v\geq 0$. Suppose that $\{W(ds,du)\}$ is a white noise on $(0,\infty)^2$ with intensity $dsdz$ and $\{N(ds,dz,du)\}$ is a Poisson random measure on $(0,\infty)^3$ with intensity $dsm(dz)du$. Let $\{X_t(v):t\geq 0,v\geq 0\}$ be the solution flow of the stochastic equation

$$\begin{aligned} X_t(v) &= v+\sigma\int_0^t\int_0^{X_{s-}(v)}W(ds,du)+b\int_0^t[\eta(v)-X_{s-}(v)]ds \\ &\quad+\int_0^t\int_0^\infty\int_0^{X_{s-}(v)} z\tilde{N}(ds,dz,du). \end{aligned}\tag{5.6}$$

By Theorem 3.11, for each $a\geq 0$ the flow $\{X_t(v):t\geq 0,v\geq 0\}$ induces an $M[0,a]$-valued immigration superprocess $\{X_t^a:t\geq 0\}$ which is the unique solution of the following martingale problem: For every $G\in C^2(\mathbb{R})$ and $f\in C[0,a]$,

$$\begin{aligned} G(\langle X_t,f\rangle) &= G(\langle\lambda,f\rangle)+b\int_0^t G'(\langle X_s,f\rangle)[\langle\eta,f\rangle-\langle X_s,f\rangle]ds \\ &\quad+\frac{1}{2}\sigma^2\int_0^t G''(\langle X_s,f\rangle)\langle X_s,f^2\rangle ds \\ &\quad+\int_0^t ds\int_0^\infty m(dz)\int_{[0,a]}\Big[G(\langle X_s,f\rangle+zf(x)) \\ &\qquad -G(\langle X_s,f\rangle)-G'(\langle X_s,f\rangle)zf(x)\Big]X_s(dx) \\ &\quad+\text{local mart.} \end{aligned}\tag{5.7}$$

Theorem 5.5 *Suppose that $kb_k\to b$, $\eta_k\to\eta$ weakly on $[0,a]$ and $k^2\sigma_k^2\delta_0(dz)+(z\wedge z^2)m_k(dz)$ converges weakly on $[0,\infty)$ to a finite measure $\sigma^2\delta_0(dz)+(z\wedge z^2)m(dz)$ as $k\to\infty$. Then $\{Y_k^a(t):t\geq 0\}$ converges to the immigration superprocess $\{X_t^a:t\geq 0\}$ in distribution on $D([0,\infty),M[0,a])$.*

For the proof of the above theorem, let us make some preparations. Since the solution of the martingale problem (5.7) is unique, it suffices to prove any weak limit point $\{Z_t^a : t \geq 0\}$ of the sequence $\{Y_k^a(t) : t \geq 0\}$ is the solution of the martingale problem. To simplify the notation we pass to a subsequence and simply assume $\{Y_k^a(t) : t \geq 0\}$ converges to $\{Z_t^a : t \geq 0\}$ in distribution. Using Skorokhod's representation theorem, we can also assume $\{Y_k^a(t) : t \geq 0\}$ and $\{Z_t^a : t \geq 0\}$ are defined on the same probability space and $\{Y_k^a(t) : t \geq 0\}$ converges a.s. to $\{Z_t^a : t \geq 0\}$ in the topology of $D([0, \infty), M[0, a])$. For $n \geq 1$ let

$$\tau_n = \inf \Big\{ t \geq 0 : \sup_{k \geq 1} \int_0^t [1 + \langle Y_k^a(s) + Z_s^a, 1\rangle^2] ds \geq n \Big\}.$$

It is easy to see that $\tau_n \to \infty$ as $n \to \infty$.

Lemma 5.6 *Suppose that $kb_k \to b$, $\eta_k \to \eta$ weakly on $[0, a]$ and $k^2\sigma_k^2\delta_0(dz) + (z \wedge z^2)m_k(dz)$ converges weakly on $[0, \infty)$ to a finite measure $\sigma^2\delta_0(dz) + (z \wedge z^2)m(dz)$ as $k \to \infty$. Let $\epsilon_k(s, z)$ be defined as in Lemma 5.1. Then for each $n \geq 1$ we have*

$$\mathbf{E}\Big[\int_0^{t\wedge\tau_n} ds \int_0^k |\epsilon_k(s, z)| m_k(dz)\Big] \to 0, \qquad k \to \infty.$$

Proof. By the mean-value theorem, we have

$$\begin{aligned} \epsilon_k(s, z) \;=\; & \frac{1}{k} z \langle Y_k^a(s), f\rangle \int_0^k \Big[G'(\langle Y_k^a(s), f\rangle + z\theta_k(s, x)) \\ & \qquad - G'(\langle Y_k^a(s), f\rangle)\Big] Y_k^a(s, dx), \end{aligned}$$

where $\theta_k(s, x)$ takes values between $f(x)$ and $f(x) - k^{-1}\langle Y_k^a(s), f\rangle$. Consequently,

$$|\epsilon_k(s, z)| \leq \frac{2}{k}\|G'\| z \langle Y_k^a(s), |f|\rangle \langle Y_k^a(s), 1\rangle \leq \frac{2}{k}\|G'\|\|f\| z \langle Y_k^a(s), 1\rangle^2.$$

Moreover, since $\langle Y_k^a(s), 1\rangle \leq k$, we get

$$\begin{aligned} |\epsilon_k(s, z)| \;\leq\; & \frac{1}{k}\|G''\| z^2 \langle Y_k^a(s), |f|\rangle \int_0^k |\theta_k(s, x))| Y_k^a(s, dx) \\ \leq\; & \frac{1}{k}\|G''\| z^2 \langle Y_k^a(s), |f|\rangle \int_0^k [|f(x)| + k^{-1}\langle Y_k^a(s), |f|\rangle] Y_k^a(s, dx) \\ \leq\; & \frac{2}{k}\|f\|^2\|G''\| z^2 \langle Y_k^a(s), 1\rangle^2. \end{aligned}$$

It follows that

$$\mathbf{E}\Big[\int_0^{t\wedge\tau_n} ds \int_0^k |\epsilon_k(s, z)| m_k(dz)\Big]$$

$$\le \frac{C}{k}\int_0^k (z\wedge z^2)m_k(dz)\mathbf{E}\Big[\int_0^{t\wedge\tau_n}\langle Y_k^a(s),1\rangle^2 ds\Big]$$
$$\le \frac{nC}{k}\int_0^k (z\wedge z^2)m_k(dz),$$

where $C = 2\|f\|(\|G'\|+\|G''\|\|f\|)$. The right-hand side goes to zero as $k\to\infty$. □

Lemma 5.7 *Suppose that $kb_k \to b$, $\eta_k \to \eta$ weakly on $[0,a]$ and $k^2\sigma_k^2\delta_0(dz) + (z\wedge z^2)m_k(dz)$ converges weakly on $[0,\infty)$ to a finite measure $\sigma^2\delta_0(dz)+(z\wedge z^2)m(dz)$ as $k\to\infty$. Let $\xi_k(s,z)$ be defined as in Lemma 5.1. Then for each $n\ge 1$ we have*

$$\mathbf{E}\Big[\int_0^{t\wedge\tau_n} ds\int_0^k |\xi_k(s,z)|m_k(dz)\Big]\to 0, \qquad k\to\infty.$$

Proof. It is elementary to see that

$$\begin{aligned}|\xi_k(s,z)| \;&\le\; k\Big|G(\langle Y_k^a(s),f\rangle - k^{-1}z\langle Y_k^a(s),f\rangle) - G(\langle Y_k^a(s),f\rangle)\\ &\qquad + k^{-1}G'(\langle Y_k^a(s),f\rangle)z\langle Y_k^a(s),f\rangle\Big|\\ &\le\; \min\Big\{2\|G'\|z\langle Y_k^a(s),|f|\rangle, \frac{1}{2k}\|G''\|z^2\langle Y_k^a(s),|f|\rangle^2\Big\}\\ &\le\; C[1+\langle Y_k^a(s),1\rangle^2](z\wedge k^{-1}z^2),\end{aligned}$$

where $C = \|f\|(2\|G'\|+\|f\|\|G''\|/2)$. Then we have

$$\begin{aligned}&\mathbf{E}\Big[\int_0^{t\wedge\tau_n} ds\int_0^k |\xi_k(s,z)|m_k(dz)\Big]\\ &\qquad\le C\int_0^k (z\wedge k^{-1}z^2)m_k(dz)\mathbf{E}\Big\{\int_0^{t\wedge\tau_n}[1+\langle Y_k^a(s),1\rangle^2]ds\Big\}\\ &\qquad\le nC\int_0^k (z\wedge k^{-1}z^2)m_k(dz).\end{aligned}$$

The right-hand side tends to zero as $k\to\infty$. □

Proof of Theorem 5.5. Let $f\in C[0,a]$. Then $\{\langle Y_k^a(t),f\rangle : t\ge 0\}$ converges a.s. to $\{\langle Z_t^a,f\rangle : t\ge 0\}$ in the topology of $D([0,\infty),\mathbb{R})$. Consequently, we have a.s. $\langle Y_k^a(t),f\rangle \to \langle Z_t^a,f\rangle$ for a.e. $t\ge 0$; see, e.g., Ethier and Kurtz (1986, page 118). By Lemma 5.1,

$$\begin{aligned}G(\langle Y_k^a(t),f\rangle) \;&=\; G(\langle\lambda,f\rangle) + kb_k\int_0^t G'(\langle Y_k^a(s),f\rangle)\langle\eta_k,f\rangle ds\\ &\qquad - kb_k\int_0^t G'(\langle Y_k^a(s),f\rangle)\langle Y_k^a(s),f\rangle ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}k^2\sigma_k^2\int_0^t G''(\langle Y_k^a(s),f\rangle)\langle Y_k^a(s),f^2\rangle ds\\
&-\frac{1}{2}k\sigma_k^2\int_0^t G''(\langle Y_k^a(s),f\rangle)\langle Y_k^a(s),f\rangle^2 ds\\
&+\int_0^t ds\int_0^k m_k(dz)\int_{[0,a]} H(x,z,\langle Z_s^a,f\rangle)Y_k^a(s,dx)\\
&+\int_0^t ds\int_0^k [\epsilon_k(s,z)+\xi_k(s,z)+\zeta_k(s,z)]m_k(dz)\\
&+\text{local mart.},
\end{aligned}
\tag{5.8}
$$

where

$$
H(x,z,u)=G(u+zf(x))-G(u)-G'(u)zf(x)
$$

and

$$
\zeta_k(s,z)=\int_{[0,a]}\Big[H(x,z,\langle Y_k^a(s),f\rangle)-H(x,z,\langle Z_s^a,f\rangle)\Big]Y_k^a(s,dx).
$$

By the mean-value theorem,

$$
|\zeta_k(s,z)|\le\int_{[0,k]}|H_u'(x,z,\theta_k(s))\langle Y_k^a(s)-Z_s^a,f\rangle|Y_k^a(s,dx),
$$

where $\theta_k(s)$ takes values between $\langle Y_k^a(s),f\rangle$ and $\langle Z_s^a,f\rangle$. For $G\in C^3(\mathbb{R})$ we have

$$
\begin{aligned}
|H_u'(x,z,\theta_k(s))| &= |G'(\theta_k(s)+zf(x))-G'(\theta_k(s))-G''(\theta_k(s))zf(x)|\\
&\le \|f\|\Big(2\|G''\|+\frac{1}{2}\|f\|\|G'''\|\Big)(z\wedge z^2).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|\zeta_k(s)| &\le \|f\|\Big(2\|G''\|+\frac{1}{2}\|f\|\|G'''\|\Big)(z\wedge z^2)\\
&\qquad\cdot\langle Y_k^a(s),1\rangle|\langle Y_k^a(s)-Z_s^a,f\rangle|.
\end{aligned}
\tag{5.9}
$$

By (5.9) and Schwarz' inequality,

$$
\begin{aligned}
&\mathbf{E}\Big[\int_0^{t\wedge\tau_n} ds\int_0^k|\zeta_k(s)|m_k(dz)\Big]\\
&\quad\le C_k(t)\bigg\{\mathbf{E}\Big[\int_0^{t\wedge\tau_n}\langle Y_k^a(s)-Z_s^a,f\rangle^2 ds\Big]\bigg\}^{1/2}\\
&\qquad\cdot\bigg\{\mathbf{E}\Big[\int_0^{t\wedge\tau_n}\langle Y_k^a(s),1\rangle^2 ds\Big]\bigg\}^{1/2}\\
&\quad\le\sqrt{n}C_k(t)\bigg\{\mathbf{E}\Big[\int_0^{t\wedge\tau_n}\langle Y_k^a(s)-Z_s^a,f\rangle^2 ds\Big]\bigg\}^{1/2},
\end{aligned}
$$

where

$$C_k(t) \;=\; \|f\|\Big(2\|G''\| + \frac{1}{2}\|G'''\|\|f\|\Big)\int_0^k (z\wedge z^2)m_k(dz).$$

Note that $\sup_{k\geq 1} C_k(t) < \infty$. It then follows that

$$\mathbf{E}\Big[\int_0^{t\wedge\tau_n} ds \int_0^k |\zeta_k(s)|m_k(dz)\Big] \to 0, \qquad k\to\infty.$$

Now letting $k\to\infty$ in (5.8) and using Lemmas 5.6 and 5.7 we obtain (5.7) for $G\in C^3(\mathbb{R})$. A simple approximation shows the martingale problem actually holds for any $G\in C^2(\mathbb{R})$. $\square$

Theorem 5.8 *Suppose that $kb_k\to b$, $\eta_k\to\eta$ weakly on $[0,a]$ and $k^2\sigma_k^2\delta_0(dz)+(z\wedge z^2)m_k(dz)$ converges weakly on $[0,\infty)$ to a finite measure $\sigma^2\delta_0(dz)+(z\wedge z^2)m(dz)$ as $k\to\infty$. Let $\{0\leq a_1<\cdots<a_n=a\}$ be an ordered set of constants. Then $\{(Y_k^{a_1}(t),\cdots,Y_k^{a_n}(t)) : t\geq 0\}$ converges to $\{(X_t^{a_1},\cdots,X_t^{a_n}) : t\geq 0\}$ in distribution on $D([0,\infty),M[0,a_1]\times\cdots\times M[0,a_n])$.*

Proof. By Lemma 5.4 the sequence $\{(Y_k^{a_1}(t),\cdots,Y_k^{a_n}(t)) : t\geq 0\}$ is tight in $D([0,\infty),\ M[0,a_1]\times\cdots\times M[0,a_n])$. Let $\{(Z_t^{a_1},\cdots,Z_t^{a_n}) : t\geq 0\}$ be a weak limit point of $\{(Y_k^{a_1}(t),\cdots,Y_k^{a_n}(t)) : t\geq 0\}$. To get the result, we only need to show $\{(Z_t^{a_1},\cdots,Z_t^{a_n}) : t\geq 0\}$ and $\{(X_t^{a_1},\cdots,X_t^{a_n}) : t\geq 0\}$ have identical distributions on $D([0,\infty),M[0,a_1]\times\cdots\times M[0,a_n])$. By passing to a subsequence and using Skorokhod's representation, we can assume $\{(Y_k^{a_1}(t),\cdots,Y_k^{a_n}(t)) : t\geq 0\}$ converges to $\{(Z_t^{a_1},\cdots,Z_t^{a_n}) : t\geq 0\}$ almost surely in the topology of $D([0,\infty),\ M[0,a_1]\times\cdots\times M[0,a_n])$. Theorem 5.5 implies $\{Z_t^{a_n} : t\geq 0\}$ is an immigration superprocess solving the martingale problem (5.7) with $a=a_n$. Let $\bar{Z}_t^{a_i}$ denote the restriction of $Z_t^{a_n}$ to $[0,a_i]$. Then $Z_t^{a_n}=\bar{Z}_t^{a_n}$ in particular. We will show $\{(Z_t^{a_1},\cdots,Z_t^{a_n}) : t\geq 0\}$ and $\{(\bar{Z}_t^{a_1},\cdots,\bar{Z}_t^{a_n}) : t\geq 0\}$ are indistinguishable. That will imply the desired result since $\{(X_t^{a_1},\cdots,X_t^{a_n}) : t\geq 0\}$ and $\{(\bar{Z}_t^{a_1},\cdots,\bar{Z}_t^{a_n}) : t\geq 0\}$ clearly have identical distributions on $D([0,\infty),M[0,a_1]\times\cdots\times M[0,a_n])$. By the general theory of càdlàg processes, the complement in $[0,\infty)$ of

$$D(Z) := \{t\geq 0 : \mathbf{P}(Z_t^{a_1}=Z_{t-}^{a_1},\cdots,Z_t^{a_n}=Z_{t-}^{a_n})=1\}$$

is at most countable; see Ethier and Kurtz (1986; page 131). For any $t\in D(Z)$ we have almost surely $\lim_{k\to\infty} Y_k^{a_i}(t)=Z_t^{a_i}$ for each $i=1,\cdots,n$; see Ethier and Kurtz (1986; page 118). By an elementary property of weak convergence, for any $t\in D(Z)$ we almost surely have

$$\begin{aligned} Z_t^{a_i}([0,a_i]) \;&=\; \lim_{k\to\infty} Y_k^{a_i}(t,[0,a_i]) = \lim_{k\to\infty} Y_k^{a_n}(t,[0,a_i]) \\ &\leq\; Z_t^{a_n}([0,a_i]) = \bar{Z}_t^{a_n}([0,a_i]) = \bar{Z}_t^{a_i}([0,a_i]). \end{aligned}$$

Since Theorem 5.5 implies $\{Z_t^{a_i} : t \geq 0\}$ is equivalent to $\{\bar{Z}_t^{a_i} : t \geq 0\}$, we have

$$\mathbf{E}[Z_t^{a_i}([0, a_i])] = \mathbf{E}[\bar{Z}_t^{a_i}([0, a_i])].$$

It then follows that almost surely

$$\lim_{k\to\infty} Y_k^{a_i}(t, [0, a_i]) = \bar{Z}_t^{a_i}([0, a_i]). \tag{5.10}$$

On the other hand, since $Y_k^{a_n}(t) \to \bar{Z}_t^{a_n}$, for any closed set $C \subset [0, a_i]$ we have

$$\limsup_{k\to\infty} Y_k^{a_i}(t, C) = \lim_{k\to\infty} Y_k^{a_n}(t, C) \leq \bar{Z}_t^{a_n}(C) = \bar{Z}_t^{a_i}(C). \tag{5.11}$$

By (5.10) and (5.11) we have $Z_t^{a_i} = \lim_{k\to\infty} Y_k^{a_i}(t) = \bar{Z}_t^{a_i}$. Then $\{Z_t^{a_i} : t \geq 0\}$ and $\{\bar{Z}_t^{a_i} : t \geq 0\}$ are indistinguishable since both processes are càdlàg. $\square$

Let $\mathscr{M}$ be the space of Radon measures on $[0, \infty)$ furnished with a metric compatible with the vague convergence. The result of Theorem 5.8 clearly implies the convergence of $\{Y_k(t) : t \geq 0\}$ in distribution on $D([0, \infty), \mathscr{M})$ with the Skorokhod topology. From Theorem 5.8 we can also derive the following generalization of a result of Bertoin and Le Gall (2006); see also Bertoin and Le Gall (2000) for an earlier result.

Corollary 5.9 *Suppose that $kb_k \to b$, $\eta_k \to \eta$ weakly on $[0, a]$ and $k^2\sigma_k^2\delta_0(dz) + (z \wedge z^2)m_k(dz)$ converges weakly on $[0, \infty)$ to a finite measure $\sigma^2\delta_0(dz) + (z \wedge z^2)m(dz)$ as $k \to \infty$. Let $\{0 \leq a_1 < \cdots < a_n\}$ be an ordered set of constants. Then $\{(Y_k(t, a_1), \cdots, Y_k(t, a_n)) : t \geq 0\}$ converges to $\{(X_t(a_1), \cdots, X_t(a_n)) : t \geq 0\}$ in distribution on $D([0, \infty), \mathbb{R}_+^n)$.*

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