Published in: *Probability Theory and Related Fields* **153** (2012), 3/4: 441–469

Joint continuity for the solutions to a class of nonlinear SPDEs[1]

Zenghu Li, Hao Wang, Jie Xiong and Xiaowen Zhou

Abstract

For a one-dimensional superprocess in random environment, a nonlinear SPDE was derived by Dawson et al [3] for its density process. The time-space joint continuity of the density process was left as an open problem. In this paper we give an affirmative answer to this problem.

Keywords: Superprocess, random environment, stochastic partial differential equation.

AMS 2000 subject classifications: Primary 60G57, 60H15; secondary 60J80.

1 Introduction

Suppose that in a system of k_n particles each particle has an independent exponential clock with parameter n. Before any of these exponential times is up, the particles with initial locations $(x_1^n, \cdots, x_{k_n}^n) \in \mathbb{R}^{k_n}$ move according to the following system of stochastic differential equations (SDE):

$$x_i^n(t) = x_i^n + B^i(t) + \int_0^t \int_{\mathbb{R}} h(y - x_i^n(s)) W(dsdy), \; i = 1, 2, \cdots, k_n, \tag{1.1}$$

where $h \in L^2(\mathbb{R})$ and $(B^1, \cdots, B^{k_n})$ is an k_n-dimensional Brownian motions independent of the Brownian sheet W on $\mathbb{R}_+ \times \mathbb{R}$. The W can be regarded as the random environment for the particle system. For convenience we assume that

$$\rho(0) \equiv \int_{\mathbb{R}} h(x)^2 dx = 1.$$

When its clock rings the particle either splits into two or dies with equal probabilities. The new particles will inherit their mother's position together with new independent exponential clocks. This pattern of motion-splitting/dying then continues as before.

[1]Research of ZL is supported partially by NSFC (10525103 and 10721091) and CJSP, JX by NSF DMS-0906907, XZ by NSERC.

In general, each particle in the system can be denoted by a multi-index $\alpha = \alpha_1\alpha_2\cdots\alpha_k$ with $\alpha_1 = 1,\cdots,k_n$ and $\alpha_i = 1,2$ for $i \geq 2$. For example, $\alpha = (3,1)$ represents the oldest daughter of the third particle in the first generation. Write $\alpha \sim t$ if particle α is alive at time t. For each n we define a measure-valued stochastic process

$$X_t^n = \frac{1}{n}\sum_{\alpha\sim t}\delta_{x_\alpha^n(t)},\ \ t\geq 0,$$

This model is first studied by Wang ([15], [16]). Write $\mathcal{M}_F(\mathbb{R})$ for the space of finite measures on E with the topology of weak convergence. Under suitable conditions, it is proved by Wang [16] and Dawson et al [2] that as $n\to\infty$, X^n converges weakly in $D([0,T],\mathcal{M}_F(\mathbb{R}))$ to the unique solution X of the following martingale problem (MP):

$$M_t^\phi \equiv \langle X_t,\phi\rangle - \langle \mu,\phi\rangle - \int_0^t \langle X_s,\Delta\phi\rangle\, ds,\ \ \forall\phi\in C_b^2(\mathbb{R}) \tag{1.2}$$

is a continuous martingale with quadratic variation process

$$\left\langle M^\phi\right\rangle_t = \int_0^t \left\langle X_s,\phi^2\right\rangle ds + \int_0^t\int_{\mathbb{R}^2}\rho(x-y)\phi'(x)\phi'(y)X_s(dx)X_s(dy)ds, \tag{1.3}$$

where $\mu\in\mathcal{M}_F(\mathbb{R})$ is the initial measure and

$$\rho(x-y) = \int_{\mathbb{R}} h(z-x)h(z-y)dz. \tag{1.4}$$

Here $\Delta\phi\equiv\phi''$ is the second derivative of ϕ. Similarly, we shall use both $\nabla\phi$ and ϕ' to denote the first derivative of ϕ.

It is proved by Dawson et al [3] and Wang [15] that X_t is absolutely continuous with respect to Lebesgue measure and its density, denoted by $X_t(x)$, solves SPDE

$$\begin{aligned} X_t(x) &= \mu(x) + \int_0^t \Delta X_s(x)ds - \int_0^t\int_{\mathbb{R}}\nabla_x(h(y-x)X_s(x))W(dsdy) \\ &\quad + \int_0^t \sqrt{X_s(x)}\frac{B(dsdx)}{dx}, \end{aligned} \tag{1.5}$$

where B is a Brownian sheet on $\mathbb{R}_+\times\mathbb{R}$ independent of W. The joint continuity of $(t,x)\mapsto X_t(x)$ is left as an open problem in [3].

When the third term on the RHS of (1.5) is replaced by $\int_0^t\int_{\mathbb{R}}\nabla(h(x)X_s(x))d\tilde{W}(s)$ with a real-valued Brownian motion $\tilde{W}$, the SPDE is satisfied by the density process of a measure-valued process for a related model studied by Skoulakis and Adler [14].

For that model, Lee et al [11] proves the continuity in x for Lebesgue almost all fixed t using Krylov's (cf. Krylov [8]) L_p theory for linear SPDE.

The goal of this paper is to prove the joint continuity of $X_t(x)$ in Theorem 1.1. For $k \in \mathbb{R}, p \geq 1$ the space H_p^k with norm $\|\cdot\|_{k,p}$ will be introduced in Section 2. We always make the following assumption (I) on the initial measure X_0.
Assumption (I): X_0 has a bounded density $\mu \in H_2^1$.

Theorem 1.1 *Suppose that $h \in H_2^2$, $\|h\|_{1,2}^2 < 2$ and X_0 satisfies the condition (I). Then the measure-valued process X_t has a density $X_t(x)$ which is almost surely jointly Hölder continuous. Furthermore, for fixed t its Hölder exponent in x is in* $(0, 1/2)$*; for fixed x its Hölder exponent in t is in* $(0, 1/10)$.

We now describe the major difficulties and sketch our approaches for the main result. When $h = 0$, X becomes the well known Dawson-Watanabe process with the joint continuity for its density studied by Konno and Shiga [6] and Reimers [13] via a convolution technique. If we adopt the same technique here, then the density can be represented as

$$\begin{aligned} X_t(x) &= \int \varphi_t(x-y)\mu(y)dy + \int_0^t \int_{\mathbb{R}} \sqrt{X_s(y)}\varphi_{t-s}(x-y)B(dsdy) \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} h(y-z)X_s(z)\partial_z\varphi_{t-s}(x-z)dzW(dsdy), \end{aligned} \tag{1.6}$$

where φ is the heat kernel with generator Δ. However, the third term on the RHS of the above equation is (for some suitable function g) roughly equal to

$$\int_0^t \int_{\mathbb{R}} (t-s)^{-1/2} g(z) W(dsdz),$$

which does *not* converge. Therefore, the convolution argument of Konno and Shiga fails in our model. It actually means that the SPDE (1.5) does not have a *mild* solution.

Since it is the term containing W that causes the problem, we want to absorb it to the kernel by considering a stochastic transition function. For this purpose let $p^W(s, x; t, y)$ be the conditional transition function of a single particle with W given (to be made precise in Section 3). We will prove that

$$X_t(y) = \int_{\mathbb{R}} p^W(0, x; t, y)\mu(x)dx + \int_0^t \int_{\mathbb{R}} \sqrt{X_s(x)}p^W(s, x; t, y)B(dsdx).$$

The first term in the above equation is easy to deal with. So we focus on the second term. We will apply Kolmogorov's criteria to obtain the joint continuity. To this end, we need the following estimates: for any $y_1,\ y_2 \in \mathbb{R}$,

$$\mathbb{E}\left|\int_0^t\int_{\mathbb{R}}|p^W(s,x,t,y_1)-p^W(s,x,t,y_2)|^2dxds\right|^p \le K|y_1-y_2|^{2+\epsilon} \tag{1.7}$$

and for $y\in\mathbb{R}$ and $t_1<t_2$,

$$\mathbb{E}\left|\int_0^{t_1}\int_{\mathbb{R}}|p^W(s,x,t_2,y)-p^W(s,x,t_1,y)|^2dxds\right|^p \le K|t_1-t_2|^{2+\epsilon}, \tag{1.8}$$

for some $\epsilon>0$ and suitable $p>0$.

To obtain (1.7) we fix t and let $u_s(x)=p^W(t-s,x,t,y_1)-p^W(t-s,x,t,y_2)$. Then u satisfies the following linear SPDE

$$u_t(x)=u_0(x)+\int_0^t\Delta u_s(x)ds+\int_0^t\int_{\mathbb{R}}\nabla u_s(x)h(y-x)\tilde{W}(dsdy) \tag{1.9}$$

with initial condition $u_0=\delta_{y_1}-\delta_{y_2}$, where $\tilde{W}$ is a Brownian sheet defined by W with its time reversed (to be made precise later). We shall derive an estimate of u_s in terms of u_0 in the spirit of Kurtz and Xiong [10] and obtain (1.7).

For (1.8), we note that $\tilde{u}_s(x)=p^W(t_1-s,x,t_2,y)-p^W(t_1-s,x,t_1,y)$ is a solution to the linear SPDE (1.9) with initial condition $\tilde{u}_0=p^W(t_1,\cdot,t_2,y)-\delta_y$. The LHS of (1.8) is then bounded by $\mathbb{E}\ \|\tilde{u}_0\|^{2p}_{-1,2}$, where $\|\cdot\|_{-1,2}$ is a Sobolev norm to be defined later. To estimate this quantity, we further define $v_t(x)=p^W(t_2-t,x,t_2,y)$ which solves SPDE (1.9) with initial $v_0(x)=\delta_y(x)$, and then estimate $\mathbb{E}\ \|v_{t_2-t_1}-\delta_y\|^{2p}_{-1,2}$. Similar to what we mentioned above for the convolution (1.6), we cannot directly apply the convolution with kernel φ_t to (1.9). We shall use a partial convolution by kernel φ_{t^α} where $\alpha\in(0,1)$ is a constant to be decided later. Then

$$\begin{aligned} v_t(z) &= \varphi_{t^\alpha}(z-y)+\int_0^t\int_{\mathbb{R}}\Delta v_{t-r}(x)\varphi_{r^\alpha}(z-x)dxdr \\ &\quad +\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\nabla v_{t-r}(x)h(y-x)\varphi_{r^\alpha}(z-x)dx\tilde{W}(drdy) \\ &\quad -\alpha\int_0^t\int_{\mathbb{R}}\Delta v_{t-r}(x)\varphi_{r^\alpha}(z-x)dxr^{\alpha-1}dr. \end{aligned} \tag{1.10}$$

The main difficulty now lies in the fourth term because, due to the integrability, we can not apply integration by parts to move Δ completely to φ. Instead, we have to transform a fraction Δ^β of Δ to φ with $\beta<1$ to be decided (together with α).

The novelty of this article is as follows. Firstly, to the best of our knowledge the joint continuity for the solution to SPDE was only previously studied when the mild solution for that equation can be defined. The current paper appears to be the first attempt for such a problem when the SPDE does not allow a mild solution. Secondly, a fractional integration by parts technique is introduced to obtain estimates for the solution to SPDE. We believe this technique will be useful in studying other SPDEs. Thirdly, a stochastic convolution technique is implemented, which provides the solution to SPDE with a "conditionally mild" representation. This technique will be applicable to other SPDEs arising from particle systems in random environments.

Besides the continuity of the solution, mild representation has been used by many authors to derive various properties for the solution of the SPDE. For example, Foondun and Khoshnevisan [4] use this representation to study the intermittency. We believe that the methods we develop in this paper can be applied to study other properties of the SPDEs for which the mild representations are not available.

The rest of the paper is organized as follows. In Section 2 we establish some estimates for the solutions to a class of linear SPDEs. Then in Section 3 we derive a representation of the density $X_t(x)$ in terms of a random transition function. Based on this representation, we estimate the spatial-increments of $X_t(x)$ in Section 4 and the time-increments in Section 5. We conclude the proof of Theorem 1.1 in Section 5.

The following conventions will be used throughout the paper. We use K to represent a positive constant whose value can be different from place to place. We use I or J with a subscript to represent a term in the quantity to be evaluated. Again, what I_1 stands for can be different from place to place.

2 Two SPDE estimates

In this section we study the SPDE (1.9) where u_0 is either a real or a generalized function for different purposes. To this end, we need to introduce some notation taken from Krylov [8]. For $\alpha \in (0,1)$ and generalized function u on $\mathbb{R}$, let

$$(I-\Delta)^{\alpha}u = c(\alpha)\int_0^\infty \frac{e^{-t}T_tu-u}{t^{\alpha+1}}dt, \tag{2.1}$$

and

$$(I-\Delta)^{-\alpha}u = d(\alpha)\int_0^\infty t^{\alpha-1}e^{-t}T_tudt, \tag{2.2}$$

where $c(\alpha)$ and $d(\alpha)$ are two constants and T_t is the Brownian semigroup. As being indicated by Krylov [8], (2.1) and (2.2) are sufficient to define $(I-\Delta)^{n/2}$ consistently for any $n \in \mathbb{R}$ (cf. Krasnoselskii et al [7]). In particular, $(I-\Delta)^\alpha(I-\Delta)^\beta = (I-\Delta)^{\alpha+\beta}$ for any $\alpha, \beta \in \mathbb{R}$. In this paper we only need it for $n \in [-1, 1]$.

Let H_p^n be the spaces of Bessel potentials with norms

$$\|u\|_{n,p} \equiv \|(I-\Delta)^{n/2}u\|_p \tag{2.3}$$

where $\|\cdot\|_p$ is the norm in L_p. Note that for $n = 1$ and $p = 2$, $\|u\|_{1,2}$ coincides with the usual Sobolev norm on $H^{1,2}$.

The existence and uniqueness of the solution to (1.9) has been studied by Krylov [8] in suitable Banach spaces. In the remaining of this section, we assume this equation has a solution (the existence will be evident from the applications in later sections), and the aim of this section is to prove that, with the appropriate initial condition, the solution actually lies in the spaces which will be useful for our purpose.

Let $\beta \in [0, 1)$ and $u_0 \in H_2^{\beta-1}$. For $f \in C_0^\infty(\mathbb{R})$, i.e., f is infinitely differentiable with compact support, we have

$$\langle u_r, f\rangle = \langle u_0, f\rangle + \int_0^r \langle \Delta u_s, f\rangle\, ds + \int_0^r \int_{\mathbb{R}} \langle \nabla u_s h(y-\cdot), f\rangle\, \tilde{W}(dsdy) \tag{2.4}$$

where $\langle u, f\rangle$ stands for the duality between the Hilbert spaces H_2^{-n} and H_2^n. Applying Itô's formula to $\langle u_r, f\rangle^2$ and summing up f over a complete orthonormal system of $H_2^{1-\beta}$, by (2.4) we get

$$\begin{aligned}\|u_r\|_{\beta-1,2}^2 &= \|u_0\|_{\beta-1,2}^2 + \int_0^r 2\langle u_s, \Delta u_s\rangle_{\beta-1,2}\, ds + \int_0^r \int_{\mathbb{R}} \|\nabla u_s h(y-\cdot)\|_{\beta-1,2}^2 dyds \\ &\quad + \int_0^r \int_{\mathbb{R}} 2\langle u_s, \nabla u_s h(y-\cdot)\rangle_{\beta-1,2}\, \tilde{W}(dsdy). \end{aligned} \tag{2.5}$$

We first apply (2.5) for $\beta = 0$. The following lemmas will be used in Theorem 2.3.

Lemma 2.1 *If $h \in H_2^1$, then for any $u \in H_2^0$,*

$$\int_{\mathbb{R}} \|\nabla u h(y-\cdot)\|_{-1,2}^2 dy \le \|h\|_{1,2}^2 \|u\|_{0,2}^2.$$

Proof: Note that

$$\int_{\mathbb{R}} \|\nabla u h(y-\cdot)\|_{-1,2}^2 dy = \int_{\mathbb{R}} \sup_{\|f\|_{1,2}\le 1} \langle \nabla u, h(y-\cdot) f\rangle^2\, dy$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \sup_{\|f\|_{1,2}\le 1} \langle u, f'h(y-\cdot) - fh'(y-\cdot)\rangle_{0,2}^2 \, dy \\
&\le \int_{\mathbb{R}} \sup_{\|f\|_{1,2}\le 1} \left(\|uh(y-\cdot)\|_{0,2}\|f'\|_{0,2} + \|uh'(y-\cdot)\|_{0,2}\|f\|_{0,2}\right)^2 dy \\
&\le \int_{\mathbb{R}} \left(\|uh(y-\cdot)\|_{0,2}^2 + \|uh'(y-\cdot)\|_{0,2}^2\right) dy \\
&= \|h\|_{1,2}^2 \|u\|_{0,2}^2,
\end{aligned}
$$

where the first inequality follows from the definition of the norm using duality. ∎

Lemma 2.2 *If $h \in H_2^2$, then for any $u \in H_2^{-1}$,*

$$
\int_{\mathbb{R}} \langle u, \nabla u h(y-\cdot)\rangle_{-1,2}^2 \, dy \le K\|u\|_{-1,2}^2.
$$

Proof: Let $g = (I-\Delta)^{-1}u$. By integration by parts, we get

$$
\begin{aligned}
\langle u, f\nabla u\rangle_{-1,2} &= \left\langle (I-\Delta)^{-1}u, f\nabla u\right\rangle_{0,2} \qquad (2.6)\\
&= -\langle f'g, g\rangle_{0,2} - \langle (f+f'')g, g'\rangle_{0,2} + \langle fg', g''\rangle_{0,2} - \langle f'g', g'\rangle_{0,2}.
\end{aligned}
$$

By Lemma 3.2 in [9], we get

$$
\int_{\mathbb{R}} \langle h(y-\cdot)g', g''\rangle_{0,2}^2 \, dy \le K\|h'\|_{0,2}^2 \|g'\|_{0,2}^2.
$$

Applying Cauchy-Schwartz inequality to the other terms of (2.6) with f replaced by $h(y-\cdot)$, we have

$$
\int_{\mathbb{R}} \langle u, \nabla u h(y-\cdot)\rangle_{-1,2}^2 \, dy \le K\|h\|_{2,2}^2 \|g'\|_{0,2}^2. \qquad (2.7)
$$

As

$$
\|g'\|_{0,2} = \|\nabla(I-\Delta)^{-1}u\|_{0,2} \le K\|(I-\Delta)^{-\frac{1}{2}}u\|_{0,2} = K\|u\|_{-1,2},
$$

the conclusion of the lemma then follows from (2.7), where the last inequality follows from the boundedness of the operator $\nabla(I-\Delta)^{-\frac{1}{2}}$ (cf. [8]). ∎

Theorem 2.3 *For $p \ge 1$, $u_0 \in H_2^{-1}$, $h \in H_2^2$ and $\|h\|_{1,2}^2 < 2$, we have*

$$
\mathbb{E} \sup_{t\le T} \|u_t\|_{-1,2}^{2p} + \mathbb{E}\left(\int_0^T \|u_t\|_{0,2}^2 dt\right)^p \le K\|u_0\|_{-1,2}^{2p}. \qquad (2.8)
$$

Proof: In this proof we adapt the arguments of Kurtz and Xiong [9] for the norm given by (2.3). Using a smoothing technique as in [9] if necessary, we may assume that $u_t \in H_2^0$. Note that for $u \in H_2^0$. By (2.3) we have

$$\langle u, \Delta u\rangle_{-1,2} = \langle u, \Delta u - u\rangle_{-1,2} + \|u\|_{-1,2}^2 = -\|u\|_{0,2}^2 + \|u\|_{-1,2}^2. \tag{2.9}$$

By Lemma 2.1 we then have

$$2\,\langle u, \Delta u\rangle_{-1,2} + \int_{\mathbb{R}} \|\nabla u h(y-\cdot)\|_{-1,2}^2 dy \le -\left(2 - \|h\|_{1,2}^2\right)\|u\|_{0,2}^2 + 2\|u\|_{-1,2}^2. \tag{2.10}$$

It follows from Lemma 2.2 that

$$\begin{aligned}
&\mathbb{E}\sup_{t\le r}\left|\int_0^t\int_{\mathbb{R}} 2\,\langle u_s, \nabla u_s h(y-\cdot)\rangle_{-1,2}\,\tilde{W}(dsdy)\right|^p \\
\le\;& K\mathbb{E}\left(\int_0^r\int_{\mathbb{R}} \langle u_s, \nabla u_s h(y-\cdot)\rangle_{-1,2}^2\,dsdy\right)^{p/2} \\
\le\;& K\mathbb{E}\int_0^r \|u_s\|_{-1,2}^{2p} ds.
\end{aligned} \tag{2.11}$$

Using (2.5) with $\beta = 0$, together with (2.10) and (2.11) we have

$$\mathbb{E}\sup_{s\le r}\|u_s\|_{-1,2}^{2p} + \mathbb{E}\left(\int_0^r \|u_s\|_{0,2}^2 ds\right)^p \le K\mathbb{E}\,\|u_0\|_{-1,2}^{2p} + K\int_0^r \mathbb{E}\,\|u_s\|_{-1,2}^{2p} ds. \tag{2.12}$$

Removing the second term on the LHS of (2.12), we get

$$\mathbb{E}\sup_{s\le r}\|u_s\|_{-1,2}^{2p} \le K_1\|u_0\|_{-1,2}^{2p} + K_2\int_0^r \mathbb{E}\,\|u_s\|_{-1,2}^{2p} ds.$$

It follows from Gronwall's inequality that

$$\mathbb{E}\sup_{s\le r}\|u_s\|_{-1,2}^{2p} \le K_1\|u_0\|_{-1,2}^{2p} e^{K_2 r}.$$

Removing the first term on the LHS of (2.12), we get

$$\mathbb{E}\left(\int_0^r \|u_s\|_{0,2}^2 ds\right)^p \le K_1\|u_0\|_{-1,2}^{2p} + K_2\int_0^r K_1\|u_0\|_{-1,2}^{2p} e^{K_2 r} ds \le K_3\|u_0\|_{-1,2}^{2p}.$$

∎

In most applications, we shall take $u_0 = \delta_y$ or $u_0 = \delta_{y_1} - \delta_{y_2}$. It is well known that for any $y \in \mathbb{R}$, $\delta_y \in H_2^{-\alpha}$ for any $\alpha > \frac{1}{2}$ (cf. Example 1 of Section 5.2 in the book of Barros-Neto [1]). This justifies the applicability of the last theorem.

We will prove a stronger version of Theorem 2.3 which is useful in estimating time-increment of the random field $X_t(y)$. To this end, we need the following two lemmas.

Lemma 2.4 *For $h \in H_2^1$ and for any $\alpha \in (0, \frac{1}{2}]$ there exists a constant K such that*

$$\int_{\mathbb{R}} \|\nabla u h(y-\cdot)\|_{-2\alpha,2}^2 dy \leq \frac{3}{2}\|\nabla u\|_{-2\alpha,2}^2 + K\|u\|_{0,2}^2.$$

Proof: Note that

$$\begin{aligned}
I(t,x,y) &\equiv T_t(u'h(y-\cdot))(x) - T_t(u')(x)T_t h(y-\cdot)(x) \\
&= \frac{1}{2}\int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u'(z_1)-u'(z_2))\left(h(y-z_1)-h(y-z_2)\right)\varphi_t(x-z_1)\varphi_t(x-z_2) \\
&= \frac{1}{2}\int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)-u(z_2))\left(h'(y-z_1)-h'(y-z_2)\right)\varphi_t(x-z_1)\varphi_t(x-z_2) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)-u(z_2))\left(h(y-z_1)-h(y-z_2)\right)\frac{z_1-z_2}{t}\varphi_t(x-z_1)\varphi_t(x-z_2).
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy |I(t,x,y)|^2 \\
\leq\ & K\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)-u(z_2))^2 \left(h'(y-z_1)-h'(y-z_2)\right)^2 \varphi_t(x-z_1)\varphi_t(x-z_2) \\
& + K\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)-u(z_2))^2 \left|\frac{1}{z_1-x}\int_x^{z_1} h'(y-z)dz\right|^2 \\
& \times \frac{|z_1-z_2|^2}{t}\varphi_t(x-z_1)\varphi_t(x-z_2) \\
\leq\ & K\int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)^2+u(z_2)^2)\varphi_{2t}(z_1-z_2) + K\int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)^2+u(z_2)^2)\varphi_{(2+\epsilon)t}(z_1-z_2) \\
=\ & K\|u\|_{0,2}^2,
\end{aligned}$$

where the constant K depends on $\|h'\|_{0,2}^2$ and we used the inequality

$$\begin{aligned}
\frac{x^2}{t}\varphi_t(x)\varphi_{(1+\epsilon)t}(x)^{-1} &= \frac{x^2}{t}\sqrt{1+\epsilon}\exp\left(\frac{x^2}{2(1+\epsilon)t} - \frac{x^2}{4t}\right) \\
&= \sqrt{1+\epsilon}\frac{x^2}{t}\exp\left(-\frac{\epsilon x^2}{4t(1+\epsilon)}\right) \\
&\leq \sqrt{1+\epsilon}\sup_{y\geq 0}\left(y\exp\left(-\frac{\epsilon}{2(1+\epsilon)}y\right)\right) \equiv K(\epsilon).
\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} dxdy \left|\int_0^\infty t^{\alpha-1}e^{-t}T_t(u')(x)T_t h(y-\cdot)(x)dt\right|^2$$

$$
\begin{aligned}
&= \int_{\mathbb{R}}\int_{\mathbb{R}} dxdy \int_0^\infty \int_0^\infty dsdt (ts)^{\alpha-1} e^{-(t+s)} T_s(u')(x) T_t(u')(x) T_t h(y-\cdot)(x) T_s h(y-\cdot)(x) \\
&\le \rho(0) \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty dsdt (ts)^{\alpha-1} e^{-(t+s)} T_s(u')(x) T_t(u')(x) \\
&= \|\nabla u\|^2_{-2\alpha,2}.
\end{aligned}
$$

By the triangular inequality, we have

$$
\begin{aligned}
\left(\int_{\mathbb{R}} \|\nabla u h(y-\cdot)\|^2_{-2\alpha,2} dy\right)^{1/2} &\le \left(\int_{\mathbb{R}}\int_{\mathbb{R}} dxdy \int_0^\infty t^{\alpha-1} e^{-t} I(t,x,y)^2 dt\right)^{1/2} + \|\nabla u\|_{-2\alpha,2} \\
&\le K\|u\|_{0,2} + \|\nabla u\|_{-2\alpha,2}.
\end{aligned}
$$

The conclusion then follows from the elementary inequality $(a+b)^2 \le \frac{3}{2}a^2 + 3b^2$. ∎

Lemma 2.5 *For $h \in H_2^1$, there exists a constant K such that for any $0 \le u \in H_2^0$,*

$$
\int_{\mathbb{R}} \langle u, \nabla u h(y-\cdot)\rangle^2_{-2\alpha,2} \, dy \le K \|u\|^2_{-2\alpha,2} \|u\|^2_{0,2}.
$$

Proof: Note that

$$
\begin{aligned}
&\left(\int_{\mathbb{R}} \langle u, \nabla u h(y-\cdot)\rangle^2_{-2\alpha,2} \, dy\right)^{1/2} \\
&= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dtds T_t u(x) T_s(u'h(y-\cdot))(x)\right)^2 dy\right)^{1/2} \\
&\le \sqrt{I_1} + \sqrt{I_2},
\end{aligned}
$$

where

$$
I_1 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dtds T_t u(x) T_s u'(x) h(y-x)\right)^2 dy
$$

and

$$
\begin{aligned}
I_2 = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dtds \\
\times T_t u(x) \left(T_s(u'h(y-\cdot))(x) - T_s u'(x) h(y-x)\right) \Big)^2 dy.
\end{aligned}
$$

By integration by parts and changing the order of T_s and ∇, we get

$$
\begin{aligned}
&\int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dtds T_t u(x) T_s u'(x) h(y-x) \\
&= \frac{1}{2} \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dtds T_t u(x) h'(y-x) T_s u(x).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_1 &\leq K \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) T_s u(x) \\
&\qquad \times \int_{\mathbb{R}} dx' \int_0^\infty \int_0^\infty (t's')^{\alpha-1} e^{-(t'+s')} dt' ds' T_{t'} u(x') T_{s'} u(x') \\
&= K \|u\|_{-2\alpha,2}^4 \leq K \|u\|_{-2\alpha,2}^2 \|u\|_{0,2}^2.
\end{aligned}
$$

Note that we used the non-negativity of $u(x)$ in the inequality above.

Now we estimate I_2. Note that

$$
I_2 = \int_{\mathbb{R}^2} d(x,x') \int_{(0,\infty)^4} d(t,s,t',s')(tst's')^{\alpha-1} e^{-(t+s+t'+s')} T_t u(x) T_{t'} u(x') J(x,x'),
$$

where

$$
\begin{aligned}
J(x,x') &= \int_{\mathbb{R}} dy \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z)(h(y-z)-h(y-x))u'(z) \\
&\qquad \times \varphi_{s'}(x'-z')(h(y-z')-h(y-x'))u'(z') \\
&= \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z)\varphi_{s'}(x'-z')u'(z)u'(z') \\
&\qquad \times \left(\rho(z-z') - \rho(x-z') - \rho(z-x') + \rho(x-x')\right).
\end{aligned}
$$

By integration by parts again, we can continue with

$$
\begin{aligned}
J(x,x') &\equiv \int_{\mathbb{R}^2} dz dz' \nabla_z \varphi_s(x-z) \nabla_{z'} \varphi_{s'}(x'-z') u(z) u(z') \\
&\qquad \times \left(\rho(z-z') - \rho(x-z') - \rho(z-x') + \rho(x-x')\right) \\
&\quad + \int_{\mathbb{R}^2} dz dz' \nabla_z \varphi_s(x-z) \varphi_{s'}(x'-z') \left(\rho'(x-z') - \rho'(z-z')\right) u(z) u(z') \\
&\quad + \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) \nabla_{z'} \varphi_{s'}(x'-z') \left(\rho'(z-z') - \rho'(z-x')\right) u(z) u(z') \\
&\quad - \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) \varphi_{s'}(x'-z') \rho''(z-z') u(z) u(z') \\
&\equiv J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

Note that for some $\epsilon > 0$ we have

$$
J_2 \leq K \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) \frac{|x-z|^2}{s} \varphi_{s'}(x'-z') u(z) u(z') \leq K_1 T_{(1+\epsilon)s} u(x) T_{s'} u(x').
$$

Let $\mathcal{G}u(x) = \int_0^\infty s^{\alpha-1} e^{-s} T_{(1+\epsilon)s} u(x) ds$. Then the corresponding term of J_2 in I_2 is bounded (up to a constant multiplication) by

$$
\|u\|_{-2\alpha,2}^2 \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) T_{(1+\epsilon)s} u(x)
$$

$$
\begin{aligned}
&= \|u\|^2_{-2\alpha,2}\left\langle (I-\Delta)^{-\alpha}u, \mathcal{G}u\right\rangle_{0,2}\\
&\leq \|u\|^3_{-2\alpha,2}\|\mathcal{G}u\|_{0,2}\\
&\leq \|u\|^3_{-2\alpha,2}\left(\int_0^\infty\int_0^\infty (ts)^{\alpha-1}e^{-(t+s)}dtds\|u\|^2_{0,2}\right)^{1/2}\\
&\leq K\|u\|^3_{-2\alpha,2}\|u\|_{0,2} \leq K\|u\|^2_{-2\alpha,2}\|u\|^2_{0,2}.
\end{aligned}
$$

The other terms can be estimated similarly. ∎

The following estimate will be used in the arguments in Section 5.

Theorem 2.6 *Suppose that the conditions of Theorem 2.3 hold. Then for* $\beta \in [0, 1/2], u_0 = \delta_z$ *and* $p \geq 1$ *we have*

$$
\mathbb{E}\sup_{t\leq T}\|u_t\|^{2p}_{\beta-1,2} + \mathbb{E}\left(\int_0^T \|u_t\|^2_{\beta,2}dt\right)^p \leq K\|\delta_z\|^{2p}_{\beta-1,2}. \tag{2.13}
$$

Proof: Similar to Theorem 2.3 we may assume that $u_t \in H^\beta_2$ a.s.. Further, using a stopping argument if necessary we may and will assume that the LHS of (2.13) is finite. Denote $1-\beta = 2\alpha$ for simplicity. By (2.5) and Lemma 2.4 we get

$$
\begin{aligned}
\|u_r\|^2_{-2\alpha,2} &\leq \|u_0\|^2_{-2\alpha,2} - \frac{1}{2}\int_0^r \|\nabla u_s\|^2_{-2\alpha,2}ds + 3\int_0^r \|u_s\|^2_{0,2}ds\\
&\quad + \int_0^r\int_{\mathbb{R}} 2\left\langle u_s, \nabla u_s h(y-\cdot)\right\rangle_{\beta-1,2}\tilde{W}(dsdy)\\
&\leq \|u_0\|^2_{-2\alpha,2} - \frac{1}{2}\int_0^r \|u_s\|^2_{1-2\alpha,2}ds + \frac{7}{2}\int_0^r \|u_s\|^2_{0,2}ds\\
&\quad + \int_0^r\int_{\mathbb{R}} 2\left\langle u_s, \nabla u_s h(y-\cdot)\right\rangle_{\beta-1,2}\tilde{W}(dsdy),
\end{aligned}
$$

where the last inequality follows from

$$
\begin{aligned}
\|\nabla u\|^2_{-2\alpha,2} &= d(\alpha)^2\left\langle \int_0^\infty t^{\alpha-1}e^{-t}T_tu'dt, \int_0^\infty t^{\alpha-1}e^{-t}T_tu'dt\right\rangle_{0,2}\\
&= -d(\alpha)^2\left\langle \int_0^\infty t^{\alpha-1}e^{-t}T_tudt, \Delta\int_0^\infty t^{\alpha-1}e^{-t}T_tudt\right\rangle_{0,2}\\
&= d(\alpha)^2\left\langle \int_0^\infty t^{\alpha-1}e^{-t}T_tudt, (I-\Delta)\int_0^\infty t^{\alpha-1}e^{-t}T_tudt\right\rangle_{0,2}\\
&\quad -d(\alpha)^2\left\langle \int_0^\infty t^{\alpha-1}e^{-t}T_tudt, \int_0^\infty t^{\alpha-1}e^{-t}T_tudt\right\rangle_{0,2}\\
&= \|(I-\Delta)^{\frac{1}{2}}(I-\Delta)^{-\alpha}u\|^2_2 - \|(I-\Delta)^{-\alpha}u\|^2_2\\
&= \|u\|^2_{1-2\alpha,2} - \|u\|^2_{-2\alpha,2} \geq \|u\|^2_{1-2\alpha,2} - \|u\|^2_{0,2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \mathbb{E} \sup_{t\le r} \|u_t\|_{\beta-1,2}^{2p} + \mathbb{E} \left(\int_0^r \|u_t\|_{\beta,2}^2 dt \right)^p \\
\le \; & K\|u_0\|_{-2\alpha,2}^{2p} + K\mathbb{E} \left(\int_0^r \|u_s\|_{0,2}^2 ds \right)^p + K\mathbb{E} \left(\int_0^r \int_{\mathbb{R}} 2 \left\langle u_s, \nabla u_s h(y-\cdot) \right\rangle_{\beta-1,2}^2 dyds \right)^{p/2} \\
\le \; & K_1\|u_0\|_{-2\alpha,2}^{2p} + K\mathbb{E} \left(\int_0^r \int_{\mathbb{R}} 2 \left\langle u_s, \nabla u_s h(y-\cdot) \right\rangle_{\beta-1,2}^2 dyds \right)^{p/2},
\end{aligned}
$$

where the last inequality follows from Theorem 2.3 and the fact $\|u_0\|_{-1,2} \le \|u_0\|_{-2\alpha,2}$.

By Lemma 2.5, we get

$$
\begin{aligned}
& \mathbb{E} \sup_{s\le r} \|u_s\|_{\beta-1,2}^{2p} + \mathbb{E} \left(\int_0^r \|u_s\|_{\beta,2}^2 ds \right)^p \\
\le \; & K\|\delta_z\|_{\beta-1,2}^{2p} + K\mathbb{E} \left(\int_0^r \|u_s\|_{\beta-1,2}^2 \|u_s\|_{0,2}^2 ds \right)^{p/2} \\
\le \; & K\|\delta_z\|_{\beta-1,2}^{2p} + K\mathbb{E} \left(\sup_{s\le r} \|u_s\|_{\beta-1,2}^2 \int_0^r \|u_s\|_{0,2}^2 ds \right)^{p/2} \\
\le \; & K\|\delta_z\|_{\beta-1,2}^{2p} + \frac{1}{2}\mathbb{E} \sup_{s\le r} \|u_s\|_{\beta-1,2}^{2p} + 8K^2\mathbb{E} \left(\int_0^r \|u_s\|_{0,2}^2 ds \right)^p \\
\le \; & K\|\delta_z\|_{\beta-1,2}^{2p} + \frac{1}{2}\mathbb{E} \sup_{s\le r} \|u_s\|_{\beta-1,2}^{2p} + K_1\|\delta_z\|_{-1,2}^{2p}.
\end{aligned}
$$

The conclusion then follows from easy calculations. ∎

3 A convolution representation

In this section, we establish a convolution representation for the density $X_t(x)$ in terms of a random transition function. We first define the random transition function by considering the spatial motion of a typical particle in the system, which satisfies

$$\xi_t = \xi_0 + B_t + \int_0^t \int_{\mathbb{R}} h(y-\xi_s) W(dsdy).$$

For $r \le t$ and $x \in \mathbb{R}$ fixed, we define the conditional transition probability

$$p_t^{r,x,W}(\cdot) \equiv p^W(r,x;t,\cdot) \equiv P^W(\xi_t \in \cdot | \xi_r = x).$$

Then for r and x fixed $p_t^{r,x,W}$ can be regarded as the optimal filter with vanishing observation function. Thus, it is a $\mathcal{P}(\mathbb{R})$-valued process satisfying the Zakai equation

$$\left\langle p_t^{r,x,W}, f \right\rangle = f(x) + \int_r^t \left\langle p_s^{r,x,W}, \Delta f \right\rangle ds + \int_r^t \int_{\mathbb{R}} \left\langle p_s^{r,x,W}, \nabla f h(y-\cdot) \right\rangle W(dsdy), \quad (3.1)$$

where $\mathcal{P}(\mathbb{R})$ is the space of Borel probability measures on $\mathbb{R}$. We refer the reader to the books of Kallianpur [5] and Xiong [17] for an introduction to nonlinear filtering and the related Zakai equation.

Next, we consider the dual equation on $C_b(\mathbb{R})$:

$$T_{r,t}(x) = f(x) + \int_r^t \Delta T_{s,t}(x)ds + \int_r^t \int_{\mathbb{R}} \nabla T_{s,t}(x)h(y-x)W(\hat{d}sdy), \qquad (3.2)$$

where $\hat{d}s$ stands for the backward Itô integral. We refer to Li et al [12] for the definition of the backward Itô integral. We also denote $T_{r,t}(x)$ by $T^f_{r,t}(x)$ to indicate the dependence on f. Similar to Corollary 6.22 in Xiong [17] it is easy to show that

$$T^f_{s,t}(x) = \int_{\mathbb{R}} f(y)p^W(s,x;t,dy) = \mathbb{E}\,^W_{s,x} f(\xi_t), \qquad (3.3)$$

where $\mathbb{E}\,^W_{s,x}$ denotes the conditional expectation given W and $\xi_s = x$.

The following convolution representation is the key in proving the joint continuity of $X_t(y)$. We shall denote $Z(dsdx) \equiv \sqrt{X_s(x)}B(dsdx)$.

Lemma 3.1 *Suppose that X_0 satisfies condition (I) and $f \in C^2_b(\mathbb{R})$. Then we have*

$$\langle X_t, f\rangle = \langle X_0, T_{0,t}\rangle + \int_0^t \int_{\mathbb{R}} T_{s,t}(x)Z(dsdx). \qquad (3.4)$$

Proof: Similar to Theorem 2.1 in Lee et al [11], we can prove that $X_t \in H^0_2$ and

$$\sup_{s\le t} \mathbb{E}\, \|X_s\|^2_{0,2} < \infty.$$

Denote the RHS of (3.4) by $\langle Y_t, f\rangle$. It is easy to show that Y_t is an H^0_2-valued process. Note that for $f \in C^2_b(\mathbb{R})$ we have

$$\begin{aligned}
&\langle Y_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle Y_s, \Delta f\rangle\, ds - \int_0^t \int_{\mathbb{R}} \langle Y_s, h(y-\cdot)\nabla f\rangle\, W(dsdy) \\
= \;& \left\langle X_0, T^f_{0,t}\right\rangle + \int_0^t \int_{\mathbb{R}} T^f_{s,t}(x)Z(dsdx) - \langle X_0, f\rangle \\
& - \int_0^t \left\{ \left\langle X_0, T^{\Delta f}_{0,s}\right\rangle + \int_0^s \int_{\mathbb{R}} T^{\Delta f}_{r,s}(x)Z(drdx)\right\} ds \\
& - \int_0^t \int_{\mathbb{R}} \left\{ \left\langle X_0, T^{h(y-\cdot)\nabla f}_{0,s}\right\rangle + \int_0^s \int_{\mathbb{R}} T^{h(y-\cdot)\nabla f}_{r,s}(x)Z(drdx)\right\} W(dsdy) \\
= \;& \left\langle X_0, T^f_{0,t} - f - \int_0^t T^{\Delta f}_{0,s} ds - \int_0^t \int_{\mathbb{R}} T^{h(y-\cdot)\nabla f}_{0,s} W(dsdy)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}}\mathbb{E}\,{}^W_{s,x}f(\xi_t)Z(dsdx)-\int_0^t\int_{\mathbb{R}}\int_r^t\mathbb{E}\,{}^W_{r,x}\Delta f(\xi_s)dsZ(drdx)\\
&-\int_0^t\int_{\mathbb{R}}\int_r^t\int_{\mathbb{R}}\mathbb{E}\,{}^W_{r,x}\left(h(y-\xi_s)\nabla f(\xi_s)\right)W(dsdy)Z(drdx)\\
= &\int_{\mathbb{R}}X_0(dx)\mathbb{E}\,{}^W_{0,x}\left(f(\xi_t)-f(x)-\int_0^t\Delta f(\xi_s)ds-\int_0^t\int_{\mathbb{R}}h(y-\xi_s)\nabla f(\xi_s)W(dsdy)\right)\\
&+\int_0^t\int_{\mathbb{R}}Z(dsdx)\mathbb{E}\,{}^W_{s,x}\left\{f(\xi_t)-\int_s^t\Delta f(\xi_r)dr-\int_s^t\int_{\mathbb{R}}h(y-\xi_r)\nabla f(\xi_r)W(drdy)\right\}\\
= &\int_0^t\int_{\mathbb{R}}f(x)Z(dsdx).
\end{aligned}
$$

Let $\tilde{X}_t=X_t-Y_t$. By (1.5) $\tilde{X}$ is an H_2^0-valued solution to the following linear SDE

$$\left\langle\tilde{X}_t,f\right\rangle=\int_0^t\left\langle\tilde{X}_s,\Delta f\right\rangle ds+\int_0^t\int_{\mathbb{R}}\left\langle\tilde{X}_s,h(y-\cdot)\nabla f\right\rangle W(dsdy). \tag{3.5}$$

By Theorem 3.5 in Kurtz and Xiong [9] we have that $\tilde{X}=0$. ∎

4 An estimate in spatial increment

In this section we estimate spatial increment of the density $X_t(y)$. As a consequence, we shall see that for $t>0$ fixed, $X_t(y)$ is Hölder continuous with exponent $1/2-\epsilon$.

Applying Theorem 2.3 to (3.1), we see that $p^W(s,x;t,\cdot)$ has a density, denote it by $p^W(s,x;t,y)$. By Lemma 3.1, $X_t(y)$ can be represented as

$$X_t(y)=\int_{\mathbb{R}}\mu(x)p^W(0,x;t,y)dx+\int_0^t\int_{\mathbb{R}}p^W(s,x;t,y)Z(dsdx)\equiv X_t^1(y)+X_t^2(y). \tag{4.1}$$

To prove the joint continuity by Kolmogorov's criteria, we need the following estimate.

Lemma 4.1 *Suppose that Condition (I) holds. Then $\forall\ p\geq 1$,*

$$
\begin{aligned}
&\mathbb{E}\left|\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))Z(dsdx)\right|^{2p}\\
\leq\ &K\left(\mathbb{E}\left|\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))^2dxds\right|^{2p-1}\right)^{\frac{p}{2p-1}}.
\end{aligned} \tag{4.2}
$$

Proof: By BDG inequality, we have

$$
\begin{aligned}
L\ \equiv\ &\mathbb{E}\left|\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))Z(dsdx)\right|^{2p}\\
\leq\ &K\mathbb{E}\,\mathbb{E}^W\left|\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))^2X_s(x)dxds\right|^p.
\end{aligned}
$$

For $2=(2p-1)/p+1/p$, applying the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
L &\leq K\mathbb{E}\left(\left|\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))^2dxds\right|^{\frac{2p-1}{2}}\right.\\
&\qquad\left.\times\left|\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))^2X_s(x)^{2p}dxds\right|^{\frac{1}{2}}\right)\\
&\leq K\left(\mathbb{E}\left|\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))^2dxds\right|^{2p-1}\right)^{\frac{1}{2}}\\
&\qquad\times\left(\mathbb{E}\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))^2X_s(x)^{2p}dxds\right)^{\frac{1}{2}}\\
&\equiv KI\times J.
\end{aligned}
$$

Since μ is bounded, it is easy to show that

$$\sup_{t,x}\langle\mu,\varphi_t(x-\cdot)\rangle<\infty. \tag{4.3}$$

It then follows from the same arguments as in the proof of Lemma 3.1 of Lee et al [11] that $\mathbb{E}\ X_s(x)^{2p}$ is bounded. Therefore,

$$J\leq K\left(\mathbb{E}\int_0^t\int_{\mathbb{R}}(p^W(s,x;t,y_1)-p^W(s,x;t,y_2))^2dxds\right)^{\frac{1}{2}}\leq KI^{1/(2p-1)}.$$

Thus, $L\leq KI^{2p/(2p-1)}$ which coincides with the RHS of (4.2). ∎

As a consequence of Theorem 2.3, we get

Proposition 4.2 *Suppose the conditions of Theorem 1.1 hold. Let $t\in[0,T]$ and $p\geq 1$ be fixed. Then, there exists a constant $K=K(p,T)$ such that*

$$\mathbb{E}\ |X_t^2(y_1)-X_t^2(y_2)|^{2p}\leq K|y_1-y_2|^p,\qquad\forall\ y_1,\ y_2\in\mathbb{R}. \tag{4.4}$$

Consequently, for $t>0$ fixed X_t^2 is Hölder continuous with exponent $1/2-\epsilon$ for any $\epsilon>0$.

Proof: Let $u_s(x)=p^W(t-s,x,t,y_1)-p^W(t-s,x,t,y_2)$. Then u solves equation (1.9) with $u_0=\delta_{y_1}-\delta_{y_2}$. For any $f\in H_2^1$ we have

$$|\langle u_0,f\rangle|=|f(y_1)-f(y_2)|=\left|\int_{y_1}^{y_2}f'(s)ds\right|\leq\sqrt{|y_2-y_1|}\|f\|_{1,2}.$$

Thus,

$$\|u_0\|_{-1,2} \le \sqrt{|y_2 - y_1|}. \tag{4.5}$$

By Theorem 2.3 we get

$$\mathbb{E} \left(\int_0^t \int_{\mathbb{R}} |p^W(s,x,t,y_1) - p^W(s,x,t,y_2)|^2 dxds \right)^p \le K|y_1 - y_2|^p.$$

Inequality (4.4) then follows from Lemma 4.1. ∎

Finally, we consider $X_t^1(y)$.

Proposition 4.3 *Suppose the conditions of Theorem 1.1 hold. Let $t \in [0,T]$. Then, for $p \ge 1$, there exists a constant $K = K(p,T)$ such that*

$$\mathbb{E} \ |X_t^1(y_1) - X_t^1(y_2)|^{2p} \le K|y_1 - y_2|^p.$$

Proof: Note that

$$\begin{aligned} \mathbb{E} \ |X_t^1(y_1) - X_t^1(y_2)|^{2p} &= \mathbb{E} \left| \int_{\mathbb{R}} \left(p^W(0,x;t,y_1) - p^W(0,x;t,y_2) \right) \mu(x) dx \right|^{2p} \\ &\le \mathbb{E} \ \|p^W(0,\cdot;t,y_1) - p^W(0,\cdot;t,y_2)\|_{-1,2}^{2p} \|\mu\|_{1,2}^{2p} \\ &\le K_1 \|\delta_{y_1} - \delta_{y_2}\|_{-1,2}^{2p} \|\mu\|_{1,2}^{2p}. \end{aligned}$$

The conclusion then follows from (4.5). ∎

5 Estimates in time increment

In this section we consider time-increments of the types of

$$\int_0^{t_1} \int_{\mathbb{R}} \left(p^W(s,x;t_2,y) - p^W(s,x;t_1,y) \right) Z(dsdx) \tag{5.1}$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}} p^W(s,x;t_2,y) Z(dsdx). \tag{5.2}$$

For the type of (5.1), we first use Theorem 2.3 to obtain a preliminary estimate by $\mathbb{E} \ \|u_{t_2-t_1} - \delta_y\|_{-1,2}^{2p}$, where u_t is a solution to SDE (1.9) with $u_0 = \delta_y$. To further estimate this quantity, we need to develop two major techniques, i.e., the partial convolution by kernel φ_{r^α} and the partial integration by parts introduced in Section 1. For the type of (5.2), we will use a technique developed by Xiong and Zhou [18].

Lemma 5.1 *For any $t_1 < t_2$ and $y \in \mathbb{R}$, we have*

$$\mathbb{E}\left(\int_0^{t_1}\int_{\mathbb{R}}\left(p^W(s,x;t_2,y)-p^W(s,x;t_1,y)\right)Z(dsdx)\right)^{2p} \le K\mathbb{E}\left\|p^W(t_1,\cdot;t_2,y)-\delta_y\right\|_{-1,2}^{2p}.$$

Proof: Note that $p^W(t_1-s,x;t_2,y)-p^W(t_1-s,x;t_1,y)$ is the solution of SPDE (1.9) with initial condition $p^W(t_1,\cdot;t_2,y)-\delta_y$ and hence,

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{t_1}\int_{\mathbb{R}}\left(p^W(s,x;t_2,y)-p^W(s,x;t_1,y)\right)Z(dsdx)\right)^{2p}\\
\le\ & K\left(\mathbb{E}\left(\int_0^{t_1}\int_{\mathbb{R}}\left(p^W(s,x;t_2,y)-p^W(s,x;t_1,y)\right)^2dsdx\right)^{2p-1}\right)^{\frac{p}{2p-1}}\\
\le\ & K\mathbb{E}\left\|p^W(t_1,\cdot;t_2,y)-\delta_y\right\|_{-1,2}^{2p}.
\end{aligned}$$

∎

Let $u_s(x)=p^W(t_2-s,x;t_2,y)$. Then u solves (1.9) with $u_0=\delta_y$. As Δu_s is not in H_2^{-1} we cannot use (1.9) directly to get an estimate on $\mathbb{E}\left\|u_{t_2-t_1}-\delta_y\right\|_{-1,2}^{2p}$. Instead, fixing t and taking differential of $\int_{\mathbb{R}}u_{t-r}(x)\varphi_{r^\alpha}(z-x)dx$ with respect to r, and then taking integral we get (1.10). Denote the second and the third term on the RHS by I_2 and I_3, respectively. Write the fourth term by I_4-I_5 with

$$I_4=\alpha\int_0^t\int_{\mathbb{R}}(I-\Delta)u_{t-r}(x)\varphi_{r^\alpha}(z-x)dxr^{\alpha-1}dr$$

and

$$I_5=\alpha\int_0^t\int_{\mathbb{R}}u_{t-r}(x)\varphi_{r^\alpha}(z-x)dxr^{\alpha-1}dr.$$

Then

$$u_t(z)-\delta_y(z)=I_1+I_2+I_3+I_4-I_5.$$

We now estimate $I_j,\ j=1,2,\cdots,5$, separately. Although the following result can be implied directly from the analyticity of Δ on $L^2(\mathbb{R})$, we give a brief and elementary proof for the convenience of the reader.

Lemma 5.2 *For $\beta\in(0,1)$ there is a constant such that for $r\in(0,T)$ we have*

$$\int_{\mathbb{R}}\left|(I-\Delta)^\beta\varphi_r(x)\right|dx\le Kr^{-\beta}. \tag{5.3}$$

Proof: Note that the integral in the definition of $(I-\Delta)^\beta$ can be split up into two parts: I_1 denotes the part from 0 to r and I_2 from r to ∞. Then

$$\int_{\mathbb{R}} |I_2(x)|dx \le \int_r^\infty \frac{e^{-t}+1}{t^{1+\beta}} dt \le \frac{2}{\beta} r^{-\beta}.$$

For $t \le r$, we have

$$\begin{aligned}
& \left| e^{-t}\varphi_{t+r}(x) - \varphi_r(x) \right| \varphi_{t+r}(x)^{-1} \\
= & \left| e^{-t} - \sqrt{\frac{t+r}{r}} \exp\left(-\frac{x^2}{2r} + \frac{x^2}{2(t+r)} \right) \right| \\
\le & \left| e^{-t} - 1 \right| + \left| 1 - \sqrt{\frac{t+r}{r}} \right| + \sqrt{\frac{t+r}{r}} \left| 1 - \exp\left(-\frac{tx^2}{2r(t+r)} \right) \right| \\
\le & \sqrt{2} \left(\frac{tx^2}{2r(r+t)} + t + \frac{t}{r} \right).
\end{aligned}$$

Multiplying both sides by $\varphi_{t+r}(x)$ and taking integral we see that $\int_{\mathbb{R}} |I_1(x)|dx \le Kr^{-\beta}$. ∎

Now we estimate I_4. Note that

$$\begin{aligned}
\|I_4\|_{-1,2} \le & \alpha \int_0^t \left\| \int_{\mathbb{R}} (I-\Delta)^{\frac{1+\beta}{2}} u_{t-r}(\cdot - x)(I-\Delta)^{\frac{1-\beta}{2}} \varphi_{r^\alpha}(x) dx \right\|_{-1,2} r^{\alpha-1} dr \\
\le & K \int_0^t \left\| (I-\Delta)^{\frac{1+\beta}{2}} u_{t-r} \right\|_{-1,2} \int_{\mathbb{R}} \left| (I-\Delta)^{\frac{1-\beta}{2}} \varphi_{r^\alpha}(x) \right| dx r^{\alpha-1} dr \\
\le & K \int_0^t \left\| (I-\Delta)^{\frac{1+\beta}{2}} u_{t-r} \right\|_{-1,2} r^{-\frac{\alpha}{2}(1-\beta)} r^{\alpha-1} dr \\
\le & K \left(\int_0^t \left\| (I-\Delta)^{\frac{1+\beta}{2}} u_{t-r} \right\|_{-1,2}^2 dr \right)^{\frac{1}{2}} \left(\int_0^t r^{\alpha(1+\beta)-2} dr \right)^{\frac{1}{2}} \\
= & K \left(\int_0^t \|u_r\|_{\beta,2}^2 dr \right)^{\frac{1}{2}} t^{(\alpha(1+\beta)-1)/2}
\end{aligned}$$

where $\beta \in (0,1/2)$ is chosen such that $\alpha(1+\beta) > 1$. Thus, $\mathbb{E}\, \|I_4\|_{-1,2}^{2p} \le Kt^{(\alpha(1+\beta)-1)p}$. I_2 and I_5 can be estimated similarly (easier).

Next, we estimate I_3. Note that

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} \nabla u_{t-r}(x) h(y-x) \varphi_{r^\alpha}(\cdot - x) dx \right\|_{-1,2}^2 drdy \\
= & \int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} u_{t-r}(x) \nabla h(y-x) \varphi_{r^\alpha}(\cdot - x) dx \right\|_{-1,2}^2 drdy
\end{aligned}$$

$$+\int_0^t\int_{\mathbb{R}}\left\|\int_{\mathbb{R}}u_{t-r}(x)h(y-x)\nabla\varphi_{r^\alpha}(\cdot-x)dx\right\|_{-1,2}^2 drdy \equiv I_{31}+I_{32}.$$

We calculate

$$\begin{aligned}
I_{32} &= \int_0^t\int_{\mathbb{R}}\left\|\int_{\mathbb{R}}u_{t-r}(\cdot-x)h(y+x-\cdot)\nabla\varphi_{r^\alpha}(x)dx\right\|_{-1,2}^2 drdy \\
&= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\langle u_{t-r}(\cdot-x)h(y+x-\cdot),u_{t-r}(\cdot-x')h(y+x'-\cdot)\rangle_{-1,2} \\
&\qquad \times\nabla\varphi_{r^\alpha}(x)\nabla\varphi_{r^\alpha}(x')dxdx'dydr \\
&= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\infty\int_0^\infty (uv)^{-1/2}e^{-(u+v)} \\
&\qquad \times\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi_u(z-z_1)u_{t-r}(z_1-x)h(y+x-z_1)dz_1 \\
&\qquad\quad \times\int_{\mathbb{R}}\varphi_v(z-z_2)u_{t-r}(z_2-x')h(y+x'-z_2)dz_2dzdudv \\
&\qquad\qquad \times\nabla\varphi_{r^\alpha}(x)\nabla\varphi_{r^\alpha}(x')dxdx'dydr \\
&\le K\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\langle u_{t-r}(\cdot-x),u_{t-r}(\cdot-x')\rangle_{-1,2}|\nabla\varphi_{r^\alpha}(x)||\nabla\varphi_{r^\alpha}(x')|dxdx'dr \\
&\le K\int_0^t\left(\int_{\mathbb{R}}\|u_{t-r}(\cdot-x)\|_{-1,2}|\nabla\varphi_{r^\alpha}(x)|dx\right)^2 dr \\
&\le K\sup_{r\le t}\|u_r\|_{-1,2}^2\int_0^t r^{-\alpha}dr \le K\sup_{r\le t}\|u_r\|_{-1,2}^2 t^{1-\alpha},
\end{aligned}$$

where in the first inequality we used the identity (1.4) and $\rho(x)\le 1$. I_{31} can be estimated similarly. Estimation for I_1 is easy. To summarize, we get

Proposition 5.3 *For $p\ge 1$, $\alpha\in(0,1)$ and $\beta\in(0,1/2)$ satisfying $\alpha(1+\beta)>1$, there exists a constant K such that $\forall\ t_1<t_2$, we have*

$$\begin{aligned}
&\mathbb{E}\left(\int_0^{t_1}\int_{\mathbb{R}}\left(p^W(s,x;t_2,y)-p^W(s,x;t_1,y)\right)^2 Z(dsdx)\right)^p \\
&\le K\max\left(|t_2-t_1|^{(\alpha(1+\beta)-1)p},|t_2-t_1|^{(1-\alpha)p}\right).
\end{aligned}$$

Finally, we estimate

$$\mathbb{E}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}}p^W(s,x,t_2,y)^2Z(dsdx)\right)^{2p}.$$

Similar to Section 4, the above moment is bounded by

$$\left(\mathbb{E}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}}p^W(s,x,t_2,y)^2dxds\right)^{2p-1}\right)^{\frac{p}{2p-1}}$$

which we shall estimate using the method of Xiong and Zhou [18].

The key identity proved in [18] is given in the following lemma. We sketch the proof for convenience of the reader since [18] is not easily accessible.

Lemma 5.4 *For any $k \in \mathbb{N}$, $s < t$ and $x, y \in \mathbb{R}^k$, we have*

$$\mathbb{E}\ \Pi_{i=1}^k p^W(s, x_i, t, y_i) = P_k(t-s, x, y),$$

where P_k is the transition function of the k-dimensional Markov process consisting of the motion of k particles of the branching particles system introduced in Section 1.

Sketch of the proof Let t and y be fixed. We define $u_r^i(x_i) = p^W(t-r, x^i, t, y)$, $i = 1, 2, \cdots, k$. Then u^i is a solution to (1.9) with initial δ_y. Applying Itô's formula to the product and taking expectation, we get

$$\frac{d}{dr}\mathbb{E}\ \Pi_{i=1}^k u_r^i(x^i) = A_k \mathbb{E}\ \Pi_{i=1}^k u_r^i(x^i)$$

where A_k is the generator of the k-dimensional Markov process consisting of the motion of k particles of the branching particles system. The conclusion of the lemma then follows easily. ∎

Lemma 5.5 *For any integer $n \geq 1$, we have*

$$\mathbb{E}\ \left(\int_{t_1}^{t_2}\int_{\mathbb{R}} p^W(s, x, t_2, y)^2 dx ds\right)^n \leq K|t_2 - t_1|^{n/2}. \tag{5.4}$$

Proof: Let $t_1 = 0$ and $t_2 = t$ for simplicity. The LHS of (5.4) is estimated as follows.

$$\begin{aligned}
L &\equiv n!\mathbb{E}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx_1\cdots dx_n \Pi_{i=1}^n p^W(s_i, x_i, t, y)^2 \\
&= n!\mathbb{E}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx_1\cdots dx_n \Pi_{i=2}^n p^W(s_i, x_i, t, y)^2 \\
&\quad \times \int_{\mathbb{R}} p^W(s_1, x_1, s_2, x_{11}) p^W(s_2, x_{11}, t, y) dx_{11} \int_{\mathbb{R}} p^W(s_1, x_1, s_2, x_{12}) p^W(s_2, x_{12}, t, y) dx_{12} \\
&= n!\mathbb{E}\int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx_1\cdots dx_n \Pi_{i=2}^n p^W(s_i, x_i, t, y)^2 \\
&\quad \times \int_{\mathbb{R}}\int_{\mathbb{R}} P_2(s_2 - s_1, (x_1, x_1), (x_{11}, x_{12})) p^W(s_2, x_{11}, t, y) p^W(s_2, x_{12}, t, y) dx_{11} dx_{12},
\end{aligned}$$

where the last equality follows from Lemma 5.4. Note that

$$P_2(s_2-s_1,(x_1,x_1),(x_{11},x_{12})) \le \frac{K}{\sqrt{s_2-s_1}}\varphi_{s_2-s_1}(x_1-x_{11}).$$

We now continue the estimate with

$$\begin{aligned}
L &\le K\mathbb{E}\int_0^t \frac{ds_1}{\sqrt{s_2-s_1}}\int_{s_1}^t ds_2\cdots\int_{s_{n-1}}^t ds_n\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx_{11}dx_{12}dx_2\cdots dx_n\\
&\quad \times p^W(s_2,x_{11},t,y)p^W(s_2,x_{12},t,y)\Pi_{i=2}^n p^W(s_i,x_i,t,y)^2\\
&= K\mathbb{E}\int_0^t \frac{ds_1}{\sqrt{s_2-s_1}}\int_{s_1}^t ds_2\cdots\int_{s_{n-1}}^t ds_n\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx_{11}dx_{12}dx_2\cdots dx_n\\
&\quad \times\int_{\mathbb{R}} p^W(s_2,x_{11},s_3,x'_{11})p^W(s_3,x'_{11},t,y)dx'_{11}\\
&\quad \times\int_{\mathbb{R}} p^W(s_2,x_{12},s_3,x'_{12})p^W(s_3,x'_{12},t,y)dx'_{12}\\
&\quad \times\int_{\mathbb{R}} p^W(s_2,x_2,s_3,x_{21})p^W(s_3,x_{21},t,y)dx_{21}\\
&\quad \times\int_{\mathbb{R}} p^W(s_2,x_2,s_3,x_{22})p^W(s_3,x_{22},t,y)dx_{22}\Pi_{i=3}^n p^W(s_i,x_i,t,y)^2\\
&= K\mathbb{E}\int_0^t \frac{ds_1}{\sqrt{s_2-s_1}}\int_{s_1}^t ds_2\cdots\int_{s_{n-1}}^t ds_n\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx_{11}dx_{12}dx_2\cdots dx_n\\
&\quad \times\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} dx'_{11}dx'_{12}dx_{21}dx_{22}P_4(s_3-s_2,(x_{11},x_{12},x_2,x_2),(x'_{11},x'_{12},x_{21},x_{22}))\\
&\quad \times p^W(s_3,x'_{11},t,y)p^W(s_3,x'_{12},t,y)p^W(s_3,x_{21},t,y)p^W(s_3,x_{22},t,y)\Pi_{i=3}^n p^W(s_i,x_i,t,y)^2,
\end{aligned}$$

where the last equality follows again from Lemma 5.4. Note that

$$\begin{aligned}
&P_4(s_3-s_2,(x_{11},x_{12},x_2,x_2),(x'_{11},x'_{12},x_{21},x_{22}))\\
\le\ & \frac{K}{\sqrt{s_3-s_2}}\varphi_{s_3-s_2}(x'_{11}-x_{11})\varphi_{s_3-s_2}(x'_{12}-x_{12})\varphi_{s_3-s_2}(x_{21}-x_2).
\end{aligned}$$

Finally, we continue to estimate the LHS of (5.4) with

$$\begin{aligned}
L &\le K\mathbb{E}\int_0^t \frac{ds_1}{\sqrt{s_2-s_1}}\int_{s_1}^t \frac{ds_2}{\sqrt{s_3-s_2}}\cdots\int_{s_{n-1}}^t ds_n\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx'_{11}dx'_{12}dx_{21}dx_{22}dx_3\cdots dx_n\\
&\quad \times p^W(s_3,x'_{11},t,y)p^W(s_3,x'_{12},t,y)p^W(s_3,x_{21},t,y)p^W(s_3,x_{22},t,y)\Pi_{i=3}^n p^W(s_i,x_i,t,y)^2.
\end{aligned}$$

Continue this procedure, we see that

$$L \le K\mathbb{E}\int_0^t \frac{ds_1}{\sqrt{s_2-s_1}}\int_{s_1}^t \frac{ds_2}{\sqrt{s_3-s_2}}\cdots\int_{s_{n-1}}^t \frac{ds_n}{\sqrt{t-s_n}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} dx_{11}dx_{12}\cdots dx_{n1}dx_{n2}$$

$$
\begin{aligned}
&\Pi_{i=1}^{n} p^W(s_n, x_{i1}, t, y) p^W(s_n, x_{i2}, t, y) \\
\leq\;& K\mathbb{E} \int_0^t \frac{ds_1}{\sqrt{s_2 - s_1}} \int_{s_1}^t \frac{ds_2}{\sqrt{s_3 - s_2}} \cdots \int_{s_{n-1}}^t \frac{ds_n}{\sqrt{t - s_n}} \\
\leq\;& K t^{n/2}.
\end{aligned}
$$

Thus we finish the proof by replacing t by $t_2 - t_1$. ∎

To summarize, we get

Proposition 5.6 *Suppose the conditions of Theorem 1.1 hold. Then, there exist integer $p \geq 1$ and real numbers $\epsilon > 0$ and $K > 0$ such that $\forall\ t_1 < t_2$ and $y \in \mathbb{R}$, we have*

$$
\mathbb{E}\, |X_{t_1}^2(y) - X_{t_2}^2(y)|^{2p} \leq K |t_1 - t_2|^{2+\epsilon}. \tag{5.5}
$$

Proof: Choose $p \geq 2$, $\alpha \in (0,1)$ and $\beta \in \left(0, \frac{1}{2}\right)$ such that

$$
\min \left\{ (\alpha(1+\beta) - 1)p,\ (1-\alpha)p,\ \frac{p}{2} \right\} \geq 2 + \epsilon.
$$

By Proposition 5.3 and Lemma 5.5, we see that (5.5) holds. ∎

Note that

$$
\begin{aligned}
\mathbb{E}\, |X_{t_1}^1(y) - X_{t_2}^1(y)|^{2p} &= \mathbb{E} \left| \int_{\mathbb{R}} \left(p^W(0, x; t_2, y) - p^W(0, x; t_1, y) \right) \mu(x) dx \right|^{2p} \\
&\leq \mathbb{E}\, \| p^W(0, \cdot; t_2, y) - p^W(0, \cdot; t_1, y) \|_{-1,2}^{2p} \|\mu\|_{1,2}^{2p}.
\end{aligned}
$$

Similar to the proof for $X_t^2(y)$, we get

Proposition 5.7 *Suppose the conditions of Theorem 1.1 hold. Then, there exist integer $p \geq 1$ and real numbers $\epsilon > 0$ and $K > 0$ such that $\forall\ t_1 < t_2$ and $y \in \mathbb{R}$, we have*

$$
\mathbb{E}\, |X_{t_1}^1(y) - X_{t_2}^1(y)|^{2p} \leq K |t_1 - t_2|^{2+\epsilon}.
$$

Remark 5.8 *It is conjectured by Yaozhong Hu and David Nualart that for x fixed, $X_t(x)$ should be Hölder continuous in t with exponent $1/4 - \epsilon$. However, the method in this paper cannot confirm this conjecture. Instead, it follows from Proposition 5.3 that $X_t(x)$ is Hölder continuous in t with exponent $\min\left(\alpha(1+\beta) - 1, 1 - \alpha\right)/2 - \epsilon$. Since $\alpha < 1$ and $\beta < 1/2$, the best Hölder exponent we can get here is $1/10 - \epsilon$.*

Proof of Theorem 1.1: Combining Propositions 4.2, 4.3, 5.6 and 5.7, we get

$$\mathbb{E}\,|X_{t_1}(y_1) - X_{t_2}(y_2)|^{2p} \le K|(t_1, y_1) - (t_2, y_2)|^{2+\epsilon}.$$

The joint continuity then follows from Kolmogorov's criteria. ∎

Acknowledgement: The third author would like to thank Yaozhong Hu and David Nualart for discussions during his talk at University of Kansas. We would like to thank two anonymous referees and an associate editor for constructive suggestions which lead to substantial improvement of the paper.

References

[1] J. Barros-Neto (1973). *An Introduction to the Theory of Distributions.* Marcel Dekker, Inc. New York.

[2] D. A. Dawson, Z. Li and H. Wang (2001). Superprocesses with dependent spatial motion and general branching densities. *Electron. J. Probab.* **6**, 1–33.

[3] D.A. Dawson, J. Vaillancourt and H. Wang (2000). Stochastic partial differential equations for a class of interacting measure-valued diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **36**,2:167–180.

[4] M. Foondum and D. Khoshnevisan (2009). Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.* **14**, 548–568.

[5] G. Kallianpur (1980). *Stochastic Filtering Theory.* Springer-Verlag.

[6] N. Konno and T. Shiga (1988). Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Related Fields* **79** , 201-225.

[7] M.A. Krasnoselskii, E.I. Pustylnik, P.E. Sobolevski and P.P. Zabrejko (1976). *Integral Operators in Spaces of Summable Functions*, Nauka, Moscow, 1966 in Russian; English translation: Noordhoff International Publishing, Leyden.

[8] N.V. Krylov (1999). An analytic approach to SPDEs, Stochastic partial differential equations: six perspectives, *Math. Surveys Monogr.*, 64, 185-242, Amer. Math. Soc., Providence, RI.

[9] T. Kurtz and J. Xiong (1999). Particle representations for a class of nonlinear SPDEs. *Stochastic Process. Appl.* **83**, 103–126.

[10] T. Kurtz and J. Xiong (2004). A stochastic evolution equation arising from the fluctuation of a class of interacting particle systems. *Commun. Math. Sci.* **2**, 325–358.

[11] K.J. Lee, C. Mueller and J. Xiong, Some properties for superprocess over a stochastic flow. To appear in *Ann. Inst. Henri Poincaré Probab. Stat.* **45**, 477-490.

[12] Z. Li, H. Wang and J. Xiong (2005). Conditional log-Laplace functionals of superprocesses with dependent spatial motion. *Acta Appl. Math.* **88**, 143–175.

[13] M. Reimers (1989). One-dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Related Fields* **81**, 319–340.

[14] G. Skoulakis and R.J. Adler (2001). Superprocesses over a stochastic flow. *Ann. Appl. Probab.* **11**, 488–543.

[15] H. Wang (1997). State classification for a class of measure-valued branching diffusions in a Brownian medium. *Probab. Theory Related Fields* **109**, 39–55.

[16] H. Wang (1998). A class of measure-valued branching diffusions in a random medium. *Stoch. Anal. Appl.* **16**, 753–786

[17] J. Xiong (2008). *An Introduction to Stochastic Filtering Theory*, Oxford Graduate Texts in Mathematics **18**. Oxford University Press, Oxford.

[18] J. Xiong and X. Zhou (2004). Superprocess over a stochastic flow with superprocess catalyst. *Int. J. Pure Appl. Math.* **17**, 353–382.

Zenghu Li: Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R. China. E-mail: `lizh@bnu.edu.cn`

Hao Wang: Department of Mathematics, University of Oregon, Eugene OR 97403-1222, U.S.A. E-mail: `haowang@uoregon.edu`

Jie Xiong: Corresponding author. Department of Mathematics, Hebei Normal University, Shijiazhuang 050016, P.R. China, and, Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1300, U.S.A. E-mail: jxiong@math.utk.edu

Xiaowen Zhou: Department of Mathematics and Statistics, Concordia University, Montreal, Quebec H3G 1M8, Canada. E-mail: xzhou@mathstat.concordia.ca