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Strong solutions for stochastic differential equations with jumps

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Abstract. General stochastic equations with jumps are studied. We provide criteria for the uniqueness and existence of strong solutions under non-Lipschitz conditions of Yamada-Watanabe type. The results are applied to stochastic equations driven by spectrally positive Lévy processes.

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1 Introduction

The question of pathwise uniqueness for one-dimensional stochastic differential equations driven by one-dimensional Brownian motions has been resolved a long time ago by Yamada and Watanabe [8]; see also Barlow [1]. The same question can also be asked for stochastic differential equations driven by discontinuous Lévy noises. Let us consider the equation

$$dx(t) = F(x(t-))dL_t, \qquad t \geq 0. \tag{1.1}$$

Bass [2] and Komatsu [6] showed that if $\{L_t\}$ is a symmetric stable process with exponent $\alpha \in (1,2)$ and if $x \mapsto F(x)$ is a bounded function with modulus of continuity $z \mapsto \rho(z)$ satisfying

$$\int_{0+} \frac{1}{\rho(z)^\alpha} dz = \infty, \tag{1.2}$$

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then (1.1) admits a strong solution and the solution is pathwise unique. This condition is the analogue of the Yamada-Watanabe criterion for the diffusion coefficient. In particular, if F is Hölder continuous with exponent $1/\alpha$, then the pathwise uniqueness holds for (1.1). The required Hölder exponent tends to 1/2 as $\alpha \to 2$ and it tends to 1 (Lipschitz condition) as $\alpha \to 1$. When the integral in (1.2) is finite, Bass [2] constructed a continuous function $x \mapsto \phi(x)$ having continuity modulus $x \mapsto \rho(x)$ for which the pathwise uniqueness for (1.1) fails; see also [3].

The pathwise uniqueness and strong solutions for stochastic differential equations driven by spectrally positive Lévy noises were studied in [4]. Those equations arise naturally in the study of branching processes. A typical special continuous state branching process is the non-negative solution to the stochastic differential equation

$$dx(t) = \sqrt[\alpha]{x(t-)}dL_t, \qquad t \geq 0, \tag{1.3}$$

where $\{L_t\}$ is a Brownian motion (for $\alpha = 2$) or a spectrally positive α-table process (for $1 < \alpha < 2$). Note that the coefficient $x \mapsto \sqrt[\alpha]{x}$ in (1.3) is non-decreasing, non-Lipschitz and degenerate at the origin. More general stochastic equations with similar structures arise naturally in limit theorems of branching processes with interactions or/and immigration.

In this paper we consider a class of stochastic differential equations with jumps, which generalizes the equation (1.3). This exploration can be regarded as a continuation of [4]. We extend the results of [4] in two directions. First of all, we notice that the pathwise uniqueness results proved in [4] for non-negative càdlàg solutions can easily be extended to any càdlàg solutions. This extended result is given in Proposition 3.1. Its proof, which is in fact the most involved stochastic part behind the results in this paper, goes through along the same lines as in [4].

The second direction is to apply the above result to formulate some criteria for the pathwise uniqueness and existence of strong solutions to general stochastic differential equations with jumps. We consider this to be the main part of this paper. The proofs in this part involve some analytical arguments that allow us to apply the general pathwise uniqueness criterion of Proposition 3.1. From those results we derive sufficient conditions for the existence and uniqueness of non-negative strong solutions under suitable additional assumptions.

We also give applications of our main results to stochastic equations driven by spectrally positive Lévy processes. These extend and improve substantially the results of [4]. As a consequence of one of those results we get the following counterpart of the theorem of Bass [2]:

Theorem 1.1 *Let $\{L_t\}$ be a spectrally positive stable process with exponent $\alpha \in (1, 2)$ that is, there exists c_α such that*

$$\mathbf{E}\left[e^{-uL(t)}\right] = e^{-c_\alpha u^\alpha t}, \qquad t \geq 0, u \geq 0.$$

Let F be a non-decreasing function on $\mathbb{R}$ with modulus of continuity $z \mapsto \rho(z)$ satisfying

$$\int_{0+} \frac{1}{\rho(z)^{\alpha/(\alpha-1)}}\, dz = \infty. \tag{1.4}$$

Also assume that there is a constant $K \geq 0$ such that

$$|F(x)| \leq K(1 + |x|), \quad x \in \mathbb{R}.$$

Then there is a pathwise unique strong solution to (1.1).

By the above theorem, if F is a non-decreasing function Hölder continuous with exponent $1-1/\alpha$, then the pathwise uniqueness holds for (1.1). The required Hölder exponent tends to 0 as $\alpha \to 1$, which differs sharply from the criterion of Bass [2] for a symmetric stable noise. Note that this result is also consistent with the Yamada-Watanabe result in the sense that as $\alpha \to 2$ the critical Hölder exponent converges to $1/2$.

The organization of the paper is as follows. The main theorem is stated in Section 2. Its proof is provided in Section 3. In Section 4 a number of particular cases is considered, for example, SDE's with stable Lévy noises. Theorem 1.1 is a consequence of one of the results obtained in that section. Throughout this paper, we make the conventions

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)} \qquad \text{for} \quad b \geq a \in \mathbb{R}.$$

2 Main strong uniqueness and existence results

Suppose that $\mu_0(du)$ and $\mu_1(du)$ are σ-finite measures on the complete separable metric spaces U_0 and U_1, respectively. Let $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. Let $\{B(t)\}$ be a standard $(\mathscr{G}_t)$-Brownian motion and let $\{p_0(t)\}$ and $\{p_1(t)\}$ be $(\mathscr{G}_t)$-Poisson point processes on U_0 and U_1 with characteristic measures $\mu_0(du)$ and $\mu_1(du)$, respectively. Suppose that $\{B(t)\}$, $\{p_0(t)\}$ and $\{p_1(t)\}$ are independent of each other. Let $N_0(ds, du)$ and $N_1(ds, du)$ be the Poisson random measures associated with $\{p_0(t)\}$ and $\{p_1(t)\}$, respectively. Suppose in addition that

- $x \mapsto \sigma(x)$ is a continuous function on $\mathbb{R}$;
- $x \mapsto b(x)$ is a continuous function on $\mathbb{R}$ having the decomposition $b = b_1 - b_2$ with b_2 being continuous and non-decreasing;
- $(x, u) \mapsto g_0(x, u)$ is a Borel function on $\mathbb{R} \times U_0$ such that $x \mapsto g_0(x, u)$ is non-decreasing for every $u \in U_0$;
- $(x, u) \mapsto g_1(x, u)$ is a Borel function on $\mathbb{R} \times U_1$.

Let $\tilde{N}_0(ds, du)$ be the compensated measure of $N_0(ds, du)$. By a *solution of* the stochastic equation

$$\begin{aligned} x(t) \;=\; & x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(ds, du) \\ & + \int_0^t b(x(s-))ds + \int_0^t \int_{U_1} g_1(x(s-), u)N_1(ds, du) \end{aligned} \tag{2.1}$$

we mean a càdlàg and $(\mathscr{G}_t)$-adapted real process $\{x(t)\}$ that satisfies the equation almost surely for every $t \geq 0$. Since $x(s-) \neq x(s)$ for at most countably many $s \geq 0$, we can also use $x(s)$ instead of $x(s-)$ for the integrals with respect to $dB(s)$ and ds on the right hand side of (2.1). We say *pathwise uniqueness* holds for (2.1) if for any two solutions $\{x_1(t)\}$ and $\{x_2(t)\}$ of the equation satisfying $x_1(0) = x_2(0)$ we have $x_1(t) = x_2(t)$ almost surely for every $t \geq 0$. Let $(\mathscr{F}_t)_{t\geq 0}$ be the augmented natural filtration generated by $\{B(t)\}$, $\{p_0(t)\}$ and $\{p_1(t)\}$. A solution $\{x(t)\}$ of (2.1) is called a *strong solution* if $x(t)$ is measurable with respect to $\mathscr{F}_t$ for every $t \geq 0$; see [5, p.163] or [7, p.76].

Lemma 2.1 *Suppose that $(z \wedge z^2)\nu(dz)$ is a finite measure on $(0, \infty)$ and define*

$$\alpha_\nu = \inf\Big\{\beta > 1 : \lim_{x\to 0+} x^{\beta-1}\int_x^\infty z\nu(dz) = 0\Big\}. \tag{2.2}$$

Then $1 \leq \alpha_\nu \leq 2$ and, for any $\alpha > \alpha_\nu$,

$$\lim_{x\to 0+} x^{\alpha-2}\int_0^x z^2\nu(dz) = 0. \tag{2.3}$$

Proof. By (2.2) it is clear that $\alpha_\nu \geq 1$. For $x > 0$ let

$$G(x) = \int_x^\infty z\nu(dz) \quad \text{and} \quad H(x) = \int_0^x z^2\nu(dz).$$

Given $\varepsilon > 0$, choose $a > 0$ so that $H(a) < \varepsilon$. Then for $a \geq x > 0$ we have

$$xG(x) = x\int_x^a z\nu(dz) + xG(a) \leq \int_x^a z^2\nu(dz) + xG(a) \leq \varepsilon + xG(a).$$

It follows that $\limsup_{x\to 0+} xG(x) \leq \varepsilon$. That proves $\lim_{x\to 0+} xG(x) = 0$, and so $\alpha_\nu \leq 2$. Clearly, (2.3) holds for any $\alpha \geq 2$. By integration by parts,

$$H(x) = -\int_0^x z dG(z) = -xG(x) + \int_0^x G(z)dz. \tag{2.4}$$

Thus we have

$$\lim_{x\to 0+}\int_0^x G(z)dz = \lim_{x\to 0+} H(x) + \lim_{x\to 0+} xG(x) = 0.$$

Now suppose that $\alpha_\nu < \alpha < 2$. In view of (2.2), for any $\varepsilon > 0$ there exists $b > 0$ so that $x^{\alpha-1}G(x) < \varepsilon$ for all $0 < x \leq b$. Then (2.4) implies

$$x^{\alpha-2}H(x) \leq x^{\alpha-2}\int_0^x G(z)dz \leq x^{\alpha-2}\int_0^x \varepsilon z^{1-\alpha}dz = \varepsilon(2-\alpha)^{-1},$$

and hence $\lim_{x\to 0+} x^{\alpha-2}H(x) = 0$. □

Let us consider a set $U_2 \subset U_1$ satisfying $\mu_1(U_1 \setminus U_2) < \infty$. As in the proof of Proposition 2.2 in [4] one can show that the uniqueness/existence of strong solutions for (2.1) can be reduced to the same question for the equation with U_1 replaced by U_2. Then in what follows all conditions for the ingredients of (2.1) only involve U_2 instead of U_1. As usual, let us consider some growth conditions on the coefficients:

(2.a) There is a constant $K \geq 0$ such that

$$\sigma(x)^2 + \int_{U_0} g_0(x,u)^2 \mu_0(du) + \int_{U_2} g_1(x,u)^2 \mu_1(du) \\ + b(x)^2 + \Big(\int_{U_2} |g_1(x,u)|\mu_1(du)\Big)^2 \leq K(1+x^2), \quad x \in \mathbb{R}.$$

We next introduce our main conditions on the modulus of continuity that are particularly useful in applications to stochastic equations driven by Lévy processes. The conditions are given as follows:

(2.b) For each $m \geq 1$ there is a non-decreasing and concave function $z \mapsto r_m(z)$ on $\mathbb{R}_+$ such that $\int_{0+} r_m(z)^{-1}\, dz = \infty$ and

$$|b_1(x) - b_1(y)| + \int_{U_2} |l_1(x,y,u)|\mu_1(du) \leq r_m(|x-y|)$$

for $|x|, |y| \leq m$, where $l_1(x,y,u) = g_1(x,u) - g_1(y,u)$.

(2.c) For each $m \geq 1$ there is a constant $p_m > 0$, a non-decreasing function $z \mapsto \rho_m(z)$ on $\mathbb{R}_+$ and a function $u \mapsto f_m(u)$ on U_0 such that

$$\int_{0+} \rho_m(z)^{-2} dz = \infty, \qquad \int_{U_0} [f_m(u) \wedge f_m(u)^2]\mu_0(du) < \infty$$

and

$$|\sigma(x) - \sigma(y)| \leq \rho_m(|x-y|), \quad |g_0(x,u) - g_0(y,u)| \leq \rho_m(|x-y|)^{2p_m} f_m(u)$$

for all $|x|, |y| \leq m$ and $u \in U_0$.

For each $m \geq 1$ and the function f_m defined in (2.c) we define the constant

$$\alpha_m := \inf\Big\{\beta > 1 : \lim_{x\to 0+} x^{\beta-1} \int_{U_0} f_m(u) 1_{\{f_m(u)\geq x\}} \mu_0(du) = 0\Big\}.$$

By Lemma 2.1 we have $1 \leq \alpha_m \leq 2$. Our first main theorem of this paper is the following theorem.

Theorem 2.2 *Suppose that conditions (2.a,b,c) hold with*

$$p_m > 1 - 1/\alpha_m \text{ for } \alpha_m < 2, \quad \text{or } p_m = 1/2 \text{ for } \alpha_m = 2. \tag{2.5}$$

Then for any given $x(0) \in \mathbb{R}$, there exists a pathwise unique strong solution $\{x(t)\}$ to (2.1).

From the above theorem we may derive some results on non-negative solutions of (2.1). For that purpose let us consider the following conditions:

(2.d) $\sigma(0) = 0$, $b(0) \geq 0$ and $g_0(0, u) = 0$ for $u \in U_0$, and $g_1(x, u) + x \geq 0$ for $x \in \mathbb{R}_+$ and $u \in U_1$;

(2.e) There is a constant $K \geq 0$ such that

$$b(x) + \int_{U_2} |g_1(x, u)|\mu_1(du) \leq K(1 + x), \qquad x \geq 0;$$

(2.f) There is a non-decreasing function $x \mapsto L(x)$ on $\mathbb{R}_+$ so that

$$\sigma(x)^2 + \int_{U_0} [|g_0(x, u)| \wedge g_0(x, u)^2]\mu_0(du) \leq L(x), \qquad x \geq 0.$$

By Proposition 2.1 of [4], under condition (2.d) any solution of (2.1) with non-negative initial value remains non-negative forever.

Theorem 2.3 *Suppose that conditions (2.b,c,d,e,f) hold with (2.5). Then for any given* $x(0) \in \mathbb{R}_+$*, there exists a pathwise unique non-negative strong solution* $\{x(t)\}$ *to (2.1).*

Remark 2.4 *Under the conditions of Theorem 2.3 we can actually conclude that for any given* $x(0) \in \mathbb{R}_+$ *there is a pathwise unique strong solution to (2.1) and the solution is non-negative. That follows from Proposition 2.1 of [4].*

Remark 2.5 *Note that when* $\alpha_m < 2$ *the assumptions of Theorem 2.2 and 2.3 are strictly weaker than Theorems 2.5 and 5.3 of [4]. In some particular cases the condition (2.5) can be weakened to* $p_m \geq 1 - 1/\alpha_m$*, as in the case of stable driving noise. This is done in Theorem 4.2.*

3 Proofs of Theorems 2.2 and 2.3

The crucial part of the proof of Theorem 2.2 is verifying the pathwise uniqueness for (2.1). As we have mentioned already it is enough to consider the equation

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(ds, du) \\ &\quad + \int_0^t b(x(s-))ds + \int_0^t \int_{U_2} g_1(x(s-), u)N_1(ds, du) \end{aligned} \tag{3.1}$$

For a function f defined on the real line $\mathbb{R}$, note

$$\Delta_z f(x) = f(x + z) - f(x) \quad \text{and} \quad D_z f(x) = \Delta_z f(x) - f'(x)z.$$

We shall need the next result, which provides a criterion for the pathwise uniqueness. It extends the criterion of Theorem 3.1 in [4], where it was formulated just for non-negative solutions.

Proposition 3.1 *Suppose that condition (2.b,c) holds. Then the pathwise uniqueness of solution to (3.1) holds if for each $m \geq 1$ there exists a sequence of non-negative and twice continuously differentiable functions $\{\phi_k\}$ with the following properties:*

(i) $\phi_k(z) \mapsto |z|$ *non-decreasingly as* $k \to \infty$*;*

(ii) $0 \leq \phi_k'(z) \leq 1$ *for* $z \geq 0$ *and* $-1 \leq \phi_k'(z) \leq 0$ *for* $z \leq 0$*;*

(iii) $\phi_k''(z) \geq 0$ *for* $z \in \mathbb{R}$ *and as* $k \to \infty$,

$$\phi_k''(x-y)[\sigma(x)-\sigma(y)]^2 \to 0$$

uniformly on $|x|, |y| \leq m$*;*

(iv) as $k \to \infty$,

$$\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) \to 0$$

uniformly on $|x|, |y| \leq m$*, where* $l_0(x,y,u) = g_0(x,u) - g_0(y,u)$*.*

Proof. For non-negative solutions the result was given in Theorem 3.1 of [4]. In what follows we will show that the proof in [4] goes through for any càdlàg solutions. Let $\{x_1(t)\}$ and $\{x_2(t)\}$ be any two solutions of (3.1) starting at $x_1(0) = x_2(0) = x_0$. For each $m \geq 1$ define $\tau_m = \inf\{t \geq 0 : \ |x_1(t)| \geq m \text{ or } |x_2(t)| \geq m\}$ and $\zeta(t) = x_1(t) - x_2(t)$. Recall that $l_i(x,y,u) = g_i(x,u) - g_i(y,u)$, $i = 0, 1$. By (3.1) and the Itô formula one can show

$$\begin{aligned}
\phi_k(\zeta(t\wedge\tau_m)) \;=\; & \int_0^{t\wedge\tau_m} \phi_k'(\zeta(s-))[b(x_1(s-)) - b(x_2(s-))]\, ds \\
& + \frac{1}{2}\int_0^{t\wedge\tau_m} \phi_k''(\zeta(s-))[\sigma(x_1(s-)) - \sigma(x_2(s-))]ds \\
& + \int_0^{t\wedge\tau_m} ds \int_{U_2} \Delta_{l_1(x_1(s-),x_2(s-),u)}\phi_k(\zeta(s-))\mu_1(du) \\
& + \int_0^{t\wedge\tau_m} ds \int_{U_0} D_{l_0(x_1(s-),x_2(s-),u)}\phi_k(\zeta(s-))\mu_0(du) \\
& + M_m(t),
\end{aligned}$$

where

$$\begin{aligned}
M_m(t) \;=\; & \int_0^{t\wedge\tau_m} \phi_k'(\zeta(s-))[\sigma(x_1(s-)) - \sigma(x_2(s-))]dB(s) \\
& + \int_0^{t\wedge\tau_m}\int_{U_2} \Delta_{l_1(x_1(s-),x_2(s-),u)}\phi_k(\zeta(s-))\tilde{N}_1(ds,du) \\
& + \int_0^{t\wedge\tau_m}\int_{U_0} \Delta_{l_0(x_1(s-),x_2(s-),u)}\phi_k(\zeta(s-))\tilde{N}_0(ds,du).
\end{aligned}$$

Under conditions (2.b,c) it is easy to show that $\{M_m(t)\}$ is a martingale. Therefore, we can follow the same argument as in the proof of Theorem 3.1 of [4] to get that, as $k \to \infty$,

$$\mathbf{E}[|\zeta(t \wedge \tau_m)|] \leq \int_0^t r_m(\mathbf{E}[|\zeta(s \wedge \tau_m)|])ds.$$

From this by standard argument we have $\mathbf{E}[|\zeta(t \wedge \tau_m)|] = 0$ for every $t \geq 0$. Since $\{x_1(t)\}$ and $\{x_2(t)\}$ are càdlàg, we have that $\tau_m \to \infty$ as $m \to \infty$. Hence letting $m \to \infty$ and using the right continuity of $\{\zeta(t)\}$ we get the result. □

To prove the pathwise uniqueness for (3.1) we need to introduce more notation and prove a lemma which will play a crucial role in the proofs. For each integer $m \geq 1$ we shall construct a sequence of functions $\{\phi_k\}$ that satisfies the properties required in Proposition 3.1. Although main ideas are similar to those in the proof of Theorem 3.2 of [4], we will go through the details for the sake of completeness. Let $1 = a_0 > a_1 > a_2 > \ldots > 0$ be defined by

$$\int_{a_k}^{a_{k-1}} \rho_m(z)\, dz = k.$$

Let $x \mapsto \psi_k(x)$ be a non-negative continuous function on $\mathbb{R}$ satisfying $\int_{a_k}^{a_{k-1}} \psi_k(x)\, dx = 1$ and

$$0 \leq \psi_k(x) \leq 2k^{-1}\rho_m(x)^{-2}1_{(a_k,a_{k-1})}(x). \tag{3.2}$$

For each $k \geq 1$ we define the non-negative and twice continuously differentiable function

$$\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x)\, dx, \quad z \in \mathbb{R}.$$

Note that although the sequences $\{a_k\}$, $\{\phi_k\}$ and $\{\psi_k\}$ also depend on $m \geq 1$, we do not put this additional index to simplify the notation.

Lemma 3.2 *Suppose that condition (2.c) holds. Fix $m \geq 1$ and let a_k, ϕ_k and ψ_k be defined as above. Then the sequence $\{\phi_k\}$ satisfies properties (i)–(iii) in Proposition 3.1 and for any $h > 0$,*

$$\begin{aligned}&\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du)\\ &\quad\leq k^{-1}\rho_m(|x-y|)^{4p_m-2}1_{\{|x-y|\leq a_{k-1}\}}\int_{U_0} f_m(u)^2 1_{\{f_m(u)\leq h\}}\mu_0(du)\\ &\qquad+\rho_m(|x-y|)^{2p_m}1_{\{|x-y|\leq a_{k-1}\}}\int_{U_0} f_m(u)1_{\{f_m(u)>h\}}\mu_0(du).\end{aligned} \tag{3.3}$$

Proof. By definition, the sequence $\{\phi_k\}$ satisfies properties (i) and (ii) in Proposition 3.1. Moreover, by (3.2) we get

$$\phi_k''(x) = \psi_k(|x|) \leq 2k^{-1}\rho_m(|x|)^{-2}1_{(a_k,a_{k-1})}(|x|) \tag{3.4}$$

for all $x \in \mathbb{R}$. This together with condition (2.c) implies

$$\phi_k''(x-y)[\sigma(x)-\sigma(y)]^2 \le \psi_k(|x-y|)\rho_m(|x-y|)^2 \le 2/k$$

for $|x|, |y| \le m$. Thus $\{\phi_k\}$ also satisfies property (iii) in Proposition 3.1. Observe that

$$D_z\phi_k(x-y) = \Delta_z\phi_k(x-y) - \phi_k'(x-y)z \le |z|1_{\{|x-y|\le a_{k-1}\}} \tag{3.5}$$

when $(x-y)z \ge 0$. By Taylor's expansion,

$$D_z\phi_k(x-y) = z^2\int_0^1 \phi_k''(x-y+tz)(1-t)dt = z^2\int_0^1 \psi_k(|x-y+tz|)(1-t)dt.$$

Then (3.4) and the monotonicity of $\zeta \mapsto \rho_m(\zeta)$ imply

$$\begin{aligned} D_z\phi_k(x-y) &\le 2k^{-1}z^2\int_0^1 \frac{(1-t)1_{(a_k,a_{k-1})}(|(x-y)+tz|)}{\rho_m(|(x-y)+tz|)^2}dt \\ &\le k^{-1}z^2\rho_m(|x-y|)^{-2}1_{\{|x-y|\le a_{k-1}\}} \end{aligned} \tag{3.6}$$

when $(x-y)z \ge 0$ and $|x|, |y| \le m$. Recall that $l_0(x,y,u) = g_0(x,u) - g_0(y,u)$. Since $x \mapsto g_0(x,u)$ is non-decreasing, for $|x|, |y| \le m$ we get by (3.5) and (2.c) that

$$D_{l_0(x,y,u)}\phi_k(x-y) \le |l_0(x,y,u)|1_{\{|x-y|\le a_{k-1}\}} \le \rho_m(|x-y|)^{2p_m}f_m(u)1_{\{|x-y|\le a_{k-1}\}}.$$

Similarly, by (3.6) and (2.c) we have

$$\begin{aligned} D_{l_0(x,y,u)}\phi_k(x-y) &\le k^{-1}\rho_m(|x-y|)^{-2}l_0(x,y,u)^2 1_{\{|x-y|\le a_{k-1}\}} \\ &\le k^{-1}\rho_m(|x-y|)^{4p_m-2}f_m(u)^2 1_{\{|x-y|\le a_{k-1}\}}. \end{aligned}$$

Then (3.3) follows immediately. □

Proposition 3.3 *Under the conditions (2.b,c) and (2.5), the pathwise uniqueness holds for equation (3.1).*

Proof. For $\alpha_m = 2$ and $p_m = 2$, the result was essentially proved in Theorem 3.3 of [4] for non-negative solutions. It follows along the same lines for all solutions. So we here only consider the case of $\alpha_m < 2$ and $p_m > 1 - 1/\alpha_m$. By Lemma 3.2 we get that the sequence $\{\phi_k\}$ satisfies properties (i)—(iii) in Proposition 3.1. Moreover for any $\beta > 0$ we can take $h = \rho_m(|x-y|)^{2\beta}$ in (3.3) to get

$$\begin{aligned} &\int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) \\ &\quad\le k^{-1}\rho_m(|x-y|)^{2(2p_m-1)}1_{\{|x-y|\le a_{k-1}\}}\int_{U_0} f_m(u)^2 1_{\{f_m(u)\le\rho_m(|x-y|)^{2\beta}\}}\mu_0(du) \\ &\qquad + \rho_m(|x-y|)^{2p_m}1_{\{|x-y|\le a_{k-1}\}}\int_{U_0} f_m(u)1_{\{f_m(u)>\rho_m(|x-y|)^{2\beta}\}}\mu_0(du). \end{aligned}$$

Since $\lim_{k\to\infty} a_k = 0$ and $\lim_{z\to 0+}\rho(z) = 0$, for $\alpha_m < \alpha < 2$ we use Lemma 2.1 to see

$$\begin{aligned}\int_{U_0} & D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) \\ &\le k^{-1}\rho_m(|x-y|)^{2(2p_m-1)}\rho_m(|x-y|)^{2\beta(2-\alpha)}1_{\{|x-y|\le a_{k-1}\}} \\ &\quad +\rho_m(|x-y|)^{2p_m}\rho_m(|x-y|)^{2\beta(1-\alpha)}1_{\{|x-y|\le a_{k-1}\}}\end{aligned} \tag{3.7}$$

when $k \ge 1$ is sufficiently large. If we can choose β and α in the way that

$$2(2p_m - 1) + 2\beta(2-\alpha) > 0 \quad \text{and} \quad 2p_m + 2\beta(1-\alpha) > 0,$$

the value on the right hand side of (3.7) will tend to zero as $k \to \infty$. The requirement is equivalent to

$$\frac{1-2p_m}{2-\alpha} < \beta < \frac{p_m}{\alpha - 1},$$

which can be done as long as

$$\frac{1-2p_m}{2-\alpha} < \frac{p_m}{\alpha-1}$$

or, equivalently, $p_m > 1-1/\alpha$. For that purpose it sufficient to have $p_m > 1-1/\alpha_m$. This gives property (iv) in Proposition 3.1 and hence the pathwise uniqueness for (3.1). □

Proposition 3.4 *Suppose that conditions (2.a) hold. Let $\{x(t)\}$ be a solution of (3.1) with $\mathbf{E}[x(0)^2] < \infty$. Then we have*

$$\mathbf{E}\Big[1 + \sup_{0\le s\le t} x(s)^2\Big] \le (1 + 6\mathbf{E}[x(0)^2])\exp\{6K(4+t)t\}. \tag{3.8}$$

Proof. Let $\tau_m = \inf\{t \ge 0 : |x(t)| \ge m\}$ for $m \ge 1$. Since $\{x(t)\}$ has càdlàg sample paths, we have $\tau_m \to \infty$ as $m \to \infty$. Let us rewrite (3.1) into

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t\int_{U_0} g_0(x(s-),u)\tilde N_0(ds,du) \\ &\quad + \int_0^t b(x(s-))ds + \int_0^t\int_{U_2} g_1(x(s-),u)\tilde N_1(ds,du) \\ &\quad + \int_0^t ds\int_{U_2} g_1(x(s-),u)\mu_1(du).\end{aligned}$$

By Doob's martingale inequalities we have

$$\begin{aligned}\mathbf{E}\Big[\sup_{0\le s\le t} x(s\wedge\tau_m)^2\Big] &\le 6\mathbf{E}[x(0)^2] + 24\mathbf{E}\Big[\int_0^{t\wedge\tau_m}\sigma(x(s-))^2 ds\Big] \\ &\quad + 6\mathbf{E}\Big[\Big(\int_0^{t\wedge\tau_m}|b(x(s-))|ds\Big)^2\Big]\end{aligned}$$

$$
\begin{aligned}
&+ 24\mathbf{E}\Big[\int_0^{t\wedge\tau_m} ds \int_{U_0} g_0(x(s-),u)^2\mu_0(du)\Big]\\
&+ 24\mathbf{E}\Big[\int_0^{t\wedge\tau_m} ds \int_{U_2} g_1(x(s-),u)^2\mu_1(du)\Big]\\
&+ 6\mathbf{E}\Big[\Big(\int_0^{t\wedge\tau_m} ds \int_{U_2} |g_1(x(s-),u)|\mu_1(du)\Big)^2\Big]\\
\le\ & 6\mathbf{E}[x(0)^2] + 6K(4+t)\mathbf{E}\Big[\int_0^{t\wedge\tau_m}(1+x(s-)^2)ds\Big].
\end{aligned}
$$

Then it is easy to see that

$$
t \mapsto F_m(t) := \mathbf{E}\Big[\sup_{0\le s\le t} x(s\wedge\tau_m)^2\Big]
$$

is locally bounded on $[0,\infty)$. Since $s\mapsto x(s)$ has at most a countable number of jumps, from the above inequality we obtain

$$
\begin{aligned}
1+F_m(t) &\le 1+6\mathbf{E}[x(0)^2] + 6K(4+t)\mathbf{E}\Big[\int_0^{t\wedge\tau_m}(1+x(s)^2)ds\Big]\\
&\le 1+6\mathbf{E}[x(0)^2] + 6K(4+t)\int_0^t [1+F_m(s)]ds.
\end{aligned}
$$

By Gronwall's inequality,

$$
\mathbf{E}\Big[1+\sup_{0\le s\le t} x(s\wedge\tau_m)^2\Big] \le (1+6\mathbf{E}[x(0)^2])\exp\{6K(4+t)t\}.
$$

Then (3.8) follows by Fatou's lemma. □

Proof of Theorem 2.2 Step 1) Suppose that conditions (2.b,c) and (2.5) hold. Instead of condition (2.a), we here assume there is a constant $K\ge 0$ such that

$$
\begin{aligned}
&\sigma(x)^2 + b(x)^2 + \sup_{u\in U_0}|g_0(x,u)| + \int_{U_0} g_0(x,u)^2\mu_0(du)\\
&\quad + \int_{U_2} g_1(x,u)^2\mu_1(du) + \Big(\int_{U_2}|g_1(x,u)|\mu_1(du)\Big)^2 \le K, \quad x\in\mathbb{R}. \qquad (3.9)
\end{aligned}
$$

Let $\{V_n\}$ be a non-decreasing sequence of Borel subsets of U_0 so that $\cup_{n=1}^\infty V_n = U_0$ and $\mu_0(V_n)<\infty$ for every $n\ge 1$. By the result on continuous-type stochastic equations, there is a weak solution to

$$
\begin{aligned}
x(t) &= x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t b(x(s))ds\\
&\quad - \int_0^t ds\int_{V_n} g_0(x(s),u)\mu_0(du); \qquad (3.10)
\end{aligned}
$$

see, e.g., Ikeda and Watanabe (1989, p.169). By Proposition 3.3, the pathwise uniqueness holds for (3.10), thus the equation has a pathwise unique strong solution. Let $\{W_n\}$ be a non-decreasing sequence of Borel subsets of U_1 so that $\cup_{n=1}^{\infty} W_n = U_2$ and $\mu_1(W_n) < \infty$ for every $n \geq 1$. Then for every integer $n \geq 1$ there is a unique strong solution $\{x_n(t) : t \geq 0\}$ to

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_{V_n} g_0(x(s-), u)\tilde{N}_0(ds, du) \\ &\quad + \int_0^t b(x(s-))ds + \int_0^t \int_{W_n} g_1(x(s-), u)N_1(ds, du). \end{aligned}$$

As in the proof of Lemma 4.3 of [4] one can see the sequence $\{x_n(t)\}$ is tight in $D([0, \infty), \mathbb{R})$, the space of càdlàg functions with the Skorohod topology. Following the proof of Theorem 4.4 of [4] it is easy to show that any limit point of the sequence is a weak solution to (3.1). This and Proposition 3.3 imply the existence and uniqueness of the strong solution to (3.1); see, e.g., [7, p.104].

Step 2) Suppose that the original conditions (2.a,b,c) and (2.5) hold. For each $m \geq 1$ let

$$\chi_m(x) = \begin{cases} x, & \text{if } |x| \leq m, \\ m, & \text{if } x > m, \\ -m, & \text{if } x < -m. \end{cases}$$

We consider the equation

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(\chi_m(x(s-)))dB(s) + \int_0^t b_m(\chi_m(x(s-)))ds \\ &\quad + \int_0^t \int_{U_0} \chi_m \circ g_0(\chi_m(x(s-)), u)\tilde{N}_0(ds, du) \\ &\quad + \int_0^t \int_{U_2} g_1(\chi_m(x(s-)), u)N_1(ds, du), \end{aligned} \tag{3.11}$$

where

$$b_m(x) = b(x) - \int_{U_0} [g_0(x, u) - \chi_m \circ g_0(x, u)]\mu_0(du).$$

By the first step, there is a unique strong solution to (3.11). Then using Proposition 3.4 one can show as in the proof of Proposition 2.4 of [4] that there is a pathwise unique strong solution to (3.1). Hence as we have mentioned above, there is a pathwise unique strong solution to (2.1) (see Proposition 2.2 of [4] and its proof for analogous result). □

Proof of Theorem 2.3 By Proposition 2.1 of [4] and Theorem 2.2 there is a pathwise unique non-negative strong solution $\{x_m(t)\}$ to the equation

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma\left(\chi_m(x(s-) \vee 0)\right) dB(s) + \int_0^t b(\chi_m(x(s-) \vee 0))ds \\ &\quad + \int_0^t \int_{U_0} \chi_m \circ g_0(\chi_m(x(s-) \vee 0), u)\tilde{N}_0(ds, du) \\ &\quad + \int_0^t \int_{U_2} \chi_m \circ g_1(x(s-) \vee 0, u)N_1(ds, du). \end{aligned} \tag{3.12}$$

By Proposition 2.3 of [4] the first moment of $\{x_m(t)\}$ is dominated by a locally bounded function on $[0,\infty)$ independent of $m \geq 1$. Then one can follow the proof of Proposition 2.4 of [4] to show there is a pathwise unique non-negative strong solution to

$$\begin{aligned} x(t) \;=\; & x(0) + \int_0^t \sigma(x(s-)\vee 0)dB(s) + \int_0^t b(x(s-)\vee 0)ds \\ & + \int_0^t \int_{U_0} g_0((x(s-)\vee 0), u)\tilde{N}_0(ds,du) \\ & + \int_0^t \int_{U_2} g_1(x(s-)\vee 0, u)N_1(ds,du). \end{aligned} \tag{3.13}$$

Now note that the non-negative solution to (3.13) is also the non-negative solution to (3.1). This and Proposition 3.3 imply that there is a pathwise unique non-negative strong solution to (3.1). This again as in the proof of Theorem 2.2 implies that there is a pathwise unique non-negative strong solution to (2.1). □

Remark 3.5 *The above proofs show it is unnecessary to assume the existence of the sequence $\{V_n\}$ in (4.b) of [4]. As a consequence, condition (5.b) of [4] is also unnecessary for the results in Section 5 of that paper.*

4 Stochastic equations with Lévy noises

In this section, we give some applications of our main results to stochastic equations driven by Lévy processes. Let (σ, b) be given as in Section 2 and let $\nu_0(dz)$ and $\nu_1(dz)$ be σ-finite Borel measures on $(0,\infty)$ satisfying

$$\int_0^\infty (z\wedge z^2)\nu_0(dz) + \int_0^\infty (1\wedge z)\nu_1(dz) < \infty.$$

Let α_0 be the constant defined by (2.2) for the measure $\nu_0(dz)$. In addition, we suppose that

- $x \mapsto h_0(x)$ is a continuous and non-decreasing function on $\mathbb{R}$;
- $x \mapsto h_1(x)$ is a continuous function on $\mathbb{R}$.

Suppose we have a filtered probability space $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbf{P})$ satisfying the usual hypotheses. Let $\{B(t)\}$ be an $(\mathscr{G}_t)$-Brownian motion and let $\{L_0(t)\}$ and $\{L_1(t)\}$ be $(\mathscr{G}_t)$-Lévy processes with exponents

$$u \mapsto \int_0^\infty (e^{iuz} - 1 - iuz)\nu_0(dz) \quad \text{and} \quad u \mapsto \int_0^\infty (e^{iuz} - 1)\nu_1(dz),$$

respectively. Suppose that $\{B(t)\}$, $\{L_0(t)\}$ and $\{L_1(t)\}$ are independent of each other. Note that $\{L_0(t)\}$ is centered and $\{L_1(t)\}$ is non-decreasing. We introduce the conditions:

(4.a) There is a constant $K \geq 0$ such that

$$|\sigma(x)| + |b(x)| + |h_0(x)| + |h_1(x)| \leq K(1 + |x|), \quad x \in \mathbb{R};$$

(4.b) There exists a non-decreasing and concave function $z \mapsto r(z)$ on $\mathbb{R}_+$ such that $\int_{0+} r(z)^{-1}\, dz = \infty$ and

$$|b(x) - b(y)| + |h_1(x) - h_1(y)| \leq r(|x - y|), \qquad x, y \in \mathbb{R};$$

(4.c) There is a constant $p > 0$ and a non-decreasing function $z \mapsto \rho(z)$ on $\mathbb{R}_+$ such that $\int_{0+} \rho(z)^{-2}\, dz = \infty$ and

$$|\sigma(x) - \sigma(y)| + |h_0(x) - h_0(y)|^{1/2p} \leq \rho(|x - y|), \qquad x, y \in \mathbb{R};$$

(4.d) $\sigma(0) = h_0(0) = 0$, $b(0) \geq 0$, and $h_1(x) \geq 0$ for $x \in \mathbb{R}_+$;

(4.e) There is a constant $K \geq 0$ such that

$$b(x) + h_1(x) \leq K(1 + x), \qquad x \geq 0.$$

Theorem 4.1 (i) *If conditions (4.a,b,c) are satisfied with $p > 1 - 1/\alpha_0$, then for any given $x(0) \in \mathbb{R}$ there is a pathwise unique strong solution to*

$$dx(t) = \sigma(x(t))dB(t) + h_0(x(t-))dL_0(t) + b(x(t))dt + h_1(x(t-))dL_1(t). \tag{4.1}$$

(ii) *If conditions (4.b,c,d,e) are satisfied with $p > 1 - 1/\alpha_0$, then for any given $x(0) \in \mathbb{R}_+$ there is a pathwise unique non-negative strong solution to (4.1).*

Proof. By Lévy-Itô decompositions, the Lévy processes have the following representations

$$\begin{aligned} L_0(t) &= \int_0^t \int_0^1 z\tilde{N}_0(ds, dz) - \int_0^t ds \int_1^\infty z\nu_0(dz) \\ &\quad + \int_0^t \int_1^\infty zN_1(ds, dz, \{0\}), \\ L_1(t) &= \int_0^t \int_0^\infty zN_1(ds, dz, \{1\}), \end{aligned}$$

where $N_0(ds, dz)$ and $N_1(ds, dz, du)$ are Poisson random measures with intensities

$$1_{\{z \leq 1\}} ds\nu_0(dz) \quad \text{and} \quad ds[1_{\{z>1\}}\nu_0(dz)\delta_0(du) + \nu_1(dz)\delta_1(du)],$$

respectively, and $\tilde{N}_0(ds, dz)$ is the compensated measure of $N_0(ds, dz)$. Here $N_0(ds, dz)$ and $N_1(ds, dz, du)$ are independent and they are independent of $\{B(t)\}$. By applying Theorem 2.2 with

$$U_0 = (0, 1],\ U_1 = [(1, \infty) \times \{0\}] \cup [(0, \infty) \times \{1\}] \quad \text{and} \quad U_2 = (0, 1] \times \{1\},$$

we see that there is a pathwise unique strong solution to

$$\begin{aligned} x(t) \;=\; & x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t \int_0^1 h_0(x(s-))z\tilde{N}_0(ds, dz) \\ & + \int_0^t \left(b(x(s)) - h_0(x(s)) \int_1^\infty z\nu_0(dz) \right) ds \\ & + \int_0^t \int_{U_1} g_1(x(s-), z, u)N_1(ds, dz, du), \end{aligned}$$

where

$$g_1(x, z, u) = h_0(x)z1_{\{z>1,u=0\}} + h_1(x)z1_{\{u=1\}}.$$

However, this is just another form of the equation (4.1) and hence part (i) of the theorem follows. The proof of part (ii) is similar. □

Theorem 4.2 *Suppose that $\{B(t)\}$, $\{L_0(t)\}$ and $\{L_1(t)\}$ are given as the above with $\nu_0(dz) = z^{-1-\alpha}dz$ for $1 < \alpha < 2$. Then we have:*
(i) *If conditions (4.a,b,c) are satisfied with $p \geq 1 - 1/\alpha$, then for any given $x(0) \in \mathbb{R}$ there is a pathwise unique strong solution to (4.1);*
(ii) *If conditions (4.b,c,d,e) are satisfied with $p \geq 1 - 1/\alpha$, then for any given $x(0) \in \mathbb{R}_+$ there is a pathwise unique non-negative strong solution to (4.1).*

Proof. Let $\{a_k\}$, $\{\phi_k\}$ and $\{\psi_k\}$ be defined as before Lemma 3.2 with $\rho_m = \rho$. Then we can easily apply Lemma 3.2 to get that $\{\phi_k\}$ satisfies properties (i)-(iii) in Proposition 3.1. Moreover, using again Lemma 3.2 with $\mu_0(du) = u^{-1-\alpha}du$, $p_m = p = (\alpha - 1)/\alpha$, $\rho_m = \rho$ and $f_m(u) = u$ we can rewrite (3.3) as

$$\begin{aligned} & \int_0^\infty D_{l_0(x,y,u)}\phi_k(x - y)\mu_0(du) \\ & \quad \leq\; k^{-1}\rho(|x - y|)^{4p-2} \int_0^h u^{1-\alpha}\, du + \rho(|x - y|)^{2p} \int_h^\infty u^{-\alpha}\, du \\ & \quad =\; k^{-1}(2 - \alpha)^{-1}\rho(|x - y|)^{4(\alpha-1)/\alpha-2}h^{2-\alpha} + (\alpha - 1)^{-1}\rho(|x - y|)^{2(\alpha-1)/\alpha}h^{1-\alpha}. \end{aligned}$$

Take $h = \rho(|x - y|)^{2/\alpha}v_k$, where v_k is a sequence such that $v_k \to \infty$ and $v_k^{2-\alpha}k^{-1} \to 0$. Then one can check that

$$\int_0^\infty D_{l_0(x,y,u)}\phi_k(x - y)\mu_0(du) \leq k^{-1}(2 - \alpha)^{-1}v_k^{2-\alpha} + (\alpha - 1)^{-1}v_k^{1-\alpha},$$

which tends to zero as $k \to \infty$. Now since all the properties in Proposition 3.1 are satisfied we get the pathwise uniqueness for (4.1). The existence of the solution follows by a modification of the proof of Theorem 2.2. That gives part (i) of the theorem. The proof of part (ii) can be given in a similar way. □

Corollary 4.3 *Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $1 \leq r \leq 2$, $1 < \alpha < 2$, $q \geq 1$ and β be constants. Suppose that $\{B(t)\}$, $\{L_0(t)\}$ and $\{L_1(t)\}$ are given as the above with $\nu_0(dz) = z^{-1-\alpha}dz$. If $1/q + 1/\alpha \geq 1$, then for any given $x(0) \in \mathbb{R}_+$ there is a pathwise unique strong solution to*

$$dx(t) = \sqrt[r]{a|x(t)|}dB(t) + \mathrm{sign}(x(t-))\sqrt[q]{c|x(t-)|}dL_0(t) + (\beta x(t) + b)dt + dL_1(t), \quad (4.2)$$

and this solution is non-negative.

Proof. One can choose $\rho(z) = \sqrt[r]{z}$ and $p = 1/q$ in (4.c) and hence by Theorem 4.2, there is a pathwise unique strong solution to (4.2) which is non-negative. □

In the special case where $r = 2$ and $q = \alpha$, the solution of (4.2) is a continuous state branching process with immigration and the strong existence and uniqueness for (4.2) were obtained in [4].

Remark 4.4 *Theorem 1.1 follows immediately from Theorem 4.2.*

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