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Ergodic theory for a superprocess over a stochastic flow

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Abstract. We study the longtime limiting behavior of the occupation time of the superprocess over a stochastic flow introduced by Skoulakis and Adler (2001). The ergodic theorems for dimensions $d = 2$ and $d \geq 3$ are established. The proofs depend heavily on a characterization of the conditional log-Laplace equation of the occupation time process.

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1 Introduction

A superprocess over a stochastic flow were constructed by Skoulakis and Adler [13]. Let $\sigma_1 = (\sigma_1^{ij}(x))$ and $\sigma_2 = (\sigma_2^{ij}(x))$ be $d \times d$ matrices defined on $\mathbb{R}^d$. Suppose that $\{W(t)\}$ and $\{B_1(t)\}, \{B_2(t)\}, \cdots$ are independent d-dimensional Brownian motions. We consider a branching particle system on $\mathbb{R}^d$ described as follows. Between its branchings the motion of the ith particle is defined by the stochastic differential equation

$$d\xi_i(t) = \sigma_1(\xi_i(t))dW(t) + \sigma_2(\xi_i(t))dB_i(t). \tag{1.1}$$

The particle splits into two or dies with equal probabilities when its standard exponential life time runs out, independent of others. By the result of Skoulakis and Adler [13], a suitable scaling limit of the above system gives a continuous superprocess $\{X_t\}$ with state space $M(\mathbb{R}^d)$, finite Borel measures on $\mathbb{R}^d$. (Those authors considered a diagonal form of σ_2, but their arguments carry over to the present situation.) Let

$$(a^{ij}) = (\sigma_1^{ij})^*(\sigma_1^{ij}) + (\sigma_2^{ij})^*(\sigma_2^{ij}),$$

where "$*$" denote the transpose of the matrix. Let $C_0^2(\mathbb{R}^d)$ be the collection of twice continuously differentiable functions on $\mathbb{R}^d$ with compact supports. We define the differential operator L by

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 f}{\partial x_i x_j}(x), \qquad x \in \mathbb{R}^d, f \in C_0^2(\mathbb{R}^d). \tag{1.2}$$

Throughout this paper, we assume the following conditions:

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(A1) *the entries of* $\sigma_2 = (\sigma_2^{ij}(x))$ *have bounded continuous derivatives up to the second order and those of* $\sigma_1 = (\sigma_1^{ij}(x))$ *have bounded continuous derivatives up to the third order;*

(A2) $\sigma_2^*\sigma_2 = (\sigma_2^{ij}(x))^*(\sigma_2^{ij}(x))$ *is uniformly positive definite on* $\mathbb{R}^d$.

Let $\langle \mu, f\rangle$ and $\mu(f)$ denote the integral of the function f with respect to the measure μ. Then the superprocess $\{X_t : t \geq 0\}$ over the stochastic flow is uniquely characterized by the following martingale problem: For every $f \in C_0^2(\mathbb{R}^d)$,

$$M_t(f) := \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle X_s, Lf\rangle ds \tag{1.3}$$

is a continuous martingale with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t (\langle X_s, 2f^2\rangle + \langle X_s, \sigma_1^*\nabla f\rangle^2) ds. \tag{1.4}$$

It is easy to see that $\{X_t\}$ reduces to a classical critical branching superprocess when $\sigma_1 = 0$. Otherwise, it has properties very different from the later; see, e.g., Xiong [15, 16]. A similar model was studied in [4, 14].

Following Xiong [15, 16] we can construct the superprocess $\{X_t\}$ and the Brownian motions $\{W(t)\}$ and $\{B_1(t)\}, \{B_2(t)\}, \cdots$ on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Throughout the paper, we use the superscript "W" to denote the conditional law given $\{W(t)\}$. Then the superprocess $\{X_t\}$ can also be characterized by the following conditional martingale problem: Under the conditional probability $\mathbf{P}^W$, for every $f \in C_0^2(\mathbb{R}^d)$,

$$N_t(f) := \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle X_s, Lf\rangle ds - \int_0^t \langle X_s, \sigma_1^*\nabla f\rangle dW(s) \tag{1.5}$$

is a continuous martingale with quadratic variation process

$$\langle N(f)\rangle_t = \int_0^t \langle X_s, 2f^2\rangle ds. \tag{1.6}$$

The log-Laplace functional has been used for classical superprocesses by many authors to study their asymptotic behaviors. In particular, the persistence property of the super-stable motion was proved in Dawson [3]. Iscoe [6] gave a characterization of the log-Laplace functional for the occupation time of the super-stable motion and studied its central limit theorems. The ergodic theory and local time for super-Brownian motion were studied in Iscoe [7]. In Xiong [15], the conditional log-Laplace functional of $\{X_t\}$ given $\{W(t)\}$ was characterized as the solution to a nonlinear stochastic partial differential equation (SPDE) driven by the later.

To explain the tools used in the exploration, we need some notation and results for SPDE's from Krylov [9]. Let H_p^n for $p > 1$ and $n \in \mathbb{R}$ denote the Soblev space on $\mathbb{R}^d$ with fractional derivatives (cf. [9, p.186]). Let H_∞ be the Banach space of bounded measurable functions equipped on $\mathbb{R}^d$ with the supremum norm and let H_∞^+ be its subset consisting of the non-negative elements. Let C_b denote the set of bounded continuous functions on $\mathbb{R}^d$. We note that $H_p^n \subset C_b$ when $np > d$ (cf. [1] or [17, p.113]). It follows that $\mathcal{X} := \cap_{p\geq 2} H_p^2 \cap H_\infty^+ \subset C_b$. For fixed $t \geq 0$ and $f \in \mathcal{X}$ we consider the nonlinear SPDE:

$$v_{r,t}(x) = f(x) + \int_r^t [Lv_{s,t}(x) - v_{s,t}^2(x)] ds + \int_r^t \sigma_1^*(x)\nabla v_{s,t}(x)\hat{d}W(s), \quad 0 \leq r \leq t, \tag{1.7}$$

where $\hat{d}W(s)$ denotes the backward Itô integral defined by

$$\int_r^t g(s)\hat{d}W(s) = \lim_{|\Delta|\to 0}\sum_{i=1}^n g(r_i)(W(r_i) - W(r_{i-1})).$$

The limit here is taken in $L^2(\Omega, \mathbf{P})$ and $|\Delta|$ is the maximum length of the subintervals of the partition $\Delta = \{r = r_0 < r_1 < \cdots < r_n = t\}$. Note that we have used the right endpoints in the Riemann sum approximation of the stochastic integral. That is the reason we call it the backward stochastic integral. We need to use this version of the stochastic integral in the SDE (1.7) because that equation is defined with the time t fixed and the time $r \le t$ varies.

For $r \ge 0$ and $\nu \in M(\mathbb{R}^d)$ let $\mathbf{P}_{r,\nu}$ denote the conditional law given $X_r = \nu$. The following theorem was essentially established by Xiong [15, Theorem 1.4 and Lemma 2.5]; see also Xiong [16].

Theorem 1.1 *Suppose that conditions (A1,2) hold. Then for any $t \ge 0$ and $f \in \mathcal{X}$ there is a unique $\mathcal{X}$-valued solution $r \mapsto v_{r,t}$ to (1.7). Moreover, for any $0 \le r \le t$ and $\nu \in M(\mathbb{R}^d)$ we have*

$$\mathbf{P}_{r,\nu}^W \exp\{-\langle X_t, f\rangle\} = \exp\{-\langle \nu, v_{r,t}\rangle\}. \tag{1.8}$$

Using the above conditional log-Laplace functional as a tool, Xiong [16] proved the persistent property of $\{X_t\}$ in high spatial dimensions $d \ge 3$. Following Xiong [16] for fixed $t \ge 0$ and $f \in \mathcal{X}$ we consider the linear stochastic integral equation

$$T_{r,t}f(x) = f(x) + \int_r^t LT_{s,t}f(x)ds + \int_r^t \sigma_1^*(x)\nabla T_{s,t}f(x)\hat{d}W(s), \quad 0 \le r \le t. \tag{1.9}$$

The solution of the above equation can be represented as

$$T_{r,t}f(x) = \int_{\mathbb{R}^d} f(y)p^W(r,x,t,dy), \qquad 0 \le r \le t, \tag{1.10}$$

for a random kernel $p^W(r,x,t,dy)$, which is intuitively the conditional transition probability of $\{\xi_i(t)\}$ given $\{W(t)\}$. It was proved in Xiong [16] that the solution of (1.7) is also the unique non-negative solution of

$$v_{r,t}(x) + \int_r^t ds \int_{\mathbb{R}^d} v_{s,t}^2(y)p^W(r,x,s,dy) = \int_{\mathbb{R}^d} f(y)p^W(r,x,t,dy), \qquad 0 \le r \le t. \tag{1.11}$$

A similar characterization of the conditional log-Laplace functional of the model of [4, 14] was given in [11]. The next theorem characterizes the conditional log-Laplace functional of the weighted occupation time of $\{X_t\}$.

Theorem 1.2 *Suppose that conditions (A1,2) hold. Let $s \mapsto f_s$ be a mapping from $[0,\infty)$ to $\mathcal{X}$ continuous in the supremum norm. Then for any $r \le t$ we have*

$$\mathbf{P}_{r,\nu}^W \exp\Big\{-\int_r^t \langle X_s, f_s\rangle ds\Big\} = \exp\{-\langle \nu, u_{r,t}\rangle\}, \tag{1.12}$$

where $r \mapsto u_{r,t}$ is the unique $\mathcal{X}$-valued solution to the equation

$$u_{r,t}(x) = \int_r^t ([Lu_{s,t}(x) - u_{s,t}^2(x) + f_s(x)]ds + \int_r^t \sigma_1^*(x)\nabla u_{s,t}(x)\hat{d}W_s, \quad 0 \le r \le t. \tag{1.13}$$

Following the proof of Xiong [16, Lemma 8] one can show that $r \mapsto u_{r,t}$ is also uniquely characterized by the following equation:

$$u_{r,t}(x) + \int_r^t ds \int_{\mathbb{R}^d} u_{s,t}^2(y) p^W(r, x, s, dy) = \int_r^t ds \int_{\mathbb{R}^d} f_s(y) p^W(r, x, s, dy), \quad r \le t. \tag{1.14}$$

In the sequel, we need an extension of the state space of the superprocess. For $p > 0$ let $M_p(\mathbb{R}^d) = \{\nu : \langle \nu, \phi_p \rangle < \infty\}$, where $\phi_p(x) = e^{-p|x|}$ and $|\cdot|$ denotes the Euclidean norm. Clearly, the Lebesgue measure λ on $\mathbb{R}^d$ is included in $M_p(\mathbb{R}^d)$. It was explained in Xiong [16, pp.45-46] that the state space of the superprocess $\{X_t\}$ can be extended to $M_p(\mathbb{R}^d)$ with the above martingale problem characterization remaining valid. The results of Theorems 1.1 and 1.2 can also be extended to this situation. The occupation time of the superprocess is defined as

$$Y_t = \int_0^t X_s ds, \qquad t \ge 0.$$

In the following theorems we assume in addition that

(B1) $\mu \in M_p(\mathbb{R}^d)$ *is an absolutely continuous measure with bounded density* $x \mapsto \mu(x)$ *and is invariant for the conditional transition function* $p^W(s, x, t, dy)$*, namely,*

$$\int_{\mathbb{R}^d} p^W(s, x, t, \cdot) \mu(dx) = \mu$$

for all $s < t$ *and almost all given* $\{W(t)\}$.

The existence of such a measure has been studied by Xiong [16]. Here we state this result briefly for the convenience of the reader. Let

$$\bar{b}^i = -\frac{1}{2} \sum_{j,k=1}^d \sigma_1^{kj} \frac{\partial}{\partial x_k} \sigma_1^{ij}$$

and

$$\bar{L} f = \sum_{i=1}^d \bar{b}^i \frac{\partial}{\partial x_i} f + \frac{1}{2} \sum_{i,j=1}^d \bar{a}^{ij} \frac{\partial^2}{\partial x_i \partial x_j} f$$

where $\bar{a}^{ij} = \sum_{k=1}^d \sigma_2^{ik} \sigma_2^{jk}$. If there exists a constant K such that

$$|\nabla \log \mu(x)| \le K(1 + |x|), \qquad \forall x \in \mathbb{R}^d, \tag{1.15}$$

and

$$\bar{L}^* \mu = 0 \text{ and } \nabla^T (\sigma_1 \mu) = 0, \tag{1.16}$$

(note that there is a typo in [16]), then μ is an invariant measure.

Now, we discuss the existence and uniqueness for the solution to the equations in (1.16). Firstly, the most interesting example is when σ_1 and σ_2 are constant matrices. In this case, the invariant measure is unique and is the Lebesgue measure. The uniqueness of the invariant measure follows from that of the positive harmonic function ($\bar{L}^* \mu = 0$). Secondly, the invariant measure is not unique in general. For example, we may fix two measures μ_1 and μ_2 such that

(1.15) holds and seek the matrices σ_1 and σ_2 satisfying (1.16). Finally, if we add a constant briefly b in the motion (1.1) with $d = 2$, such a non-uniqueness can be given explicitly if we take

$$\sigma_1 = \begin{pmatrix} b_2 & -b_1 \\ b_2 & -b_1 \end{pmatrix} \text{ and } \sigma_2 = I.$$

Then

$$d\mu_1 = dx \text{ and } d\mu_2 = e^{-b^T x} dx$$

are two invariant measures.

To prove convergence in the space $M_p(\mathbb{R}^d)$, we define a metric on it. Let $\{f_j,\ j = 1, 2, \cdots\}$ be a dense family in $\mathcal{X}$ with compact supports, and for $\nu_1,\ \nu_2 \in M_p(\mathbb{R}^d)$, we define

$$\rho(\nu_1, \nu_2) = \sum_{j=1}^{\infty} 2^{-j} \left(\langle \nu_1 - \nu_2, f_j \rangle \wedge 1 \right).$$

Theorem 1.3 *Suppose that $d \geq 3$ and conditions (A1,2) and (B1) hold. If $X_0 = \mu$, then*

$$\rho\left(t^{-1} Y_t, \mu\right) \xrightarrow{p} 0, \qquad t \to \infty,$$

where "$\xrightarrow{p}$" denotes convergence in probability.

The above theorem asserts that in high dimensions the average in time of the superprocess converges to the invariant measure μ of the conditional underlying transition function $p^W(s, x, t, dy)$. For the critical dimension $d = 2$, we need to assume the following additional conditions:

(C1) $\lim_{|x| \to \infty} \mu(x) = \mu(\infty)$, and there exist two strictly positive constants c_1, c_2 so that $c_1 \leq \mu(x) \leq c_2$ for all $x \in \mathbb{R}^2$;

(C2) there exist two constant matrices $(\tilde{\sigma}_1^{ij})$ and $(\tilde{\sigma}_2^{ij})$ so that

$$\sigma_1^{ij}(x) \to \tilde{\sigma}_1^{ij}, \quad \sigma_2^{ij}(x) \to \tilde{\sigma}_2^{ij}, \qquad |x| \to \infty,\ i, j = 1, 2.$$

Under those conditions, let $\tilde{p}^W(s, x, t, dy)$ be defined by (1.9) and (1.10) with σ_l^{ij} replaced by $\tilde{\sigma}_l^{ij}$. It is easy to see that $\tilde{p}^W(s, x, t, dy)$ is absolutely continuous with respect to the Lebesgue measure and has density $\tilde{p}^W(s, x, t, y)$ given by

$$\tilde{p}^W(s, x, t, y) = \frac{1}{(t-s)^{d/2} \det(\tilde{\sigma}_2)} g\left(\frac{\tilde{\sigma}_2^{-1}(y - x - \tilde{\sigma}_1(W(t) - W(s)))}{\sqrt{t-s}} \right), \tag{1.17}$$

where g is the density of the 2-dimensional standard normal distribution. Recall that the Lebesgue measure is denoted by λ.

Theorem 1.4 *Suppose that $d = 2$ and conditions (A1,2), (B1) and (C1,2) hold. If $X_0 = \mu$, then*

$$t^{-1} Y_t \xrightarrow{d} \xi, \qquad t \to \infty,$$

where "$\xrightarrow{d}$" denotes convergence in distribution, and ξ is a random measure with Laplace transform given by

$$\mathbf{P}\left[\exp\{-\langle \xi, f \rangle\}\right] = \mathbf{P} \exp\left\{ -\langle \mu, f \rangle + \mu(\infty) \int_0^1 \langle \lambda, v^2(s, \cdot) \rangle ds \right\}, \qquad f \in \mathcal{X}, \tag{1.18}$$

where $(r,x) \mapsto v(r,x)$ is the unique positive solution to the following equation:

$$v(r,x) + \int_r^1 ds \int_{\mathbb{R}^2} v^2(s,y)\tilde{p}^W(r,x,s,y)dy = \langle \lambda, f\rangle \int_r^1 \tilde{p}^W(r,x,s,0)ds \tag{1.19}$$

with $0 \le r \le 1$ and $x \in \mathbb{R}^2$.

Remark. (1) For $d = 1$, Xiong [16] has proved $\int_0^\infty \langle X_t, f\rangle dt < \infty$, P_μ-a.s. For the super-Brownian motion without stochastic flow, the occupation time process $\{Z_t : t \ge 0\}$ has been constructed by Iscoe [6], and its ergodicity limits were obtained by Iscoe [7]: For $d = 1$, the total weighted occupation time is finite; For the critical dimension $d = 2$, as $t \to \infty$, $\frac{1}{t}Z_t$ converges vaguely to $\zeta\lambda$ for some real random variable ζ; while for $d \ge 3$, the limit measure is λ, see [7, Theorems 1,2]. Hence, the ergodicity of the process with stochastic flow is similar to that of the classical super-Brownian motion.

(2) It is known that the underlying motion $\{\xi(t) : t \ge 0\}$ is transient if and only if $d > 2$. The asymptotic behaviors of the corresponding super-processes are mainly dependent on the behavior of underlying process. So the $d \ge 3$ and $d = 2$ dichotomy appears in the present paper and also Iscoe [7].

Under the conditions of Theorem 1.4 one can actually show that $\{T^{-1}Y_{tT} : 0 \le t \le 1\}$ converges as $T \to \infty$ to a measure-valued process $\{\xi_t : 0 \le t \le 1\}$ in finite dimensional distributions characterized by

$$\mathbf{P}\exp\Big\{-\sum_{i=1}^n \langle \xi_{t_i}, f_i\rangle\Big\} = \mathbf{P}\exp\Big\{-\sum_{i=1}^n t_i\langle \mu, f_i\rangle + \mu(\infty)\int_0^1 \langle \lambda, v^2(s,\cdot)\rangle ds\Big\},$$

where $0 \le t_1 < \cdots < t_n \le 1$, $f_1, \cdots, f_n \in \mathcal{X}$, and $v(s,x) := v(s,x; f_1, \cdots, f_n)$ is the unique positive solution to

$$v(s,x) + \int_s^1 du \int_{\mathbb{R}^2} v^2(u,y)\tilde{p}^W(s,x,u,dy) = \sum_{i=1}^n \int_s^1 \langle \lambda, f_i\rangle 1_{[0,t_i]}(u)\tilde{p}^W(s,x,u,0)du$$

with $0 \le s \le 1$ and $x \in \mathbb{R}^2$. With some additional work on tightness, one can also prove the weak convergence in the space $C([0,1], \mathbb{R}^+)$. The tightness can be established by checking Kolmogorov's criterion based on the third order moment estimate of $\frac{1}{t}\langle Y_t, f\rangle$. Suppose $(T_n)_{n=1}^\infty$ is a sequence such that $T_n \uparrow \infty$. Under the same conditions of Theorem 1.4, there exists a positive constant C_0 independent of $(T_n)_{n=1}^\infty$, such that for $0 \le t_1 \le t_2 \le 1$,

$$\mathbf{P}\left\{\left[\frac{1}{T_n}\int_{t_1 T_n}^{t_2 T_n} \langle X_s, f\rangle ds\right]^3\right\} \le C_0(t_2 - t_1)^2.$$

Therefore, the sequence $\{T_n^{-1}\langle Y_{tT_n}, f\rangle : 0 \le t \le 1\}$ is tight in $C([0,1], \mathbb{R}^+)$. That implies the tightness of $\{T_n^{-1}Y_{tT_n} : 0 \le t \le 1\}$ in $C([0,1], M_p(\mathbb{R}^2))$. The calculations are complicated while the idea is classical, so we skip them here.

Theorems 1.1 and 1.2 are proved in Section 2. The proofs of Theorems 1.3 and 1.4 are given in Sections 3 and 4, respectively. In the proofs of those results, we shall use $C, C_1, C_2, \cdots$ to denote constants which can vary from place to place. Let $\|\cdot\|_0$ denote the norm of $L^2(\mathbb{R}^d, \lambda)$.

2 The conditional log-Laplace equation

In this section we give the characterization of the conditional log-Laplace functionals of the superprocess $\{X_t\}$ and its weighted occupation times. We here assume Conditions (A1,2) hold. The results hold for all dimensions $d \geq 1$.

To prove Proposition 2.3 below, we will need to use Krylov's L_p theory for SPDE. To make our paper as self-contained as possible, we outline the main definitions and results of Krylov [9] enough for our purpose (in a less general setup).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space, $(\mathcal{F}_t, t \geq 0)$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all $\mathbf{P}$-null subsets of Ω, and $\mathcal{P}$ be the predictable σ-field generated by $(\mathcal{F}_t, t \geq 0)$. Denote $\mathbb{H}_p^n(\mathbb{R}^m) = L_p([0,T] \times \Omega, \mathcal{P}, H_p^n(\mathbb{R}^m))$ where $H_p^n(\mathbb{R}^m)$ stands for $(f_1, \cdots, f_n)$ with $f_i \in H_p^n$, $i = 1, 2, \cdots, n$. Now we define the space $\mathcal{H}_p^n$ which plays a key role in the L_p theory.

Definition 2.1 *The space $\mathcal{H}_p^n$ consists of $u \in \mathbb{H}_p^n$ such that $u(0, \cdot) \in L_p(\Omega, \mathcal{F}_0, H_p^{n-2/p})$, $u_{xx} \in \mathbb{H}_p^n(\mathbb{R}^{d\times d})$, and there exist $f \in \mathbb{H}_p^n$ and $g \in \mathbb{H}_p^n(\mathbb{R}^d)$ such that for any $\phi \in C_0^\infty$, the equality*

$$\langle u(t,\cdot), \phi\rangle = \langle u(0,\cdot), \phi\rangle + \int_0^t \langle f(s,\cdot), \phi\rangle\, ds + \sum_{k=1}^d \int_0^t \left\langle g^k(s,\cdot), \phi\right\rangle dW^k(s)$$

holds for all $t \leq T$ with probability 1, where u_{xx} is the $d \times d$ matrix consists of all second order partial derivatives of u and $W(t)$ is a d-dimensional Brownian motion.

Consider the following SPDE:

$$du(t,x) = \left[\sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(t,x) + f(u,t,x)\right] dt + \sum_{i,k=1}^d \sigma_1^{ij}(x) \frac{\partial}{\partial x_i} u(t,x) dW^k(t), \qquad (2.1)$$

where f is real-valued.

Let

$$\alpha^{ij}(x) = \frac{1}{2} \sum_{k=1}^d \sigma_1^{ik}(x) \sigma_1^{jk}(x).$$

Let $\gamma = 0$ if n is an integer; and otherwise $\gamma > 0$ is such that $|n| + \gamma$ is not an integer. Define

$$B^{|n|+\gamma} = \begin{cases} B(\mathbb{R}^d) & \text{if } n = 0, \\ C^{|n|-1,1}(\mathbb{R}^d) & \text{if } n = \pm 1,\ \pm 2,\ \cdots, \\ C^{|n|+\gamma}(\mathbb{R}^d) & \text{otherwise,} \end{cases}$$

where $B(\mathbb{R}^d)$ is the set of bounded functions, $C^{|n|-1,1}(\mathbb{R}^d)$ is the Banach space of $|n| - 1$ times continuously differentiable functions whose derivatives of $(|n| - 1)$st order satisfy the Lipschitz condition on $\mathbb{R}^d$, and $C^{|n|+\gamma}(\mathbb{R}^d)$ is the usual Hölder space. Actually, we will need only the case of $n = 0$ in the proof of Proposition 2.3 below.

The following conditions are imposed by Krylov [9].

(K1) (coercivity) For any $x \in \mathbb{R}^d$, we have

$$K|\lambda|^2 \geq \sum_{i,j=1}^d \left[a^{ij}(x) - \alpha^{ij}(x)\right] \lambda^i \lambda^j \geq \delta |\lambda|^2,$$

where K, δ are fixed strictly positive constants.

(K2) (uniform continuity of a and σ_1) For any $\epsilon > 0$, i, j, there exists a $\kappa_\epsilon > 0$ such that

$$|a^{ij}(x) - a^{ij}(y)| + |\sigma_1^{ij}(x) - \sigma_1^{ij}(y)| \leq \epsilon$$

whenever $|x - y| < \kappa_\epsilon$.

(K3) a^{ij}, $\sigma_1^{ij} \in B^{|n|+\gamma}$.

(K4) For any $u \in H_p^{n+2}$, the functions $f(u, t, x)$ as a function taking values in H_p^n.

(K5) $f(0, \cdot, \cdot) \in \mathcal{F}_p^n$.

(K6) The function f is continuous in u. Moreover, for any $\epsilon > 0$, there exists a constant K_ϵ such that for any u, $v \in H_p^{n+2}$, t, we have

$$\|f(u, t, \cdot) - f(v, t, \cdot)\|_{n,p} \leq \epsilon \|u - v\|_{n+2,p} + K_\epsilon \|u - v\|_{n,p}.$$

The following theorem is Theorem 5.1 in the book [9].

Theorem 2.2 *Let Assumptions (K1-K6) be satisfied and let*

$$u_0 \in L_p\left(\Omega, \mathcal{F}_0, H^{n+1-2/p}\right).$$

Then the Cauchy problem for equation (2.1) on $[0, T]$ with initial condition $u(0, \cdot) = u_0$ has a unique solution $u \in \mathcal{H}_p^{n+2}$.

Now we apply Krylov's result to our setup.

Proposition 2.3 *Let $c \geq 0$ be a constant. Then for any $f \in \mathcal{X}$ there is a unique solution $u \in \mathcal{X}$ to the following SPDE*

$$u(t, x) = f(x) + \int_0^t [Lu(s, x) - cu^2(s, x)]ds + \int_0^t \sigma_1^*(x) \nabla u(s, x) d\tilde{W}(s), \tag{2.2}$$

where $t \mapsto \tilde{W}(t)$ is a d-dimensional Brownian motion.

Proof. For $c = d = 1$ and f with compact support, it is proved in Xiong [15] that (2.2) has a unique solution $t \mapsto u(t, \cdot) \in H_\infty^+$. The same argument applies to $c \geq 0$, $d \geq 1$ and $f \in \mathcal{X}$. We only need to prove $u(\cdot, \cdot) \in \mathcal{H}_p^2$. Fix $u(\cdot, \cdot)$ and consider the linear SPDE:

$$v(t, x) = f(x) + \int_0^t \left[Lv(s, x) + f(v, s, x)\right] ds + \int_0^t \sigma_1^*(x) \nabla v(s, x) d\tilde{W}(s), \tag{2.3}$$

where $f(v, t, x) = -cu(t, x)v(t, x)$. Note that $f \in H_p^0$ if $v \in H_p^2$. Moreover, for $v_1, v_2 \in H_p^2$ it is easy to see

$$\|f(v_1, t, \cdot) - f(v_2, t, \cdot)\|_{0,p} \leq K_0 \|v_1 - v_2\|_{0,p}.$$

where $\|\cdot\|_{0,p}$ denote the norm in $L^p(\mathbb{R}^d, \lambda)$. The verifications of the other conditions of (K1-K5) with $n = 0$ are straight forward. Then have $v \in \mathcal{H}_p^2$. The conclusion of the proposition follows because $t \mapsto u(t, \cdot)$ is the unique solution to (2.2) taking values in H_∞^+. ∎

Corollary 2.4 *For any $t \geq 0$ and $f \in \mathcal{X}$ there is a solution $r \mapsto v_{r,t} \in \mathcal{X}$ to the backward SPDE*

$$v_{r,t}(x) = f(x) + \int_r^t [Lv_{s,t}(x) - cv_{s,t}^2(x)]ds + \int_r^t \sigma_1^*(x)\nabla v_{s,t}(x)\hat{d}W(s). \tag{2.4}$$

Proof. This follows from the above proposition applied to the Brownian motion $r \mapsto \tilde{W}(r) = W(t-r) - W(t)$. ∎

Proof of Theorem 1.1. For $d = 1$, the result was established in Xiong [15] using Wong-Zakai approximation. Here we sketch a simpler proof by adapting an argument of Mytnik and Xiong [12] to the current model. For fixed $\varepsilon > 0$ we define a measure-valued process $\{X_t^\varepsilon\}$ as follows. For $i = 0, 1, 2, \cdots$ we assume $\{X_t^\varepsilon : 2i\varepsilon \leq t \leq (2i+1)\varepsilon\}$ is a classical superprocess corresponding to the non-linear equation

$$v_{s,t}^\varepsilon(x) = f(x) - \int_s^t [Lv_{r,t}^\varepsilon(x) - 2v_{r,t}^\varepsilon(x)^2]dr,$$

where $2i\varepsilon \leq s \leq t \leq (2i+1)\varepsilon$. For $i = 0, 1, 2, \cdots$ let $\{X_t^\varepsilon : (2i+1)\varepsilon \leq t \leq 2(i+1)\varepsilon\}$ be the solution to the linear equation

$$\langle X_t^\varepsilon, f\rangle = \langle X_{(2i+1)\varepsilon}^\varepsilon, f\rangle + \int_{(2i+1)\varepsilon}^t \langle X_s^\varepsilon, Lf\rangle ds + \int_{(2i+1)\varepsilon}^t \langle X_s^\varepsilon, \sigma_1^*\nabla f\rangle dW(s).$$

Observe that $\{X_t^\varepsilon : (2i+1)\varepsilon \leq t \leq 2(i+1)\varepsilon\}$ corresponds to the backward equation

$$v_{s,t}^\varepsilon(x) = f(x) + \int_s^t Lv_{r,t}^\varepsilon(x)dr + \int_s^t \sigma_1^*(x)\nabla v_{r,t}^\varepsilon(x)\hat{d}W(r),$$

where $(2i+1)\varepsilon \leq s \leq t \leq 2(i+1)\varepsilon$. Then we claim that

$$\mathbf{P}_{r,\nu}^W \exp\{-\langle X_t^\varepsilon, f\rangle\} = \exp\{-\langle \nu, \nu_{r,t}^\varepsilon\rangle\}, \qquad t \geq r \geq 0. \tag{2.5}$$

In the case of $2k\varepsilon \leq t \leq (2k+1)\varepsilon$ for some $k \geq 0$, we observe that the behaviors of the processes $\{X_s^\varepsilon : 2k\varepsilon \leq s \leq t\}$ and $\{v_{s,t}^\varepsilon : 2k\varepsilon \leq s \leq t\}$ do not depend on $\{W(t)\}$. It follows that

$$\mathbf{P}_{r,\nu}^W\big[e^{-\langle X_t^\varepsilon, f\rangle}\big|X_{2k\varepsilon}^\varepsilon\big] = \exp\{-\langle X_{2k\varepsilon}^\varepsilon, v_{2k\varepsilon,t}^\varepsilon\rangle\},$$

and hence

$$\mathbf{P}_{r,\nu}^W \exp\{-\langle X_t^\varepsilon, f\rangle\} = \mathbf{P}_{r,\nu}^W \exp\{-\langle X_{2k\varepsilon}^\varepsilon, v_{2k\varepsilon,t}^\varepsilon\rangle\}.$$

By Xiong [17, Corollary 6.21] we have

$$\langle X_{2k\varepsilon}^\varepsilon, v_{2k\varepsilon,t}^\varepsilon\rangle = \langle X_{(2k-1)\varepsilon}^\varepsilon, v_{(2k-1)\varepsilon,t}^\varepsilon\rangle,$$

and so

$$\mathbf{P}_{r,\nu}^W \exp\{-\langle X_t^\varepsilon, f\rangle\} = \mathbf{P}_{r,\nu}^W \exp\big\{-\langle X_{(2k-1)\varepsilon}^\varepsilon, v_{(2k-1)\varepsilon,t}^\varepsilon\rangle\big\}.$$

Continuing this pattern gives (2.5). The proof of the equality in the case of $(2k+1)\varepsilon \leq t < 2(k+1)\varepsilon$ for some $k \geq 0$ is similar. The conclusion of the theorem then follows by proving the *weak convergence* of $(X^\varepsilon, W, v^\varepsilon)$ to (X, W, v) using the same techniques as in [12]. We omit the details here. ∎

Proof of Theorem 1.2. Let $s_2 \geq s_1 \geq 0$ and $f_1, f_2 \in \mathcal{X}$. For $s_1 \leq r \leq s_2$ let $\psi_{r,s_2}(x)$ be given by

$$\psi_{r,s_2}(x) = f_2(x) + \int_r^{s_2} [L\psi_{s,s_2}(x) - \psi_{s,s_2}^2(x)]ds + \int_r^{s_2} \sigma_1^*(x)\nabla\psi_{s,s_2}(x)\hat{d}W_s.$$

For $r \leq s_1$ let $\phi_{r,s_1}(x)$ be the solution to

$$\begin{aligned}\phi_{r,s_1}(x) &= f_1(x) + \psi_{s_1,s_2}(x) + \int_r^{s_1} [L\phi_{s,s_1}(x) - \phi_{s,s_1}^2(x)]ds \\ &\quad + \int_r^{s_1} \sigma_1^*(x)\nabla\phi_{r,s_1}(x)\hat{d}W_s.\end{aligned}$$

By Theorem 1.1 for $r \leq s_1 \leq s_2$ we have

$$\begin{aligned}\mathbf{P}_{r,\mu}^W \exp\big\{ - \langle X_{s_1}, f_1\rangle - \langle X_{s_2}, f_2\rangle\big\} &= \mathbf{P}_{r,\mu}^W \exp\big\{ - \langle X_{s_1}, f_1 + \psi_{s_1,s_2}\rangle\big\} \\ &= \exp\{-\langle \mu, \phi_{r,s_1}\rangle\}.\end{aligned}$$

Now we define

$$u(r,x) = \begin{cases} \psi_{r,s_2}(x), & s_1 \leq r \leq s_2, \\ \phi_{r,s_1}(x), & r < s_1. \end{cases}$$

It is easy to see that

$$\begin{aligned}u(r,x) &= f_1(x)1_{\{r<s_1\}} + f_2(x)1_{\{r<s_2\}} + \int_r^{s_2} [Lu(s,x) - u^2(s,x)]ds \\ &\quad + \int_r^{s_2} \sigma_1^*(x)\nabla u(s,x)\hat{d}W_s.\end{aligned}$$

By similar arguments as the above we get

$$\mathbf{P}_{r,\nu}^W \exp\left\{ - \sum_{i=1}^n \left\langle X_{s_i}, \frac{1}{n} f_{s_i}\right\rangle\right\} = \exp\{-\langle \nu, u_t^n(r,\cdot)\rangle\}, \tag{2.6}$$

where $s_i = it/n$ and $u_t^n(\cdot,\cdot)$ is the solution to

$$u_t^n(r,x) = f_r^n(x) + \int_r^t [Lu_t^n(s,x) - u_t^n(s,x)^2]ds + \int_r^t \sigma_1^*(x)\nabla u_t^n(s,x)\hat{d}W_s, \tag{2.7}$$

where

$$f_r^n(x) := \frac{1}{n}\sum_{i=1}^n f_{s_i}(x)1_{\{r<s_i\}} \to \int_r^t f_s(x)ds.$$

To prove the convergence of $u_t^n(r,x)$ we consider the forward version of (2.7). Setting $\bar{u}^n(s,x) = u_t^n(t-s,x)$ we have

$$\bar{u}_t^n(s,x) = f_{t-s}^n(x) + \int_0^s [L\bar{u}_t^n(r,x) - \bar{u}_t^n(r,x)^2]dr + \int_0^s \sigma_1^*(x)\nabla\bar{u}_t^n(r,x)d\tilde{W}_r,$$

where $\tilde{W}(r) = W(t) - W(t-r)$ and the stochastic integral is the usual Itô integral. Let $u_t^{n,m}(r,x) = \bar{u}_t^n(r,x) - \bar{u}_t^m(r,x)$, $f_s^{n,m}(x) = f_s^n(x) - f_s^m(x)$ and $c_s^{n,m}(x) = \bar{u}_t^n(s,x) + \bar{u}_t^m(s,x)$. Then we have

$$u_t^{n,m}(s,x) = f_{t-s}^{n,m}(x) + \int_0^s [Lu_t^{n,m}(r,x) - c_r^{n,m}(x)u_t^{n,m}(r,x)]dr$$

$$+\int_0^s \sigma_1^*(x)\nabla u_t^{n,m}(r,x)d\tilde{W}_r.$$

As in the proof of Xiong [17, Corollary 6.13], one can show there exists $C>0$ such that

$$\sup_{0\le s\le t}\mathbf{P}\Big[\|u_t^{n,m}(s,\cdot)\|_0^2\Big]\le C\sup_{0\le s\le t}\|f_s^n-f_s^m\|_0^2.$$

Then there is a random function $(s,x)\mapsto u_t(s,x)$ such that

$$\sup_{0\le r\le t}\mathbf{P}\Big[\|u_t^n(r,\cdot)-u_t(r,\cdot)\|_0^2\Big]\to 0.$$

It is easy to see that $(r,x)\mapsto u_t(r,x)$ solves (1.13). The uniqueness of the solution follows by a similar calculation. Then (1.12) follows from (2.6) for a finite measure ν. We can extend the result to the σ-finite measure ν using the same arguments as in the proof of Lee et al [10, Theorem 2.5]. ∎

3 Ergodicity for high dimensions

In this section we assume Conditions (A1,2) and (B1) hold. We shall need some estimates of the transition densities of diffusion processes. Let $(T_t)_{t\ge 0}$ denote the transition semigroup of the standard d-dimensional Brownian motion and let

$$(t,x,y)\mapsto g_t(x-y)=g(t,x-y)$$

denote the corresponding transition density.

Lemma 3.1 *For any $t_i>0$ and $x,y_i\in\mathbb{R}^d$ $(i=1,\cdots,n)$ we have*

$$\prod_{i=1}^n g(t_i,x-y_i)\le\Big(\sum_{i=1}^n\frac{t_1\cdots t_n}{t_i}\Big)^{-d/2}g\Big(\Big(\sum_{i=1}^n\frac{1}{t_i}\Big)^{-1},\ x-\Big(\sum_{i=1}^n\frac{1}{t_i}\Big)^{-1}\sum_{i=1}^n\frac{y_i}{t_i}\Big).$$

Proof. By elementary calculations,

$$\begin{aligned}\prod_{i=1}^n g(t_i,x-y_i) &= \frac{1}{(2\pi t_1\cdots t_n)^{d/2}}\exp\Big\{-\sum_{i=1}^n\frac{1}{2t_i}|x-y_i|^2\Big\}\\
&= \frac{1}{(2\pi t_1\cdots t_n)^{d/2}}\exp\Big\{-\sum_{i=1}^n\frac{|x|^2}{2t_i}+\sum_{i=1}^n\frac{xy_i}{t_i}-\sum_{i=1}^n\frac{|y_i|^2}{2t_i}\Big\}\\
&= \frac{1}{(2\pi t_1\cdots t_n)^{d/2}}\exp\Big\{-\sum_{i=1}^n\frac{1}{2t_i}\Big|x-\Big(\sum_{i=1}^n\frac{1}{t_i}\Big)^{-1}\sum_{i=1}^n\frac{y_i}{t_i}\Big|^2\Big\}\\
&\qquad\cdot\exp\Big\{\frac12\Big(\sum_{i=1}^n\frac{1}{t_i}\Big)^{-1}\Big(\Big|\sum_{i=1}^n\frac{y_i}{t_i}\Big|^2-\sum_{i=1}^n\frac{1}{t_i}\sum_{j=1}^n\frac{|y_j|^2}{t_j}\Big)\Big\}\\
&= \frac{1}{(2\pi t_1\cdots t_n)^{d/2}}\exp\Big\{-\sum_{i=1}^n\frac{1}{2t_i}\Big|x-\Big(\sum_{i=1}^n\frac{1}{t_i}\Big)^{-1}\sum_{i=1}^n\frac{y_i}{t_i}\Big|^2\Big\}\\
&\qquad\cdot\exp\Big\{\frac12\Big(\sum_{i=1}^n\frac{1}{t_i}\Big)^{-1}\Big(\sum_{i,j=1}^n\frac{y_iy_j}{t_it_j}-\sum_{i,j=1}^n\frac{|y_j|^2}{t_it_j}\Big)\Big\},\end{aligned}$$

where

$$\sum_{i,j=1}^{n} \frac{y_i y_j}{t_i t_j} \leq \sum_{i,j=1}^{n} \frac{|y_i|^2 + |y_j|^2}{2t_i t_j} = \sum_{i,j=1}^{n} \frac{|y_j|^2}{t_i t_j}.$$

Then we have the desired inequality. ∎

Next we consider d-dimensional diffusion processes generated by differential operators. Let us consider the operator A defined by

$$Af(x) = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \tag{3.1}$$

where the coefficients are β-Hölder continuous for $0 < \beta \leq 1$ and bounded by a constant $B > 0$. In addition, we assume $(a^{ij}(x))$ is a symmetric and positive definite matrix that is uniformly elliptic. More precisely, there are $C > c > 0$ so that

$$c|\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \leq C|\xi|^2, \qquad \xi \in \mathbb{R}^d.$$

It is well-known that A generates a diffusion process in $\mathbb{R}^d$ with continuous transition density $p(t, x, y)$.

Lemma 3.2 (Aronson [2] and Friedman [5, p.24]) *For any $T \geq 0$ there are constants $c_0 > 0$ and $K > k > 0$ only depending on (c, B, T) so that*

$$kg(c_0 t, x - y) \leq p(t, x, y) \leq Kg(c_0 t, x - y), \qquad 0 < t \leq T,\ x, y \in \mathbb{R}^d.$$

Corollary 3.3 *Let $p^W(r, x, t, dy)$ be defined by (1.9) and (1.10). Then for any $0 \leq r \leq t_1 \leq t_2 \leq \cdots \leq t_n$ there is $C_n > 0$ so that*

$$\mathbf{P}\Big[\prod_{i=1}^{n} \int_{\mathbb{R}^d} f_i(y_i) p^W(r, x_i, t_i, dy_i)\Big] \leq C_n \prod_{i=1}^{n} \int_{\mathbb{R}^d} f_i(y_i) g(c_0(t_i - r), x_i - y_i) dy_i \tag{3.2}$$

for all $x_1, \cdots, x_n \in \mathbb{R}^d$ and $f_1, \cdots, f_n \in B(\mathbb{R}^d)^+$.

Proof. For $i = 1, \cdots, n$ define $\{\xi_i(t) : t \geq r\}$ by (1.1) with $\xi_i(r) = x_i$. Then $\{(\xi_1(t), \cdots, \xi_n(t)) : t \geq r\}$ is an nd-dimensional diffusion with generator L_n given by

$$\begin{aligned} L_n F(x_1, \cdots, x_n) &= \frac{1}{2} \sum_{p,q=1}^{n} \sum_{i,j=1}^{d} a_1^{ij}(x_p) \frac{\partial^2 F}{\partial x_p^i \partial x_q^j}(x_1, \cdots, x_n) \\ &\quad + \frac{1}{2} \sum_{p=1}^{n} \sum_{i,j=1}^{d} a_2^{ij}(x_p) \frac{\partial^2 F}{\partial x_p^i \partial x_p^j}(x_1, \cdots, x_n), \end{aligned}$$

where

$$a_m^{ij}(x) = \sum_{k=1}^{d} \sigma_m^{ik}(x) \sigma_m^{jk}(x), \qquad m = 1, 2.$$

Conditions (A1,2) imply that the coefficient matrix of L_n is uniformly elliptic. By the arguments of Xiong and Zhou [18] it is simple to see that

$$\mathbf{P}\bigg[\prod_{i=1}^n \int_{\mathbb{R}^d} f_i(y_i) p^W(r, x_i, t_i, dy_i)\bigg] = \mathbf{P}_{r,(x_1,\cdots,x_n)}\bigg[\prod_{i=1}^n f_i(\xi_i(t_i))\bigg]. \tag{3.3}$$

By Lemma 3.2 we get (3.2) for $t_1 = \cdots = t_n$. In the general case $0 \leq r \leq t_1 \leq t_2 \leq \cdots \leq t_n$, we prove the result by induction in $n \geq 1$. For $n = 1$ this is trivial. Suppose the result holds for $n-1$. Then

$$\begin{aligned} \mathbf{P}_{r,(x_1,\cdots,x_n)}\bigg[\prod_{i=1}^n f_i(\xi_i(t_i))\bigg] &= \mathbf{P}_{r,(x_1,\cdots,x_n)}\bigg\{f_1(\xi_1(t_1))\mathbf{P}_{r,(\xi_2(t_1),\cdots,\xi_n(t_1))}\bigg[\prod_{i=2}^n f_i(\xi_i(t_i))\bigg]\bigg\} \\ &\leq C_{n-1}\mathbf{P}_{r,(x_1,\cdots,x_n)}\bigg\{f_1(\xi_1(t_1))\bigg[\prod_{i=2}^n T_{c_0(t_i-t_1)} f_i(\xi_i(t_1))\bigg]\bigg\} \\ &\leq C_n \prod_{i=1}^n T_{c_0(t_i-r)} f_i(x_i) \end{aligned}$$

by the semigroup property of $(T_t)_{t\geq 0}$. That gives the desired inequality. ∎

Lemma 3.4 *Let A and A_n be differential operators of the form (3.1) with coefficients (a^{ij}) and (a_n^{ij}), respectively. Let $p(t,x,y)$ and $p_n(t,x,y)$ denote the transition densities of the corresponding diffusion processes. Suppose that $F \subset \mathbb{R}^d$ is a set of zero Lebesgue measure and $\lim_{n\to\infty} a_n^{ij}(x) = a^{ij}(x)$ for all $x \in F^c$. Then for any $t > 0$ and $x \in F^c$ we have*

$$\lim_{n\to\infty} p_n(t,x,y) = p(t,x,y), \qquad y \in B \tag{3.4}$$

uniformly for each bounded set $B \subset \mathbb{R}^d$.

Proof. We need a construction of the transition density $p(t,x,y)$ given in Friedman [5]. Let $(\alpha^{ij}(x))$ be the inverse matrix to $(a^{ij}(x))$. For $t > 0$ and $x, y \in \mathbb{R}^d$ let

$$Z(t,x,y) = \frac{\det(\alpha^{ij}(x))^{1/2}}{(2\pi t)^{d/2}} \exp\bigg\{-\frac{1}{2t}\sum_{i,j=1}^d \alpha^{ij}(x)(y_i - x_i)(y_j - x_j)\bigg\}.$$

Then define

$$(LZ)_1(t,x,y) = \frac{1}{2}\sum_{i,j=1}^d [a^{ij}(y) - a^{ij}(x)]\frac{\partial^2 Z}{\partial y_i \partial y_j}(t,x,y)$$

and define inductively

$$(LZ)_{m+1}(t,x,y) = \int_0^t ds \int_{\mathbb{R}^d} (LZ)_m(s,x,\xi)(LZ)_1(t-s,\xi,y)d\xi.$$

By [5, p.23, Theorem 10] we have

$$p(t,x,y) = Z(t,x,y) + \int_0^t ds \int_{\mathbb{R}^d} F(s,x,\xi)Z(t-s,\xi,y)d\xi, \tag{3.5}$$

where

$$F(t,x,y) = \sum_{m=1}^{\infty} (LZ)_m(t,x,y).$$

A similar construction can be given for $p_n(t,x,y)$. Fix $t>0$ and $x \in F^c$. If $y_n \to y$ as $n \to \infty$, one can use (3.5) and dominated convergence to see $p_n(t,x,y_n) \to p(t,x,y)$. The estimates to justify the application of the dominated convergence can be found in [5]. Then we have the desired result. ■

Proof of Theorem 1.3. By Theorem 1.2 and (1.14), for any $\theta \geq 0$ we have

$$\mathbf{P}\exp\big\{-t^{-1}\langle Y_t, \theta f\rangle\big\} = \mathbf{P}\exp\big\{-\langle \mu, u_t(0,\cdot;\theta)\rangle\big\}, \tag{3.6}$$

where $(r,x) \mapsto u_t(r,x;\theta)$ is the unique positive solution to

$$u(r,x) + \int_r^t ds \int_{\mathbb{R}^d} u(s,y)^2 p^W(r,x,s,dy) = \frac{\theta}{t}\int_r^t ds \int_{\mathbb{R}^d} f(y) p^W(r,x,s,dy). \tag{3.7}$$

Recalling that $\mu(dx)$ is an invariant measure of $p^W(r,x,t,dy)$ we obtain

$$\mathbf{P}\exp\big\{-t^{-1}\langle Y_t, \theta f\rangle\big\} = \mathbf{P}\exp\bigg\{-\theta\langle \mu, f\rangle + \int_0^t dr \int_{\mathbb{R}^d} u_t^2(r,x)\mu(dx)\bigg\},$$

The inequality $|e^{-x} - e^{-y}| \leq |x-y|$, $x,y \geq 0$, together with (3.6) and (3.7), implies that

$$\Big|\mathbf{P}\exp\big\{-t^{-1}\langle Y_t, \theta f\rangle\big\} - \exp\{-\theta\langle\mu, f\rangle\}\Big| \leq \mathbf{P}[\varepsilon(t)], \tag{3.8}$$

where

$$\varepsilon(t) = \int_0^t dr \int_{\mathbb{R}^d} u_t^2(r,x)\mu(dx).$$

In view of (3.7) we have

$$\begin{aligned}
\mathbf{P}[\varepsilon(t)] &\leq \frac{\theta^2}{t^2}\mathbf{P}\bigg\{\int_0^t dr \int_{\mathbb{R}^d}\bigg[\int_r^t ds \int_{\mathbb{R}^d} f(y)p^W(r,x,s,dy)\bigg]^2 \mu(dx)\bigg\} \\
&= \frac{\theta^2}{t^2}\mathbf{P}\bigg[\int_0^t dr \int_{\mathbb{R}^d}\mu(dx)\int_r^t ds_1 \int_r^t ds_2 \int_{\mathbb{R}^d} f(y_1)p^W(r,x,s_1,dy_1) \\
&\qquad \int_{\mathbb{R}^d} f(y_2)p^W(r,x,s_2,dy_2)\bigg] \\
&\leq \frac{C}{t^2}\mathbf{P}\bigg[\int_0^t dr \int_{\mathbb{R}^d} dx \int_r^t ds_1 \int_{s_1}^t ds_2 \int_{\mathbb{R}^{2d}} f(y_1)f(y_2) \\
&\qquad p^W(r,x,s_1,dy_1)p^W(r,x,s_2,dy_2)\bigg].
\end{aligned}$$

By Corollary 3.3 we get

$$\begin{aligned}
\mathbf{P}[\varepsilon(t)] &\leq \frac{C}{t^2}\int_0^t dr \int_r^t ds_1 \int_{s_1}^t ds_2 \int_{\mathbb{R}^{2d}} g_{s_1+s_2-2r}(y_1-y_2)f(y_1)f(y_2)dy_1dy_2 \\
&\leq \frac{C}{t^2}\int_0^t dr \int_r^t ds_1 \int_{s_1}^t \{1 \wedge (s_1+s_2-2r)^{-\frac{d}{2}}\}ds_2
\end{aligned}$$

$$\le \frac{C}{t^2}\int_0^t dr\int_r^t ds_1\int_{s_1}^t \{1\wedge(s_1+s_2-2r)^{-\frac{3}{2}}\}ds_2$$
$$\le \frac{C}{t^2}\int_0^t dr\int_r^t (s_1-r)^{-\frac{1}{2}}ds_1 \le \frac{C}{t^2}\int_0^t (t-r)^{\frac{1}{2}}dr,$$

which tends to zero as $t\to\infty$. Then the result follows by (3.8). ∎

Remark. When both σ_1 and σ_2 are constant matrixes, the conditional transition function $p^W(r,x;s,y)$ can be expressed by (1.17). In this case, we can prove Theorem 1.3 along Iscoe's line as [7, Page 203–204]. But Theorem 1.4 can not be proved in this way even if σ_1 and σ_2 are constant; see the Remark after the proof of Lemma 4.5.

4 Ergodicity for dimension two

In this section, we give the proof of the ergodic theorem for the critical dimension $d=2$. We assume Conditions (A1,2), (B1) and (C1,2) hold. Let $\{W(t)\}$ and $\{B_1(t)\},\{B_2(t)\},\cdots$ be independent standard 2-dimensional Brownian motions and let $\{\xi_i^T(t)\}$ be defined by

$$d\xi_i^T(t)=\sigma_1^T(\xi_i^T(t))dW(t)+\sigma_2^T(\xi_i^T(t))dB_i(t), \tag{4.1}$$

where $\sigma_i^T(x)=\sigma_i(\sqrt{T}x)$. Let $p^{W,T}(r,x,t,dy)$ denote the conditional transition probability of $\{\xi_i^T(t)\}$ given $\{W(t)\}$. Let $\{\tilde{\xi}_i(t)\}$ be the Brownian motion defined by

$$d\tilde{\xi}_i(t)=\tilde{\sigma}_1 dW(t)+\tilde{\sigma}_2 dB_i(t), \tag{4.2}$$

Let $\tilde{p}^W(r,x,t,dy)$ denote the conditional transition probability of $\{\tilde{\xi}_i(t)\}$ given $\{W(t)\}$. Note that both $p^{W,T}(r,x,t,dy)$ and $\tilde{p}^W(r,x,t,dy)$ are independent of $i=1,2,\cdots$. The following result gives a conditional scaling limit theorem of the process defined by (4.1).

Proposition 4.1 *For $0\le r\le 1$, $T\ge 1$ and $f\in\mathcal{X}$, let*

$$\varepsilon_0(r,T)=\int_{\mathbb{R}^2}\mathbf{P}\Big\{\Big[\int_r^1\Big(T\mathbf{P}_{r,x}^W f(\sqrt{T}\xi_1^T(s))-\langle\lambda,f\rangle\tilde{p}^W(r,x,s,0)\Big)ds\Big]^2\Big\}dx.$$

Then $\sup_{0\le r\le 1}\varepsilon_0(r,T)\to 0$ *as* $T\to\infty$.

Proof. For $T\ge 1$ and $y\in\mathbb{R}^2$ write $y^T=T^{-1/2}y$. By a change of the integral variable we have

$$T\mathbf{P}_{r,x}^W f(\sqrt{T}\xi_1^T(s))=T\int_{\mathbb{R}^2} f(\sqrt{T}y)p^{W,T}(r,x,s,dy)=\int_{\mathbb{R}^2} f(y)p^{W,T}(r,x,s,dy^T).$$

It is simple to see

$$\begin{aligned}\varepsilon_0(r,T) &= \int_{\mathbb{R}^2}\mathbf{P}\Big\{\int_r^1\int_r^1\Big(T\mathbf{P}_{r,x}^W f(\sqrt{T}\xi_1^T(s_1))-\langle\lambda,f\rangle\tilde{p}^W(r,x,s_1,0)\Big)\\ &\qquad \Big(T\mathbf{P}_{r,x}^W f(\sqrt{T}\xi_1^T(s_2))-\langle\lambda,f\rangle\tilde{p}^W(r,x,s_2,0)\Big)ds_1ds_2\Big\}dx\\ &= \int_r^1 ds_1\int_r^1 ds_2\int_{\mathbb{R}^4} F^T(r,s_1,s_2,x)dx\end{aligned}$$

$$= \quad 2\int_r^1 ds_1 \int_{s_1}^1 ds_2 \int_{\mathbb{R}^4} F^T(r,s_1,s_2,x)dx,$$

where for $r \le \min(s_1,s_2)$,

$$\begin{aligned} F^T(r,s_1,s_2,x) \quad = \quad & \mathbf{P}\Big[\int_{\mathbb{R}^4} f(y_1)f(y_2)p^{W,T}(r,x,s_1,dy_1^T)p^{W,T}(r,x,s_2,dy_2^T)\Big] \\ & -\mathbf{P}\Big[\int_{\mathbb{R}^4} f(y_1)f(y_2)p^{W,T}(r,x,s_1,dy_1^T)\tilde{p}^W(r,x,s_2,0)dy_2\Big] \\ & -\mathbf{P}\Big[\int_{\mathbb{R}^4} f(y_1)f(y_2)\tilde{p}^W(r,x,s_1,0)p^{W,T}(r,x,s_2,dy_2^T)dy_1\Big] \\ & +\mathbf{P}\Big[\int_{\mathbb{R}^4} f(y_1)f(y_2)\tilde{p}^W(r,x,s_1,0)\tilde{p}^W(r,x,s_2,0)dy_1dy_2\Big]. \end{aligned} \tag{4.3}$$

By the property of independent increments of $\{W(t)\}$, for $r < s_1 < s_2 \le 1$ we have

$$\begin{aligned} & \mathbf{P}\big[p^{W,T}(r,x,s_1,dy_1^T)p^{W,T}(r,x,s_2,dy_2^T)\big] \\ = \quad & \mathbf{P}\Big[p^{W,T}(r,x,s_1,dy_1^T)\int_{\mathbb{R}^2} p^{W,T}(r,x,s_1,dz)p^{W,T}(s_1,z,s_2,dy_2^T)\Big] \\ = \quad & \int_{\mathbb{R}^2} p_2^T(s_1-r,(x,x),(y_1^T,z))p^T(s_2-s_1,z,y_2^T)dy_2dy_1dz, \end{aligned}$$

where $p^T(t,x,y)$ is the transition density of $\{\xi_1^T(t)\}$ and $p_2^T(t,(x_1,x_2),(y_1,y_2))$ is the transition density of the diffusion process $\{(\xi_1^T(t),\xi_2^T(t))\}$. We can use similar reasoning to the other three terms in (4.3) to see

$$F^T(r,s_1,s_2,x) = \int_{\mathbb{R}^6} f(y_1)f(y_2)h^T(s_1-r,s_2-s_1,x,y_1^T,y_2^T,z)dy_2dy_1dz,$$

where

$$\begin{aligned} h^T(s,t,x,y_1^T,y_2^T,z) \quad = \quad & p_2^T(s,(x,x),(y_1^T,z))p^T(t,z,y_2^T) + \tilde{p}_2(s,(x,x),(y_2^T,z))\tilde{p}(t,z,0) \\ & - q_2^T(s,(x,x),(y_2^T,z))\tilde{p}(t,z,0) - q_2^T(s,(x,x),(y_1^T,z))\tilde{p}(t,z,0) \end{aligned}$$

where $\tilde{p}(t,x,y)$ is the transition density of $\{\tilde{\xi}_1(t)\}$, $\tilde{p}_2(t,(x_1,x_2),(y_1,y_2))$ is the transition density of $\{(\tilde{\xi}_1(t),\tilde{\xi}_2(t))\}$, and $q_2^T(t,(x_1,x_2),(y_1,y_2))$ is the transition density of $\{(\xi_1^T(t),\tilde{\xi}_2(t))\}$. Then we have

$$\varepsilon_0(r,T) = 2\int_r^1 ds_1 \int_{s_1}^1 ds_2 \int_{\mathbb{R}^8} f(y_1)f(y_2)h^T(s_1-r,s_2-s_1,x,y_1^T,y_2^T,z)dxdy_2dy_1dz.$$

Observe that $\varepsilon_0(r,T) \le 2\varepsilon_0(T)$, where

$$\varepsilon_0(T) = \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^8} f(y_1)f(y_2)|h^T(s,t,x,y_1^T,y_2^T,z)|dxdy_2dy_1dz.$$

However, an application of Lemma 3.2 shows

$$|h^T(s,t,x,y_1^T,y_2^T,z)| \le Cg_{c_0s}(x-z)g_{c_0s}(x-y_1^T)g_{c_0t}(z-y_2^T).$$

By dominate convergence,

$$\int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^8} f(y_1)f(y_2)g_{c_0s}(x-z)g_{c_0s}(x-y_1^T)g_{c_0t}(z-y_2^T)dxdy_1dy_2dz$$

$$
\begin{aligned}
&= \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^4} f(y_1)f(y_2)g(c_0(2s+t), y_1^T + y_2^T)dy_1dy_2 \\
&\to \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^4} f(y_1)f(y_2)g(c_0(2s+t), 0)dy_1dy_2 \\
&= \int_0^1 ds \int_0^1 dt \int_{\mathbb{R}^8} f(y_1)f(y_2)g_{c_0s}(x-z)g_{c_0s}(x)g_{c_0t}(z)dxdy_1dy_2dz.
\end{aligned}
$$

Applying Lemma 3.4 by setting $F = \{0\}$ therein, it is easy to show $|h^T(s,t,x,y_1^T,y_2^T,z)| \to 0$ for $x \neq 0$. Then another application of dominated convergence shows $\varepsilon_0(T) \to 0$. ∎

Now let us consider a rescaled version of the equation (1.14). Given $f \in \mathcal{X}$ and let $(r,x) \mapsto v_T(r,x)$ be the solution to

$$
v_T(r,x) + \int_r^1 ds \int_{\mathbb{R}^2} v_T^2(s,y)p^{W,T}(r,x,s,dy) = \int_r^1 ds \int_{\mathbb{R}^2} Tf(\sqrt{T}y)p^{W,T}(r,x,s,dy), \tag{4.4}
$$

where $0 \leq r \leq 1$ and $x \in \mathbb{R}^2$.

Lemma 4.2 *For any $n \geq 1$ there is $C_n > 0$ so that*

$$
\mathbf{P}\Big[\prod_{i=1}^n v_T(r,x_i)\Big] \leq C_n \prod_{i=1}^n \int_0^1 ds \int_{\mathbb{R}^2} f(z)g_{c_0s}\Big(x_i - \frac{z}{\sqrt{T}}\Big)dz.
$$

Proof. From (4.4) and Corollary 3.3 we have

$$
\begin{aligned}
\mathbf{P}\Big[\prod_{i=1}^n v_T^n(r,x_i)\Big] &\leq T^n \int_r^1 ds_1 \cdots \int_r^1 \mathbf{P}\Big[\prod_{i=1}^n \int_{\mathbb{R}^2} f(\sqrt{T}y_i)p^{W,T}(r,x_i,s_i,dy_i)\Big]ds_n \\
&\leq CT^n \int_r^1 ds_1 \cdots \int_r^1 ds_n \int_{\mathbb{R}^{2n}} \prod_{i=1}^n f(\sqrt{T}y_i)g_{c_0(s_i-r)}(x_i-y_i)dy_1\cdots dy_n \\
&\leq C \int_0^1 ds_1 \cdots \int_0^1 ds_n \int_{\mathbb{R}^{2n}} \prod_{i=1}^n f(z_i)g_{c_0s_i}\Big(x_i - \frac{z_i}{\sqrt{T}}\Big)dz_1\cdots dz_n \\
&= C\prod_{i=1}^n \int_0^1 ds \int_{\mathbb{R}^2} f(z)g_{c_0s}\Big(x_i - \frac{z}{\sqrt{T}}\Big)dz.
\end{aligned}
$$

That proves the desired inequality. ∎

Lemma 4.3 *For $0 \leq r \leq 1$ and $T \geq 1$ let*

$$
\varepsilon_1(r,T) = \int_r^1 ds \int_{\mathbb{R}^2} \mathbf{P}\Big\{\Big(\int_{\mathbb{R}^2} v_T^2(s,y)\Big[p^{W,T}(r,x,s,dy) - \tilde{p}^W(r,x,s,dy)\Big]\Big)^2\Big\}dx. \tag{4.5}
$$

Then $\sup_{0\leq r\leq 1} \varepsilon_1(r,T) \to 0$ as $T \to \infty$.

Proof. By the property of independent increments of $\{W(t)\}$ we have

$$
\varepsilon_1(r,T) = \int_r^1 ds \int_{\mathbb{R}^2} \mathbf{P}\Big\{\int_{\mathbb{R}^4} v_T^2(s,y_1)v_T^2(s,y_2)\Big[p^{W,T}(r,x,s,dy_1)p^{W,T}(r,x,s,dy_2)
$$

$$
\begin{aligned}
&\quad - p^{W,T}(r,x,s,dy_1)\tilde{p}^{W}(r,x,s,dy_2) - \tilde{p}^{W}(r,x,s,dy_1)p^{W,T}(r,x,s,dy_2) \\
&\quad - \tilde{p}^{W}(r,x,s,dy_1)\tilde{p}^{W}(r,x,s,dy_2)\Big]\Big\}dx \\
&= \int_r^1 ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^4} \mathbf{P}[v_T^2(s,y_1)v_T^2(s,y_2)]R_T(s-r,x,y_1,y_2)dy_1dy_2, \qquad (4.6)
\end{aligned}
$$

where

$$
\begin{aligned}
R_T(s-r,x,y_1,y_2)dy_1dy_2 &= \mathbf{P}\big[p^{W,T}(r,x,s,dy_1)p^{W,T}(r,x,s,dy_2)\big] \\
&\quad - \mathbf{P}\big[p^{W,T}(r,x,s,dy_1)\tilde{p}^{W}(r,x,s,y_2)dy_2\big] \\
&\quad - \mathbf{P}\big[\tilde{p}^{W}(r,x,s,y_1)dy_1p^{W,T}(r,x,s,dy_2)\big] \\
&\quad + \mathbf{P}\big[\tilde{p}^{W}(r,x,s,y_1)\tilde{p}^{W}(r,x,s,y_2)dy_1dy_2\big] \\
&= p_2^T(s-r,(x,x),(y_1,y_2))dy_1dy_2 \\
&\quad - q_2^T(s-r,(x,x),(y_1,y_2))dy_1dy_2 \\
&\quad - q_2^T(s-r,(x,x),(y_2,y_1))dy_1dy_2 \\
&\quad + \tilde{p}_2(s-r,(x,x),(y_1,y_2))dy_1dy_2.
\end{aligned}
$$

Then we use Lemma 4.2 to see

$$
\mathbf{P}[v_T^2(s,y_1)v_T^2(s,y_2)] \le CF_T(y_1)F_T(y_2), \qquad (4.7)
$$

where

$$
\begin{aligned}
F_T(y) &= \int_0^1 ds_1 \int_0^1 ds_2 \int_{\mathbb{R}^4} f(z_1)f(z_2)g_{c_0s_1}\Big(y-\frac{z_1}{\sqrt{T}}\Big)g_{c_0s_2}\Big(y-\frac{z_2}{\sqrt{T}}\Big)dz_1dz_2 \\
&\le \int_0^1 ds_1 \int_0^1 \frac{1}{s_1+s_2}ds_2 \int_{\mathbb{R}^4} f(z_1)f(z_2)g\Big(\frac{cs_1s_2}{s_1+s_2}, y-\frac{s_2z_1+s_1z_2}{\sqrt{T}(s_1+s_2)}\Big)dz_1dz_2 \\
&=: G_T(y).
\end{aligned}
$$

Here we also used Lemma 3.1 for the inequality. From (4.6) it follows that

$$
\varepsilon_1(r,T) \le C\int_0^1 ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^4} F_T(y_1)F_T(y_2)|R_T(s,x,y_1,y_2)|dy_1dy_2, \qquad (4.8)
$$

By dominated convergence, for any $y \neq 0$ we have

$$
G_T(y) \to G(y) := \int_0^1 ds_1 \int_0^1 \frac{1}{s_1+s_2}ds_2 \int_{\mathbb{R}^4} f(z_1)f(z_2)g\Big(\frac{cs_1s_2}{s_1+s_2}, y\Big)dz_1dz_2
$$

By Lemma 3.2 it is simple to see that

$$
|R_T(s,x,y_1,y_2)| \le Cg_{c_0s}(x,y_1)g_{c_0s}(x,y_2), \qquad 0 < s \le 1,\ x,y_1,y_2 \in \mathbb{R}^2.
$$

On the other hand,

$$
\int_0^1 ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^4} G_T(y_1)G_T(y_2)g_{c_0s}(x-y_1)g_{c_0s}(x-y_2)dy_1dy_2
$$

$$
\begin{aligned}
&= \int_0^1 ds \int \left[\int G_T(y) g_{c_0 s}(x-y) dy \right]^2 dx \\
&= \int_0^1 ds \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} g_{c_0 s}(x-y) dy \int_0^1 ds_1 \int_0^1 \frac{1}{s_1+s_2} ds_2 \int_{\mathbb{R}^4} f(z_1) f(z_2) \right. \\
&\qquad\qquad \left. g\left(\frac{c s_1 s_2}{s_1+s_2}, y - \frac{s_2 z_1 + s_1 z_2}{\sqrt{T}(s_1+s_2)} \right) dz_1 dz_2 \right]^2 dx \\
&= \int_0^1 ds \int_{\mathbb{R}^2} \left[\int_0^1 ds_1 \int_0^1 \frac{1}{s_1+s_2} ds_2 \int_{\mathbb{R}^4} f(z_1) f(z_2) \right. \\
&\qquad\qquad \left. g\left(cs + \frac{c s_1 s_2}{s_1+s_2}, x - \frac{s_2 z_1 + s_1 z_2}{\sqrt{T}(s_1+s_2)} \right) dz_1 dz_2 \right]^2 dx \\
&= \int_0^1 ds \int_0^1 ds_1 \cdots \int_0^1 \frac{1}{s_1+s_2} \frac{1}{s_3+s_4} ds_4 \int_{\mathbb{R}^8} f(z_1) \cdots f(z_4) \\
&\quad g\left(2cs + \frac{c s_1 s_2}{s_1+s_2} + \frac{c s_3 s_4}{s_3+s_4}, \frac{s_2 z_1 + s_1 z_2}{\sqrt{T}(s_1+s_2)} + \frac{s_4 z_3 + s_3 z_4}{\sqrt{T}(s_3+s_4)} \right) dz_1 \cdots dz_4, \qquad (4.9)
\end{aligned}
$$

where

$$
\begin{aligned}
& g\left(2cs + \frac{c s_1 s_2}{s_1+s_2} + \frac{c s_3 s_4}{s_3+s_4}, \frac{s_2 z_1 + s_1 z_2}{\sqrt{T}(s_1+s_2)} + \frac{s_4 z_3 + s_3 z_4}{\sqrt{T}(s_3+s_4)} \right) \\
&\qquad\qquad \le C \left(\frac{c s_1 s_2}{s_1+s_2} + \frac{c s_3 s_4}{s_3+s_4} \right)^{-1}.
\end{aligned}
$$

It is elementary to show

$$
\begin{aligned}
& \int_0^1 ds_1 \cdots \int_0^1 \frac{1}{s_1+s_2} \frac{1}{s_3+s_4} \left(\frac{c s_1 s_2}{s_1+s_2} + \frac{c s_3 s_4}{s_3+s_4} \right)^{-1} ds_4 \\
= & \int_0^1 ds_1 \cdots \int_0^1 \frac{1}{s_1 s_2 (s_3+s_4) + s_3 s_4 (s_1+s_2)} ds_4 < \infty.
\end{aligned}
$$

By dominated convergence we have

$$
\begin{aligned}
& \int_0^1 ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^4} G_T(y_1) G_T(y_2) g_{c_0 s}(x-y_1) g_{c_0 s}(x-y_2) dy_1 dy_2 \\
\to & \int_r^1 ds \int_0^1 ds_1 \cdots \int_0^1 \frac{1}{s_1+s_2} \frac{1}{s_3+s_4} ds_4 \int_{\mathbb{R}^8} f(z_1) \cdots f(z_4) \\
& \qquad\qquad g\left(2cs + \frac{c s_1 s_2}{s_1+s_2} + \frac{c s_3 s_4}{s_3+s_4}, 0 \right) dz_1 \cdots dz_4 \\
= & \int_r^1 ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} G(y) G(z) g_{c_0 s}(x-y) g_{c_0 s}(x-z) dz.
\end{aligned}
$$

By Lemma 3.4 we have

$$
|R_T(s, x, y_1, y_2)| \to 0, \qquad 0 < s \le 1,\ x \in \mathbb{R}^2 \setminus \{0\},\ y_1, y_2 \in \mathbb{R}^2.
$$

Then we can use dominated convergence to the right hand side of (4.8) to obtain the desired result. ∎

Lemma 4.4 *For any $n \ge 2$ we have*

$$
\sup_{T \ge 1} \sup_{0 \le r \le 1} \mathbf{P}\left[\left(\int_{\mathbb{R}^2} v_T^2(r, x) dx \right)^{n/2} \right] < \infty. \qquad (4.10)
$$

Proof. By Jensen's inequality, if we set $C_n = \left(\int_{\mathbb{R}^2} e^{-\frac{2|x|}{n}}dx\right)^{\frac{n-2}{2}}$, then

$$\begin{aligned}\mathbf{P}\left[\left(\int_{\mathbb{R}^2} v_T^2(r,x)dx\right)^{n/2}\right] &= \mathbf{P}\left[\left(\int_{\mathbb{R}^2} v_T^2(r,x)e^{\frac{2|x|}{n}}e^{-\frac{2|x|}{n}}dx\right)^{n/2}\right] \\ &\le C_n \int_{\mathbb{R}^2} \mathbf{P}\big[v_T^n(r,x)\big]e^{|x|}dx, \end{aligned} \tag{4.11}$$

where

$$\mathbf{P}\big[v_T^n(r,x)\big] \le C_n\left[\int_0^1 ds\int_{\mathbb{R}^2} f(z)g_{c_0 s}\Big(x-\frac{z}{\sqrt{T}}\Big)dz\right]^n$$

by Lemma 4.2. Let $t = \big(\sum_{i=1}^n 1/s_i\big)^{-1}$. It follows that

$$\begin{aligned}
\text{l.h.s. of (4.11)} &\le C_n\int_0^1 ds_1\cdots\int_0^1 \frac{1}{s_1\cdots s_n}ds_n\int_{\mathbb{R}^{2n}} f(z_1)\cdots f(z_n)dz_1\cdots dz_n \\
&\quad\cdot\int \exp\Big\{-\sum_{i=1}^n\frac{1}{2s_i}\Big|x-t\sum_{i=1}^n\frac{z_i}{s_i\sqrt{T}}\Big|^2\Big\}e^{|x|}dx \\
&= C_n\int_0^1 ds_1\cdots\int_0^1 \frac{1}{s_1\cdots s_n}ds_n\int_{\mathbb{R}^{2n}} f(z_1)\cdots f(z_n)dz_1\cdots dz_n \\
&\quad\cdot\int \exp\Big\{-\frac{1}{2t}|y|^2\Big\}\exp\Big\{\Big|y+t\sum_{i=1}^n\frac{z_i}{s_i\sqrt{T}}\Big|\Big\}dy \\
&\le C_n\int_0^1 ds_1\cdots\int_0^1 \frac{1}{s_1\cdots s_n}ds_n\int_{\mathbb{R}^2}\exp\Big\{-\frac{1}{2t}|y|^2+|y|\Big\}dy \\
&\quad\cdot\int_{\mathbb{R}^{2n}}\exp\Big\{t\sum_{i=1}^n\frac{|z_i|}{s_i\sqrt{T}}\Big\}f(z_1)\cdots f(z_n)dz_1\cdots dz_n \\
&\le C_n\int_0^1 ds_1\cdots\int_0^1 \frac{1}{s_1\cdots s_n}ds_n\int\exp\Big\{-\frac{1}{2t}(|y|-t)^2\Big\}dy \\
&\quad\cdot\int_{\mathbb{R}^{2n}}\exp\Big\{t\sum_{i=1}^n\frac{|z_i|}{s_i\sqrt{T}}\Big\}f(z_1)\cdots f(z_n)dz_1\cdots dz_n \\
&\le C_n\int_0^1\cdots\int_0^1\frac{1}{s_1\cdots s_n}\Big(\frac{1}{s_1}+\cdots+\frac{1}{s_n}\Big)^{-1}ds_1\cdots ds_n \\
&= C_n\int_1^\infty\cdots\int_1^\infty\frac{1}{v_1\cdots v_n(v_1+\cdots+v_n)}dv_1\cdots dv_n \\
&\le C_n\int_1^\infty\cdots\int_1^\infty\frac{1}{v_1\cdots v_n(v_1\cdots v_n)^{1/n}}dv_1\cdots dv_n<\infty,
\end{aligned}$$

where we have used the compact support property of f. ∎

Lemma 4.5 *For* $0\le r\le 1$ *and* $T_1,T_2\ge 1$ *let*

$$q(r,T_1,T_2) := \int_{\mathbb{R}^2}\mathbf{P}\big\{[v_{T_1}(r,x)-v_{T_2}(r,x)]^2\big\}dx. \tag{4.12}$$

Then $\sup_{0\le r\le 1} q(r,T_1,T_2)\to 0$ *as* $T_1,T_2\to\infty$.

Proof. *Step 1.* By considering the difference of two equations in the form of (4.4) we have

$$
\begin{aligned}
|v_{T_1}(r,x) - v_{T_1}(r,x)| \;\le\; & \int_r^1 \Big| T_1 \mathbf{P}^W_{r,x}\big[f(\sqrt{T_1}\xi_1^{T_1}(s))\big] - T_2 \mathbf{P}^W_{r,x}\big[f(\sqrt{T_2}\xi_1^{T_2}(s))\big]\Big| ds \\
& + \int_r^1 \Big| \int_{\mathbb{R}^2} v_{T_1}^2(s,y) p^{W,T_1}(r,x,s,dy) \\
& \qquad - \int_{\mathbb{R}^2} v_{T_2}^2(s,y) p^{W,T_2}(r,x,s,dy)\Big| ds.
\end{aligned}
$$

Let $\varepsilon_1(r,T_i)$ be defined by (4.5) and let

$$
\varepsilon_2(r,T_1,T_2) = \int_{\mathbb{R}^2} \mathbf{P}\bigg\{\bigg(\int_r^1 \Big[T_1 \mathbf{P}^W_{r,x} f(\sqrt{T_1}\xi_1^{T_1}(s)) - T_2 \mathbf{P}^W_{r,x} f(\sqrt{T_2}\xi_1^{T_2}(s))\Big] ds\bigg)^2\bigg\} dx.
$$

By Lemma 4.3, we have $\sup_{0\le r\le 1}\varepsilon_2(r,T_1,T_2)\to 0$ as $T_1,T_2\to\infty$. We can now write

$$
q(r,T_1,T_2) \le C[\varepsilon_1(r,T_1) + \varepsilon_1(r,T_2) + \varepsilon_2(r,T_1,T_2) + b(r,T_1,T_2)], \tag{4.13}
$$

where

$$
b(r,T_1,T_2) = \int_{\mathbb{R}^2} \mathbf{P}\bigg\{\bigg(\int_r^1 ds \int_{\mathbb{R}^2} [v_{T_1}^2(s,y) - v_{T_2}^2(s,y)]\tilde{p}^W(r,x,s,y) dy\bigg)^2\bigg\} dx.
$$

Step 2. In view of (1.17) we have

$$
\tilde{p}^W(r,x,t,z) \le \frac{C}{t-r} \quad \text{and} \quad \int_{\mathbb{R}^2} \tilde{p}^W(r,x,s,y) dx = 1.
$$

By Cauchy-Schwarz inequality,

$$
\begin{aligned}
b(r,T_1,T_2) \;=\; & 2\mathbf{P}\bigg\{\int_{\mathbb{R}^2} dx \int_r^1 ds_1 \int_{\mathbb{R}^2} [v_{T_1}^2(s_1,y_1) - v_{T_2}^2(s_1,y_1)]\tilde{p}^W(r,x,s_1,y_1) dy_1 \\
& \qquad \int_{s_1}^1 ds_2 \int_{\mathbb{R}^2} [v_{T_1}^2(s_2,y_2) - v_{T_2}^2(s_2,y_2)]\tilde{p}^W(r,x,s_2,y_2) dy_2\bigg\} \\
\le\; & C\mathbf{P}\bigg\{\int_r^1 ds_1 \int_{\mathbb{R}^2} [v_{T_1}^2(s_1,y_1) - v_{T_2}^2(s_1,y_1)] dy_1 \\
& \qquad \int_{s_1}^1 \frac{1}{s_2 - r} ds_2 \int_{\mathbb{R}^2} [v_{T_1}^2(s_2,y_2) - v_{T_2}^2(s_2,y_2)] dy_2\bigg\} \\
\le\; & C\mathbf{P}\bigg\{\int_r^1 \frac{1}{\sqrt{s_1 - r}} ds_1 \int_{\mathbb{R}^2} [v_{T_1}^2(s_1,y_1) - v_{T_2}^2(s_1,y_1)] dy_1 \\
& \qquad \int_{s_1}^1 \frac{1}{\sqrt{s_2 - r}} ds_2 \int_{\mathbb{R}^2} [v_{T_1}^2(s_2,y_2) - v_{T_2}^2(s_2,y_2)] dy_2\bigg\} \\
\le\; & C\mathbf{P}\bigg\{\bigg(\int_r^1 \frac{1}{\sqrt{s-r}} ds \int_{\mathbb{R}^2} [v_{T_1}^2(s,y) - v_{T_2}^2(s,y)] dy\bigg)^2\bigg\} \\
\le\; & C\mathbf{P}\bigg\{a(r,T_1,T_2) \int_r^1 \frac{\|v_{T_1}(s) - v_{T_2}(s)\|_0^2}{\sqrt{s-r}} ds\bigg\},
\end{aligned} \tag{4.14}
$$

where

$$
a(r,T_1,T_2) = \int_r^1 \frac{\|v_{T_1}(s) + v_{T_2}(s)\|_0^2}{\sqrt{s-r}} ds.
$$

By Lemma 4.4 we have

$$\sup_{0\le r\le 1}\sup_{T_1,T_2\ge 1}\mathbf{P}\big\{a(r,T_1,T_2)>m\big\}\le m^{-1}\sup_{0\le r\le 1}\sup_{T_1,T_2\ge 1}\mathbf{P}\big[a(r,T_1,T_2)\big]\to 0$$

as $m\to\infty$. The same lemma implies that the random variable under the expectation on the right hand side of (4.14) is uniformly integrable; see e.g. [8, p.67]. Then for any $\varepsilon>0$ there exists $m_0\ge 1$ so that

$$\begin{aligned} b(r,T_1,T_2) &\le C\mathbf{P}\bigg\{1_{\{a(r,T_1,T_2)\le m_0\}}a(r,T_1,T_2)\int_r^1\frac{\|v_{T_1}(s)-v_{T_2}(s)\|_0^2}{\sqrt{s-r}}ds\bigg\}+\varepsilon \\ &\le Cm_0\int_r^1\mathbf{P}[\|v_{T_1}(s)-v_{T_2}(s)\|_0^2]\frac{1}{\sqrt{s-r}}ds+\varepsilon. \end{aligned}$$

Step 3. Recalling (4.13) and applying the above estimate for $b(r,T_1,T_2)$ we get

$$q(r,T_1,T_2)\le\varepsilon+C[\varepsilon_2(r,T_1,T_2)+\varepsilon_1(r,T_1)+\varepsilon_1(r,T_2)]+C\int_r^1\frac{q(s,T_1,T_2)}{\sqrt{s-r}}ds.$$

For sufficiently large $T_1,T_2\ge 1$ we have $C[\varepsilon_2(r,T_1,T_2)+\varepsilon_1(r,T_1)+\varepsilon_1(r,T_2)]\le\varepsilon$ for $0\le r\le 1$. Let $h(t)=\sup_{0\le s\le t}\mathbf{P}[\|v_{T_1}(1-s)-v_{T_2}(1-s)\|_0^2]$ for $0\le t\le 1$. Then we obtain

$$h(t)\le 2\varepsilon+C\int_0^t h(s)\frac{ds}{\sqrt{t-s}}.$$

We can use the above inequality twice to get

$$\begin{aligned} h(t) &\le 2\varepsilon+C\int_0^t\left[2\varepsilon+C\int_0^s h(r)\frac{dr}{\sqrt{s-r}}\right]\frac{ds}{\sqrt{t-s}} \\ &\le 2\varepsilon+2\varepsilon C\int_0^t\frac{ds}{\sqrt{t-s}}+C^2\int_0^t h(r)dr\int_r^t\frac{ds}{\sqrt{t-s}\sqrt{s-r}} \\ &\le 2\varepsilon+4\varepsilon C+2\sqrt{2}C^2\int_0^t h(r)\frac{dr}{\sqrt{t-r}}\int_r^{(r+t)/2}\frac{ds}{\sqrt{s-r}} \\ &\le 2\varepsilon+4\varepsilon C+4C^2\int_0^t h(r)dr. \end{aligned}$$

Then we obtain (4.12) by a standard application of Gronwall's inequality. ∎

Remark. Our proof of Theorem 1.4 is different from that of Iscoe [7, Theorem 2]. To prove Theorem 1.4 , the key step is Lemma 4.5. Note that in (4.14), $v_{T_1}^2(s_1,y_1)-v_{T_2}^2(s_1,y_1)$ depends on the path $\{W_u: s_1\le u\le t\}$, and not independent of $\tilde{p}^W(r,x;s_2,y_2)$ $(s_1\le s_2)$. The four terms under $\int_{\mathbb{R}^2}dx\int_r^1 ds_1\int_{s_1}^1 ds_2\int_{\mathbb{R}^2}dy_1\int_{\mathbb{R}^2}dy_2$ are intervolved, where $[v_{T_1}^2(s_1,y_1)-v_{T_2}^2(s_1,y_1)]$ and $[v_{T_1}^2(s_2,y_2)-v_{T_2}^2(s_2,y_2)]$ can not be separated with $\tilde{p}^W(r,x;s_1,y_1)$ and $\tilde{p}^W(r,x;s_2,y_2)$ when we take the expectation by $\mathbf{P}$. Hence we can not integrate $\tilde{p}^W(r,x;s_1,y_1)\cdot\tilde{p}^W(r,x;s_2,y_2)$ by $\int dx$ and thereafter use the scaling property of $\tilde{p}^W$ as in Iscoe [7, the proof of Theorem 2]. Even if σ_1 and σ_2 are constant, the above problems still exist.

Lemma 4.6 *There is a unique positive solution $r\mapsto v(r):=v(r,x;\theta)$ to (1.19). Moreover, as $T\to\infty$, we have*

$$\sup_{0\le r\le 1}\mathbf{P}\|v_T(r)-v(r)\|_0^2]\to 0. \tag{4.15}$$

Proof. By Lemma 4.5 there exists a random function $(r,x)\mapsto v(r,x)$ so that (4.15) holds. By Proposition 4.1 the right hand side of (4.4) converges to that of (1.19) in $L^2(\Omega\times\mathbb{R}^2,\mathbf{P}\times\lambda)$. Then we only need to prove the convergence of the second term on the left hand side of (4.4). Observe that

$$\bigg|\int_{\mathbb{R}^2} v_T^2(s,y)p^{W,T}(r,x,s,dy)-\int_{\mathbb{R}^2} v^2(s,y)\tilde{p}^W(r,x,s,dy)\bigg|\le\eta_2^T(r,s,x)+\eta_3^T(r,s,x),$$

where

$$\eta_2^T(r,s,x)=\bigg|\int_{\mathbb{R}^2} v_T^2(s,y)\big[p^{W,T}(r,x,s,dy)-\tilde{p}^W(r,x,s,dy)\big]\bigg|$$

and

$$\eta_3^T(r,s,x)=\bigg|\int_{\mathbb{R}^2}\big[v_T^2(s,y)-v^2(s,y)\big]\tilde{p}^W(r,x,s,dy)\bigg|.$$

By Lemma 4.3 we have

$$\sup_{0\le r\le 1}\int_r^1 ds\int_{\mathbb{R}^2}\mathbf{P}[\eta_2^T(r,s,x)^2]dx\to 0.$$

Arguing as in the proof of Lemma 4.5 one can prove

$$\sup_{0\le r\le 1}\int_r^1 ds\int_{\mathbb{R}^2}\mathbf{P}[\eta_3^T(r,s,x)^2]dx\to 0.$$

Then $(r,x)\mapsto v(r,x)$ is a solution to (1.19). Now suppose $(r,x)\mapsto\bar{v}(r,x)$ is another solution to (1.19). Then we have

$$\begin{aligned}|v_T(r,x)-\bar{v}(r,x)|\;&\le\;\int_r^1\Big|T_1\mathbf{P}_{r,x}^W\big[f(\sqrt{T_1}\xi_1^{T_1}(s))\big]-\tilde{p}^W(r,x,s,0)\Big|ds\\&\quad+\eta_2^T(r,s,x)+\bar{\eta}_3^T(r,s,x),\end{aligned}$$

where $\bar{\eta}_3^T(r,s,x)$ is defined from v_T and $\bar{v}$. Then the above arguments shows (4.15) also holds when $v(r)$ is replaced by $\bar{v}(r)$, which implies the uniqueness of the solution to (1.19). ∎

Proof of Theorem 1.4. By Theorem 1.2, for $f\in\mathscr{X}$ we have

$$\mathbf{P}\exp\big\{-\big\langle T^{-1}Y_T,f\big\rangle\big\}=\mathbf{P}\exp\big\{-\langle\mu,u_T(0,\cdot)\rangle\big\},\tag{4.16}$$

where $u^T(\cdot,\cdot)$ is the solution to

$$u^T(r,x)+\int_r^T\mathbf{P}_{r,x}^W[u^T(s,\xi_s)^2]ds=\int_r^T\mathbf{P}_{r,x}^W[T^{-1}f(\xi_s)]ds,\qquad 0\le r\le T.$$

It is not difficult to prove that $(r,x)\mapsto v^T(r,x):=Tu^T(Tr,\sqrt{T}x)$ is the solution to

$$v^T(r,x)+\int_r^1\mathbf{P}_{r,x}^W[v^T(s,\xi^T(s))^2]ds=\int_r^1\mathbf{P}_{r,x}^W[Tf(\sqrt{T}\xi^T(s))]ds,\qquad 0\le r\le 1,$$

where $\xi^T(t)=T^{-1/2}\xi_1(Tt)$ satisfies

$$d\xi^T(t)=\sigma_1(\sqrt{T}\xi^T(t))dW^T(t)+\sigma_2(\sqrt{T}\xi^T(t))dB^T(t)$$

for independent standard Brownian motions $W^T(t) := T^{-1/2}W(Tt)$ and $B^T(t) := T^{-1/2}B_1(Tt)$. Then $\{\xi^T(t)\}$ is a weak solution of (4.1). Since $\mu(\sqrt{T}dx)$ is an invariant measure of $\{\xi^T(t)\}$, from (4.16) we have

$$\begin{aligned}
\mathbf{P}\exp\big\{-\big\langle T^{-1}Y_T, f\big\rangle\big\} &= \mathbf{P}\exp\Big\{-\int_{\mathbb{R}^2} v^T(0,x)\mu(\sqrt{T}x)dx\Big\} \\
&= \mathbf{P}\exp\bigg\{-\langle \mu, f\rangle + \int_0^1 ds\int_{\mathbb{R}^2} v^T(s,x)^2\mu(\sqrt{T}x)dx\bigg\}.
\end{aligned}$$

This together with (1.18) implies

$$\begin{aligned}
&\Big|\mathbf{P}\exp\big\{-\big\langle T^{-1}Y_T, f\big\rangle\big\} - \mathbf{P}\big[\exp\{-\langle \xi, f\rangle\}\big]\Big| \\
\le\;& \mathbf{P}\Big[\int_0^1 ds\int_{\mathbb{R}^2}\big|v^T(s,x)^2\mu(\sqrt{T}x) - v(s,x)^2\mu(\infty)\big|dx\Big] \\
\le\;& \mathbf{P}\Big[\int_0^1 ds\int_{\mathbb{R}^2}\big|v^T(s,x)^2 - v(s,x)^2\big|\mu(\sqrt{T}x)dx\Big] \\
&+\mathbf{P}\Big[\int_0^1 ds\int_{\mathbb{R}^2} v(s,x)^2\big|\mu(\sqrt{T}x) - \mu(\infty)\big|dx\Big].
\end{aligned}\tag{4.17}$$

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\mathbf{P}\Big[\int_0^1 ds\int_{\mathbb{R}^2}\big|v^T(s,x)^2 - v(s,x)^2\big|dx\Big] \\
\le\;& \Big\{\mathbf{P}\Big[\int_0^1 ds\int_{\mathbb{R}^2}\big|v^T(s,x) - v(s,x)\big|^2 dx\Big]\Big\}^{\frac{1}{2}} \\
&\cdot\Big\{\mathbf{P}\Big[\int_0^1 ds\int_{\mathbb{R}^2}\big|v^T(s,x) + v(s,x)\big|^2 dx\Big]\Big\}^{\frac{1}{2}},
\end{aligned}$$

which tends to zero by Lemma 4.6. Then we can apply dominated convergence to the right hand side of (4.17) to see

$$\Big|\mathbf{P}\exp\big\{-\big\langle T^{-1}Y_T, f\big\rangle\big\} - \mathbf{P}\big[\exp\{-\langle \xi, f\rangle\}\big]\Big| \to 0.$$

That proves the desired result. ∎

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