Published in *"Communications in Stochastic Analysis"* **2** (2008), 3, 469-477.

DISTRIBUTION AND PROPAGATION PROPERTIES OF SUPERPROCESSES WITH GENERAL BRANCHING MECHANISMS

ZENGHU LI* AND XIAOWEN ZHOU**

ABSTRACT. A theorem of Evans and Perkins (1991) on the absolute continuity of distributions of Dawson-Watanabe superprocesses is extended to general branching mechanisms. This result is then used to establish the propagation properties of the superprocesses following Evans and Perkins (1991) and Perkins (1990).

1. Introduction

Suppose that E is a Lusin topological space. Let $M(E)$ be the space of finite Borel measures on E endowed with the topology of weak convergence. Let $B(E)$ denote the set of bounded Borel functions on E and $B(E)^+$ the subset of non-negative elements. For $f \in B(E)$ and $\mu \in M(E)$ write $\mu(f)$ for $\int f d\mu$. We consider a conservative Borel right process $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$ in E with transition semigroup $(P_t)_{t\geq 0}$ and a *branching mechanism* on E given by

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (e^{-zu} - 1 + zu)m(x,du), \tag{1.1}$$

where $b \in B(E)$, $c \in B(E)^+$ and $(u \wedge u^2)m(x,du)$ is a bounded kernel from E to $(0,\infty)$. Given $f \in B(E)^+$ let $t \mapsto V_t f$ be the unique locally bounded non-negative solution to

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_E \phi(y, V_s f(y)) P_{t-s}(x,dy), \qquad t \geq 0, x \in E. \tag{1.2}$$

The *Dawson-Watanabe superprocess* with parameters (ξ, ϕ), or simply a (ξ,ϕ)-*superprocess*, is a Markov process in $M(E)$ with transition semigroup $(Q_t)_{t\geq 0}$ defined by

$$\int_{M(E)} e^{-\nu(f)} Q_t(\mu, d\nu) = e^{-\mu(V_t f)}, \qquad t \geq 0, \mu \in M(E), f \in B(E)^+. \tag{1.3}$$

The operators $(V_t)_{t\geq 0}$ constitute a semigroup, which is called the *cumulant (or log-Laplace) semigroup* of the (ξ,ϕ)-superprocess; see, e.g., Dawson [2] and Fitzsimmons [7, 8].

2000 *Mathematics Subject Classification.* Primary 60J80, 60J30; Secondary 60G57.

Key words and phrases. Dawson-Watanabe superprocess, absolute continuity of distribution, Lévy process, symmetric stable process, propagation of support.

* Supported by NSFC (10525103 and 10721091) and CJSP grants.

** Supported by an NSERC grant.

The branching mechanism is said to be *spatially constant* if it is independent of $x \in E$. In particular, we say the superprocess has *binary branching* if $\phi(x,z) \equiv cz^2$ for some constant $c > 0$. For a spatially constant branching mechanism $z \mapsto \phi^*(z)$ we often consider the condition:

(A1) There is some constant $\theta > 0$ so that

$$\phi^*(z) > 0 \text{ for } z \geq \theta \text{ and } \int_\theta^\infty \phi^*(z)^{-1} dz < \infty.$$

A theorem on the absolute continuity of distributions of Dawson-Watanabe superprocesses with binary branching was proved in Evans and Perkins [6]. As applications of the theorem, they established some results on support preparations of super Lévy processes. Using a different approach, Evans and Perkins [6] also proved some preparation results for superprocesses with general Feller spatial motion processes and branching mechanisms given by (1.1) under a stronger integral condition on the kernel $m(x, du)$; see also Perkins [11, 12]. The purpose of this note is to extend some of those results to superprocesses with general local branching mechanisms. This work was brought up by the recent paper Bojdecki et al. [1]. Although the arguments follow closely those of Evans and Perkins [6] and Perkins [12], we provide the details for the convenience of the reader.

2. Main theorem

Let $X = (W, \mathcal{G}, \mathcal{G}_t, X_t, \mathbf{Q}_\mu)$ be a canonical Borel right realization of the (ξ, ϕ)-superprocess, where W is the space of right continuous paths from $[0, \infty)$ to $M(E)$. The existence of such a realization was proved in Fitzsimmons [7, 8]. It is well-known that the first moment of the superprocess is given by

$$\int_{M(E)} \nu(f) Q_t(\mu, d\nu) = \mu(P_t^b f), \qquad t \geq 0, \mu \in M(E), f \in B(E), \tag{2.1}$$

where $(P_t^b)_{t\geq 0}$ is the semigroup of kernels on E defined by

$$P_t^b f(x) = \mathbf{P}_x\big[e^{-\int_0^t b(\xi_s)ds} f(\xi_t)\big].$$

By (1.3), (2.1) and Jensen's inequality it is simple to see that

$$V_t f(x) \leq P_t^b f(x), \qquad t \geq 0, x \in E, f \in B(E)^+. \tag{2.2}$$

If $(x, z) \mapsto \phi(x, z)$ is bounded below by a spatially constant branching mechanism $z \mapsto \phi^*(z)$ satisfying (A1), the cumulant semigroup $(V_t)_{t\geq 0}$ of the (ξ, ϕ)-superprocess admits the canonical representation:

$$V_t f(x) = \int_{M(E)^\circ} \big(1 - e^{-\nu(f)}\big) R_t(x, d\nu), \qquad t > 0, x \in E, f \in B(E)^+, \tag{2.3}$$

where $R_t(x, d\nu)$ is a bounded kernel from E to $M(E)^\circ := M(E) \setminus \{0\}$. In fact, we have

$$h_t(x) := R_t(x, 1) \leq \bar{v}_t, \qquad t > 0, x \in E, \tag{2.4}$$

where $t \mapsto \bar{v}_t$ is the minimal non-negative solution of the differential equation

$$\frac{d}{dt}\bar{v}_t = -\phi^*(\bar{v}_t), \qquad t > 0$$

with the singular initial condition $\bar{v}_{0+} = \infty$; see Dawson [2, Section 11.5] or Li [10, Sections 3.2 and 8.1]. From (1.3), (2.1) and (2.3) it is simple to see that

$$P_t^b f(x) = \int_{M(E)^\circ} \nu(f) R_t(x, d\nu), \qquad t > 0, x \in E, f \in B(E)^+. \tag{2.5}$$

Then $R_t(x, 1) > 0$ for all $t > 0$ and $x \in E$. For any $0 < r \le t$ and $x \in E$ we can use (1.3), (2.3) and the semigroup property of $(V_t)_{t\ge 0}$ to see

$$R_t(x, \cdot) = \int_{M(E)^\circ} R_r(x, d\mu) Q_{t-r}(\mu, \cdot) \tag{2.6}$$

as measures on $M(E)^\circ$. For $t > 0$ and $\mu \in M(E)$ write

$$R_t(\mu, \cdot) = \int_E \mu(dx) R_t(x, \cdot).$$

In view of (1.3) and (2.3), we have the following

Lemma 2.1. *Suppose that $(x, z) \mapsto \phi(x, z)$ is bounded below by a spatially constant branching mechanism $z \mapsto \phi^*(z)$ satisfying (A1). If Ξ_t^μ is a Poisson random measure on $M(E)^\circ$ with intensity $R_t(\mu, \cdot)$, then*

$$X_t^\mu := \int_{M(E)^\circ} \nu \, \Xi_t^\mu(d\nu) \tag{2.7}$$

has distribution $Q_t(\mu, \cdot)$ on $M(E)$. In particular, if $\mu \in M(E)^\circ$, for any Borel set $F \subset M(E)^\circ$ we have

$$Q_t(\mu, F) \ge \mathbf{P}\big\{X_t^\mu \in F, \Xi_t^\mu(M(E)^\circ) = 1\big\} = e^{-R_t(\mu,1)} R_t^*(\mu, F), \tag{2.8}$$

where $R_t^*(\mu, \cdot) = R_t(\mu, \cdot)/R_t(\mu, 1)$.

The representation (2.7) is known as the *cluster decomposition* of the random measure. For $t > 0$ let $(f, g) \mapsto V_t(f, g)$ be the operator from $B(E)^+ \times B(E)^+$ to $B(E)^+$ defined by

$$V_t(f, g)(x) = \int_{M(E)^\circ} \nu(g) e^{-\nu(f)} R_t(x, d\nu), \qquad x \in E, f, g \in B(E)^+. \tag{2.9}$$

We have

$$V_t(f, g)(x) = P_t g(x) - \int_0^t ds \int_E \psi(y, V_s f(y), V_s(f, g)(y)) P_{t-s}(x, dy), \tag{2.10}$$

where

$$\psi(x, y, z) = b(x)z + c(x)yz + \int_0^\infty uz\big(1 - e^{-uy}\big) m(x, du);$$

see Li [10, Section 2.4]. From (2.5) and (2.9) we get

$$V_t(f, g)(x) \le V_t(0, g)(x) \le P_t^b g(x), \qquad t \ge 0, x \in E. \tag{2.11}$$

The next theorem was first proved in Evans and Perkins [6, Theorem 1.1] for a binary branching; see also Perkins [12, Theorem III.2.2]. The ideas of the proof given below follow those in the two references.

Theorem 2.2. *Suppose that $(x,z) \mapsto \phi(x,z)$ is bounded below by a spatially constant branching mechanism $z \mapsto \phi^*(z)$ satisfying (A1). Then for any fixed $\mu_1, \mu_2 \in M(E)$ the following properties are equivalent:*

(i) $\mu_1 P_r \ll \mu_2 P_t$ *for all* $0 < r \le t$;
(ii) $R_r(\mu_1, \cdot) \ll R_t(\mu_2, \cdot)$ *on* $M(E)^\circ$ *for all* $0 < r \le t$;
(iii) $Q_r(\mu_1, \cdot) \ll Q_t(\mu_2, \cdot)$ *on* $M(E)$ *for all* $0 < r \le t$;
(iv) $\mathbf{Q}_{\mu_1}(X_{r+\cdot} \in \cdot) \ll \mathbf{Q}_{\mu_2}(X_{t+\cdot} \in \cdot)$ *on* $(W, \mathcal{G})$ *for all* $0 < r \le t$.

Proof. "(iii) $\Leftrightarrow$ (iv)" The implication (iv) $\Rightarrow$ (iii) is trivial and the implication (iii) $\Rightarrow$ (iv) follows from the Markov property:

$$\mathbf{Q}_\mu(X_{r+\cdot} \in \cdot) = \int_{M(E)} Q_r(\mu, d\nu)\mathbf{Q}_\nu(X_\cdot \in \cdot).$$

"(i) $\Rightarrow$ (ii)" Suppose that (i) holds. Let $0 < u < r < t$ and let $F \subset M(E)^\circ$ be a Borel set so that $R_t(\mu_2, F) = 0$. From (2.6) and (2.8) we have

$$R_t(\mu_2, \cdot) \ge \int_{M(E)^\circ} R_{t-u}(\mu_2, d\mu) e^{-R_u(\mu,1)} R_u^*(\mu, \cdot).$$

It follows that

$$0 = \int_{M(E)^\circ} R_{t-u}(\mu_2, d\mu) \int_E \mu(dx) R_u(x, F). \tag{2.12}$$

By (2.5) and (2.12),

$$R_u(x, F) = 0 \text{ for } \mu_2 P_{t-u}^b\text{-a.a. } x \in E.$$

Since $P_s(x, \cdot)$ and $P_s^b(x, \cdot)$ are equivalent for every $s \ge 0$ and $x \in E$, property (i) implies

$$R_u(x, F) = 0 \text{ for } \mu_1 P_{r-u}^b\text{-a.a. } x \in E.$$

Then we can reverse the above steps to conclude

$$0 = \int_{M(E)^\circ} R_{r-u}(\mu_1, d\mu) \int_E \mu(dx) R_u(x, F).$$

Consequently,

$$\begin{aligned} 0 &= \int_{M(E)^\circ} R_{r-u}(\mu_1, d\mu) e^{-R_u(\mu,1)} R_u^*(\mu, F) \\ &= \int_{M(E)^\circ} R_{r-u}(\mu_1, d\mu) \mathbf{P}\big\{X_u^\mu \in F, \Xi_u^\mu(M(E)^\circ) = 1\big\} \\ &\ge \int_{M(E)^\circ} R_{r-u}(\mu_1, d\mu) \Big[\mathbf{P}\big\{X_u^\mu \in F\big\} - \mathbf{P}\big\{\Xi_u^\mu(M(E)^\circ) \ge 2\big\}\Big] \\ &= R_r(\mu_1, F) - \int_{M(E)^\circ} \mathbf{P}\big\{\Xi_u^\mu(M(E)^\circ) \ge 2\big\} R_{r-u}(\mu_1, d\mu), \end{aligned} \tag{2.13}$$

where for the inequality we have used the fact that $X_u^\mu \in F \subset M(E)^\circ$ implies $\Xi_u^\mu(M(E)^\circ) \ge 1$. From (2.3), (2.4) and (2.9) we see that the last term on the right hand side of (2.13) is equal to

$$\int_{M(E)^\circ} \Big[1 - e^{-R_u(\mu,1)} - R_u(\mu, 1) e^{-R_u(\mu,1)}\Big] R_{r-u}(\mu_1, d\mu)$$

$$\begin{aligned}
&= \int_{M(E)^\circ} \Big[1 - e^{-\mu(h_u)} - \mu(h_u)e^{-\mu(h_u)}\Big] R_{r-u}(\mu_1, d\mu) \\
&= \mu_1\Big(V_{r-u}h_u - V_{r-u}(h_u, h_u)\Big) \\
&= \int_0^{r-u} ds \int_E \psi(y, V_s h_u(y), V_s(h_u, h_u)(y))\mu_1 P_{t-s}(dy) \\
&\quad - \int_0^{r-u} ds \int_E \phi(y, V_s h_u(y))\mu_1 P_{t-s}(dy),
\end{aligned}$$

where we also used (1.2) and (2.10) for the last equality. Using (2.2), (2.4) and (2.11) it is elementary to see that the right hand side of the equation above tends to zero as $u \to r$. By (2.13) we conclude $R_r(\mu_1, F) = 0$.

"(ii) $\Rightarrow$ (iii)" By the cluster decomposition of the superprocess we see that $Q_r(\mu_1, \cdot)$ and $Q_t(\mu_2, \cdot)$ are the laws of $\sum_{i=1}^{\eta_1} \nu_i^1$ and $\sum_{i=1}^{\eta_2} \nu_i^2$, respectively, where η_1 and η_2 are Poissonian random variables with means $R_r(\mu_1, 1)$ and $R_t(\mu_2, 1)$, respectively, and conditional on η_1 and η_2 the sequences $\{\nu_i^1 : i \geq 1\}$ and $\{\nu_i^2 : i \geq 1\}$ are i.i.d. with laws $R_r^*(\mu_1, \cdot)$ and $R_t^*(\mu_2, \cdot)$, respectively. Clearly, (ii) implies the n-fold product of $R_r^*(\mu_1, \cdot)$ is absolutely continuous to the n-fold product of $R_t^*(\mu_2, \cdot)$. Therefore we can sum over the values of η_1 and η_2 to obtain $Q_r(\mu_1, \cdot) \ll Q_t(\mu_2, \cdot)$ as required.

"(iii) $\Rightarrow$ (i)" From (2.1) we see that (iii) implies $\mu_1 P_r^b \ll \mu_2 P_t^b$ for all $0 < r \leq t$, which in turn implies $\mu_1 P_r \ll \mu_2 P_t$ for all $0 < r \leq t$. □

We remark that Dynkin [4, Theorem 6.2] gave a result on the equivalence of the exit distributions from a domain for a superdiffusion with branching mechanism given by (1.1) with $b(x) = 0$ and with $u^n m(x, du)$ being a bounded kernel from E to $(0, \infty)$ for every $n \geq 2$.

3. Propagations of supports

It is well-known that a (ξ, ϕ)-superprocess $\{X_t : t \geq 0\}$ has no negative jumps. The following characterizations of the superprocess can be derived from the results of Fitzsimmons [7, 8]; see El Karoui and Roelly [5] for the results under Feller assumptions. Let $\mathbf{F}$ be the set of functions $f \in B(E)$ that are finely continuous relative to ξ. Fix $\beta > 0$ and let $(A, \mathcal{D}(A))$ be the weak generator of $(P_t)_{t \geq 0}$ defined by $\mathcal{D}(A) = U^\beta \mathbf{F}$ and $Af = \beta f - g$ for $f = U^\beta g \in \mathcal{D}(A)$. Let $N(ds, d\nu)$ be the optional random measure on $[0, \infty) \times M(E)^\circ$ defined by

$$N(ds, d\nu) = \sum_{s>0} 1_{\{\Delta X_s \neq 0\}} \delta_{(s, \Delta X_s)}(ds, d\nu),$$

where $\Delta X_s = X_s - X_{s-}$, and let $\hat{N}(ds, d\nu)$ denote the predictable compensator of $N(ds, d\nu)$. Then $\hat{N}(ds, d\nu) = dsK(X_{s-}, d\nu)$ with the kernel defined by

$$\int_{M(E)^\circ} F(\nu) K(\mu, d\nu) = \int_E \mu(dx) \int_0^\infty F(u\delta_x) m(x, du).$$

Let $\tilde{N}(ds, d\nu) = N(ds, d\nu) - \hat{N}(ds, d\nu)$. For any $f \in \mathcal{D}(A)$ we have

$$X_t(f) = X_0(f) + M_t^c(f) + M_t^d(f) + \int_0^t X_s(Af - bf) ds, \tag{3.1}$$

where

$$t \mapsto M_t^d(f) = \int_0^t \int_{M(E)^\circ} \nu(f) \tilde{N}(ds, d\nu)$$

is a purely discontinuous local martingale and $t \mapsto M_t^c(f)$ is a continuous local martingale with quadratic variation $2X_t(cf^2)dt$. Let $\{X_{t-}(f) : t > 0\}$ denote the left limit process of the semimartingale $\{X_t(f) : t \geq 0\}$. (In general, the limit measures $\{X_{t-} : t > 0\}$ only exist in $M(\bar{E})$ with $\bar{E}$ being the Ray-Knight completion of E with respect to ξ.)

Now we prove a supporting property of the (ξ, ϕ)-superprocess. The following theorem was obtained in Evans and Perkins [6, Proof of Theorem 5.1] for binary branching; see also Perkins [12, Theorem III.2.1].

Theorem 3.1. *Let $\mathcal{D}(A)^+$ denote the set of non-negative elements in $\mathcal{D}(A)$. Then for any $\mu \in M(E)$ and $f \in \mathcal{D}(A)^+$ we have*

$$\mathbf{Q}_\mu\{X_s(Af) \neq 0 \textit{ implies } X_s(f) > 0 \textit{ for Lebesgue a.a. } s \geq 0\} = 1.$$

Proof. Let $l_t \equiv l_t^0(X(f))$ denote the local time of the semimartingale $t \mapsto X_t(f)$ at zero. For any $\epsilon > 0$ and $t \geq 0$ we have

$$\begin{aligned}
&\mathbf{Q}_\mu\left[\epsilon^{-1}\int_0^t 1_{\{0<X_s(f)\leq\epsilon\}} d\langle M^c(f)\rangle_s\right] \\
&\quad\leq\ \mathbf{Q}_\mu\left[\epsilon^{-1}\int_0^t 1_{\{0<X_s(f)\leq\epsilon\}} X_s(cf^2)ds\right] \\
&\quad\leq\ \|cf\|\mathbf{Q}_\mu\left[\epsilon^{-1}\int_0^t 1_{\{0<X_s(f)\leq\epsilon\}} X_s(f)ds\right] \\
&\quad\leq\ \|cf\|\mathbf{Q}_\mu\left[\int_0^t 1_{\{0<X_s(f)\leq\epsilon\}} ds\right],
\end{aligned}$$

which goes to zero as $\epsilon \to 0$. Then $l_t = 0$ for all $t \geq 0$. Since f is non-negative, Tanaka's Formula implies

$$\begin{aligned}
X_t(f) &= X_0(f) + \int_0^t 1_{\{X_{s-}(f)>0\}} dM_s^c(f) + \int_0^t 1_{\{X_{s-}(f)>0\}} dM_s^d(f) \\
&\quad + \int_0^t 1_{\{X_{s-}(f)>0\}} X_s(Af - bf)ds + \sum_{0<s\leq t} 1_{\{X_{s-}(f)=0\}} X_s(f); \quad (3.2)
\end{aligned}$$

see, e.g., Dellacherie and Meyer [3, Section VIII.29]. As $s \mapsto X_s$ has at most countably many jumps, we have

$$\mathbf{Q}_\mu\left[\left|\int_0^t 1_{\{X_{s-}(f)=0\}} X_s(bf)ds\right|\right] \leq \|b\|\mathbf{Q}_\mu\left[\int_0^t 1_{\{X_{s-}(f)=0\}} X_s(f)ds\right] = 0$$

and

$$\mathbf{Q}_\mu\left[\left(\int_0^t 1_{\{X_{s-}(f)=0\}} dM_s^c(f)\right)^2\right] = \mathbf{Q}_\mu\left[\int_0^t 1_{\{X_{s-}(f)=0\}} X_s(cf^2)ds\right] = 0.$$

Observe also that

$$1_{\{X_{s-}(f)=0\}} X_s(f) = 1_{\{X_{s-}(f)=0\}} \Delta X_s(f) = 1_{\{X_{s-}(f)=0\}} \Delta M_s^d(f).$$

Then (3.2) implies

$$X_t(f) = X_0(f) + M_t^c(f) + M_t^d(f) + \int_0^t 1_{\{X_{s-}(f)>0\}} X_s(Af)ds - \int_0^t X_s(bf)ds.$$

From this and (3.1) it follows that

$$\int_0^t 1_{\{X_{s-}(f)=0\}} X_s(Af)ds = 0.$$

Since $X_{s-}(f) \neq X_s(f)$ for at most countably many $s \geq 0$, we get

$$\int_0^t 1_{\{X_s(f)=0\}} X_s(Af)ds = 0.$$

Then $1_{\{X_s(f)=0\}} X_s(Af) = 0$ for Lebesgue a.a. $s \geq 0$, and the theorem is proved. □

In the sequel we consider the special case of $E = \mathbb{R}^d$. Given a σ-finite Borel measure ν on $\mathbb{R}^d$ let $S(\nu)$ denote the closed support of ν and let $M_\nu(\mathbb{R}^d) = \{\eta \in M(\mathbb{R}^d) : S(\nu * \eta) \subset S(\eta)\}$, where "$*$" stands for convolution of measures. It is easy to see that $\eta \in M_\nu(\mathbb{R}^d)$ holds if and only if $S(\nu^{*k} * \eta) \subset S(\eta)$ for $k = 1, 2, \cdots$. In particular, if

$$\bigcup_{k=1}^{\infty} S(\nu^{*k}) = \mathbb{R}^d, \tag{3.3}$$

then $\eta \in M_\nu(\mathbb{R}^d)$ implies $S(\eta) = \mathbb{R}^d$ or $\emptyset$. The following results provide some propagation properties of super Lévy processes.

Theorem 3.2. *Suppose that ξ is a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Then for every $\mu \in M(\mathbb{R}^d)$ we have $\mathbf{Q}_\mu\{X_s \in M_\nu(\mathbb{R}^d)\} = 1$ for Lebesgue a.a. $s > 0$.*

Proof. The generator A of ξ has domain $\mathcal{D}(A)$ containing $C_K^\infty(\mathbb{R}^d)$, the set of infinitely differentiable functions on $\mathbb{R}^d$ with compact support. Let B be an open ball in $\mathbb{R}^d$ and choose $f \in C_K^\infty(\mathbb{R}^d)^+$ such that $\{f > 0\} = B$. Then

$$Af(x) = \int_{\mathbb{R}^d} f(x+y)\nu(dy), \qquad x \notin B;$$

see, e.g., Gikhman and Skorokhod [9, Theorem IV.4.1]. This means that $X_s(B) = 0$ implies

$$X_s(Af) = \int_{\mathbb{R}^d} X_s(dx) \int_{\mathbb{R}^d} f(x+y)\nu(dy) = X_s * \nu(f).$$

By Theorem 3.1 we conclude $\mathbf{Q}_\mu\{X_s * \nu(B) > 0$ implies $X_s(B) > 0$ for Lebesgue a.a. $s \geq 0\} = 1$. Taking a union over balls with rational radii and centers we get $\mathbf{Q}_\mu\{S(X_s * \nu) \subset S(X_s)$ for Lebesgue a.a. $s \geq 0\} = 1$. Then the result follows by Fubini's theorem. □

Corollary 3.3. *Suppose that ξ is a Lévy process on $\mathbb{R}^d$ with Lévy measure ν and that $(x, z) \mapsto \phi(x, z)$ is bounded below by a spatially constant branching mechanism $z \mapsto \phi^*(z)$ satisfying (A1). If any one of the properties (i) – (iv) in Theorem 2.2 holds for $\mu_1 = \mu_2 = \mu$, then for every $t > 0$ we have $\mathbf{Q}_\mu\{X_t \in M_\nu(\mathbb{R}^d)\} = 1$.*

Proof. By Theorem 3.2 there is a sequence $r_n \to \infty$ so that $\mathbf{Q}_\mu\{X_{r_n} \in M_\nu(\mathbb{R}^d)\} = 1$ for each $n \geq 1$. But, Theorem 2.2 implies that $\mathbf{Q}_\mu(X_t \in \cdot)$ is absolutely continuous with respect to $\mathbf{Q}_\mu(X_{r_n} \in \cdot)$ for any $0 < t \leq r_n$. Then the desired result holds. □

The results of the above theorem and its corollary were already obtained in Evans and Perkins [6, Theorem 5.1] and Perkins [11, Theorem 1.5 and Corollary 1.5] for binary branching. The proofs here follow those of Evans and Perkins [6, pp.674-676] and Perkins [12, pp.202-203]. A different approach for general Feller spatial motion processes was also provided in Evans and Perkins [6, Corollaries 5.3 and 5.4], which implies the above results when $(u \vee u^2)m(x, du)$ is a bounded kernel from E to $(0, \infty)$. Some simple consequences of those results are given as follows.

Theorem 3.4. *Suppose that ξ is a Lévy process on $\mathbb{R}^d$ with Lévy measure ν and absolutely continuous transition semigroup, that is,*

$$P_t(x, dy) = p_t(x, y)dy, \qquad t > 0,\ x, y \in \mathbb{R}^d.$$

If $(x, z) \mapsto \phi(x, z)$ is bounded below by a spatially constant branching mechanism $z \mapsto \phi^*(z)$ satisfying (A1), then for every $t > 0$ and $\mu \in M(\mathbb{R}^d)$ we have $\mathbf{Q}_\mu\{X_t \in M_\nu(\mathbb{R}^d)\} = 1$.

Proof. Note that μP_t is absolutely continuous with respect to λ for any $t > 0$. Choose $\eta \in M(\mathbb{R}^d)$ that is equivalent to the Lebesgue measure λ. It is easy to see that $(\mu + \epsilon\eta)P_t$ is equivalent to λ for every $t > 0$ and $\epsilon > 0$. By Corollary 3.3 we have $\mathbf{Q}_{\mu+\epsilon\eta}\{X_t \in M_\nu(\mathbb{R}^d)\} = 1$ for every $t > 0$. Note that X_t under $\mathbf{Q}_{\mu+\epsilon\eta}$ has the same distribution on $M(\mathbb{R}^d)$ as $X_t^\mu + X_t^{\epsilon\eta}$, where $\{X_t^\mu : t \geq 0\}$ and $\{X_t^{\epsilon\eta} : t \geq 0\}$ are independent (ξ, ϕ)-superprocesses with initial values μ and $\epsilon\eta$, respectively. It is easy to see that

$$\mathbf{P}\{X_t^{\epsilon\eta} = 0\} = \lim_{\theta\to\infty} \exp\{-\epsilon\eta(V_t\theta)\} = \exp\{-\epsilon R_t(\eta, 1)\} = \exp\{-\epsilon\eta(h_t)\},$$

which tends to one as $\epsilon \to 0$. Then we have

$$\begin{aligned}
\mathbf{P}\{X_t^\mu \in M_\nu(\mathbb{R}^d)\} &= \lim_{\epsilon\to 0} \mathbf{P}(\{X_t^\mu \in M_\nu(\mathbb{R}^d)\} \cap \{X_t^{\epsilon\eta} = 0\}) \\
&= \lim_{\epsilon\to 0} \mathbf{P}(\{X_t^\mu + X_t^{\epsilon\eta} \in M_\nu(\mathbb{R}^d)\} \cap \{X_t^{\epsilon\eta} = 0\}) \\
&= \lim_{\epsilon\to 0} \mathbf{P}\{X_t^\mu + X_t^{\epsilon\eta} \in M_\nu(\mathbb{R}^d)\} = 1,
\end{aligned}$$

giving the desired result. □

Corollary 3.5. *Suppose that ξ is a symmetric α-stable process with index $0 < \alpha < 2$ and $(x, z) \mapsto \phi(x, z)$ is bounded below by a spatially constant branching mechanism $z \mapsto \phi^*(z)$ satisfying (A1). Then for every $t > 0$ and $\mu \in M(\mathbb{R}^d)$ we have $\mathbf{Q}_\mu\{S(X_t) = \mathbb{R}^d$ or $\emptyset\} = 1$.*

Proof. The spatial motion process ξ now has Lévy measure $\nu(dx) = c|x|^{-d-\alpha}dx$ for some constant $c > 0$. Since ν has full support, we have (3.3) and the desired result follows from Theorem 3.4. □

Finally, we point out that assuming $(x,z) \mapsto \phi(x,z)$ is bounded below by a spatially constant branching mechanism $z \mapsto \phi^*(z)$ satisfying (A1) is an artifact of the cluster representation approach in the proof of Theorem 2.2. This condition was not required in Evans and Perkins [6, Corollaries 5.3 and 5.4]. Nevertheless, the condition is satisfied if $\phi(x,z) \equiv \gamma(x)z^{1+\beta}$ for a constant $0 < \beta < 1$ and a function $\gamma \in B(E)^+$ bounded away from zero, which was excluded from Evans and Perkins [6].

Acknowledgments. We are grateful to Professor L.G. Gorostiza for suggesting us the problem of support propagations for general branching mechanisms. We give our sincere thanks to Professor E.A. Perkins and an anonymous referee for their very helpful comments.

References

1. Bojdecki, T.; Gorostiza, L.G. and Talarczyk, A.: Occupation times of branching systems with initial inhomogeneous Poisson states and related superprocesses, *Preprint* (2008).
2. Dawson, D.A.: Measure-Valued Markov Processes, in: *Ecole d'Eté de Probabilités de Saint-Flour XXI-1991*, Hennequin, P.L. ed., Lecture Notes Math. **1541** (1993), 1-260, Springer, Berlin.
3. Dellacherie, C. and Meyer, P.A.: *Probabilites and Potential*, Chapters V-VIII, North-Holland, Amsterdam, 1982.
4. Dynkin, E.B.: Harmonic functions and exit boundary of superdiffusion, *J. Funct. Anal.* **206** (2004), 33-68
5. El Karoui, N. and Roelly, S.: Propriétés de martingales, explosion et representation de Lévy-Khintchine d'une classe de processus de branchement à valeurs mesures, *Stochastic Process. Appl.* **38** (1991), 239-266.
6. Evans, S. N. and Perkins, E.: Absolute continuity results for superprocesses with some applications, *Trans. Amer. Math. Soc.* **325** (1991), 661-681.
7. Fitzsimmons, P.J.: Construction and regularity of measure-valued Markov branching processes, *Israel J. Math.* **64** (1988), 337-361.
8. Fitzsimmons, P.J.: On the martingale problem for measure-valued Markov branching processes, in: *Seminar on Stochastic Processes* **1991** (1992), 39-51, Cinlar, E. et al eds, Birkhäuser Boston, Inc., Boston, MA.
9. Gikhman, I.I. and Skorokhod, A.V.: *The Theory of Stochastic Processes*, Vol. 2. Springer, Berlin, 1975.
10. Li, Z.H.: *Measure-Valued Branching Markov Processes*, Forthcoming book, 2008. (Peprint form available at: `http://math.bnu.edu.cn/~lizh`.
11. Perkins, E.A.: Polar sets and multiple points for super-Brownian motion, *Ann. Probab.* **18** (1990), 453-491.
12. Perkins, E.A.: Dawson-Watanabe Superprocesses and Measure-Valued Diffusions, in: *Ecole d'Eté de Probabilités de Saint-Flour XXIX-1999*, Bernard, P. ed., Lecture Notes Math. **1781** (2002), 125-329, Springer, Berlin.

Zenghu Li: Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China
E-mail address: `lizh@bnu.edu.cn`
URL: `http://math.bnu.edu.cn/~lizh`

Xiaowen Zhou: Department of Mathematics and Statistics, Concordia University, 7141 Sherbrooke Street West, Montreal, Canada, H4B 1R6
E-mail address: `zhou@alcor.concordia.ca`
URL: `http://alcor.concordia.ca/~zhou`