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Conditional entrance laws for superprocesses with dependent spatial motion

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We identify a class of conditional entrance laws for superprocesses with dependent spatial motion (SDSM). Those entrance laws are used to characterize some conditional excursion laws. As an application of the results, we give a sample path decomposition of the SDSM and that of a related immigration superprocess. The main tool used here is the conditional log-Laplace functional technique that handles the difficulty of the loss of the multiplicative property due to the interactions in the spatial motions.

Keywords: superprocess, non-linear SPDE, conditional log-Laplace equation, conditional entrance law, conditional excursion law, sample path decomposition.

AMS Subject Classifications: Primary 60J80, 60G57; Secondary 60J35

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1 Introduction

A class of superprocesses with dependent spatial motion (SDSM) over the real line $\mathbb{R}$ were introduced and constructed by Wang [16, 17]. A generalization of the model was then given in [3]. The SDSM arises as the weak limit of critical branching particle systems with dependent spatial motion. Let $c \in C_b^2(\mathbb{R})$ and $h \in C_b^2(\mathbb{R})$ and assume both h and h' are square-integrable. Let

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \qquad x \in \mathbb{R},$$

and $a(x) = c(x)^2 + \rho(0)$. Consider a family of independent Brownian motions $\{B_i(t) : t \geq 0, i = 1, 2, \cdots\}$, the individual noises, and a time-space white noise $\{W(dt, dy) : t \geq 0, y \in \mathbb{R}\}$, the common noise. The migration of a particle in the approximating system with label i is defined by the stochastic equation

$$dx_i(t) = c(x_i(t))dB_i(t) + \int_{\mathbb{R}} h(y - x_i(t))W(dt, dy), \qquad t \geq 0. \tag{1.1}$$

We denote by $M(\mathbb{R})$ the space of finite Borel measures on $\mathbb{R}$ endowed with a metric compatible with its topology of weak convergence. For $f \in C_b(\mathbb{R})$ and $\mu \in M(\mathbb{R})$ denote $\langle f, \mu \rangle = \int f d\mu$. Let $\sigma \in C_b^2(\mathbb{R})$ be a non-negative function. A typical SDSM $\{X_t : t \geq 0\}$ is characterized by the following stochastic equation: For each $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned} \langle \phi, X_t \rangle &= \langle \phi, X_0 \rangle + \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds + \int_0^t \int_{\mathbb{R}} \phi(y) Z(ds, dy) \\ &\quad + \int_0^t \int_{\mathbb{R}} \langle h(y - \cdot)\phi', X_s \rangle W(ds, dy), \end{aligned} \tag{1.2}$$

where $W(ds, dy)$ is a time-space white noise and $Z(ds, dy)$ is an orthogonal martingale measure that is orthogonal to $W(ds, dy)$ and has covariation measure $\sigma(y)X_s(dy)ds$. We refer the reader to [15] for the theory of stochastic integrals relative to martingale measures and white noises.

Clearly, the SDSM defined by (1.2) reduces to a usual critical branching Dawson-Watanabe superprocess if $h(\cdot) \equiv 0$; see, e.g., [1, 20]. It is known that for a Dawson-Watanabe superprocess, the multiplicative property and the log-Laplace functional give a class of σ-finite excursion laws, with which one can decompose the sample paths of the superprocess into excursions. The decomposition gives an explicit representation of the family structures of the superprocess; see, e.g., [6, 10, 11, 13]. For the SDSM, however, the multiplicative property fails and log-Laplace functional cannot be expressed in a convenient way. For this reason, the sample path decomposition is much harder. In the degenerate case $c(x) \equiv 0$, the SDSM is purely atomic; see [2, 4, 16, 18]. Based on this property, a reconstruction of the degenerate SDSM was given in [2] by one-dimensional excursions carried by a stochastic flow.

In this work, we are interested in the reconstruction of the SDSM in terms of excursions in the non-degenerate case, namely, where the coefficient $c(x)$ can be non-trivial. In this case, the SDSM can be absolutely continuous; see [3, 5, 16, 18]. Therefore, we cannot use the method in [2] for our purpose. In view of (1.1) and (1.2), given $\{W(ds, dy)\}$ the solution $\{X_t : t \geq 0\}$ should be a generalized inhomogeneous Dawson-Watanabe superprocess, where

$$\int_{\mathbb{R}} h(y - \cdot)W(dt, dy)$$

gives a generalized drift in the underlying migration. This observation was confirmed to some extend in [12] by characterizing the conditional log-Laplace functional of $\{X_t : t \geq 0\}$ given $\{W(ds, dy)\}$. Some similar results were obtained earlier in [21, 22] for the model of Skoulakis and Adler [14]. However, the conditional log-Laplace functional in [12] does not give automatically a decomposition of the SDSM. The main difficulty is that under the conditional probability given $\{W(ds, dy)\}$ we only have the a.s. Markov property of the finite dimensional distributions of the SDSM, which does not imply immediately the full conditional Markov property. A precise description of the situation is given in section 3. Nevertheless, we shall see that the conditional log-Laplace functions still give a class of conditional entrance laws in a weak sense. Those conditional entrance laws can be used to characterize some conditional excursion laws, which can then be used in a reconstruction of the SDSM. We expect that the conditional log-Laplace functional would also serve as a basic tool for a series of investigations. See [21, 22, 12] for more motivations and details of them.

The paper is organized as follows: Some basic properties of the stochastic log-Laplace functional are proved in Section 2. In Section 3, a comparison theorem of the solutions of the stochastic log-Laplace equation is established. In Section 4, we identify conditional transition semigroup of the SDSM and its first and the second moments. In Section 5, we describe a class of conditional entrance laws, with which we characterize the conditional excursion laws. Constructions of the SDSM and a related immigration superprocess are given in Section 6.

2 Conditional log-Laplace functionals

Let (a, c, h, σ) be given as in the introduction. Suppose that $W(ds, dx)$ is a time-space white noise. We consider the following non-linear backward SPDE:

$$\begin{aligned}\psi_{r,t}(x) &= \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_x^2\psi_{s,t}(x) - \frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\right]ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_{s,t}(x)\cdot W(ds, dy), \qquad t \geq r \geq 0,\end{aligned} \tag{2.1}$$

where "$\cdot$" denotes the backward stochastic integral. Let $\{H_k(\mathbb{R}) : k = 0, \pm 1, \pm 2, \cdots\}$ denote the Sobolev spaces on $\mathbb{R}$.

Theorem 2.1 ([12]) *For any $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$, equation (2.1) has a unique $H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$-valued strong solution $(\psi_{r,t})_{r \leq t}$. Furthermore, we have a.s. $\|\psi_{r,t}\| \leq \|\phi\|$ for all $t \geq r \geq 0$, where $\|\cdot\|$ denotes the supremum norm of the Banach space $C_b(\mathbb{R})$.*

Theorem 2.2 *Let $(\psi_{r,t})_{r \leq t}$ be defined by (2.1). Then for any $t \geq u \geq r \geq 0$ and $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ we have a.s. $\psi_{r,t}(\cdot, \phi) = \psi_{r,u}(\cdot, \psi_{u,t}(\cdot, \phi))$.*

Proof. As usual, we denote $\psi_{s,t}(x) = \psi_{s,t}(x, \phi)$. According to the assumption, for $0 \leq s \leq t$ we have

$$\begin{aligned}\psi_{s,t}(x, \phi) &= \phi(x) + \int_s^t \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_{v,t}(x, \phi) - \frac{1}{2}\sigma(x)\psi_{v,t}(x, \phi)^2\right]dv \\ &\quad + \int_s^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_{v,t}(x, \phi)\cdot W(dv, dy)\end{aligned} \tag{2.2}$$

and for $0 \le s \le u$ we have

$$\begin{aligned}
\psi_{s,u}(x, \psi_{u,t}) &= \psi_{u,t}(x) + \int_s^u \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_{v,u}(x, \psi_{u,t}) - \frac{1}{2}\sigma(x)\psi_{v,u}(x, \psi_{u,t})^2\right] dv \\
&\quad + \int_s^u \int_{\mathbb{R}} h(y - x)\partial_x\psi_{v,u}(x, \psi_{u,t}) \cdot W(dv, dy). \qquad (2.3)
\end{aligned}$$

Now we define $(\psi^*_{s,t})_{s\le t}$ by

$$\psi^*_{s,t}(x) = \begin{cases} \psi_{s,t}(x, \phi) & \text{for } u \le s \le t, \\ \psi_{s,u}(x, \psi_{u,t}) & \text{for } 0 \le s \le u. \end{cases}$$

Clearly, $(\psi^*_{r,t})_{r\le t}$ satisfies (2.1) for $u \le r \le t$. For $0 \le r \le u$ we may use (2.2) and (2.3) to see that

$$\begin{aligned}
\psi^*_{r,t}(x) &= \psi_{u,t}(x) + \int_r^u \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_{v,u}(x, \psi_{u,t}) - \frac{1}{2}\sigma(x)\psi_{v,u}(x, \psi_{u,t})^2\right] dv \\
&\quad + \int_r^u \int_{\mathbb{R}} h(y - x)\partial_x\psi_{v,u}(x, \psi_{u,t}) \cdot W(dv, dy) \\
&= \phi(x) + \int_u^t \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_{v,t}(x, \phi) - \frac{1}{2}\sigma(x)\psi_{v,t}(x, \phi)^2\right] dv \\
&\quad + \int_u^t \int_{\mathbb{R}} h(y - x)\partial_x\psi_{v,t}(x, \phi) \cdot W(dv, dy) \\
&\quad + \int_r^u \left[\frac{1}{2}a(x)\partial_{xx}^2\psi^*_{v,t}(x) - \frac{1}{2}\sigma(x)\psi^*_{v,t}(x)^2\right] dv \\
&\quad + \int_r^u \int_{\mathbb{R}} h(y - x)\partial_x\psi^*_{v,t}(x) \cdot W(dv, dy) \\
&= \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_{xx}^2\psi^*_{v,t}(x) - \frac{1}{2}\sigma(x)\psi^*_{v,t}(x)^2\right] dv \\
&\quad + \int_r^t \int_{\mathbb{R}} h(y - x)\partial_x\psi^*_{v,t}(x) \cdot W(dv, dy). \qquad (2.4)
\end{aligned}$$

Then we have a.s. $\psi_{r,t}(\cdot, \phi) = \psi^*_{r,t}(\cdot) = \psi_{r,u}(\cdot, \psi_{u,t}(\cdot, \phi))$ by the uniqueness of the solution of (2.1). □

We remark that the exceptional null set in Theorem 2.1 depends on the (t, u, r). In this sense, we shall say the random operators $(\psi_{r,t})_{r\le t}$ satisfy the *a.s. semigroup property*. In particular, the linear equation

$$\begin{aligned}
T_{r,t}\phi(x) &= \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_x^2 T_{s,t}\phi(x)\right] ds \\
&\quad + \int_r^t \int_{\mathbb{R}} h(y - x)\partial_x T_{s,t}\phi(x) \cdot W(ds, dy) \qquad (2.5)
\end{aligned}$$

defines a family of random linear operators $(T_{r,t})_{r\le t}$ on $H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ with the a.s. semigroup property. Given the stochastic semigroup of linear operators $(T_{r,t})_{r\le t}$ let us consider the equation

$$\psi_{r,t}(x) = T_{r,t}\phi(x) - \frac{1}{2}\int_r^t T_{r,s}(\sigma\psi^2_{s,t})(x) ds, \qquad t \ge r \ge 0. \qquad (2.6)$$

Theorem 2.3 *For any $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$, equations* (2.1) *and* (2.6) *are equivalent. Consequently,* (2.6) *also has $(\psi_{r,t})_{r\leq t}$ as the unique strong solution.*

Proof. The existence of a solution of (2.6) follows by a standard iteration argument as in the deterministic case; see, e.g., [1]. Suppose that $\psi_{r,t}(x)$ is an arbitrary solution of (2.6). By the stochastic Fubini theorem, we have

$$\begin{aligned}
\psi_{r,t}(x) &= T_{r,t}\phi(x) - \frac{1}{2}\int_r^t T_{r,s}(\sigma\psi_{s,t}^2)(x)ds \\
&= \phi(x) + \frac{1}{2}\int_r^t a(x)\partial_{xx}^2 T_{u,t}\psi(x)du + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{u,t}\psi(x)\cdot W(du,dy) \\
&\quad -\frac{1}{2}\int_r^t \sigma(x)\psi_{s,t}(x)^2 ds - \frac{1}{2}\int_r^t \left\{\frac{1}{2}\int_r^s a(x)\partial_{xx}^2 T_{u,s}(\sigma\psi_{s,t}^2)(x)du\right\} ds \\
&\quad -\frac{1}{2}\int_r^t \left\{\int_r^s \int_{\mathbb{R}} h(y-x)\partial_x T_{u,s}(\sigma\psi_{s,t}^2)(x)\cdot W(du,dy)\right\} ds \\
&= \phi(x) + \frac{1}{2}\int_r^t a(x)\partial_{xx}^2 T_{u,t}\psi(x)du - \frac{1}{2}\int_r^t \sigma(x)\psi_{s,t}(x)^2 ds \\
&\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{u,t}\psi(x)\cdot W(du,dy) \\
&\quad -\frac{1}{2}\int_r^t \left\{\frac{1}{2}\int_u^t a(x)\partial_{xx}^2 T_{u,s}(\sigma\psi_{s,t}^2)(x)ds\right\} du \\
&\quad -\frac{1}{2}\int_r^t \int_{\mathbb{R}} \left\{\int_u^t h(y-x)\partial_x T_{u,s}(\sigma\psi_{s,t})^2(x)ds\right\}\cdot W(du,dy) \\
&= \phi(x) + \frac{1}{2}\int_r^t a(x)\partial_{xx}^2 \psi_{u,t}(x)du - \frac{1}{2}\int_r^t \sigma(x)\psi_{s,t}(x)^2 ds \\
&\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x \psi_{u,t}(x)\cdot W(du,dy),
\end{aligned}$$

where we have used (2.6) twice for the last equality. That means that $\psi_{r,t}(x)$ is also a solution of (2.1). The uniqueness for the solution of (2.6) follows from that of (2.1). □

Equation (2.6) looks very much like the log-Laplace equation of a standard Dawson-Watanabe superprocess; see, e.g., Dawson [1]. We shall refer the solution $(\psi_{r,t})_{r\leq t}$ of this equation as a *stochastic* or *conditional log-Laplace semigroup.*

Proposition 2.4 *For any $t \geq r \geq 0$, the operators $\phi \mapsto T_{r,t}\phi$ and $\phi \mapsto \psi_{r,t}(\cdot,\phi)$ are contractive in the uniform norm. Moreover, there is a locally bounded non-negative function $(a,T) \mapsto C(a,T)$ on $[0,\infty)^2$ so that*

$$\|\psi_{r,t}(\cdot,\phi_1) - \psi_{r,t}(\cdot,\phi_2)\| \leq C(a,T)\|\phi_1 - \phi_2\|$$

for all $0 \leq r \leq t \leq T$ and $\phi_1,\phi_2 \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ satisfying $0 \leq \|\phi_1\|, \|\phi_2\| \leq a$.

Proof. The first assertion follows by Theorem 2.1 and the second one follows by (2.6) and Gronwall's inequality. □

By the above proposition we can extend the operators $\phi \mapsto T_{r,t}\phi$ and $\phi \mapsto \psi_{r,t}(\cdot,\phi)$ to $\phi \in C_b(\mathbb{R})^+$ by uniform convergence. Then we can extend $\phi \mapsto T_{r,t}\phi$ to all $\phi \in C_b(\mathbb{R})$ by linearity. We shall use those extensions whenever they are necessary. To conclude this section, we prove two results on the continuity of $(\psi_{r,t})_{r\leq t}$.

Lemma 2.5 *For any $\lambda \geq 1$, $\mu \in M(\mathbb{R})$ and $\phi \in H_1(\mathbb{R}) \cap C_b^2(\mathbb{R})^+$ we have*

$$\mathbf{E}\Big(|\langle \mu, \psi_{r,t_1}\rangle - \langle \mu, \psi_{r,t_2}\rangle|^{2\lambda}\Big) \leq C(\lambda, \phi, \langle \mu, 1\rangle)|t_1 - t_2|^\lambda, \qquad t_2 \geq t_1 \geq r \geq 0, \tag{2.7}$$

where $C(\lambda, \phi, \langle \mu, 1\rangle) \geq 0$ is a constant. In particular, for any $x \in \mathbb{R}$ the mapping $t \mapsto \psi_{r,t}(x, \phi)$ has a continuous modification.

Proof. Since $\|\psi_{r,t}\| \leq \|\phi\|$, it is easy to see that

$$\mathbf{E}\Big(|\langle \mu, \psi_{r,t_1}\rangle - \langle \mu, \psi_{r,t_2}\rangle|^{2\lambda}\Big) \leq c\mathbf{E}\Big(\Big|e^{-\langle \mu, \psi_{r,t_1}\rangle} - e^{-\langle \mu, \psi_{r,t_2}\rangle}\Big|^{2\lambda}\Big), \tag{2.8}$$

where $c = \exp\{2\lambda\|\phi\|\langle \mu, 1\rangle\}$. Now we consider a new probability space on which the following equation is realized: For $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned} \langle \phi, X_t\rangle &= \langle \phi, \mu\rangle + \frac{1}{2}\int_r^t \langle a\phi'', X_s\rangle ds + \int_r^t \int_{\mathbb{R}} \phi(y) Z(ds, dy) \\ &\quad + \int_r^t \int_{\mathbb{R}} \langle h(y - \cdot)\phi', X_s\rangle W(ds, dy), \qquad t \geq r, \end{aligned} \tag{2.9}$$

where $W(ds, dx)$ is a time-space white noise and $Z(ds, dy)$ is an orthogonal martingale measure which is orthogonal to $W(ds, dy)$ and has covariation measure $\sigma(y)X_s(dy)ds$. The governing probability measure $\mathbf{P}_{r,\mu}$ satisfies $\mathbf{P}_{r,\mu}\{X_r = \mu\} = 1$. Then

$$\begin{aligned} \mathbf{E}\Big(\Big|e^{-\langle \mu, \psi_{r,t_1}\rangle} - e^{-\langle \mu, \psi_{r,t_2}\rangle}\Big|^{2\lambda}\Big) &= \mathbf{E}_{r,\mu}\Big(\Big|e^{-\langle \mu, \psi_{r,t_1}\rangle} - e^{-\langle \mu, \psi_{r,t_2}\rangle}\Big|^{2\lambda}\Big) \\ &= \mathbf{E}_{r,\mu}\Big(\Big|\mathbf{E}_{r,\mu}^W\Big(e^{-\langle X_{t_1}, \phi\rangle} - e^{-\langle X_{t_2}, \phi\rangle}\Big)\Big|^{2\lambda}\Big) \\ &\leq \mathbf{E}_{r,\mu}\Big[\mathbf{E}_{r,\mu}^W\Big(\Big|e^{-\langle X_{t_1}, \phi\rangle} - e^{-\langle X_{t_2}, \phi\rangle}\Big|^{2\lambda}\Big)\Big] \\ &\leq \mathbf{E}_{r,\mu}\Big(|\langle X_{t_1} - X_{t_2}, \phi\rangle|^{2\lambda}\Big) \\ &\leq C(\lambda, \phi, \langle \mu, 1\rangle)|t_1 - t_2|^\lambda, \end{aligned}$$

where $C(\lambda, \phi, \langle \mu, 1\rangle) \geq 0$ is a constant and the last inequality follows by a standard argument applied to (2.9). By (2.8) and an adjustment of the constant we get (2.7). For any $x \in \mathbb{R}$, letting $\mu = \delta_x$ and $\lambda = 2$ we see that $t \mapsto \psi_{r,t}(x, \phi)$ has a continuous modification. □

Proposition 2.6 *Suppose that (a, c, h, σ) satisfy the conditions specified in the introduction. Then for any $\phi \in H_1(\mathbb{R}) \cap C_b^2(\mathbb{R})^+$, the mapping $(r, t) \mapsto \psi_{r,t}(\cdot, \phi) \in H_1(\mathbb{R})$ has a continuous modification.*

Proof. Let $\{h_i : i = 1, 2, \cdots\} \subset C_c^2(\mathbb{R})$ be a sequence which is dense in the set

$$\Big\{h \in H_1(\mathbb{R}) : \int_{\mathbb{R}} h(x)dx \leq 1, \|h\| \leq 1, \|h'\| \leq 1, \|h''\| \leq 1\Big\}$$

by the norm of $H_1(\mathbb{R})$. We can define a metric ρ on $H_1(\mathbb{R})$ by

$$\rho(\phi, \psi) = \sum_{i=1}^\infty 2^{-i}(|\langle \phi - \psi, h_i\rangle_0| \wedge 1).$$

Let $\lambda \geq 1$. For $t_2 \geq t_1 \geq r \geq 0$ we apply (2.7) to $\mu(dx) = h_i^{\pm}(x)dx$ to see that

$$\begin{aligned}
&\mathbf{E}\Big(|\langle \psi_{r,t_1} - \psi_{r,t_2}, h_i\rangle_0|^{2\lambda}\Big) \\
&\quad \leq 2^{2\lambda-1}\mathbf{E}\Big(|\langle \psi_{r,t_1} - \psi_{r,t_2}, h_i^{+}\rangle_0|^{2\lambda} + |\langle \psi_{r,t_1} - \psi_{r,t_2}, h_i^{-}\rangle_0|^{2\lambda}\Big) \\
&\quad \leq 2^{2\lambda-1}C(\lambda, \phi, 1)|t_1 - t_2|^{\lambda}.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\mathbf{E}[\rho(\psi_{r,t_1}, \psi_{r,t_2})^{2\lambda}] &\leq \sum_{i=1}^{\infty} 2^{-i}\mathbf{E}[\langle \psi_{r,t_1} - \psi_{r,t_2}, h_i\rangle_0^{2\lambda} \wedge 1] \\
&\leq \sum_{i=1}^{\infty} 2^{-i}C(\lambda, \phi, 1)|t_1 - t_2|^{\lambda} \\
&\leq C(\lambda, \phi, 1)|t_1 - t_2|^{\lambda}. \qquad (2.10)
\end{aligned}$$

For $t \geq r_2 \geq r_1 \geq 0$ we get from (2.1) that

$$\begin{aligned}
\langle \psi_{r_1,t} - \psi_{r_2,t}, h_i\rangle &= \frac{1}{2}\int_{r_1}^{r_2} \Big\langle \psi_{s,t}, \partial_{xx}^2(ah_i) - \sigma\psi_{s,t}h_i\Big\rangle ds \\
&\quad - \int_{r_1}^{r_2}\int_{\mathbb{R}} \Big\langle \psi_{s,t}, \partial_x(h(y-\cdot)h_i)\Big\rangle \cdot W(ds, dy).
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
\mathbf{E}\Big[\langle \psi_{r_1,t} - \psi_{r_2,t}, h_i\rangle^{2\lambda}\Big] &\leq \frac{1}{2}\mathbf{E}\Big[\Big(\int_{r_1}^{r_2} \Big\langle \psi_{s,t}, \partial_{xx}^2(ah_i) - \sigma\psi_{s,t}h_i\Big\rangle ds\Big)^{2\lambda}\Big] \\
&\quad + 2^{2\lambda-1}\mathbf{E}\Big[\Big(\int_{r_1}^{r_2}\int_{\mathbb{R}} \Big\langle \psi_{s,t}, \partial_x(h(y-\cdot)h_i)\Big\rangle \cdot W(ds, dy)\Big)^{2\lambda}\Big] \\
&\leq \frac{1}{2}(r_2 - r_1)^{2\lambda-1}\mathbf{E}\Big[\int_{r_1}^{r_2} \Big\langle \psi_{s,t}, \partial_{xx}^2(ah_i) - \sigma\psi_{s,t}h_i\Big\rangle^{2\lambda} ds\Big] \\
&\quad + 2^{2\lambda-1}\mathbf{E}\Big[\Big(\int_{r_1}^{r_2} ds\int_{\mathbb{R}} \Big\langle \psi_{s,t}, \partial_x(h(y-\cdot)h_i)\Big\rangle^2 dy\Big)^{\lambda}\Big].
\end{aligned}$$

Since $\|\psi_{s,t}\| \leq \|\phi\|$, there is a constant $C(\lambda, \phi) \geq 0$ so that

$$\mathbf{E}\Big[\langle \psi_{r_1,t} - \psi_{r_2,t}, h_i\rangle^{2\lambda}\Big] \leq C(\lambda, \phi)[(r_2 - r_1)^{2\lambda} + (r_2 - r_1)^{\lambda}].$$

By calculations similar to those in (2.10) we have

$$\mathbf{E}[\rho(\psi_{r_1,t}, \psi_{r_2,t})^{2\lambda}] \leq C(\lambda, \phi, 1)[(r_2 - r_1)^{2\lambda} + (r_2 - r_1)^{\lambda}].$$

By taking sufficiently large $\lambda \geq 1$ we see that $(r, t) \mapsto \psi_{r,t}(\cdot, \phi) \in H_1(\mathbb{R})$ has a continuous modification. $\square$

3 A stochastic comparison theorem

A comparison theorem for the stochastic log-Laplace equation (2.1) is provided by the following

Theorem 3.1 *Let (a, c, h, σ_1) and (a, c, h, σ_2) be two sets of parameters satisfying the conditions specified in the introduction. Suppose that $\sigma_1(x) \le \sigma_2(x)$ for all $x \in \mathbb{R}$. For $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ let $(\psi^i_{r,t})_{r \le t}$ be the unique strong solution of (2.1) with σ replaced by σ_i. Then for any $t \ge r \ge 0$ we have a.s. $\psi^1_{r,t}(x) \ge \psi^2_{r,t}(x)$ simultaneously for all $x \in \mathbb{R}$.*

Proof. Let $u_{s,t}(x) = \psi^1_{s,t}(x) - \psi^2_{s,t}(x)$. It is easy to check that

$$\begin{aligned} u_{r,t}(x) &= \int_r^t \left[\frac{1}{2}a(x)\partial_x^2 u_{s,t}(x) - \frac{1}{2}d_s(x)u_{s,t}(x)\right]ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x u_{s,t}(x) \cdot W(dy, ds) \\ &\quad + \int_r^t c_s(x)ds, \end{aligned} \tag{3.1}$$

where

$$d_s(x) = \sigma_1(x)(\psi^1_{s,t}(x) + \psi^2_{s,t}(x)) \ge 0,$$

and

$$c_s(x) = \frac{1}{2}(\sigma_2(x) - \sigma_1(x))(\psi^2_{s,t}(x))^2 \ge 0.$$

Let $v_{s,t}(x)$ be the difference of any two solutions of (3.1). We have

$$\begin{aligned} v_{r,t}(x) &= \int_r^t \left(\frac{1}{2}a(x)\partial_x^2 v_{s,t}(x) - \frac{1}{2}d_s(x)v_{s,t}(x)\right)ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x v_{s,t}(x) \cdot W(dy, ds). \end{aligned}$$

By an argument similar to the proof of Lemma 4.2 in [12], we can show that there exists a constant $K \ge 0$ such that

$$\mathbf{E}\|v_{r,t}\|_0^2 \le K \int_r^t \mathbf{E}\|v_{s,t}\|_0^2 ds.$$

An application of Gronwall's inequality yields $v = 0$. Then (3.1) has at most one solution. In the sequel, we show that (3.1) has a non-negative strong solution. To this end, for $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ we consider the equation

$$\begin{aligned} U_{r,t}(x) &= \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_{xx}^2 U_{s,t}(x) - \frac{1}{2}d_s(x)U_{s,t}(x)\right]ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x U_{s,t}(x) \cdot W(dy, ds). \end{aligned} \tag{3.2}$$

By [9], (3.2) has a non-negative solution $U_{r,t}(x) = U_{r,t}(x, \phi)$. We claim that

$$\theta_{r,t}(x) = \int_r^t U_{r,u}(x, c_u)du \ge 0$$

is a solution of (3.1). Indeed, from (3.2) we have

$$\begin{aligned} U_{r,u}(x,c_u) &= c_u(x) + \int_r^u \left[\frac{1}{2}a(x)\partial_{xx}^2 U_{v,u}(x,c_u) - \frac{1}{2}d_v(x)U_{v,u}(x,c_u)\right] dv \\ &\quad + \int_r^u \int_{\mathbb{R}} h(y-x)\partial_x U_{v,u}(x,c_u)\cdot W(dy,dv). \end{aligned}$$

It follows that

$$\begin{aligned} \theta_{r,t}(x) &= \int_r^t c_u(x)du + \int_r^t \int_r^u \left[\frac{1}{2}a(x)\partial_{xx}^2 U_{v,u}(x,c_u) - \frac{1}{2}d_v(x)U_{v,u}(x,c_u)\right] dvdu \\ &\quad + \int_r^t \int_r^u \int_{\mathbb{R}} h(y-x)\partial_x U_{v,u}(x,c_u)\cdot W(dy,dv)du \\ &= \int_r^t c_u(x)du + \int_r^t \int_v^t \left[\frac{1}{2}a(x)\partial_{xx}^2 u_{v,u}(x,c_u) - \frac{1}{2}d_v(x)U_{v,u}(x,c_u)\right] dudv \\ &\quad + \int_r^t \int_{\mathbb{R}} \int_v^t h(y-x)\partial_x U_{v,u}(x,c_u)du\cdot W(dy,dv) \\ &= \int_r^t c_u(x)du + \int_r^t \left[\frac{1}{2}a(x)\partial_{xx}^2 \theta_{v,t}(x) - \frac{1}{2}d_v(x)\theta_{v,t}(x)\right] dv \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x \theta_{v,t}(x)\cdot W(dy,dv). \end{aligned}$$

This finishes the proof of the claim, and hence, the proof of the lemma. □

4 The conditional transition semigroup

Let us first give a more precise formulation of the stochastic equation (1.2). Let (a,c,h,σ) be given as in the introduction. For an arbitrary measure $\mu \in M(\mathbb{R})$, we established in [12] the joint existence of the continuous measure-valued process $X = \{X_t : t \geq 0\}$ and the time-space white noise $\{W(ds,dy)\}$ so that $X_0 = \mu$ and (1.2) defines an orthogonal martingale measure $\{Z(ds,dy)\}$ that is orthogonal to $\{W(ds,dy)\}$ and has covariation measure $\sigma(y)X_s(dy)ds$. Those give the weak existence of the solution of (1.2).

It is well-known that the white noise $\{W(ds,dy)\}$ can be obtained from a continuous process $W = \{W_t : t \geq 0\}$ taking values in a suitable weighted Sobolev space over $\mathbb{R}$; see [15]. Let $(\Omega, \mathscr{F}, \mathscr{F}_t)$ be the canonical space of (X,W). By the results of [12] we may actually construct the family of probability measures $\{\mathbf{P}_K : K \text{ is a probability measure on } M(\mathbb{R})\}$, where $\mathbf{P}_K$ is the probability measure on $(\Omega,\mathscr{F})$ under which (1.2) is realized with X_0 distributed according to K. Let $\mathbf{P}_K^W$ denote the conditional probability given $\{W(ds,dy)\}$. Write $\mathbf{P}_\mu = \mathbf{P}_{\delta_\mu}$ and $\mathbf{P}_\mu^W = \mathbf{P}_{\delta_\mu}^W$ for $\mu \in M(\mathbb{R})$. For $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ let $\psi_{r,t} = \psi_{r,t}^W$ and $T_{r,t} = T_{r,t}^W$ be defined by (2.1) and (2.5), respectively. The following result from [12] gives the characterization of transition probabilities of the SDSM, which implies the uniqueness in law of the superprocess.

Theorem 4.1 ([12]) *For every $t \geq r \geq 0$, every $\phi \in C_b(\mathbb{R})^+$ and every initial distribution K we have a.s.*

$$\mathbf{E}_K^W\{e^{-\langle \phi, X_t\rangle}|\mathscr{F}_r\} = \exp\Big\{ -\langle \psi_{r,t}^W, X_r\rangle\Big\}. \tag{4.1}$$

Consequently, $\{X_t : t \geq 0\}$ is a diffusion process with Feller transition semigroup $(Q_t)_{t\geq 0}$ given by

$$\int_{M(\mathbb{R})} e^{-\langle \phi,\nu\rangle} Q_t(\mu, d\nu) = \mathbf{E}_K \exp\Big\{ - \langle \psi_{0,t}^W, \mu\rangle \Big\}.$$

Proof. For $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ the equalities were proved in [12]. By the comments following Proposition 2.4 they can be extended to all $\phi \in C_b(\mathbb{R})^+$. $\square$

Theorem 4.2 *For any $t \geq r \geq 0$ we can define a unique random probability kernel $Q_{r,t}^W(\mu, d\nu)$ on $M(\mathbb{R})$ by*

$$\int_{M(\mathbb{R})} e^{-\langle \phi,\nu\rangle} Q_{r,t}^W(\mu, d\nu) = \exp\Big\{ - \langle \psi_{r,t}^W, \mu\rangle \Big\}, \qquad \phi \in C_b(\mathbb{R})^+. \tag{4.2}$$

Moreover, for any $t \geq s \geq r \geq 0$ and $\mu \in M(\mathbb{R})$ we have a.s.

$$Q_{r,t}^W(\mu, d\nu) = \int_{M(\mathbb{R})} Q_{r,s}^W(\mu, d\gamma) Q_{s,t}^W(\gamma, d\nu). \tag{4.3}$$

Proof. By Proposition 2.4 the stochastic operator $\phi \mapsto \psi_{r,t}(\cdot, \phi)$ is uniquely determined by its operation on a countable number of functions $\phi \in C_b(\mathbb{R})^+$. By applying Theorem 4.1 with $r = 0$ and $X_0 = \mu$, we have

$$\mathbf{E}_\mu^W \{e^{-\langle \phi, X_t\rangle}\} = \exp\Big\{ - \langle \psi_{0,t}^W, \mu\rangle \Big\}.$$

Then the right hand side of the above equation defines a random kernel $Q_{0,t}^W(\mu, d\nu)$ on $M(\mathbb{R})$. By the property of independent and stationary increments of the time-space white noise, $\psi_{r,t}(\cdot, \phi)$ is identically distributed with $\psi_{0,t-r}(\cdot, \phi)$. By a shifting argument we see that (4.2) defines a unique random kernel $Q_{r,t}^W(\mu, d\nu)$ on $M(\mathbb{R})$. The uniqueness of $\psi_{r,t}^W$ implies that of $Q_{r,t}^W(\mu, d\nu)$. Equation (4.3) follows from the a.s. semigroup property of $(\psi_{r,t}^W)_{r\leq t}$. $\square$

From Theorems 4.1 and 4.2, one might expect $\{X_t : t \geq 0\}$ conditioned upon $\{W(ds, dy)\}$ is an inhomogeneous diffusion process with transition semigroup $(Q_{r,t}^W)_{r\leq t}$. Indeed, for any $t \geq r \geq 0$ we have a.s.

$$\mathbf{E}^W \{e^{-\langle \phi, X_t\rangle} | \mathscr{F}_r\} = \int_{M(\mathbb{R})} e^{-\langle \phi,\nu\rangle} Q_{r,t}^W(X_r, d\nu). \tag{4.4}$$

However, the conditional Markov property does not follow so easily since the null exceptional set of (4.4) depends on (t, r). Of course, it can be made that (4.4) is a.s. true simultaneously for all r and t (with $t \geq r$) in some fixed countable dense subset U of $[0, \infty)$. In this case, the restricted process $\{X_t : t \in U\}$ becomes an inhomogeneous Markov process under the conditional probability given $\{W(ds, dy)\}$.

In particular, if $\sigma(x) \equiv 0$, the corresponding SDSM $\{\hat{X}_t : t \geq 0\}$ does not involve branching and from (4.2) its conditional transition semigroup is given by

$$\int_{M(\mathbb{R})} e^{-\langle \phi,\nu\rangle} \hat{Q}_{r,t}^W(\mu, d\nu) = \exp\Big\{ - \langle T_{r,t}^W \phi, \mu\rangle \Big\}, \qquad \phi \in C_b(\mathbb{R})^+. \tag{4.5}$$

Thus given $\hat{X}_r = \delta_x$ we have $\hat{X}_t(dy) = T^W_{r,t}(x, dy)$ for all $t \geq r$. That is, the evolution of the SDSM is completely determined by the linear semigroup $(T^W_{r,t})_{r\leq t}$. Observe that given $\{W(ds, dy)\}$ the solutions of (1.1) are independent of each other. Then by a (conditional) law of large numbers and the construction of the SDSM based on the small particle approximation we see that a.s.

$$T^W_{r,t}\phi(x) = \langle \phi, \hat{X}_t\rangle = \mathbf{E}^W[\phi(x(t))], \tag{4.6}$$

where $\{x(t) : t \geq r\}$ is the unique solution of

$$x(t) = x + \int_r^t c(x(s))dB(s) + \int_r^t \int_{\mathbb{R}} h(y - x(s))W(ds, dy), \qquad t \geq r, \tag{4.7}$$

and $\{B(t) : t \geq 0\}$ is a Brownian motion independent of $\{W(ds, dy)\}$. Thus $(T^W_{r,t})_{r\leq t}$ is roughly the conditional transition semigroup of the diffusion process $\{x(t) : t \geq r\}$ given $\{W(ds, dy)\}$. In this sense, we may regard $\{x(t) : t \geq r\}$ as a diffusion process in random environments. Unfortunately, the null exceptional set of (4.6) also depends on (t, r), so the full conditional Markov property of $\{x(t) : t \geq r\}$ still remains as an open problem.

The following theorem gives representations of some conditional moments of the process in terms of $(T^W_{r,t})_{r\leq t}$.

Theorem 4.3 *For any $t \geq r \geq 0$ and $\phi \in C_b(\mathbb{R})$ we have a.s.*

$$\int_{M(\mathbb{R})} \langle \phi, \nu\rangle Q^W_{r,t}(\mu, d\nu) = \langle T^W_{r,t}\phi, \mu\rangle \tag{4.8}$$

and

$$\int_{M(\mathbb{R})} \langle \phi, \nu\rangle^2 Q^W_{r,t}(\mu, d\nu) = \langle T^W_{r,t}\phi, \mu\rangle^2 + \int_r^t \langle T^W_{r,s}(\sigma(T^W_{s,t}\phi)^2), \mu\rangle ds. \tag{4.9}$$

Proof. Let $\phi \in H_1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ be fixed. For any $\lambda \geq 0$, let $T^\lambda_{r,t}(x) \equiv T^W_{r,t}(\lambda\phi)(x)$ be the unique strong solution of

$$T^\lambda_{r,t}(x) = \lambda\phi(x) + \frac{1}{2}a(x)\int_r^t \partial^2_{xx}T^\lambda_{s,t}(x)ds + \int_r^t \int_{\mathbb{R}} h(y - x)\partial_x T^\lambda_{s,t}(x) \cdot W(ds, dy).$$

By the uniqueness of the solution, we have $T^W_{r,t}(\lambda\phi)(x) = \lambda T^W_{r,t}\phi(x)$. It follows that

$$\frac{\partial}{\partial\lambda}T^W_{r,t}(\lambda\phi)(x) = T^W_{r,t}\phi(x) \quad \text{and} \quad \frac{\partial^2}{\partial\lambda^2}T^W_{r,t}(\lambda\phi)(x) \equiv 0$$

Let $\psi^\lambda_{r,t}(x) = \psi_{r,t}(x, \lambda\phi)$ be the unique strong solution of

$$\psi^\lambda_{r,t}(x) = T^W_{r,t}(\lambda\phi)(x) - \frac{1}{2}\int_r^t T^W_{r,s}(\sigma(\psi^\lambda_{s,t})^2)(x)ds. \tag{4.10}$$

Clearly, we have $\psi^\lambda_{r,t}(x)|_{\lambda=0} = 0$. Now we prove that $\psi^\lambda_{r,t}(x)$ is twice differentiable in probability with respect to λ in the supremum norm $\|\cdot\|$. Let $Z^\lambda_{s,t}(x) = \lambda^{-1}\psi^\lambda_{s,t}(x) - T^W_{s,t}\phi(x)$. According to (4.10), we have

$$\begin{aligned}
Z^\lambda_{r,t}(x) &= -\frac{1}{2\lambda}\int_r^t T^W_{r,s}(\sigma(\psi^\lambda_{s,t})^2)(x)ds \\
&= -\frac{1}{2}\int_r^t T^W_{r,s}(\sigma\lambda^{-1}(\psi^\lambda_{s,t})^2 - \sigma\psi^\lambda_{s,t}T^W_{s,t}\phi + \sigma\psi^\lambda_{s,t}T^W_{s,t}\psi)(x)ds \\
&= -\frac{1}{2}\int_r^t T^W_{r,s}(\sigma\psi^\lambda_{s,t}Z^\lambda_{s,t})(x)ds - \frac{1}{2}\int_r^t T^W_{r,s}(\sigma\psi^\lambda_{s,t}T^W_{s,t}\phi)(x)ds.
\end{aligned}$$

By Theorem 2.1 we have $\|\psi^{\lambda}_{s,t}\| \leq \lambda\|\phi\|$ and $\|T^W_{r,s}\phi\| \leq \|\phi\|$. Then an application of Gronwall's inequality yields

$$\mathbf{E}[\|Z^{\lambda}_{r,t}\|^2] \to 0 \qquad \text{as} \quad \lambda \to 0. \tag{4.11}$$

That proves $\frac{\partial \psi^{\lambda}_{r,t}(x)}{\partial \lambda}|_{\lambda=0} = T^W_{r,t}\phi(x)$. Now let

$$u_{r,t}(x) = -\int_r^t T^W_{r,s}(\sigma(T^W_{s,t}\psi)^2)(x)ds$$

and

$$u^{\lambda}_{s,t}(x) = \lambda^{-2}[\psi^{2\lambda}_{s,t}(x) - 2\psi^{\lambda}_{s,t}(x)] - u_{s,t}(x).$$

By elementary calculations based on (4.10) we get

$$\begin{aligned} u^{\lambda}_{r,t}(x) &= -\frac{1}{2\lambda^2}\int_r^t T^W_{r,s}(\sigma(\psi^{2\lambda}_{s,t})^2)(x)ds + \frac{1}{\lambda^2}\int_r^t T^W_{r,s}(\sigma(\psi^{\lambda}_{s,t})^2)(x)ds \\ &\quad + \int_r^t T^W_{r,s}(\sigma(T^W_{s,t}\phi)^2)(x)ds \\ &= \int_r^t T^W_{r,s}\Big\{\sigma\Big[(T^W_{s,t}\phi)^2 + \Big(\frac{\psi^{\lambda}_{s,t}}{\lambda}\Big)^2 - 2\Big(\frac{\psi^{2\lambda}_{s,t}}{2\lambda}\Big)^2\Big]\Big\}(x)ds. \end{aligned}$$

Then we can use (4.11) to get $\mathbf{E}[\|u^{\lambda}_{r,t}\|^2] \to 0$ as $\lambda \to 0$. It then follows that

$$\frac{\partial^2 \psi^{\lambda}_{r,t}(x)}{\partial \lambda^2}|_{\lambda=0} = -\int_r^t T^W_{r,s}(\sigma(T^W_{s,t}\phi)^2)(x)ds.$$

Now we can get (4.8) and (4.9) by taking derivatives with respect to λ in

$$\int_{M(\mathbb{R})} e^{-\langle \lambda\psi, \nu\rangle} Q^W_{r,t}(\mu, d\nu) = \exp\Big\{-\langle \psi^{\lambda}_{r,t}, \mu\rangle\Big\}$$

and letting $\lambda = 0$. The extensions of (4.8) and (4.9) to an arbitrary $\phi \in C_b(\mathbb{R})$ are immediate. □

5 Conditional entrance laws and excursion laws

In this section, we assume there is a constant $\sigma_0 > 0$ such that $\sigma(x) \geq \sigma_0$ for all $x \in \mathbb{R}$. We shall characterize a class of conditional entrance laws of the SDSM, from which we deduce the existence of some conditional excursion laws. Suppose that $\{W(ds, dy)\}$ is a time-space white noise defined on a standard probability space and let $(\psi^W_{r,t})_{r\leq t}$ be the stochastic log-Laplace semigroup defined by (2.1).

Proposition 5.1 *For any $t \geq r \geq 0$ and $\phi \in C_b(\mathbb{R})^+$ we have a.s.*

$$\|\psi^W_{r,t}(\cdot, \phi)\| \leq \frac{\|\phi\|}{1 + \sigma_0(t-r)\|\phi\|/2}. \tag{5.1}$$

Proof. Let $\bar{\psi}^W_{r,t}(x,\phi)$ be defined by (2.1) with $\sigma(x)$ replaced by σ_0. By Theorem 3.1 we have $\psi^W_{r,t}(x,\phi) \le \bar{\psi}^W_{r,t}(x,\phi)$. Observe that the stochastic operators $\phi \mapsto \bar{\psi}^W_{r,t}(x,\phi)$ corresponds to an SDSM $\{X^0_t : t \ge 0\}$ satisfying the SPDE: For any $\phi \in C^2_b((\mathbb{R})$,

$$\begin{aligned}\langle \phi, X^0_t\rangle &= \langle \phi,\mu\rangle + \frac{1}{2}\int_0^t \langle a\phi'', X^0_s\rangle ds + \int_0^t\int_{\mathbb{R}} \phi(y) Z(ds,dy) \\ &\quad + \int_0^t\int_{\mathbb{R}} \langle h(y-\cdot)\phi', X^0_s\rangle W(ds,dy), \end{aligned} \tag{5.2}$$

where $W(ds,dx)$ is a space-time white noise and $Z(ds,dy)$ is an orthogonal martingale measure which is orthogonal to $W(ds,dy)$ and has covariation measure $\sigma_0 X^0_s(dy)ds$. (If it is necessary, we may construct a new probability space on which all those random elements are defined.) From (5.2) we see that $\{\langle 1, X^0_t\rangle : t \ge 0\}$ is a continuous martingale which is orthogonal to $W(ds,dy)$ and has quadratic variation $\sigma_0\langle X^0_s, 1\rangle ds$. Then, by considering an extension of the original probability space, we have

$$\langle 1, X^0_t\rangle = \langle \phi,\mu\rangle + \int_0^t \sqrt{\sigma_0\langle 1, X^0_s\rangle} dB(s),$$

where $\{B(t) : t \ge 0\}$ is a Brownian motion orthogonal to and hence independent of $W(ds,dy)$. In other words, $\{\langle 1, X^0_t\rangle : t \ge 0\}$ is a Feller's branching diffusion with constant branching rate σ_0. Since the above equation has a unique strong solution, we conclude that $\{\langle 1, X^0_t\rangle : t \ge 0\}$ is independent of $W(ds,dy)$. Then we have

$$\exp\Big\{ -\psi^W_{0,t}(x,\phi)\Big\} \ge \exp\Big\{ -\bar{\psi}^W_{0,t}(x,\phi)\Big\} \ge \mathbf{E}^W_{\delta_x}\Big[e^{-\|\phi\|\langle 1, X^0_t\rangle}\Big] = \mathbf{E}_{\delta_x}\Big[e^{-\|\phi\|\langle 1, X^0_t\rangle}\Big].$$

By a well-known result on the characterization of Laplace transform of the Feller's branching diffusion we have

$$\mathbf{E}_{\delta_x}\Big[e^{-\lambda\langle 1, X^0_t\rangle}\Big] = \exp\Big\{ -\frac{\lambda}{1+\sigma_0 t\lambda/2}\Big\}$$

for any $\lambda \ge 0$; see, e.g., [8, pp.235-236]. Since $\psi^W_{r,t}(\cdot,\phi)$ is identically distributed with $\psi^W_{0,t-r}(\cdot,\phi)$, we have the desired inequality. □

Let $(Q^W_{r,t})_{r\le t}$ be defined by (4.2) and let $(Q^{\circ,W}_{r,t})_{r\le t}$ be the restriction of $(Q^W_{r,t})_{r\le t}$ on $M(\mathbb{R})^\circ := M(\mathbb{R}) \setminus \{0\}$. The following theorem specifies a useful class of *entrance laws* of the restricted conditional semigroup $(Q^{\circ,W}_{r,t})_{r\le t}$.

Theorem 5.2 *For any $x \in \mathbb{R}$ and $t > r \ge 0$, there is a unique finite random measure $L^W_{r,t}(x,d\nu)$ on $M(\mathbb{R})^\circ$ such that for any $\phi \in C_b(\mathbb{R})^+$ we have a.s.*

$$\int_{M(\mathbb{R})^\circ} (1-e^{-\langle\phi,\nu\rangle}) L^W_{r,t}(x,d\nu) = \psi^W_{r,t}(x,\phi). \tag{5.3}$$

Furthermore, for any $t > s > r \ge 0$ we have a.s.

$$L^W_{r,t}(x,d\nu) = \int_{M(\mathbb{R})^\circ} L^W_{r,s}(x,d\mu) Q^{\circ,W}_{s,t}(\mu,d\nu). \tag{5.4}$$

Proof. By Proposition 2.4 the stochastic operator $\phi \mapsto \psi_{r,t}(\cdot, \phi)$ is uniquely determined by its operation on a countable number of functions $\phi \in C_b(\mathbb{R})^+$. By Theorem 4.2 we have a.s.

$$\int_{M(\mathbb{R})} e^{-\langle \phi, \nu \rangle} Q^W_{r,t}(q\delta_x, d\nu) = \exp\Big\{ - q\psi^W_{r,t}(x, \phi)\Big\}$$

simultaneously for all rationals $q \geq 0$ and all functions ϕ in a countable dense subset of $C_b(\mathbb{R})^+$. The above expression implies that the probability measure $Q^W_{r,t}(\delta_x, \cdot)$ on $M(\mathbb{R})^\circ$ is a.s. infinitely divisible. Then we have the canonical representation

$$\psi^W_{r,t}(x, \phi) = \int_{\mathbb{R}} \phi(y)\lambda^W_{r,t}(x, dy) + \int_{M(\mathbb{R})^\circ} (1 - e^{-\langle \phi, \nu \rangle}) L^W_{r,t}(x, d\nu), \tag{5.5}$$

where $\lambda^W_{r,t}(x, \cdot)$ is a finite measure on $\mathbb{R}$ and $L^W_{r,t}(x, \cdot)$ is a σ-finite measure on $M(\mathbb{R})^\circ$. From Theorem 3.1 and Proposition 5.1 it follows that

$$\lim_{\lambda \to \infty} \psi^W_{r,t}(x, \lambda) \leq \frac{2}{\sigma_0(t-r)} < \infty \tag{5.6}$$

for any $t > r \geq 0$. Then we have a.s. $\lambda^W_{r,t}(x, \mathbb{R}) = 0$ and $L^W_{r,t}(x, M(\mathbb{R})^\circ) < \infty$. Thus representation (5.3) follows. The uniqueness of $\psi^W_{r,t}$ implies that of $L^W_{r,t}(x, d\nu)$. The relation (5.4) follows by the a.s. semigroup property of $(\psi^W_{r,t})_{r \leq t}$. $\square$

Let $\mathbf{W} = \{w \in C([0, \infty), M(\mathbb{R})) :$ there is a non-empty interval $(\alpha(w), \beta(w)) \subset [0, \infty)$ such that $w(t) \in M(\mathbb{R})^\circ$ for $t \in (\alpha(w), \beta(w))$ and $w_t = 0$ otherwise$\}$. For any $r \geq 0$ let $\mathscr{G}$ be the σ-algebra on $\mathbf{W}$ generated by the coordinate process. Let $\mathbf{W}^r$ be the subset of $\mathbf{W}$ comprising of paths $\{w_t : t \geq 0\}$ such that $\alpha(w) = r$ and let $\mathscr{G}^r$ be the trace of $\mathscr{G}$ on $\mathbf{W}^r$.

Theorem 5.3 *For any $r \geq 0$ there is a σ-finite random measure $Q^{x,W}_r$ on $(\mathbf{W}^r, \mathscr{G}^r)$ such that for every finite sequence $t_n > \cdots > t_1 > r$ we have a.s.*

$$Q^{x,W}_r\{w_{t_1} \in d\nu_1, \cdots, w_{t_n} \in d\nu_n\} = L^W_{r,t_1}(x, d\nu_1) Q^{\circ,W}_{t_1,t_2}(\nu_1, d\nu_2) \cdots Q^{\circ,W}_{t_{n-1},t_n}(\nu_{n-1}, d\nu_n). \tag{5.7}$$

In particular, for any $t > r$ and $\phi \in C_b(\mathbb{R})^+$ we have a.s.

$$\int_{\mathbf{W}^r} (1 - e^{-\langle \phi, w_t \rangle}) Q^{x,W}_r(dw) = \psi^W_{r,t}(x, \phi).$$

Proof. Let $\mathbf{C}^r = C([r, \infty), M(\mathbb{R}))$ and let $\mathscr{F}^r$ be the σ-algebra on $\mathbf{C}^r$ generated by the coordinate process. Recall that $(\Omega, \mathscr{F}, \mathscr{F}_t)$ is the canonical space of (X, W) and $\mathbf{P}_\mu$ is the probability measure on $(\Omega, \mathscr{F})$ under which (1.2) is realized with $X_0 = \mu$. Let Q^W_μ denote the regular conditional distribution of $X = \{X_t : t \geq 0\}$ under $\mathbf{P}^W_\mu$ given $W = \{W_t : t \geq 0\}$. For any $s > r$, let $Q^W_{s,\mu}$ denote the image of $Q^{W_{s+\cdot}}_\mu$ under the map $X \mapsto X_{s+\cdot}$ from $\mathbf{C}^0$ to $\mathbf{C}^s$. Observe that $Q^W_{s,\mu}$ is a probability measure on $(\mathbf{C}^s, \mathscr{F}^s)$, which is intuitively the conditional distribution of $\{X_t : t \geq s\}$ given $\{W_t : t \geq s\}$ and $X_s = \mu$. In the obvious way, we may regard $Q^W_{s,\mu}$ as a random probability measure on $(\mathbf{C}^r, \mathscr{F}^{r,s})$, where $\mathscr{F}^{r,s} = \sigma(\{w_t : t \geq s\})$. Then for each integer $k \geq 1$ we can define a random measure $\tilde{Q}^{x,W}_{r+1/k}$ on $(\mathbf{C}^r, \mathscr{F}^{r,r+1/k})$ by

$$\tilde{Q}^{x,W}_{r+1/k}(dw) = \int_{M(\mathbb{R})^\circ} L^W_{r,r+1/k}(x, d\mu) Q^W_{r+1/k,\mu}(dw).$$

By (5.4) and the a.s. semigroup property of $(Q^{\circ,W}_{r,t})_{r\leq t}$ it is easy to show that a.s.

$$\tilde{Q}^{x,W}_{r+1/k}(dw_{t_1}\in d\nu_1,\cdots,w_{t_n}\in d\nu_n)=L^W_{r,t_1}(x,d\nu_1)Q^{\circ,W}_{t_1,t_2}(\nu_1,d\nu_2)\cdots Q^{\circ,W}_{t_{n-1},t_n}(\nu_{n-1},d\nu_n)$$

for any $t_n>\cdots>t_1>r+1/k$ and $\nu_n,\cdots,\nu_1\in M(\mathbb{R})^\circ$. Let us equip $C((r,\infty),M(\mathbb{R}))$ with the natural σ-algebra. By an argument based on the inverse limit similar to that of Getoor and Glover (1987), it is not hard to show that there is random measure $Q^{x,W}_r$ on $C((r,\infty),M(\mathbb{R}))$ satisfying (5.7). It is σ-finite since $L^W_{r,t}(x,M(\mathbb{R})^\circ)<\infty$ a.s. for any $t>r$. From (4.2) it is easily seen that 0 is a trap for $(Q^W_{r,t})_{r\leq t}$. Moreover, by (4.2) and (5.1) we have

$$Q^W_{r,t}(\mu,\{0\})=\lim_{\lambda\to\infty}\exp\{-\langle\psi^W_{r,t}(\cdot,\lambda),\mu\rangle\}\geq\exp\Big\{-\frac{2\langle 1,\mu\rangle}{\sigma_0(t-r)}\Big\}.$$

Thus $Q^W_{r,t}(\mu,\{0\})\to 1$ as $t\to\infty$. Let H_r be a countable dense subset of (r,∞). Clearly, we can assume (5.7) a.s. holds for all ordered subset $\{t_n>\cdots>t_1\}\subset H_r$. Then it is easy to show that for $Q^{x,W}_r$-a.a. paths $\{w_t:t>r\}$ in $C((r,\infty),M(\mathbb{R}))$ we have $\beta(w):=\inf\{s>r:w_s=0\}<\infty$ and $w_t=0$ for $t\geq\beta(w)$. Following the proof of [10, Theorem 5.1] one shows that $w_{r+}=0$ for $Q^{x,W}_r$-a.a. paths $\{w_t:t>r\}$. Then $Q^{x,W}_r$ is actually supported by $\mathbf{W}^r$. □

The property (5.7) suggests that, roughly speaking, under $Q^{x,W}_r$ the coordinate process $\{w_t:t>r\}$ of $\mathbf{W}^r$ is a diffusion process with transition semigroup $(Q^{\circ,W}_{s,t})_{s\leq t}$ and one-dimensional distributions $(L^W_{r,t}(x,\cdot))_{t>r}$. However, the exceptional set of (5.7) depends on the sequence $t_n>\cdots>t_1>r$. This is similar to the situation explained after (4.4). For convenience we sometimes think $Q^{x,W}_r$ as a σ-finite random measure on the enlarged space $(\mathbf{W},\mathscr{G})$.

6 Construction of the superprocesses

Since the quadratic variation process of Z depends on X, we cannot expect a strong solution of (1.2) in the usual sense. In other words, the time-space white noise $\{W(ds,dy)\}$ does not contain sufficient information to determine the SDSM. However, based on the conditional excursion law constructed in the last section, we can construct an SDSM from $\{W(ds,dy)\}$ and an additional Poisson noise.

We assume there is a constant $\sigma_0>0$ so that $\sigma(x)\geq\sigma_0$ for all $x\in\mathbb{R}$. Then the conditional excursion law $Q^{W,x}_0(dw)$ exists by Theorem 5.3. Let $\mu\in M(\mathbb{R})$ and suppose on a standard probability space we are given the two random elements W and N, where $W(ds,dy)$ is a time-space white noise and, conditioned upon W, $N(dx,dw)$ is a Poisson random measure on $\mathbb{R}\times\mathbf{W}^0$ with intensity $\mu(dx)Q^{W,x}_0(dw)$. Let

$$X_t:=\int_{\mathbb{R}}\int_{\mathbf{W}^0}w_tN(dx,dw),\qquad t>0. \tag{6.1}$$

By Theorem 5.3 we have a.s. $L^W_{r,t}(x,M(\mathbb{R})^\circ)<\infty$, so the right hand side of (6.1) contains only a finite number of non-trivial terms. Therefore, $\{X_t:t>0\}$ is a.s. continuous.

Theorem 6.1 *Let $X_0=\mu$ and let $\{X_t:t>0\}$ be defined by (6.1). Then for any $t_n>\cdots>t_1>t_0=0$ and $\{\phi_0,\phi_1,\cdots,\phi_n\}\subset C_b(\mathbb{R})^+$ we have*

$$\begin{aligned}&\mathbf{E}^W\exp\Big\{-\sum_{i=0}^n\langle\phi_i,X_{t_i}\rangle\Big\}\\&\quad=\exp\Big\{-\langle\phi_0+\psi^W_{0,t_1}(\phi_1+\cdots+\psi^W_{t_{n-1},t_n}(\phi_n)\cdots),\mu\rangle\Big\}.\end{aligned} \tag{6.2}$$

Consequently, $\{X_t : t \geq 0\}$ is a realization of the SDSM defined by (1.2).

Proof. By the formula for the Laplace transform of the Poisson random measure we have

$$\begin{aligned}
&\mathbf{E}^W \exp\Big\{ - \sum_{i=0}^n \langle \phi_i, X_{t_i}\rangle \Big\} \\
&\quad = \mathbf{E}^W \exp\Big\{ - \langle \phi_0, \mu\rangle - \int_{\mathbb{R}} \int_{\mathbf{W}^0} \sum_{i=1}^n \langle \phi_i, w_{t_i}\rangle N(dx, dw) \Big\} \\
&\quad = \exp\Big\{ - \langle \phi_0, \mu\rangle - \int_{\mathbb{R}} \mu(dx) \int_{\mathbf{W}^0} \Big[1 - \exp\Big(\sum_{i=1}^n \langle \phi_i, w_{t_i}\rangle \Big)\Big] Q_0^{W,x}(dw) \Big\}.
\end{aligned}$$

From (5.7) it follows that

$$\begin{aligned}
&\int_{\mathbf{W}^0} \Big[1 - \exp\Big(- \sum_{i=1}^n \langle \phi_i, w_{t_i}\rangle \Big)\Big] Q_0^{W,x}(dw) \\
&= \int_{M(\mathbb{R})} L_{0,t_1}^W(x, d\nu_1) \int_{M(\mathbb{R})} Q_{t_1,t_2}^W(\nu_1, d\nu_2) \cdots \int_{M(\mathbb{R})} Q_{t_{n-2},t_{n-1}}^W(\nu_{n-2}, d\nu_{n-1}) \\
&\quad \int_{M(\mathbb{R})} \Big[1 - \exp\Big(- \sum_{i=1}^n \langle \phi_i, \nu_i\rangle \Big)\Big] Q_{t_{n-1},t_n}^W(\nu_{n-1}, d\nu_n) \\
&= \int_{M(\mathbb{R})} L_{0,t_1}^W(x, d\nu_1) \int_{M(\mathbb{R})} Q_{t_1,t_2}^W(\nu_1, d\nu_2) \cdots \int_{M(\mathbb{R})} Q_{t_{n-3},t_{n-2}}^W(\nu_{n-3}, d\nu_{n-2}) \\
&\quad \int_{M(\mathbb{R})} \Big[1 - \exp\Big(- \sum_{i=1}^{n-1} \langle \phi_i, \nu_i\rangle - \langle \psi_{t_{n-1},t_n}^W(\phi_n), \nu_{n-1}\rangle \Big)\Big] Q_{t_{n-2},t_{n-1}}^W(\nu_{n-2}, d\nu_{n-1}) \\
&= \psi_{0,t_1}^W(x, \phi_1 + \psi_{t_1,t_2}^W(\phi_2 + \cdots + \psi_{t_{n-2},t_{n-1}}^W(\phi_{n-1} + \psi_{t_{n-1},t_n}^W(\phi_n)) \cdots)).
\end{aligned}$$

Then we have (6.2), from which it is easy to show that $X_t \to \mu$ in probability as $t \to 0$. From (6.2) we see that $\{X_t : t \geq 0\}$ has the same conditional finite-dimensional distributions given $\{W(ds, dy)\}$ as the SDSM; see Theorem 4.1. Then the non-conditional finite-dimensional distributions of $\{X_t : t \geq 0\}$ also coincide with those of the SDSM. □

An immigration superprocess can be constructed in a similar way. Let $m \in M(\mathbb{R})$. Suppose on a standard probability space we have two random elements W and N, where $W(ds, dy)$ is a time-space white noise and, conditioned upon W, $N(ds, dx, dw)$ be a Poisson random measure on $[0, \infty) \times \mathbb{R} \times \mathbf{W}$ with intensity $dsm(dx)Q_s^{W,x}(dw)$. Then we can construct an $M(\mathbb{R})$-valued process

$$Y_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbf{W}} w_{t-s} N(ds, dx, dw), \qquad t \geq 0. \tag{6.3}$$

Theorem 6.2 *For any $t \geq r \geq 0$ and $\phi \in C_b(\mathbb{R})^+$ we have a.s.*

$$\mathbf{E}^W \Big[\exp\{ - \langle \phi, Y_t\rangle\} \Big| \mathscr{F}_r \Big] = \exp\Big\{ - \langle \psi_{r,t}^W, Y_r\rangle - \int_r^t \langle \psi_{s,t}^W, m\rangle ds \Big\}.$$

Consequently, $\{Y_t : t \geq 0\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$ given by

$$\int_{M(\mathbb{R})} e^{-\langle \phi, \nu\rangle} Q_t(\mu, d\nu) = \mathbf{E} \exp\Big\{ - \langle \psi_{0,t}^W, \mu\rangle - \int_0^t \langle \psi_{s,t}^W, m\rangle ds \Big\}.$$

This can be proved in a similar way as Theorem 6.1. Indeed, $\{Y_t : t \geq 0\}$ is an immigration superprocess associated with the SDSM; see [12, Theorem 5.1].

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