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Catalytic Discrete State Branching Models and Related Limit Theorems

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Abstract. Catalytic discrete state branching processes with immigration are defined as strong solutions of stochastic integral equations. We provide main limit theorems of those processes using different scalings. The class of limit processes of the theorems includes essentially all continuous state catalytic branching processes and spectrally positive regular affine processes.

Key words. Catalytic branching process, affine process, immigration, white noise, Poisson random measure, limit theorem.

AMS 2000 subject classifications. Primary 60J35; secondary 60J80, 60H20, 60K37.

1 Introduction

Catalytic branching processes were introduced by Dawson and Fleischmann [7] in the measure-valued setting and have been studied by many authors; see, e.g., [8, 9] for the surveys on the topic. One motivation of the study came from the modeling of biochemical reactions such as glycolysis. In the reaction involving two types of particles called 'catalyst' and 'reactant' respectively, the catalyst particles propagate autonomously, but they catalyze the reactant particles. Let σ_1 and σ_2 be two real constants and let $B_1(\cdot)$ and $B_2(\cdot)$ be two one-dimensional Brownian motions. Following the idea of [7], we define a class of catalytic continuous state branching processes (catalytic CB-processes) by the stochastic differential equations

$$dx(t) = \sigma_1\sqrt{x(t)}dB_1(t) \quad \text{and} \quad dy(t) = \sigma_2\sqrt{x(t)y(t)}dB_2(t), \tag{1.1}$$

where $x(\cdot)$, the catalyst, is a CB-process, and $y(\cdot)$, the reactant, is a CB-process with random branching rate proportional to the catalyst. In contrast to the conventional set-up of catalytic branching models, here the underlying Brownian noises that drive the corresponding branching mechanism may be or may not be independent. As a useful and realistic modification, which allows for immigration into the catalyst and reactant as well as for an additional branching mechanism for the reactant that is independent of the catalyst, the catalytic CB-processes with immigration (catalytic CBI-processes) were introduced and studied in Dawson and Li [11]. In terms of fluctuation limit theorems, they established a connection between the catalytic CBI-processes and affine Markov processes which have been used widely in mathematical finance as natural models of asset prices, interest rates and so on; see, e.g., Duffie *et al.* [12] and the references therein.

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This paper originated from the curiosity of finding the proper formulation of the discrete state counterpart of the catalytic CBI-processes and their connections with the affine processes. The reason of the curiosity is that discrete state processes arise more naturally in describing branching phenomena (see [3]) and the understanding of those processes is the final objective of the theoretical work. We shall introduce a class of processes which we call catalytic discrete state branching processes with immigration (catalytic DBI-processes). This model is justified by the fact that their fluctuation limit theorems lead to the same class of spectrally positive affine processes as in [11], which is a subclass of the processes studied in [12]. However, their high density limits give catalytic CBI-processes different from the models of [11]. To characterize the new catalytic CBI-processes we introduce stochastic equations driven by time-space white noises instead of finite-dimensional Brownian motions. In view of the connection with the discrete models, the new catalytic CBI-processes seem more natural than those of [11]. In this sense, our results provide new perspectives into the branching and the affine structures and the connections between them. On one hand the results give an interpretation of the catalytic CBI-processes in terms of the more realistic catalytic DBI-processes, and on the other hand the limit theorems imply that the catalytic DBI-processes can be approximated by the catalytic CBI-processes or the affine processes via suitable transformations. Those affine process approximations show that catalytic branching mechanisms may have potential applications in finance. We refer the reader to [11, 12] for the characterizations of the catalytic CBI-processes and the affine processes, and to [17, 22] for the limit results of DB- and DBI-processes. We also mention that the fluctuation limit theorems proved here can be regarded as a succession of several results connecting branching processes and Lévy processes; see, e.g., [4, 19, 20].

In the sequel of this introduction we give some descriptions of the general pictures of the processes mentioned above and the connections between them. To avoid involving too many technical details, we first consider the models without immigration. Let $l_1 \geq 0$ be a constant and let $\{p_i : i = 0, 1, 2, \cdots\}$ be a discrete distribution on $\mathbb{N} := \{0, 1, 2, \cdots\}$. By a *DB-process*, we mean an $\mathbb{N}$-valued Markov chain with Q-matrix (q_{ij}) defined by

$$q_{ij} = \begin{cases} l_1 i p_{j-i+1} & \text{if } j \geq i-1 \text{ and } j \neq i, \\ l_1 i(p_1 - 1) & \text{if } j = i, \\ 0 & \text{others.} \end{cases} \tag{1.2}$$

A DB-process models the size of a population of particles in an isolated island that propagate according to stochastic laws; see, e.g., Athreya and Ney [3]. Let μ_1 be the probability measure on $\mathbb{N}$ defined by $\mu_1(\{i\}) = p_i$ and suppose that $N_1(ds, dz, du)$ is a Poisson random measure on $(0, \infty) \times \mathbb{N} \times (0, \infty)$ with intensity $ds\mu_1(dz)du$. Given the initial value $\xi(0) \in \mathbb{N}$, we consider the stochastic integral equation

$$\xi(t) = \xi(0) + \int_0^t \int_{\mathbb{N}} \int_0^{l_1\xi(s-)} (z-1) N_1(ds, dz, du), \qquad t \geq 0. \tag{1.3}$$

Here and in the sequel we make the convention that $\int_r^t = -\int_t^r = \int_{(r,t]}$ for $r \leq t$. Under a suitable moment condition on μ_1, it is easy to show that (1.3) has a unique strong solution and the solution is a strong Markov process with state space $\mathbb{N}$; see, e.g., [11, Theorems 5.1 and 5.2]. Since $\{\xi(t) : t \geq 0\}$ is a step process, for any bounded function f on $\mathbb{N}$ we have

$$\begin{aligned} f(\xi(t)) &= f(\xi(0)) + \int_0^t \int_{\mathbb{N}} \int_0^{l_1\xi(s-)} [f(\xi(s-) + z - 1) - f(\xi(s-))] \tilde{N}_1(ds, dz, du) \\ &\quad + \int_0^t ds \int_{\mathbb{N}} l_1\xi(s)[f(\xi(s) + z - 1) - f(\xi(s))]\mu_1(dz) \\ &= f(\xi(0)) + \text{mart.} + \int_0^t \Big[\sum_{j=0}^\infty l_1\xi(s)p_j f(\xi(s) + j - 1) - l_1\xi(s) f(\xi(s))\Big] ds, \end{aligned}$$

where $\tilde{N}_1(ds,dz,du) = N_1(ds,dz,du) - ds\mu_1(dz)du$. Then $\xi(\cdot)$ is a realization of the DB-process with Q-matrix (q_{ij}) given by (1.2). The stochastic integral on the r.h.s. of (1.3) means that the propagation of each particle occurs at rate l_1 and the propagations of all the particles are determined by the Poisson random measure $N_1(ds,dz,du)$.

Let $l_2 \geq 0$ and let μ_2 be another probability measure on $\mathbb{N}$. Suppose that $N_2(ds,dz,du)$ is a Poisson random measure on $(0,\infty)\times\mathbb{N}\times(0,\infty)$ with intensity $ds\mu_2(dz)du$. Given $\eta(0)\in\mathbb{N}$, we may define another $\mathbb{N}$-valued process $\eta(\cdot)$ by

$$\eta(t) = \eta(0) + \int_0^t \int_{\mathbb{N}} \int_0^{l_2\xi(s-)\eta(s-)} (z-1) N_2(ds,dz,du), \qquad t \geq 0. \tag{1.4}$$

Under a moment condition on μ_2, this equation has a unique solution. The process $\eta(\cdot)$ can be interpreted similarly as $\xi(\cdot)$, except that the number of its jumps are dominated by the latter. Following [7], we call $(\xi(\cdot),\eta(\cdot))$ a catalytic discrete state branching process (catalytic DB-process), where $\xi(\cdot)$ is the catalyst process and $\eta(\cdot)$ is the reactant process. In this formulation, the Poisson random measures N_1 and N_2 may be or may not be independent. If they are really independent, $\xi(\cdot)$ is independent of N_2 and we may regard $\eta(\cdot)$ as a DB-process in a random environment determined by $\xi(\cdot)$.

In applications, it is natural to consider the general situation where N_1 and N_2 are not necessarily independent. A convenient reformulation is to assume they are the projections of a Poisson random measure on $(0,\infty)\times\mathbb{N}^2\times(0,\infty)$. Let μ be a probability measure on $\mathbb{N}^2$ and let $N(ds,dz,du)$ be a Poisson random measure on $(0,\infty)\times\mathbb{N}^2\times(0,\infty)$ with intensity $ds\mu(dz)du$. By a catalytic DB-process, we mean the solution of the system of stochastic integral equations

$$\xi(t) = \xi(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{l_1\xi(s-)} (z_1-1) N(ds,dz,du) \tag{1.5}$$

and

$$\eta(t) = \eta(0) + \int_0^t \int_{\mathbb{N}^2} \int_0^{l_2\xi(s-)\eta(s-)} (z_2-1) N(ds,dz,du), \tag{1.6}$$

where $z=(z_1,z_2)\in\mathbb{N}^2$ and $(\xi(0),\eta(0))$ is an $\mathbb{N}^2$-valued random variable independent of N. Since (1.5) and (1.6) contain a common Poisson noise, the catalyst and the reactant may involve dependent branching mechanism. It is not hard to show that the solution $(\xi(\cdot),\eta(\cdot))$ is a strong Markov process with generator A defined by

$$\begin{aligned} Af(i,j) \;=\; & (l_1 i \wedge l_2 ij) \int_{\mathbb{N}^2} [f(i+z_1-1, j+z_2-1) - f(i,j)]\mu(dz) \\ & + [l_1 i - (l_1 i \wedge l_2 ij)] \int_{\mathbb{N}^2} [f(i+z_1-1, j) - f(i,j)]\mu(dz) \\ & + [l_2 ij - (l_1 i \wedge l_2 ij)] \int_{\mathbb{N}^2} [f(i, j+z_2-1) - f(i,j)]\mu(dz), \end{aligned} \tag{1.7}$$

where the first term on the r.h.s. represents the common propagation of the system, and the second and third terms reflect the independent propagations of the catalyst and reactant, respectively. The form of the generator is not as simple as one might have expected from equations (1.5) and (1.6). Compared with characterizations using the generator, the stochastic equations give a simple formulation of the catalytic DB-process with clear intuitive meanings.

Unfortunately, the high density limits of the catalytic DB-processes defined by (1.5) and (1.6) can not always be represented by (1.1) even in the critical branching diffusion case (see Remark 2.1 for details). We meet here some difficulties brought about by the first term (covariation term) on the r.h.s. of (1.7), which comes from the dependence of the branching mechanism. To find an appropriate representation for the limit process, we first define a Feller branching diffusion $x(\cdot)$ by

$$x(t) = x(0) + \int_0^t \int_0^{x(s)} \sigma_1 W(ds, du), \tag{1.8}$$

where $W(ds, du)$ is a time-space white noise with intensity $dsdu$. Then we consider another process $y(\cdot)$ defined by

$$y(t) = y(0) + \int_0^t \int_0^{x(s)y(s)} \sigma_2 W(ds, du), \tag{1.9}$$

and $y(\cdot)$ is a 'branching diffusion' with random branching rate proportional to $x(\cdot)$. It turns out that the high density limit of the catalytic DB-processes defined by (1.5) and (1.6) are typically solutions of equations of the forms (1.8) and (1.9). See Theorems 2.1 and 2.2 for general results on the high density limits of general catalytic DBI-processes.

Our next objective is to investigate the high density fluctuation limits of catalytic DBI-processes. To do so, we choose a sequence of catalytic DBI-processes denoted by $(\xi_n(\cdot), \eta_n(\cdot))$ with a slight abuse of notation and consider the rescaled sequence

$$X_n(\cdot) := \frac{\xi_n(\cdot)}{n} \quad \text{and} \quad Y_n(\cdot) := \frac{\eta_n(\cdot) - n^2}{n}. \tag{1.10}$$

It is worthwhile to notice that although the scaling for $\xi_n(\cdot)$ in (1.10) is standard, that for $\eta_n(\cdot)$ is of a higher order than the standard one. We shall see that the class of limit processes of sequences of form (1.10) coincides with the regular affine process $(X(\cdot), Y(\cdot))$ with non-negative jumps (see Theorem 2.3). In the diffusion case, the process is described by the following stochastic equations

$$\begin{aligned} X(t) &= X(0) + \int_0^t (b_1 + \beta_{11} X(s))\, ds + \int_0^t \sigma_{11} \sqrt{X(s)}\, dB_1(s) \\ &\quad + \int_0^t \sigma_{12} \sqrt{X(s)}\, dB_2(s), \end{aligned} \tag{1.11}$$

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t (b_2 + \beta_{21} X(s) + \beta_{22} Y(s))\, ds + \int_0^t \sqrt{2a}\, dB_0(s) \\ &\quad + \int_0^t \sigma_{21} \sqrt{X(s)}\, dB_1(s) + \int_0^t \sigma_{22} \sqrt{X(s)}\, dB_2(s), \end{aligned} \tag{1.12}$$

for real constants $a, b_1 \geq 0$ and $b_2, \beta_{11}, \beta_{21}, \beta_{22}, (\sigma_{ij})$, where $B(\cdot) = \big(B_0(\cdot), B_1(\cdot), B_2(\cdot)\big)$ is a three-dimensional Brownian motion. The above process arises in financial applications as a general two-factor affine interest rate model, where $Y(\cdot)$ is the interest rate with its stochastic volatility factor $X(\cdot)$, and it is computationally tractable and flexible in capturing many of empirical features of the actual interest rate dynamics (see [12] and the references therein). However, there is an obvious defect in this model as the last three terms on the r.h.s. of (1.12) may bring the interest rate $Y(t)$ to the negative half line, which is not desirable. The fluctuation limit theorem shows that $(X(\cdot), Y(\cdot))$ is connected with the sequence of non-negative processes $(\xi_n(\cdot), \eta_n(\cdot))$ via (1.10). In other words, the above affine interest rate model behaves approximately as some rescaled catalytic CBI- or DBI-process representing a two-type population system. This implies that catalytic branching mechanisms may have applications in developing a non-negative interest rate model which is outside but closely connected with the affine class. See Example 2.1 for a detailed discussion.

The remainder of this paper is organized as follows. In section 2, we first give a very brief introduction to the regular affine processes, and then the precise formulation of the catalytic DBI-processes. At the end of this section the main limit theorems are presented. The subsequent three sections are devoted to the proofs of the main results.

2 Definitions and main results

We first introduce some notations and definitions. For $z \in \mathbb{R}$ set $l_1(z) = |z|$, $l_{12}(z) = |z| \wedge |z|^2$ and $\chi(z) = (1 \wedge z) \vee (-1)$. For $z = (z_1, z_2) \in \mathbb{R}^2$ define $\chi(z) = (\chi(z_1), \chi(z_2))$. Let $C(D)$ be the Banach space of bounded and continuous functions on a domain $D \subset \mathbb{R}^2$ endowed with the supremum norm $\|\cdot\|$. For $f \in C(D)$ let $\Delta_{(z_1,z_2)} f(x,y) = f(x+z_1, y+z_2) - f(x,y)$ if the r.h.s. is defined. Let $C^k(D)$ be the space of k times differentiable functions with partial derivatives up to order k belonging to $C(D)$. For $f \in C^1(D)$ let $\nabla f(x,y) = (f_1'(x,y), f_2'(x,y))$.

Let us consider the regular affine process in two-dimensional case. Suppose that $\mathbb{R}_+ = [0,\infty)$ and let $D = \mathbb{R}_+ \times \mathbb{R}$. The 'affine property' refers to the fact that the logarithm of the characteristic function of the transition distribution $p_t((x,y),\cdot)$ of the process is given by an affine transformation of the initial state $(x,y) \in D$ (see [12]). A set of parameters $(a, (\alpha_{ij}), (b_1, b_2), (\beta_{ij}), m, \mu)$ is said to be *admissible for a regular affine process* if

(a.1) $a \in \mathbb{R}_+$ is a constant;

(a.2) (α_{ij}) is a symmetric non-negative definite (2×2)- matrix;

(a.3) $(b_1, b_2) \in D$ is a vector;

(a.4) (β_{ij}) is a (2×2)-matrix with $\beta_{12} = 0$;

(a.5) $m(dz)$ is a σ-finite measure on D supported by $D \backslash \{0\}$ such that

$$\int_D [l_1(z_1) + l_{12}(z_2)] m(dz) < \infty;$$

(a.6) $\mu(dz)$ is a σ-finite measure on D supported by $D \backslash \{0\}$ such that

$$\int_D [l_{12}(z_1) + l_{12}(z_2)] \mu(dz) < \infty.$$

The regular affine process is a Feller process taking values in D characterized by the above set of admissible parameters in terms of the generator A given by

$$\begin{aligned} Af(x,y) &= \alpha_{11} x f_{11}''(x,y) + 2\alpha_{12} x f_{12}''(x,y) + \alpha_{22} x f_{22}''(x,y) + a f_{22}''(x,y) \\ &\quad + (b_1 + \beta_{11} x) f_1'(x,y) + (b_2 + \beta_{21} x + \beta_{22} y) f_2'(x,y) \\ &\quad + \int_D (\Delta_{(z_1,z_2)} f(x,y) - f_2'(x,y) z_2) m(dz) \\ &\quad + \int_D (\Delta_{(z_1,z_2)} f(x,y) - \langle \nabla f(x,y), z \rangle) x \mu(dz), \end{aligned} \tag{2.1}$$

where $f \in C^2(D)$. We say the corresponding regular affine process is *spectrally positive* if both μ and m are supported by $\mathbb{R}_+^2 \backslash \{0\}$. A regular affine process can also be described as the strong solution of a system of stochastic equations with non-Lipschitz coefficients and Poisson-type integrals over some random sets (see [11]).

Let us give a formal definition of the catalytic DBI-process mentioned in the introduction. A set of parameters $(\theta, (l_1, l_2), r; m_0, \mu_0, \nu_0)$ is said to be *admissible for a catalytic DBI-process* if

(b.1) θ, l_1, l_2 and r are non-negative constants;

(b.2) $m_0(dz)$, $\mu_0(dz)$ and $\nu_0(dz_2)$ are probability measures on $\mathbb{N}^2$, $\mathbb{N}^2$ and $\mathbb{N}$ respectively such that

$$\int_{\mathbb{N}^2}(z_1+z_2)m_0(dz)+\int_{\mathbb{N}^2}(z_1+z_2)\mu_0(dz)+\int_{\mathbb{N}}z_2\nu_0(dz_2)<\infty,$$

where $z=(z_1,z_2)$.

Let $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbf{P})$ be a filtered probability space satisfying the usual hypotheses. Suppose that on this probability space the following objects are defined:

(c.1) a Poisson random measure $N_0(ds,dz)$ on $(0,\infty)\times\mathbb{N}^2$ with intensity $\theta ds m_0(dz)$;

(c.2) a Poisson random measure $N_1(ds,dz,du)$ on $(0,\infty)\times\mathbb{N}^2\times(0,\infty)$ with intensity $ds\mu_0(dz)du$;

(c.3) a Poisson random measure $N_2(ds,dz_2,du)$ on $(0,\infty)\times\mathbb{N}\times(0,\infty)$ with intensity $ds\nu_0(dz_2)du$.

Suppose that N_0, N_1 and N_2 are independent of each other. Given any $\mathbb{N}^2$-valued $\mathcal{F}_0$-measurable random variable $(\xi(0),\eta(0))$ independent of N_0, N_1 and N_2, we consider the following system of stochastic integral equations:

$$\xi(t)=\xi(0)+\int_0^t\int_{\mathbb{N}^2}z_1N_0(ds,dz)+\int_0^t\int_{\mathbb{N}^2}\int_0^{l_1\xi(s-)}(z_1-1)N_1(ds,dz,du) \tag{2.2}$$

and

$$\begin{aligned}\eta(t)\;=\;&\eta(0)+\int_0^t\int_{\mathbb{N}^2}z_2N_0(ds,dz)+\int_0^t\int_{\mathbb{N}^2}\int_0^{l_2\xi(s-)\eta(s-)}(z_2-1)N_1(ds,dz,du)\\&+\int_0^t\int_{\mathbb{N}}\int_0^{r\eta(s-)}(z_2-1)N_2(ds,dz_2,du).\end{aligned} \tag{2.3}$$

It can be proved as in [11] that the equation system (2.2)-(2.3) has a unique solution and we call the solution $(\xi(\cdot),\eta(\cdot))$ a *catalytic DBI-process with parameters* $(\theta,(l_1,l_2),r;m_0,\mu_0,\nu_0)$. For the convenience of statement, we also refer $(\theta,(l_1,l_2),r;h_0,g_0,f_0)$ as the parameters, where h_0, g_0 and f_0 are the generating functions corresponding to m_0, μ_0 and ν_0, respectively. As described intuitively in the introduction, ξ-particles branch at rate l_1 with the offspring generating function $g_0(\cdot,1)$, while some of η-particles, controlled by ξ, branch at rate $l_2\xi(t)$ at time t with the offspring generating function $g_0(1,\cdot)$, and others, not controlled by ξ, branch independently at rate r with the offspring generating function $f_0(\cdot)$. Meanwhile, particles from an outside source immigrate into the system at rate θ with the immigration-size generating function $h_0(\cdot,\cdot)$.

Now let us consider a sequence of catalytic DBI-processes $(\xi_n(\cdot),\eta_n(\cdot))$ with parameters $(\theta_n,(\gamma_n,$ $\gamma_n/n)$, r_n; m_n, μ_n, $\nu_n)$ or with equivalent parameters $(\theta_n,(\gamma_n,\,\gamma_n/n),\,r_n;\,h_n,\,g_n,\,f_n)$. Clearly, a realization of $(\xi_n(\cdot),\,\eta_n(\cdot))$ can be given by (2.2)-(2.3) with the parameters depending on the index n in suitable ways. For $0\le\lambda_1,\lambda_2\le n$, set

$$R_n(\lambda_1,\lambda_2)=n\gamma_n\Big[g_n\Big(1-\frac{\lambda_1}{n},1-\frac{\lambda_2}{n}\Big)-\Big(1-\frac{\lambda_1}{n}\Big)\Big(1-\frac{\lambda_2}{n}\Big)\Big] \tag{2.4}$$

and

$$H_n(\lambda_1,\lambda_2)=\theta_n\Big[h_n\Big(1-\frac{\lambda_1}{n},1-\frac{\lambda_2}{n}\Big)-1\Big]. \tag{2.5}$$

Let $\beta_n=f_n'(1-)-1$ and let $\sigma_n=f_n''(1-)<\infty$. Consider the following conditions:

(A) The sequence $\{R_n\}$ is uniformly Lipschitz in (λ_1, λ_2) on each bounded rectangle, and converges to a continuous function as $n \to \infty$;

(B) The sequence $\{H_n\}$ is uniformly Lipschitz in (λ_1, λ_2) on each bounded rectangle, and converges to a continuous function as $n \to \infty$;

(C) $\lim_{n\to\infty} r_n\beta_n = \beta_{22}, \quad \lim_{n\to\infty} \frac{r_n\sigma_n}{n} = \sigma_0 \quad$ and $\quad \lim_{b\to\infty} \sup_n \frac{r_n}{n} \int_{\{z_2>b\}} z_2^2\nu_n(dz_2) = 0.$

Proposition 2.1 *Under condition (A), the limit function R of $\{R_n\}$ has representation*

$$\begin{aligned} R(\lambda_1, \lambda_2) \;&=\; -\beta_{11}\lambda_1 - \beta_{21}\lambda_2 + \alpha_{11}\lambda_1^2 + 2\alpha_{12}\lambda_1\lambda_2 + \alpha_{22}\lambda_2^2 \\ &\quad + \int_{\mathbb{R}_+^2} \Big(e^{-\langle\lambda,z\rangle} - 1 + \langle\lambda, z\rangle\Big)\mu(dz), \end{aligned} \tag{2.6}$$

where $(\beta_{11}, \beta_{21}) \in \mathbb{R}^2$, (α_{ij}) is a symmetric non-negative definite (2×2)-matrix and $\mu(dz)$ is a σ-finite measure on $\mathbb{R}_+^2$ supported by $\mathbb{R}_+^2\backslash\{0\}$ such that

$$\int_{\mathbb{R}_+^2} \big[l_{12}(z_1) + l_{12}(z_2)\big]\mu(dz) < \infty. \tag{2.7}$$

Proposition 2.2 *Under condition (B), the limit function H of $\{H_n\}$ has representation*

$$H(\lambda_1, \lambda_2) = -b_1\lambda_1 - b_2\lambda_2 + \int_{\mathbb{R}_+^2} \big(e^{-\langle\lambda,z\rangle} - 1\big)\upsilon(dz), \tag{2.8}$$

where $b_1 \geq 0$, $b_2 \geq 0$ and $\upsilon(dz)$ is a σ-finite measure on $\mathbb{R}_+^2$ supported by $\mathbb{R}_+^2\backslash\{0\}$ such that

$$\int_{\mathbb{R}_+^2} \big[l_1(z_1) + l_1(z_2)\big]\upsilon(dz) < \infty. \tag{2.9}$$

We actually obtain a set of parameters $(\sigma_0, (\alpha_{ij}), (b_1, b_2), (\beta_{11}, \beta_{21}, \beta_{22}), \upsilon, \mu)$ which will play an important role in the characterization of our limit processes. Let $(x_n(\cdot), y_n(\cdot))$ be defined by

$$x_n(t) := \frac{\xi_n(t)}{n} \quad \text{and} \quad y_n(t) := \frac{\eta_n(t)}{n}. \tag{2.10}$$

The next result gives the rescaling limit of the above catalytic DBI-processes.

Theorem 2.1 *Suppose that (A), (B) and (C) are satisfied. If $(x_n(0), y_n(0))$ converges in distribution to $(x(0), y(0))$, then $(x_n(\cdot), y_n(\cdot))$ converges in distribution on $D([0,\infty), \mathbb{R}_+^2)$ to a process $(x(\cdot), y(\cdot))$, which can be constructed as the unique pair of solutions of the following stochastic equation system*

$$\begin{aligned} x(t) \;&=\; x(0) + \int_0^t (b_1 + \beta_{11}x(s))ds + \int_0^t \int_0^{x(s-)} \sigma_{11}W_1(ds, du) \\ &\quad + \int_0^t \int_0^{x(s-)} \sigma_{12}W_2(ds, du) + \int_0^t \int_{\mathbb{R}_+^2} z_1 N_0(ds, dz) \\ &\quad + \int_0^t \int_{\mathbb{R}_+^2} \int_0^{x(s-)} z_1 \tilde{N}_1(ds, dz, du) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} y(t) &= y(0) + \int_0^t (b_2 + \beta_{21} x(s) y(s) + \beta_{22} y(s)) ds + \int_0^t \int_0^{y(s-)} \sigma_0 W_0(ds, du) \\ &\quad + \int_0^t \int_0^{x(s-)y(s-)} \sigma_{21} W_1(ds, du) + \int_0^t \int_0^{x(s-)y(s-)} \sigma_{22} W_2(ds, du) \\ &\quad + \int_0^t \int_{\mathbb{R}_+^2} z_2 N_0(ds, dz) + \int_0^t \int_{\mathbb{R}_+^2} \int_0^{x(s-)y(s-)} z_2 \tilde{N}_1(ds, dz, du), \end{aligned} \tag{2.12}$$

where (σ_{ij}) is a (2×2)-matrix satisfying $(2\alpha_{ij}) = (\sigma_{ij})(\sigma_{ij})^\tau$, $(W_i)_{i=0}^2$ are three orthogonal white noises on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dsdu$, $N_0(ds, dz)$ is a Poisson random measure on $(0, \infty) \times \mathbb{R}_+^2$ with intensity $ds\upsilon(dz)$, $N_1(ds, dz, du)$ is a Poisson random measure on $(0, \infty) \times \mathbb{R}_+^2 \times (0, \infty)$ with intensity $ds\mu(dz)du$ and $\tilde{N}_1(ds, dz, du) = N_1(ds, dz, du) - ds\mu(dz)du$. N_0 and N_1 are independent of each other.

Remark 2.1 (i) Conditions (A) and (B) originated from the sufficient conditions for the convergence of DB-process; see, e.g., Li [21] for the discussions in the setting of measure-valued processes. Those conditions describe a sequence of catalytic DBI-processes, where the offspring mean of the catalyst (reactant) tends to its critical value one. Condition (C) is for the technical purpose and we sometimes let $r_n = n$ and let f_n be the critical binary generating function.

(ii) In view of the limit theorem, $(x(\cdot), y(\cdot))$ can be naturally regarded as a catalytic CBI-process although it is slightly different from the one given by [11] (see also (1.1) for a special case). It is worthwhile to notice that the corresponding branching mechanisms of $x(\cdot)$ and $y(\cdot)$ are driven by the common time-space white noises W_1 and W_2 instead of finite-dimensional Brownian motions. By Itô's formula, $(x(\cdot), y(\cdot))$ has weak generator L by

$$\begin{aligned} Lf(x, y) &= \alpha_{11} x f_{11}''(x, y) + 2\alpha_{12}(x \wedge xy) f_{12}''(x, y) + (\alpha_{22} xy + \hat{a} y) f_{22}''(x, y) \\ &\quad + (b_1 + \beta_{11} x) f_1'(x, y) + (b_2 + \beta_{21} xy + \beta_{22} y) f_2'(x, y) + \int_{\mathbb{R}_+^2} \Delta_{(z_1, z_2)} f(x, y) \upsilon(dz) \\ &\quad + \int_{\mathbb{R}_+^2} [\Delta_{(z_1, z_2)} f(x, y) - \langle \nabla f(x, y), z \rangle](x \wedge xy) \mu(dz) \\ &\quad + \int_{\mathbb{R}_+^2} [\Delta_{(z_1, 0)} f(x, y) - f_1'(x, y) z_1][x - (x \wedge xy)] \mu(dz) \\ &\quad + \int_{\mathbb{R}_+^2} [\Delta_{(0, z_2)} f(x, y) - f_2'(x, y) z_2][xy - (x \wedge xy)] \mu(dz), \end{aligned} \tag{2.13}$$

for $\hat{a} = \frac{1}{2}\sigma_0^2$ and $f \in C^2(\mathbb{R}_+^2)$. In view of generators, the only difference between our catalytic CBI-process and the one given by [11] lies in the second term on the r.h.s. of (2.13). Because of the truncation in this covariation term, the stochastic equation for $(x(\cdot), y(\cdot))$ driven by Brownian motions would have to involve non-smooth diffusion coefficients.

(iii) Stochastic equations driven by white noises were studied by El Karoui and Méléard [13]. This type of equations also appear, for example, in some particle systems with dependent spatial motions, where there is an interaction in the term of diffusion (see [10], [28]). For the pathwise uniqueness of solutions of (2.11)-(2.12), it can be easily proved by using a method similar to the Yamada-Watanabe one with additional estimates for jumps of Poisson type (see Lemma 4.5).

(iv) A simple but interesting property for (2.11) is as follows. Let $x_1(\cdot)$ and $x_2(\cdot)$ be two solutions of (2.11) satisfying $x_2(0) \geq x_1(0)$ a.s.. Let $\zeta(\cdot) = x_2(\cdot) - x_1(\cdot)$. By the properties of stationary independent increment of white noises and Poisson processes, it is easy to check that $\zeta(\cdot)$ is a CB-process. However, in general, this result does not hold for the first equation in (1.1).

Theorem 2.2 *Suppose that σ_0, β_{11}, β_{21}, β_{22} are real constants, $(b_1, b_2) \in \mathbb{R}_+^2$, (σ_{ij}) is a (2×2)-matrix, and $\mu(dz)$ and $\upsilon(dz)$ are σ-finite measures on $\mathbb{R}_+^2$ supported by $\mathbb{R}_+^2 \setminus \{0\}$ satisfying (2.7) and (2.9) respectively. Then the corresponding equation system (2.11)-(2.12) has a unique pair of solutions, which can be approximated by a rescaled sequence of catalytic DBI-processes.*

Now let us turn to another sequence of catalytic DBI-processes $(\xi_n(\cdot), \eta_n(\cdot))$ with parameters $(n^2\theta_n$, $(\gamma_n, \gamma_n/n^2)$, θ_n; m_n, μ_n, $\nu_n)$ or with equivalent parameters $(n^2\theta_n$, $(\gamma_n, \gamma_n/n^2)$, θ_n; h_n, g_n, $f_n)$. A realization of $(\xi_n(\cdot), \eta_n(\cdot))$ is given by

$$\xi_n(t) = \xi_n(0) + \int_0^t \int_{\mathbb{N}^2} z_1 N_{n,0}(ds, dz) + \int_0^t \int_{\mathbb{N}^2} \int_0^{\gamma_n \xi_n(s-)} (z_1 - 1) N_{n,1}(ds, dz, du), \tag{2.14}$$

$$\begin{aligned}\eta_n(t) &= \eta_n(0) + \int_0^t \int_{\mathbb{N}^2} z_2 N_{n,0}(ds, dz) + \int_0^t \int_{\mathbb{N}^2} \int_0^{\frac{\gamma_n}{n^2}\xi_n(s-)\eta_n(s-)} (z_2 - 1) N_{n,1}(ds, dz, du) \\ &\quad + \int_0^t \int_{\mathbb{N}} \int_0^{\theta_n \eta_n(s-)} (z_2 - 1) N_{n,2}(ds, dz_2, du),\end{aligned} \tag{2.15}$$

where $(\xi_n(0), \eta_n(0))$, $N_{n,0}(ds, dz)$, $N_{n,1}(ds, dz, du)$ and $N_{n,2}(ds, dz_2, du)$ are given as in (2.2)-(2.3), but depend on the index n. Suppose that $a_n = 1 - f_n'(1-) \geq 0$ and $\sigma_n = f_n''(1-) < \infty$. For $0 \leq \lambda_1, \lambda_2 \leq n$, set

$$F_n(\lambda_1, \lambda_2) = n^2\theta_n \Big[h_n\Big(1 - \frac{\lambda_1}{n}, 1 - \frac{\lambda_2}{n}\Big) - \Big(1 - \frac{a_n\lambda_2}{n}\Big)\Big]. \tag{2.16}$$

In addition to (A), which concerns γ_n and g_n, we will need the following conditions:

(D1) As $n \to \infty$, we have $\theta_n a_n \to a_0$, for some $a_0 \geq 0$;

(D2) The sequence $\{F_n\}$ is uniformly Lipschitz in (λ_1, λ_2) on each bounded rectangle, and converges to a continuous function as $n \to \infty$;

(E) $\lim_{n\to\infty} \theta_n \sigma_n = \sigma$ and $\lim_{b\to\infty} \sup_n \theta_n \int_{\{z_2 > b\}} z_2^2 \nu_n(dz_2) = 0$.

Proposition 2.3 *Under conditions (D1,2), the limit function F of $\{F_n\}$ has representation*

$$F(\lambda_1, \lambda_2) = -b_1\lambda_1 - b_2\lambda_2 + \alpha\lambda_2^2 + \int_{\mathbb{R}_+^2} \Big(e^{-\langle \lambda, z\rangle} - 1 + \lambda_2 z_2\Big) m(dz), \tag{2.17}$$

where $\alpha \geq 0$, $b_1 \geq 0$, $b_2 \in \mathbb{R}$ and $m(dz)$ is a σ-finite measure on $\mathbb{R}_+^2$ supported by $\mathbb{R}_+^2 \setminus \{0\}$ such that

$$\int_{\mathbb{R}_+^2} \big[l_1(z_1) + l_{12}(z_2)\big] m(dz) < \infty. \tag{2.18}$$

Under the above conditions, let $a = \alpha + a_0 + \sigma/2$ and $\beta_{22} = -a_0$. Then $(a, (\alpha_{ij}), (b_1, b_2), (\beta_{ij}), m, \mu)$ is clearly just a set of admissible parameters for a spectrally positive regular affine process. Let $(X_n(\cdot), Y_n(\cdot))$ be defined by

$$X_n(t) := \frac{\xi_n(t)}{n} \quad \text{and} \quad Y_n(t) := \frac{\eta_n(t) - n^2}{n}. \tag{2.19}$$

The following result gives the fluctuation limit of the above catalytic DBI-processes.

Theorem 2.3 *(i) Suppose that conditions (A), (D1,2) and (E) are satisfied. If* $(X_n(0), Y_n(0))$ *converges in distribution to* $(X(0), Y(0))$*, then* $(X_n(\cdot), Y_n(\cdot))$ *converges in distribution on* $D([0,\infty), \mathbb{R}_+ \times \mathbb{R})$ *to a spectrally positive regular affine process* $(X(\cdot), Y(\cdot))$ *with parameters* $(a,\ (\alpha_{ij}),\ (b_1, b_2),\ (\beta_{ij}),\ m,\ \mu)$ *satisfying* $\beta_{22} \le 0$ *and* $a \ge -\beta_{22}$.

(ii) Conversely, if $\big(X(\cdot), Y(\cdot)\big)$ *is a spectrally positive regular affine process with admissible parameters* $(a,\ (\alpha_{ij}),\ (b_1, b_2),\ (\beta_{ij}),\ m,\ \mu)$ *satisfying* $\beta_{22} \le 0$ *and* $a \ge -\beta_{22}$*, there exist positive constants* r_1*,* r_2 *and a sequence of catalytic DBI-processes* $(\xi_n(\cdot), \eta_n(\cdot))$ *such that the sequence* $\big(r_1 X_n(\cdot), r_2 Y_n(\cdot)\big)$ *converges in distribution on* $D([0,\infty), \mathbb{R}_+ \times \mathbb{R})$ *to* $\big(X(\cdot), Y(\cdot)\big)$*, where* $\big(X_n(\cdot), Y_n(\cdot)\big)$ *is defined by (2.19).*

In view of equation-system (1.11)-(1.12), the first coordinator of a two-dimensional affine process is a CBI-process and the second one appears to be an Ornstein-Uhlenbeck type process (OU-type) whose (Lévy) driving terms are dominated by the first one. The above theorem implies that any spectrally positive 'dominated OU-type process' arises as the fluctuation limits of 'dominated DBI-processes', which goes back to Li [22] in some sense.

Example 2.1 Consider a simple example of affine processes, the CBI-diffusion applied to interest rate modeling by Cox, Ingersoll and Ross [6] (CIR), that is

$$dy(t) = \kappa(\bar{v} - y(t))dt + \sigma\sqrt{y(t)}dB(t),$$

where $y(t)$ denotes the short-term interest rate, B is a one-dimensional Brownian motion, and κ, $\bar{v}$ and σ are non-negative constants. Here $y(t)$ mean-reverts towards the level $\bar{v}$, κ measures the speed of the reversion, and the term $\sigma\sqrt{y(t)}$ gives the rate volatility. This CIR model guarantees non-negativity and captures an aspect of conditional heteroskedasticity by having volatility increase with the level of the rate, the so-called 'level-effect'. However the general affine models do not constrain the short rate to be non-negative (see (1.11)-(1.12)). To overcome this defect, inspired by the connection between catalytic branching models and affine processes, we may propose another extension of the CIR model, which evolves as a catalytic CBI-diffusion:

$$dx(t) = \kappa_1(\bar{v}_1 - x(t))dt + \sigma_1\sqrt{x(t)}dB_1(t), \tag{2.20}$$

$$dy(t) = \kappa_2(\bar{v}_2 - y(t))dt + \sigma_2\sqrt{x(t)y(t)}dB_2(t), \tag{2.21}$$

where the reactant $y(t)$ represents the short rate, the catalyst $x(t)$ represents the stochastic volatility factor, B_1 and B_2 are two one-dimensional Brownian motions, and κ_i, $\bar{v}_i$, σ_i $(i = 1, 2)$ are non-negative constants. This so-called randomly controlled CIR model incorporates both level and stochastic volatility effects and thus is outside the affine class. It assures non-negative rates by having volatility be proportional to the product of the two effects. See the model of Andersen and Lund [2] for a similar idea. Now consider a sequence of these models $(x(\cdot), y_n(\cdot))$, where $x(\cdot)$ is defined by (2.20) and $y_n(\cdot)$ is defined by

$$dy_n(t) = \kappa_2(n - y_n(t))dt + \sigma_2\sqrt{x(t)y_n(t)/n}\, dB_2(t). \tag{2.22}$$

It is shown in [11] that as $n \to \infty$ the limit of the processes $\{(x(\cdot), y_n(\cdot) - n)\}$ is an affine process $(X(\cdot), Y(\cdot))$ defined as in (1.11)-(1.12). Observe that the level effect $\sqrt{y_n(t)/n}$ goes to one. This means that for the sequence $(x(\cdot), y_n(\cdot))$ which is non-affine and less tractable, we scale the level effect away just to get the process $(X(\cdot), Y(\cdot))$ which is affine and much easier to handle analytically and computationally in pricing and estimation. As a cost, the non-negativity of $Y(\cdot)$ can not be guaranteed since it is obtained approximately by shifting $y_n(\cdot)$ by a negative quantity $-n$. In this sense, the affine model is applied as a simple approximation of the randomly controlled CIR model.

Based on similar considerations, we may consider the catalytic DBI-processes as some kind of Markov-chain interest rate models. See Mamon [24] for other related models. In some cases, the affine class also gives these models a tractable approximation. For example, consider a sequence of catalytic DBI-processes:

$$\begin{aligned}\xi_n(t) &= \xi_n(0) + \int_0^t \int_{\mathbb{N}^2} z_1 N_{n,0}(ds,dz) + \int_0^t \int_{\mathbb{N}^2} \int_0^{\gamma_n \xi_n(s-)} (z_1 - 1) N_{n,1}(ds,dz,dzu), \\ \eta_n(t) &= \eta_n(0) + \int_0^t \int_{\mathbb{N}^2} z_2 N_{n,0}(ds,dz) + \int_0^t \int_{\mathbb{N}^2} \int_0^{\frac{\gamma_n}{n^2}\xi_n(s-)\eta_n(s-)} (z_2 - 1) N_{n,1}(ds,dz,du).\end{aligned}$$

We choose $\gamma_n = 2n + \beta$ with $\beta > 0$. Let

$$g_n^{(1)}(s_1,s_2) = \frac{s_1(1+s_2^2)}{2}, \quad g_n^{(2)}(s_1,s_2) = \frac{1+s_1^2 s_2^2}{2}.$$

Let $\rho \in [0,1]$ and let the offspring generating function be defined by

$$g_n(s_1,s_2) = \frac{1}{\gamma_n}\Big[2n(1-\rho^2)g_n^{(1)}(s_1,s_2) + 2n\rho^2 g_n^{(2)}(s_1,s_2) + \beta s_2\Big].$$

For $b_1, b_2 > 0$, let $\theta_n = b_1 + b_2$ and $h_n(s_1,s_2) = (1-1/n) + (b_1 s_1 + b_2 s_2)/(n\theta_n)$. We can check that

$$\begin{aligned}R(\lambda_1,\lambda_2) &= \beta\lambda_1 + \rho^2\lambda_1^2 + 2\rho^2\lambda_1\lambda_2 + \lambda_2^2, \\ F(\lambda_1,\lambda_2) &= -b_1\lambda_1 - b_2\lambda_2.\end{aligned}$$

By Theorem 2.3, the sequence $(X_n(\cdot), Y_n(\cdot))$ defined by (2.19) converges weakly to the affine process $(X(\cdot), Y(\cdot))$ given by

$$dX(t) = (b_1 - \beta X(t))\,dt + \rho\sqrt{2x(t)}\,d\bar{B}_1(t), \tag{2.23}$$

$$dY(t) = b_2\,dt + \sqrt{2X(t)}\,(\rho\,d\bar{B}_1(t) + \sqrt{1-\rho^2}\,d\bar{B}_2(t)\,), \tag{2.24}$$

where $\bar{B}(\cdot) = \big(\bar{B}_1(\cdot), \bar{B}_2(\cdot)\big)$ is a two-dimensional Brownian motion. The above affine process was applied in Heston [15] to model an asset price $e^{Y(\cdot)}$ with stochastic volatility $X(\cdot)$. In his point, the correlation between volatility and spot assets (driven by $\bar{B}_1(\cdot)$) can capture important return skewness effects in simulation. In view of our discrete state models, the above correlation is approximately from $g_n^{(2)}$ showing that $\xi_n(\cdot)$ and its volatility $\eta_n(\cdot)$ may increase or decrease one unit with the common probability 1/2. □

Finally we shall show that a spectrally positive regular affine process may also arise as the fluctuation limit of catalytic CBI-processes obtained in Theorem 2.1. Let $(a, (\alpha_{ij}), (b_1,b_2), (\beta_{ij}), m, \mu)$ be a set of admissible parameters with $\beta_{22} \le 0$ for a spectrally positive affine process. Let $\sigma_0 = \sqrt{2a}$ and let (σ_{ij}) be a (2×2)-matrix satisfying $(2\alpha_{ij}) = (\sigma_{ij})(\sigma_{ij})^\tau$. Let $D_n := \{(z_1,z_2) \in \mathbb{R}_+^2 : z_2 > 1/n\}$. If $\beta_{22} = 0$, set the sequence $\hat{\beta}_n = -\int_{D_n} z_2\,m(dz)/n^2$ and let $\theta_n = n^2$; If $\beta_{22} < 0$, set $\hat{\beta}_n = \beta_{22}$ and choose θ_n such that $-\beta_{22}\theta_n \ge \int_{D_n} z_2\,m(dz)$, $\theta_n \ge 1$ and $\theta_n \to \infty$ as $n \to \infty$. Suppose that $x(\cdot)$ is defined by (2.11). For each $n \ge 1$, let $y_n(\cdot)$ be defined by

$$\begin{aligned}y_n(t) = y_n(0) &+ \int_0^t (-\hat{\beta}_n\theta_n + \frac{\beta_{21}}{\theta_n}x(s)y_n(s) + \frac{b_2}{\theta_n}y_n(s) + \hat{\beta}_n y_n(s))ds \\ &+ \int_0^t \int_0^{\frac{y_n(s-)}{\theta_n}} \sigma_0 W_0(ds,du) + \int_0^t \int_0^{\frac{y_n(s-)}{\theta_n}x(s-)} \sigma_{21} W_1(ds,du) \\ &+ \int_0^t \int_0^{\frac{y_n(s-)}{\theta_n}x(s-)} \sigma_{22} W_2(ds,du) + \int_0^t \int_{D_n} z_2 \tilde{N}_0(ds,d\zeta) \\ &+ \int_0^t \int_{\mathbb{R}_+^2} \int_0^{\frac{y_n(s-)}{\theta_n}x(s-)} z_2 \tilde{N}_1(ds,dz,du),\end{aligned} \tag{2.25}$$

where $(W_i)_{i=0}^2$, N_0 and N_1 are given as in Theorem 2.1 and the initial value $(x(0), y_n(0))$ is independent of $(W_i)_{i=0}^2$, N_0 and N_1. Set $X'_n = x(\cdot)$ and $Y'_n(\cdot) = y_n(\cdot) - \theta_n$.

Theorem 2.4 *If $\big(X'_n(0), Y'_n(0)\big)$ converges in distribution to $(X'(0), Y'(0))$, then $\big(X'_n(\cdot), Y'_n(\cdot)\big)$ converges in distribution on $D([0,\infty), \mathbb{R}_+ \times \mathbb{R})$ to a spectrally positive regular affine process $(X'(\cdot), Y'(\cdot))$ with the above admissible parameters $(a, (\alpha_{ij}), (b_1, b_2), (\beta_{ij}), m, \mu)$ satisfying $\beta_{22} \le 0$.*

3 Representation results for limit functions

In this section, we introduce and prove some representation results for the limit functions of (2.4), (2.5) and (2.16), which play an important role in the proof of Theorem 2.1 and Theorem 2.3. Recall that μ_n is the probability measure on $\mathbb{N}^2$ corresponding to g_n. Fix $n \ge 1$. Let $G = [-1, \infty)^2$ and $G_n = \{\big((i-1)/n, (j-1)/n\big) : i, j \in \mathbb{N}\}$. Let ρ_n be the measure defined by

$$\rho_n(\cdot) = n\gamma_n \sum_{i,j=0}^{\infty} \mu_n(\{(i,j)\})\delta_{(\frac{i-1}{n}, \frac{j-1}{n})}(\cdot). \tag{3.1}$$

Then ρ_n is a finite measure on G supported by G_n.

Proof of Proposition 2.1 Set $\psi_n(\lambda_1, \lambda_2) = n\gamma_n\big[g_n(e^{-\lambda_1/n}, e^{-\lambda_2/n}) - e^{-\langle \mathbf{1/n},\, \lambda\rangle}\big]$ and it follows from mean-value theorem that

$$\begin{aligned} R_n(\lambda_1, \lambda_2) \;&=\; \psi_n(\lambda_1, \lambda_2) + n\gamma_n\big[g'_{n,1}(\eta_{n,1}, \eta_{n,2}) - \eta_{n,2}\big](1 - \lambda_1/n - e^{-\lambda_1/n}) \\ &\qquad + n\gamma_n\big[g'_{n,2}(\eta_{n,1}, \eta_{n,2}) - \eta_{n,1}\big](1 - \lambda_2/n - e^{-\lambda_2/n}), \end{aligned} \tag{3.2}$$

where $1 - \lambda_i/n \le \eta_{n,i} \le e^{-\lambda_i/n}$ and $g'_{n,i}$ denotes the partial derivative of g_n with respect to the ith variable for $i = 1, 2$. Under condition (A), the sequences

$$\begin{aligned} |R'_{n,1}(\lambda_1, \lambda_2)| \;&=\; \gamma_n\big|(1 - \lambda_2/n) - g'_{n,1}(1 - \lambda_1/n, 1 - \lambda_2/n)\big| \\ |R'_{n,2}(\lambda_1, \lambda_2)| \;&=\; \gamma_n\big|(1 - \lambda_1/n) - g'_{n,2}(1 - \lambda_1/n, 1 - \lambda_2/n)\big| \end{aligned}$$

are uniformly bounded on each bounded rectangle $[0, c]^2$ for $c \ge 0$ and thus the sequences $\gamma_n\big|\eta_{n,2} - g'_{n,1}(\eta_{n,1}, \eta_{n,2})\big|$ and $\gamma_n\big|\eta_{n,1} - g'_{n,2}(\eta_{n,1}, \eta_{n,2})\big|$ are also uniformly bounded. By (A) and (3.2), we have $\psi_n(\lambda_1, \lambda_2) \to R(\lambda_1, \lambda_2)$, as $n \to \infty$. It is enough to consider the limit representation of ψ_n and we use Venttsel's classical method (see [27], [26] or [12]) to prove it.

Step 1: *Decomposition.* Let $\|z\|^2 = z_1^2 + z_2^2$ and let $l(z) = \|z\|^2 \wedge 1$. Set $\varrho_n = \int_G l(z)\rho_n(dz)$. If $\varrho_n > 0$, define the probability measure $P_n(dz) = (l(z)/\varrho_n)\,\rho_n(dz)$ supported by $G_n \backslash \{0\}$. Let $G^\infty = G \cup \{\infty\}$ be the one point compactification of G. Choose any subsequence denoted again by $\{P_n\}$, which converges weakly to a probability measure P on G^∞ supported by $\mathbb{R}_+^2 \cup \{\infty\}$. Let E be the set of $\varepsilon > 0$ for which $P\big(\|z\| = \varepsilon\big) = 0$. For $\varepsilon \in E$, define $Q := \{z \in G : \|z\| \le \varepsilon\}$ and Q is a P-continuity set. We have

$$e^{\langle \mathbf{1/n},\, \lambda\rangle}\psi_n(\lambda_1, \lambda_2) = -\beta_{n,1}\lambda_1 - \beta_{n,2}\lambda_2 + \varrho_n\Big(\sum_{i,j=1}^{2} a_{n,\varepsilon}(i,j)\lambda_i\lambda_j + I_{n,\varepsilon} + J_{n,\varepsilon}\Big), \tag{3.3}$$

where

$$\begin{aligned} \beta_{n,i} \;&=\; \int_G \chi(z_i)\rho_n(dz), \quad a_{n,\varepsilon}(i,j) = \frac{1}{2}\int_{Q\backslash\{0\}} \chi(z_i)\chi(z_j)(l(z))^{-1}P_n(dz), \\ h(z,\lambda) \;&=\; e^{-\langle \lambda, z\rangle} - 1 + \langle \chi(z), \lambda\rangle, \quad J_{n,\varepsilon} = \int_{G\backslash Q} h(z,\lambda)(l(z))^{-1}P_n(dz), \\ I_{n,\varepsilon} \;&=\; \int_{Q\backslash\{0\}} \big(h(z,\lambda) - \frac{1}{2}\langle \chi(z), \lambda\rangle^2\big)(l(z))^{-1}P_n(dz). \end{aligned} \tag{3.4}$$

for $i = 1, 2$ (If $\varrho_n = 0$, the last term of (3.3) is zero). It is easy to see that $\lim_{E\ni\varepsilon\downarrow 0}\lim_{n\to\infty} I_{n,\varepsilon} = 0$.

Step 2: *Limiting*. Fix $\varepsilon \in E$ for a moment. Let $k(\varepsilon) = (1\wedge\varepsilon^2)^{-1}$. Define $\Delta_n = \varrho_n + |\beta_{n,1}| + |\beta_{n,2}|$ for $n \geq 1$ and consider two cases as follows. If $\liminf_{n\to\infty}\Delta_n = 0$, in this case $\psi(\lambda_1, \lambda_2) = 0$. If $\liminf_{n\to\infty}\Delta_n > 0$, there exists a subsequence, denoted again by $\{\Delta_n\}$, converging to $\Delta \in (0, \infty]$. Then the following limits exist (passing to a subsequence if necessary)

$$\begin{aligned} &\frac{\varrho_n}{\Delta_n} \to \varrho \in [0,1], \quad \frac{\beta_{n,i}}{\Delta_n} \to \beta_i \in [-1,1], \quad a_{n,\varepsilon}(i,j) \to a_\varepsilon(i,j) \in [-1,1], \\ &\int_{G\setminus Q} \frac{1}{l(z)} P_n(dz) \to c_0 \in [0, k(\varepsilon)], \quad \int_{G\setminus Q} \frac{\chi(z_i)}{l(z)} P_n(dz) \to c_i \in [-k(\varepsilon), k(\varepsilon)], \end{aligned} \tag{3.5}$$

for $i = 1,\ 2$. Dividing both side of equation (3.3) by Δ_n we get in the limit

$$\lim_{n\to\infty}\int_{G\setminus Q} e^{-\langle\lambda, z\rangle}(l(z))^{-1}P_n(dz) = L(\lambda), \tag{3.6}$$

where $L(\lambda)$ is some function of λ and is continuous at 0. It is not hard to show that

$$\int_{G\setminus Q}(l(z))^{-1}P(dz) = \lim_{n\to\infty}\int_{G\setminus Q} e^{-\langle \frac{\mathbf{1}}{\mathbf{n}},\, z\rangle}(l(z))^{-1}P(dz) = \lim_{n\to\infty} L(\frac{\mathbf{1}}{\mathbf{n}}) = L(0). \tag{3.7}$$

It follows that $P(\{\infty\}) = 0$. Thus the subsequence $\{P_n\}$ converges to the probability distribution P on G and P is supported by $\mathbb{R}_+^2$. Define $v(\{0\}) = 0$ and $v(dz) = (l(z))^{-1}P(dz)$ on $\{z \in \mathbb{R}_+^2 : \|z\| > 0\}$, we can show that

$$\lim_{E\ni\varepsilon\downarrow 0}\lim_{n\to\infty} J_{n,\varepsilon} = \int_{\mathbb{R}_+^2} h(z,\lambda)v(dz), \tag{3.8}$$

$$\lim_{E\ni\varepsilon\downarrow 0}\liminf_{n\to\infty} a_{n,\varepsilon}(i,j) = \lim_{E\ni\varepsilon\downarrow 0}\limsup_{n\to\infty} a_{n,\varepsilon}(i,j) = a(i,j) \tag{3.9}$$

for $i, j = 1, 2$ and $\big(a(i,j)\big)$ is a non-negative definite matrix. Then we obtain a Lévy-Khintchine type representation for $\psi(\lambda)/\Delta$.

Step 3: It remains to verify that $1/\Delta > 0$. This can be proved as step 4 in the proof of [12, Lemma 4.1]. Then set $\alpha_{ij} = \varrho\Delta a(i,j)$ for $i, j = 1, 2$ and $\mu(\cdot) = \varrho\Delta v(\cdot)$. By the definition of μ and Lipschitz continuity of R, we get (2.7) and then $\int_{\mathbb{R}_+^2}(z_i - \chi(z_i))\mu(dz) < \infty$ $(i = 1, 2)$. Let $\beta_{i1} = \Delta\beta_i + \int_{\mathbb{R}_+^2}(z_i - \chi(z_i))\mu(dz) < \infty$ $(i = 1, 2)$. Thus, we have (2.6). □

Let us write $f \in C_*(\mathbb{R}^2)$ if f is bounded continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying $f(z) = o(\|z\|^2)$ when $\|z\| \to 0$.

Proposition 3.1 *Under conditions of Proposition 2.1, we have*

(i) $\displaystyle\int_G \chi(z_i)\rho_n(dz) \to \beta_{i1} - \int_{\mathbb{R}_+^2}(z_i - \chi(z_i))\mu(dz)$ *as $n \to \infty$ for $i = 1, 2$;*

(ii) $\displaystyle\int_G \chi(z_i)\chi(z_j)\rho_n(dz) \to 2\alpha_{ij} + \int_{\mathbb{R}_+^2}\chi(z_i)\chi(z_j)\mu(dz)$ *as $n \to \infty$ for $i, j = 1, 2$;*

(iii) $\displaystyle\lim_{n\to\infty}\int_G f(z)\rho_n(dz) = \int_{\mathbb{R}_+^2} f(z)\mu(dz),$ *for $f \in C_*(\mathbb{R}^2)$.*

Proof. In the proof of Proposition 2.1, we actually choose a subsequence $\{n_l\} \subseteq \{n\}$ such that $\{P_{n_l}\}$ converges weakly to P on G^∞ (step 1) and then we also choose a subsequence $\{n_k\} \subseteq \{n_l\}$ such that $\{\Delta_{n_k}\}$ converges to $\Delta \in (0, \infty]$ and (3.5) holds (step 2). In fact, the results from (3.6) to (3.9) hold via the subsequence $\{n_k\}$ and P_{n_k} converges weakly to P on G supported by $\mathbb{R}^2_+$. Recall that $\Delta > 0$ and we have $\lim_{n_k\to\infty} \varrho_{n_k} = \Delta\varrho$ and $\lim_{n_k\to\infty} \int_G \chi(z_i)\rho_{n_k}(dz) = \Delta\beta_i$ for $i = 1, 2$. Then, for $f \in C_*(\mathbb{R}^2)$, it is easy to check that

$$\int_G f(z)\rho_{n_k}(dz) = \varrho_{n_k} \int_Q \frac{f(z)}{l(z)} 1_{\{z\neq 0\}} P_{n_k}(dz)$$

which converges to $\Delta\varrho \int_Q \frac{f(z)}{l(z)} 1_{\{z\neq 0\}} P(dz)$, namely $\int_{\mathbb{R}^2_+} f(z)\mu(dz)$. Thus (i) and (iii) hold via the subsequence $\{n_k\}$, which, combining (3.9), imply that (ii) holds via $\{n_k\}$. Moreover, the uniqueness of the representation of the function R on the r.h.s. of (2.6) by $((\beta_{11}, \beta_{21}), (\alpha_{ij}), \mu)$ (see [25, Theorems 8.1]) ensures that (i), (ii) and (iii) hold for the whole sequence $\{n\}$. □

Proposition 3.2 *Given the representation (2.6) of the limit function R with admissible parameters $((\beta_{ij}), (\alpha_{ij}), \mu)$, either of the following conditions implies that there is a sequence R_n in form (2.4) satisfying condition (A):*

(i) $\alpha_{ij} \geq 0$ *for* $i, j = 1, 2$;

(ii) $\alpha_{12} < 0$ *and* $|\alpha_{12}| \leq \alpha_{ii}$ *for* $i = 1, 2$.

Proof. Firstly, suppose $|\beta_{11}|+|\beta_{21}| > 0$ (if $|\beta_{11}|+|\beta_{21}| = 0$, set $g_{1,n}(\lambda_1, \lambda_2) = \lambda_1\lambda_2$). If $\beta_{11} \geq 0$ and $\beta_{21} \geq 0$, set $g_{1,n}(\lambda_1, \lambda_2) = \frac{\beta_{11}}{\beta_{11}+\beta_{21}}\lambda_1^2\lambda_2 + \frac{\beta_{21}}{\beta_{11}+\beta_{21}}\lambda_1\lambda_2^2$. If $\beta_{11} \geq 0$ and $\beta_{21} \leq 0$, set $g_{1,n}(\lambda_1, \lambda_2) = \frac{\beta_{21}}{\beta_{21}-\beta_{11}}\lambda_1 + \frac{\beta_{11}}{\beta_{11}-\beta_{21}}\lambda_1^2\lambda_2$. If $\beta_{11} \leq 0$ and $\beta_{21} \geq 0$, set $g_{1,n}(\lambda_1, \lambda_2) = \frac{\beta_{21}}{\beta_{21}-\beta_{11}}\lambda_1\lambda_2^2 + \frac{\beta_{11}}{\beta_{11}-\beta_{21}}\lambda_2$. If $\beta_{11} \leq 0$ and $\beta_{21} \leq 0$, set $g_{1,n}(\lambda_1, \lambda_2) = \frac{\beta_{21}}{\beta_{11}+\beta_{21}}\lambda_1 + \frac{\beta_{11}}{\beta_{11}+\beta_{21}}\lambda_2$.

Secondly, suppose that $\alpha_{11} > 0$ (if $\alpha_{11} = 0$, set $g_{1,n}(\lambda_1, \lambda_2) = \lambda_1\lambda_2$). If condition (i) holds, then we set $w = (w_1, w_2)$, where $w_1 = \sqrt{2\alpha_{11}}$ and $w_2 = \sqrt{2}\alpha_{12}/\sqrt{\alpha_{11}}$. Let $\gamma_0 = 1+w_1+w_2+2\big(\alpha_{22}-\frac{\alpha_{12}^2}{\alpha_{11}}\big)$. Define the sequences $\gamma_{2,n} = n\gamma_0 + \sqrt{n}w_1 + \sqrt{n}w_2$ and

$$\begin{aligned} g_{2,n}(\lambda_1, \lambda_2) \ =\ & \frac{n\gamma_0}{\gamma_{2,n}}\Big[\lambda_1\lambda_2 + \frac{1}{\gamma_0}\Big(\alpha_{22} - \frac{\alpha_{12}^2}{\alpha_{11}}\Big)(1-\lambda_2)^2\lambda_1\Big] \\ & + \frac{1}{\gamma_{2,n}}\Big(e^{-\sqrt{n}\langle \mathbf{1}-\lambda, w\rangle} - 1\Big)\lambda_1\lambda_2 + \frac{\sqrt{n}w_2}{\gamma_{2,n}}\lambda_1 + \frac{\sqrt{n}w_1}{\gamma_{2,n}}\lambda_2. \end{aligned}$$

If condition (ii) holds, then we set $\gamma_0 = 2(\alpha_{11} + \alpha_{22})$. Define the sequence $\gamma_n = n\gamma_0$ and

$$g_{2,n}(\lambda_1, \lambda_2) = \lambda_1\lambda_2 + \frac{\alpha_{11}}{\gamma_0}(1-\lambda_1)^2\lambda_2 + \frac{\alpha_{22}}{\gamma_0}\lambda_1(1-\lambda_2)^2 + \frac{\alpha_{12}}{\gamma_0}(\lambda_1+\lambda_2)(1-\lambda_1)(1-\lambda_2).$$

Thirdly, suppose that $\mu \neq 0$ (if $\mu = 0$, set $g_{1,n}(\lambda_1, \lambda_2) = \lambda_1\lambda_2$). Let $D_n = \{(z_1, z_2) \in \mathbb{R}^2_+ : z_1 > 1/\sqrt{n},\ z_2 > 1/\sqrt{n}\,\}$ and $u_{i,n} = \int_{D_n} \big(z_i - 1/n\big)\mu(dz)$. Define the sequences $\gamma_{3,n} = (u_{1,n} + u_{2,n}) + \mu(D_n)/n$ and

$$g_{3,n}(\lambda_1, \lambda_2) = \frac{1}{n\gamma_{3,n}} \int_{D_n} e^{-n\langle \mathbf{1}-\lambda, z\rangle}\mu(dz) + \frac{u_{2,n}}{\gamma_{3,n}}\lambda_1 + \frac{u_{1,n}}{\gamma_{3,n}}\lambda_2.$$

Finally, let $\gamma_n = \gamma_{1,n} + \gamma_{2,n} + \gamma_{3,n}$ and let $g_n = \gamma_n^{-1}(\gamma_{1,n}g_{1,n} + \gamma_{2,n}g_{2,n} + \gamma_{3,n}g_{3,n})$. Then $\{R_n\}$ defined by (2.4) satisfies condition (A). □

Proof of Proposition 2.2 It can be proved with the same method as in Proposition 2.1. □

Recall that m_n is the probability measure on $\mathbb{N}^2$ corresponding to h_n and $\hat{G}_n = \{(i/n, j/n) : i, j \in \mathbb{N}\}$. Let υ_n be the measure defined by

$$\upsilon_n(\cdot) = \theta_n \sum_{i,j=0}^{\infty} m_n\{(i,j)\}\delta_{(i/n,j/n)}(\cdot). \tag{3.10}$$

Then υ_n is a finite measure on $\mathbb{R}_+^2$ supported by $\hat{G}_n$. Let us write $f \in C_\diamond(\mathbb{R}_+^2)$ if f is bounded continuous function from $\mathbb{R}_+^2$ to $\mathbb{R}$ satisfying $f(z_1, z_2) = o(|z_1| + |z_2|)$ when $\|z\| \to 0$.

Proposition 3.3 *Under (B), as $n \to \infty$ we have the following:*

(i) $\displaystyle\int_{\mathbb{R}_+^2} \chi(z_i)\upsilon_n(dz) \to b_i + \int_{\mathbb{R}_+^2} \chi(z_i)\upsilon(dz), \quad i = 1, 2;$

(ii) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}_+^2} f(z)\upsilon_n(dz) = \int_{\mathbb{R}_+^2} f(z)\upsilon(dz)., \quad \textit{for } f \in C_\diamond(\mathbb{R}_+^2).$

Conversely, given the representation (2.8) of the limit function H with parameters $((b_1, b_2), \upsilon)$, there is a sequence H_n in form (2.5) satisfying condition (B).

Proof. The first part is proved with the same method as in Proposition 3.1. We only prove the converse part. Firstly suppose that $b_1 + b_2 > 0$. Set $\theta_{1,n} = n(b_1 + b_2)$ and $h_{1,n}(\lambda_1, \lambda_2) = \frac{b_1}{b_1+b_2}\lambda_1 + \frac{b_2}{b_1+b_2}\lambda_2$. Secondly, suppose that $\upsilon \neq 0$. Let $D_n = \{(z_1, z_2) \in \mathbb{R}_+^2 : z_1 > 1/\sqrt{n},\ z_2 > 1/\sqrt{n}\}$. Define the sequences $\theta_{2,n} = \upsilon(D_n)$ and $h_{2,n}(\lambda_1, \lambda_2) = \upsilon(D_n)^{-1} \int_{D_n} e^{-n\langle 1-\lambda, z\rangle}\upsilon(du)$. Finally, we let $\theta_n = \theta_{1,n} + \theta_{2,n}$ and $h_n = \theta_n^{-1}(\theta_{1,n}h_{1,n} + \theta_{2,n}h_{2,n})$. □

Let κ_n be another measure defined by

$$\kappa_n(\cdot) = n^2\theta_n \sum_{i,j=0}^{\infty} m_n\{(i,j)\}\delta_{(i/n,j/n)}(\cdot).$$

Then κ_n is a finite measure on $\mathbb{R}_+^2$ supported by $\hat{G}_n$. Let us write $f \in C_\#(\mathbb{R}_+^2)$ if f is bounded continuous function from $\mathbb{R}_+^2$ to $\mathbb{R}$ satisfying $f(z_1, z_2) = o(|z_1| + |z_2|^2)$ when $\|z\| \to 0$.

Proposition 3.4 *Under conditions (D1,2), (2.17) holds. In addition, as $n \to \infty$ we have*

(i) $\displaystyle\int_{\mathbb{R}_+^2} \chi(z_1)\kappa_n(dz) \to b_1 + \int_{\mathbb{R}_+^2} \chi(z_1)m(dz);$

(ii) $\displaystyle\int_{\mathbb{R}_+^2} \Big(\chi(z_2) - \frac{a_n}{n}\Big)\kappa_n(dz) \to b_2 - \int_{\mathbb{R}_+^2} \big(z_2 - \chi(z_2)\big)m(dz);$

(iii) $\displaystyle\int_{\mathbb{R}_+^2} \chi^2(z_2)\kappa_n(dz) \to 2\alpha + a_0 + \int_{\mathbb{R}_+^2} \chi^2(z_2)m(dz);$

(iv) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}_+^2} f(z)\kappa_n(dz) = \int_{\mathbb{R}_+^2} f(z)m(dz), \quad \textit{for } f \in C_\#(\mathbb{R}_+^2).$

Proof. Using the same method as in Proposition 2.1, 3.1, we have (2.17) and (i)-(iv). But we still need to verify $\alpha \geq 0$. It follows from (ii), (D1) that

$$\frac{1}{n}\int_{\mathbb{R}_+^2} \chi(z_2)\kappa_n(dz) = \frac{1}{n}\int_{\mathbb{R}_+^2} \Big(\chi(z_2) - \frac{a_n}{n}\Big)\kappa_n(dz) + a_n\theta_n,$$

which tends to a_0 as $n \to \infty$. Let E be the set of $\varepsilon > 0$ for which $m\big(\|z\| = \varepsilon\big) = 0$. By the above equality and (iii), we have

$$\lim_{E\ni\varepsilon\downarrow 0}\lim_{n\to\infty}\int_{\{\|z\|<\varepsilon\}} \Big(\chi^2(z_2) - \frac{1}{n}\chi(z_2)\Big)\kappa_n(dz) = 2\alpha. \tag{3.11}$$

Note that $\chi^2(z_2) - \dfrac{1}{n}\chi(z_2) \geq 0$ if $z \in \hat{G}_n$, and the support of κ_n is $\hat{G}_n$. Then $\alpha \geq 0$. □

4 Proof of Theorem 2.1 and Theorem 2.2

The proof of weak convergence involves two steps: tightness and identification of the limit. Recall that $x_n(\cdot) = \xi_n(\cdot)/n$ and $y_n(\cdot) = \eta_n(\cdot)/n$. Then $(x_n(\cdot), y_n(\cdot))$ can be given by

$$x_n(t) = x_n(0) + \int_0^t\int_{\mathbb{N}^2} \frac{z_1}{n} N_{n,0}(ds, dz) + \int_0^t\int_{\mathbb{N}^2}\int_0^{n\gamma_n x_n(s-)} \frac{z_1 - 1}{n} N_{n,1}(ds, dz, du), \tag{4.1}$$

$$\begin{aligned} y_n(t) \;&=\; y_n(0) + \int_0^t\int_{\mathbb{N}^2} \frac{z_2}{n} N_{n,0}(ds, dz) + \int_0^t\int_{\mathbb{N}^2}\int_0^{n\gamma_n x_n(s-)y_n(s-)} \frac{z_2 - 1}{n} N_{n,1}(ds, dz, du) \\ &\quad + \int_0^t\int_{\mathbb{N}}\int_0^{nr_n y_n(s-)} \frac{z_2 - 1}{n} N_{n,2}(ds, dz_2, du). \end{aligned} \tag{4.2}$$

Lemma 4.1 *Assume (A) holds. Let E^n be the set $\{x_n(0) + y_n(0) \leq c\}$ for some $c > 0$. Let ρ_n be defined by (3.1) and let υ_n be defined by (3.10). Then we have for any $t \geq 0$,*

(i) $\sup_n \mathbf{E}[x_n(t)1_{E^n}] \leq (c + l_0 t)\exp\{l_0 t\}$,

(ii) $\sup_n \mathbf{E}[\sup_{0\leq s\leq t} x_n(s)1_{E^n}] \leq c + l_0 t + 3l_0\int_0^t (c + l_0 s)e^{l_0 s}ds + 4(l_0\int_0^t (c + l_0 s)e^{l_0 s}ds)^{\frac{1}{2}}$,

where $l_0 = \sup_n \sum_{i=1}^2 \big[|\int_G z_i\rho_n(dz)| + |\int_G \chi(z_i)\,\rho_n(dz)| + \int_G \chi^2(z_i)\rho_n(dz) + \int_{\mathbb{R}_+^2} z_i\,\upsilon_n(dz)\big]$.

Proof. This lemma can be proved by using Gronwall's inequality and standard stopping time argument. So we omit it. □

Lemma 4.2 *Under the conditions of Theorem 2.1, $(x_n(\cdot), y_n(\cdot))$ is a tight sequence of processes in $D([0,\infty), \mathbb{R}_+^2)$.*

Proof. For any $\varepsilon > 0$, we choose $c > 0$ such that $\sup_n P(x_n(0) + y_n(0) > c) < \varepsilon$. Let $\sigma_n^k = \inf\{t > 0 : x_n(t) \geq k \text{ or } x_n(t-) \geq k\}$, $x_n^k(t) = x_n(t\wedge\sigma_n^k)$ and $y_n^k(t) = y_n(t\wedge\sigma_n^k)$. Note that $\{\sigma_n^k < T\} \subseteq \{\sup_{0\leq s\leq T}|x_n(s)| \geq k - 1\}$. For any $T \geq 0$, by Lemma 4.1 (ii), we can choose k to be large enough such that $\sup_n P(\sigma_n^k < T + 1) < 2\varepsilon$. Fix k above. By condition (C), we have

$l_1 := \sup_n \big(|r_n \int_{\mathbb{N}^2}(z_2-1)\nu_n(dz_2)|+|(r_n/n)\int_{\mathbb{N}^2}(z_2-1)^2\nu_n(dz_2)|\big) < \infty$. Using Gronwall's inequality and stopping time argument we get that

$$\sup_n \mathbf{E}[y_n^k(t)1_{E^n}] \leq (c+l_0t)\exp\{(kl_0+l_1)t\},$$

for $t \geq 0$. Hence there exists a positive number $R = R(c,k,\varepsilon)$ such that $\sup_n \mathbf{P}\big(x_n(t)+y_n(t) \geq R\big) < 4\varepsilon$ for any fixed $t \leq T$.

Let $\{\tau_n\}$ be a sequence of stopping times bounded above by $T \geq 0$. By the properties of stationary independent increments of the Poisson process we obtain that

$$\mathbf{E}[1_{E^n}|x_n^k(\tau_n+t)-x_n^k(\tau_n)|] \leq (2k+1)l_0t+(kl_0t)^{\frac{1}{2}}.$$

Then there exists a positive number $\delta = \delta(c,k,\varepsilon)$ ($\delta \leq 1$) such that

$$\sup_n \sup_{t\in[0,\delta]} \mathbf{P}\big(|x_n(\tau_n+t)-x_n(\tau_n)|>\varepsilon\big) < \frac{(2k+1)l_0\delta+(kl_0\delta)^{\frac{1}{2}}}{\varepsilon}+3\varepsilon < 4\varepsilon$$

and similarly we can show that $\sup_n \sup_{s\in[0,\delta]} \mathbf{P}\big(|y_n(\tau_n+t)-y_n(\tau_n)|>\varepsilon\big) < 4\varepsilon$ for suitable δ. Thus the criterion of Aldous (see [1]) yields tightness for $\{(x_n(\cdot),y_n(\cdot))\}$. □

Let $(x(\cdot),y(\cdot))$ be any limit point of $\{(x_n(\cdot),y_n(\cdot))\}$. Without loss of generality suppose that $(x_n(\cdot),y_n(\cdot))$ converges weakly to $(x(\cdot),y(\cdot))$. By Skorokhod's theorem we may assume that on some Skorokhod space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbf{P})$, $(x_n(\cdot),y_n(\cdot)) \xrightarrow{a.s.} (x(\cdot),y(\cdot))$ in the topology of $D([0,\infty),\mathbb{R}_+^2)$.

Lemma 4.3 *Assume the conditions of Theorem 2.1 hold. Then for any fixed $\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2$,*

$$\begin{aligned} M(t) &= \exp\{i\lambda_1x(t)+i\lambda_2y(t)\}-\exp\{i\lambda_1x(0)+i\lambda_2y(0)\} \\ &\quad -\int_0^t L\exp\{i\lambda_1x(s)+i\lambda_2y(s)\}ds \end{aligned}$$

is a complex-valued local $\mathcal{F}_t$-martingale. Here the operator L is defined by (2.13).

Proof. Let $\|\cdot\|$ be the supremum of $\mathbb{R}^2$. Define the stopping times

$$\begin{aligned} \tau^a &= \inf\{t\geq 0: \|(x(t),y(t))\|\geq a \text{ or } \|(x(t-),y(t-))\|\geq a\}, \\ \tau_n^a &= \inf\{t\geq 0: \|(x_n(t),y_n(t))\|\geq a \text{ or } \|(x_n(t-),y_n(t-))\|\geq a\}. \end{aligned}$$

Let $x^a(t)=x(t\wedge\tau^a)$, $x_n^a(t)=x_n(t\wedge\tau_n^a)$, and analogously $y^a(t)$, $y_n^a(t)$. Switch to the Skorokhod space. Since $(x_n(\cdot),y_n(\cdot)) \xrightarrow{a.s.} (x(\cdot),y(\cdot))$ on this space, it follows from Jocod and Schiryaev [18, Lemma 2.10 and Proposition 2.11] that for all but countably many a,

$$\tau_n^a \xrightarrow{a.s.} \tau^a \text{ in } \mathbb{R} \quad \text{and} \quad (x_n^a(\cdot),y_n^a(\cdot)) \xrightarrow{a.s.} (x^a(\cdot),y^a(\cdot)) \tag{4.3}$$

in the topology of $D([0,\infty),\mathbb{R}_+^2)$. Define $\tau_n^a(t)=\tau_n^a\wedge t$ and $\tau^a(t)=\tau^a\wedge t$. We claim that

$$\tau_n^a(\cdot) \xrightarrow{a.s.} \tau^a(\cdot) \quad \text{in } C([0,\infty),\mathbb{R}_+), \qquad \text{as } n\to\infty. \tag{4.4}$$

In fact, since $0\leq\tau_n^a(t+\varepsilon)-\tau_n^a(t)\leq\varepsilon$ for any $t\geq 0$, the criterion of Aldous yields tightness for $\{\tau_n^a(\cdot),\ n\geq 1\}$. By (4.3) we have that (4.4) holds. Set

$$M_n(t) = e^{i\lambda_1x_n(t)+i\lambda_2y_n(t)}-e^{i\lambda_1x_n(0)+i\lambda_2y_n(0)}-\int_0^t L_ne^{i\lambda_1x_n(s)+i\lambda_2y_n(s)}\,ds, \tag{4.5}$$

where L_n is the weak generator for $(x_n(\cdot), y_n(\cdot))$ defined by (4.1)-(4.2) and can be easily obtained by using Itô's formula. Thus $M_n^a(t) := M_n(t \wedge \tau_n^a)$ is a complex-valued local martingale. Now let $M^a(t) = M(t \wedge \tau^a)$ and we will show that

$$M_n^a(\cdot) \xrightarrow{a.s.} M^a(\cdot) \quad \text{in } D([0,\infty), \mathbb{C}), \qquad \text{as } n \to \infty. \tag{4.6}$$

In fact, by (4.3), Ethier and Kurtz [14, Problem 13, P151], Jocod and Schiryaev [18, Proposition 1.23], it suffices to prove the convergence of the last integral terms in (4.5). Let f_λ be a complex-valued continuous function on $\mathbb{R}_+^2$. By (4.4), we get that

$$\int_0^{\tau_n^a(t)} f_\lambda(x_n^a(s), y_n^a(s))\, ds \to \int_0^{\tau^a(t)} f_\lambda(x^a(s), y^a(s)) ds$$

in the topology of $C([0,\infty), \mathbb{C})$. By Proposition 3.1, 3.4 and condition (C), it is not hard to show that (4.6) holds. Then for almost all $t \geq 0$, $M_n^a(t) \xrightarrow{a.s.} M^a(t)$ in $\mathbb{C}$. Fix arbitrary $T > 0$. For any $t \leq T$,

$$\Big| \int_0^{\tau_n^a(t)} e^{i\lambda_1 x_n^a(s) + i\lambda_2 y_n^a(s)}\, x_n^a(s) y_n^a(s)\, ds \Big| \leq a^2 T, \tag{4.7}$$

where the bound holds uniformly in n. Then for almost $t \leq T$, $M_n^a(t) \xrightarrow{L_1} M^a(t)$, as $n \to \infty$. Thus, by the right continuity and boundedness of $M^a(t)$ $(t \leq T)$, we have that $M^a(t)$ is a martingale. Note that $\tau^a \to \infty$ as $a \to \infty$, and hence $M(t)$ is a local martingale. □

It follows from the above lemma and Jocod and Schiryaev [18, Theorem 2.42, P86] that $(x(\cdot), y(\cdot))$ is a semimartingale and it admits the canonical representation

$$x(t) \;=\; x(0) + \bar{b}_1 + x^c(t) + \int_0^t \beta_{11} x(s) ds + \int_0^t \int_{\mathbb{R}_+^2} z_1 \tilde{J}(ds, dz), \tag{4.8}$$

$$y(t) \;=\; y(0) + \bar{b}_2 + y^c(t) + \int_0^t (\beta_{21} x(s) + \beta_{22}) y(s) ds + \int_0^t \int_{\mathbb{R}_+^2} z_2 \tilde{J}(ds, dz), \tag{4.9}$$

where $\bar{b}_i = b_i + \int_{\mathbb{R}_+^2} z_i \upsilon(dz)$, $(x^c(t), y^c(t))$ is a vector of two continuous local martingales with quadratic covariation process $\big(\int_0^t c_{ij}(s)\, ds\big)_{i,j=1}^2$ by $c_{11}(s) = 2\alpha_{11} x(s-)$, $c_{12}(s) = 2\alpha_{12} x(s-)(1 \wedge y(s-))$, and $c_{22}(s) = 2(\alpha_{22} x(s-) + a) y(s-)$. $J(dt, dz)$ is an integer-valued random measure on $(0,\infty) \times \mathbb{R}_+^2$ with compensator $K(t, dz)\, dt$ by

$$\begin{aligned} K(t, dz) \;&=\; x(t-)\big(1 \wedge y(t-)\big)\, \mu(dz) + x(t-)\big(1 - 1 \wedge y(t-)\big)\, \mu_1(dz) \\ &\quad+\; x(t-)\big(y(t-) - 1 \wedge y(t-)\big)\, \mu_2(dz) + \upsilon(dz), \end{aligned}$$

where $\mu_1(\cdot) = \mu(\cdot \cap \{z_2 = 0\})$ and $\mu_2(\cdot) = \mu(\cdot \cap \{z_1 = 0\})$. $\tilde{J}(dt, dz) = J(dt, dz) - K(t, dz) dt$.

Lemma 4.4 *There exists a standard extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{P})$ of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ supporting three orthogonal white noises $(W_i)_{i=0}^2$ on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dsdu$, a Poisson random measure N_0 on $(0,\infty) \times \mathbb{R}_+^2$ with intensity $ds\upsilon(dz)$ and a Poisson random measure N_1 on $(0,\infty) \times \mathbb{R}_+^2 \times (0,\infty)$ with intensity $ds\mu(dz)du$ (N_0 and N_1 are independent of each other), such that*

$$x^c(t) \;=\; \int_0^t \int_0^{x(s-)} \sigma_{11} W_1(ds, du) + \int_0^t \int_0^{x(s-)} \sigma_{12} W_2(ds, du), \tag{4.10}$$

$$\begin{aligned} y^c(t) \;&=\; \int_0^t \int_0^{y(s-)} \sigma_0 W_0(ds, du) + \int_0^t \int_0^{x(s-)y(s-)} \sigma_{21} W_1(ds, du) \\ &\quad+ \int_0^t \int_0^{x(s-)y(s-)} \sigma_{22} W_2(ds, du), \end{aligned} \tag{4.11}$$

and for any $E \in \mathfrak{B}(\mathbb{R}_+^2)$,

$$J((0,t]\times E) = \int_0^t \int_{\mathbb{R}_+^2\times(0,\infty)} 1_E\big(\theta(s,z,u)\big)\, N_1(ds,dz,du) + \int_0^t \int_{\mathbb{R}_+^2} 1_E(z)\, N_0(ds,dz), \qquad (4.12)$$

where $\theta(s,z,u) = \big(z_1 1_{(0,x(s-)]}(u), z_2 1_{(0,x(s-)y(s-)]}(u)\big)$, $(s,z,u)\in(0,\infty)\times\mathbb{R}_+^2\times(0,\infty)$.

Proof. By [13, Theorem III-10] (4.10) and (4.11) hold. Define the measure $V(dz,du) = \mu(dz)l(du) + \upsilon(dz)\delta_0(du)$, where $l(du)$ is the Lebesgue measure on $(0,\infty)$ and $\delta_0(du)$ is the Dirac measures at $u=0$. Since $\upsilon(dz)$ and $\mu(dz)$ are the σ-finite measure supported by $\mathbb{R}_+^2\backslash\{0\}$, we can check that

$$K(t,E) = \int_{\mathbb{R}_+^2\times[0,\infty)} 1_{\{(z,u)\,:\,\tilde\theta(t,z,u)\in E\}}\, V(dz,du), \quad \text{for } E\in\mathfrak{B}(\mathbb{R}_+^2).$$

where $\tilde\theta(s,z,u) = \big(z_1 1_{[0,x(s-)]}(u), z_2 1_{[0,x(s-)y(s-)]}(u)\big)$, $(s,z,u)\in(0,\infty)\times\mathbb{R}_+^2\times[0,\infty)$. Note that $J(ds,dz)$ is quasi-left-continuous and $\tilde\theta(s,z,u)$ is a predictable $\mathbb{R}_+^2$-valued process. From Ikeda and Watanabe [16, Theorem 7.4, P93] (see also [5]), there exists a Poisson random measure $N(dt,dz,du)$ on $(0,\infty)\times\mathbb{R}_+^2\times[0,\infty)$ with intensity $dsV(dz,du)$ such that for any $E\in\mathfrak{B}(\mathbb{R}_+^2)$,

$$J((0,t]\times E) = \int_0^t \int_{\mathbb{R}_+^2\times[0,\infty)} 1_E\big(\tilde\theta(s,z,u)\big)\, N(ds,dz,du). \qquad (4.13)$$

Set $N_0(ds,dz) = N(ds,dz,\{0\})$ and set $N_1(ds,dz,du) = N(ds,dz,du)\big|_{(0,\infty)\times\mathbb{R}_+^2\times(0,\infty)}$. Then (4.12) holds. □

Lemma 4.5 *The pathwise uniqueness of solutions holds for the equation-system (2.11)-(2.12).*

Proof. This lemma is proved with the same method as Theorem 5.1 and Theorem 6.3 in [11]. □

Proof of Theorem 2.1 By Lemma 4.5, it suffices to show that $(x(\cdot),y(\cdot))$ is a pair of solutions of (2.11)-(2.12). Applying Lemma 4.4, we obtain that

$$\int_0^t\int_{|z_1|>\varepsilon} z_1\tilde J(ds,dz) = \int_0^t\int_{|z_1|>\varepsilon}\int_0^{x(s-)} z_1\tilde N_1(ds,dz,du) + \int_0^t\int_{|z_1|>\varepsilon} z_1\tilde N_0(ds,dz),$$

for any $\varepsilon>0$. By the above equality,

$$\begin{aligned}\int_0^t\int_{|z_1|\le 1} z_1\tilde J(ds,dz) &= \lim_{\varepsilon\downarrow 0}\Big(\int_0^t\int_{\varepsilon<|z_1|\le 1}\int_0^{x(s-)} z_1\tilde N_1(ds,dz,du) + \int_0^t\int_{\varepsilon<|z_1|\le 1} z_1\tilde N_0(ds,dz)\Big)\\ &= \int_0^t\int_{|z_1|\le 1}\int_0^{x(s-)} z_1\tilde N_1(ds,dz,du) + \int_0^t\int_{|z_1|\le 1} z_1\tilde N_0(ds,dz).\end{aligned}$$

The above limit holds in L_2. By (4.8)-(4.9), (2.11)-(2.12) holds for $(x(\cdot),y(\cdot))$. □

Proof of Theorem 2.2 Let $(\alpha_{ij}) = \frac{1}{2}(\sigma_{ij})(\sigma_{ij})^\tau$. If the matrix (α_{ij}) satisfies condition (i) or (ii) of Proposition 3.2, set the sequence $r_n = 1 + n\sigma_0 + |\beta_{22}|$ and

$$f_n(\lambda_2) = \begin{cases} \big[-\beta_{22} + (1+n\sigma_0)\lambda_2 + n\sigma_0(1-\lambda_2)^2/2\big]/r_n, & \text{if } \beta_{22}\le 0;\\ \big[(1+n\sigma_0)\lambda_2 + \beta_{22}\lambda_2^2 + n\sigma_0(1-\lambda_2)^2/2\big]/r_n, & \text{if } \beta_{22}>0.\end{cases}$$

In this case we have Theorem 2.2. If not, we introduce positive constants ι_1 and ι_2 such that $|\alpha_{12}|/\alpha_{22} \le \iota_1/\iota_2 \le \alpha_{11}/|\alpha_{12}|$ ($\alpha_{12} < 0$). Define the bijection $\Gamma : \mathbb{R}_+^2 \ni (z_1, z_2) \to (z_1/\iota_1, z_2/\iota_2) \in \mathbb{R}_+^2$. Consider the parameters:

$$\hat\sigma_0 = \frac{\sigma_0}{\iota_2}; \quad \hat b_i = \frac{b_i}{\iota_i}; \quad \hat\sigma_{ij} = \frac{\sigma_{ij}}{\iota_i}; \quad \hat\beta_{11} = \beta_{11},\ \hat\beta_{21} = \iota_1\beta_{21},\ \hat\beta_{22} = \beta_{22};$$
$$\hat\upsilon(dz) = \upsilon \circ \Gamma^{-1}(dz); \quad \hat\mu(dz) = \mu \circ \Gamma^{-1}(dz).$$

for $i, j = 1, 2$. Note that $(\hat\alpha_{ij}) = \frac{1}{2}(\hat\sigma_{ij})(\hat\sigma_{ij})^\tau$ satisfies condition (ii) of Proposition 3.2. Then there exists a sequence $(\hat\xi_n(\cdot), \hat\eta_n(\cdot))$ with parameters $(\theta_n, (\gamma_n,\ \gamma_n/n),\ r_n;\ h_n,\ g_n,\ f_n)$ such that $(\hat\xi_n(\cdot)/n, \hat\eta_n(\cdot)/n)$ converges weakly to $(\hat x(\cdot), \hat y(\cdot))$ defined by (2.11)-(2.12) with parameters ($\hat\sigma_0$, $(\hat b_1, \hat b_2)$, $(\hat\sigma_{ij})$, $(\hat\beta_{11}, \hat\beta_{21}, \hat\beta_{22})$, $\hat\upsilon$, $\hat\mu$). Then we can choose the sequence of catalytic DBI-processes $(\xi_n(\cdot), \eta_n(\cdot))$ with parameters $(\theta_n,\ \iota_1\gamma_n,\ \iota_1\iota_2\gamma_n/n,\ \iota_2 r_n; h_n, g_n, f_n)$ and $(\iota_1\xi_n(\cdot)/n, \iota_2\eta(\cdot)/n)$ converges weakly to $(x(\cdot), y(\cdot))$. □

5 Proof of Theorem 2.3

Proof of Theorem 2.3 (i) It is easy to see that $\{(X_n(t), Y_n(t)), t \ge 0\}$ is a strong Markov process with values in $E_n := \{(i/n, (j - n^2)/n) : i, j \in \mathbb{N}\}$. For any bounded function f on E_n define the operator $A_n f = A_{n,1} f + A_{n,2} f$, where

$$\begin{aligned} A_{n,1}f(x,y) &= x[1 \wedge (\frac{y}{n}+1)] \int_G \Delta_{(z_1,z_2)} f(x,y)\rho_n(dz) \\ &\quad + x[1 - 1 \wedge (\frac{y}{n}+1)] \int_G \Delta_{(z_1,0)} f(x,y)\rho_n(dz) \\ &\quad + x[(\frac{y}{n}+1) - 1 \wedge (\frac{y}{n}+1)] \int_G \Delta_{(0,z_2)} f(x,y)\rho_n(dz), \end{aligned} \tag{5.1}$$

$$\begin{aligned} A_{n,2}f(x,y) &= \theta_n(ny+n^2) \int_{\mathbb{N}} \Delta_{(0,\frac{z_2-1}{n})} f(x,y)\nu_n(dz_2) \\ &\quad + \int_{\mathbb{R}_+^2} \Delta_{(z_1,z_2)} f(x,y)\kappa_n(dz). \end{aligned} \tag{5.2}$$

Let A be the generator of $(X(\cdot), Y(\cdot))$ defined by (2.1). For $f \in C^2(D)$, set

$$\begin{aligned} A_1 f(x,y) &= \beta_{11} f_1'(x,y) + \beta_{21} f_2'(x,y) + \alpha_{11} f_{11}''(x,y) \\ &\quad + 2\alpha_{12} f_{12}''(x,y) + \alpha_{22} f_{22}''(x,y) \\ &\quad + \int_{\mathbb{R}_+^2} (\Delta_{(z_1,z_2)} f(x,y) - \langle \nabla f(x,y), z\rangle)\mu(dz) \end{aligned}$$

and $A_2 f(x,y) = Af(x,y) - xA_1 f(x,y)$. Let $C_c(D)$ be the space consisting of $f \in C(D)$ with compact support. Let $C_c^\infty(D) = \bigcap_{k=1}^\infty C^k(D) \bigcap C_c(D)$. By [12, Theorem 2.7], $C_c^\infty(D)$ is a core of A. To prove Theorem 2.3 (i), by Ethier and Kurtz [14, Corollary 8.7], it suffices to show that

$$\lim_{n\to\infty} \big[A_n f(x_n, y_n) - Af(x_n, y_n)\big] = 0 \tag{5.3}$$

for all $f \in C_c^\infty(D)$ and for every sequence $(x_n, y_n)_{n\in\mathbb{N}}$ with $(x_n, y_n) \in E_n$ such that $x_n \to x \in [0, \infty]$ and $y_n \to y \in [-\infty, +\infty]$. We will consider (x_n, y_n) in three cases.

Case 1: for $(x_n, y_n) \in E_n$, $x_n \to x \in [0, \infty)$ and $y_n \to y \in (-\infty, +\infty)$. Set

$$\bar H(x,y,z) = \Delta_{(z_1,z_2)} f(x,y) - \langle \nabla f(x,y), \chi(z)\rangle - \frac{1}{2}\sum_{i,j=1}^2 f_{ij}''(x,y)\chi(z_i)\chi(z_j),$$

for $f \in C_c^\infty(D)$. Note that $\bar{H}(x,y,z) \in C_*(\mathbb{R}^2)$ as a function of $z \in \mathbb{R}^2$ for fixed $(x,y) \in D$. Then for any $\varepsilon > 0$, there exists a positive number δ $(2\delta < 1)$ such that $\|z\| \le 2\delta$ implies $\bar{H}(x,y,z) \le \varepsilon(z_1^2 + z_2^2)$ for every $(x,y) \in D$. In addition, the function $g_\delta(z) = \big(\delta\|z\| - 1\big)^+ \wedge 1$ belongs to $C_*(\mathbb{R}^2)$ and $g_\delta(z) \le \big(\delta^2 \vee 1\big)\big(\|z\|^2 \wedge 1\big)$; see, e.g., [18]. Then Proposition 3.1 (iii) implies that as $n \to \infty$,

$$\rho_n\big(\|z\| > \delta\big) \le \int_G g_{\frac{2}{\delta}}(z)\rho_n(dz) \to \int_{\mathbb{R}_+^2} g_{\frac{2}{\delta}}(z)\mu(dz) \le \mu\big(\|z\| > \delta/2\big). \tag{5.4}$$

By (2.7), $\sup_n \rho_n\big(\|z\| > \delta\big) < \infty$. By the uniform continuity of $\bar{H}$ in $(x,y) \in D$ and Proposition 3.1 (ii), it is easy to check that

$$\int_G \big|\bar{H}(x_n, y_n, z) - \bar{H}(x,y,z)\big|\rho_n(dz) \to 0 \quad \text{as } n \to \infty. \tag{5.5}$$

Based on (5.5) and Proposition 3.1, we have that

$$\begin{aligned}
&\int_G \Delta_{(z_1,z_2)} f(x_n,y_n)\rho_n(dz) \\
&= \int_G H(x,y,z)\rho_n(dz) + \int_G \big[H(x_n,y_n,z) - H(x,y,z)\big]\rho_n(dz) \\
&\quad + \int_G \langle \nabla f(x_n,y_n), \chi(z)\rangle \rho_n(dz) + \frac{1}{2}\sum_{i,j=1}^{2} f''_{ij}(x_n,y_n)\int_G \chi(z_i)\chi(z_j)\rho_n(dz)
\end{aligned} \tag{5.6}$$

which converges to $A_1 f(x,y)$, as $n \to \infty$. Hence $A_{n,1}f(x_n,y_n) \to xA_1 f(x,y)$. Let

$$K(x,y,z) = \Delta_{(z_1,z_2)} f(x,y) - f'_1(x,y)\chi(z_1) - f'_2(x,y)\chi(z_2) - \frac{1}{2}f''_{22}(x,y)\chi_2^2(z).$$

Applying Taylor's formula to the first term in (5.2), $A_{n,2}f(x_n,y_n)$ can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}_+^2} K(x,y,z)\kappa_n(dz) + \int_{\mathbb{R}_+^2} \big[K(x_n,y_n,z) - K(x,y,z)\big]\kappa_n(dz) \\
&+ \Big(\int_{\mathbb{R}_+^2} \chi(z_1)\kappa_n(dz)\Big) f'_1(x_n,y_n) + \Big(\int_{\mathbb{R}_+^2} \Big(\chi(z_2) - \frac{a_n}{n}\Big)\kappa_n(dz) - a_n\theta_n y_n\Big) f'_2(x_n,y_n) \\
&+ \frac{1}{2}\Big(\int_{\mathbb{R}_+^2} \chi_2^2(z)\kappa_n(dz) + \theta_n\big(1 + \frac{y_n}{n}\big)(\sigma_n + a_n)\Big) f''_{22}(x_n,y_n) + \frac{1}{2}\Big(1 + \frac{y_n}{n}\Big) I_n,
\end{aligned} \tag{5.7}$$

where

$$I_n = \theta_n \int_{\mathbb{N}_+^2} \Big[f''_{22}\Big(x_n, y_n + \vartheta\frac{z_2 - 1}{n}\Big) - f''_{22}(x_n,y_n)\Big](z_2 - 1)^2\nu_n(dz_2), \quad 0 \le \vartheta \le 1.$$

We obtain as in the proof of (5.5) that the second term of (5.7) converges to 0. By the uniform continuity of f''_{22}, condition (D1) and (E), we deduce that I_n tends to 0 as $n \to \infty$. Note that $K(x,y,z) \in C_\#(\mathbb{R}^2)$ as a function of $z \in \mathbb{R}^2$ for fixed $(x,y) \in D$. Then by Proposition 3.4, $A_{n,2}f(x_n,y_n) \to A_2 f(x,y)$. Hence, (5.3) holds in this case.

Case 2: For $(x_n,y_n) \in E_n$, $x_n \to +\infty$ or $y_n \to +\infty$. Since $f \in C_c^\infty(D)$, we denote by M a constant such that $\|(x,y)\| \ge M$ implies $f(x,y) = 0$. Choose n to be large enough such that $x_n > M + 1$ or $y_n > M + 1$, then $A_n f(x_n,y_n) = Af(x_n,y_n) = 0$.

Case 3: For $(x_n, y_n) \in E_n$, $x_n \to x < +\infty$ and $y_n \to -\infty$. Let n be large enough such that $y_n < -M$. Then $f(x_n + z_1, y_n + z_2) \neq 0$ or $f(x_n, y_n + z_2) \neq 0$ implies that $y_n + z_2 > -M$. We have

$$\big|A_{n,1} f(x_n, y_n)\big| \leq x_n \Big(\frac{y_n}{n} + 1\Big) \|f\| \rho_n \Big(z_2 > -M - y_n\Big). \tag{5.8}$$

If $y_n \to -\infty$, then $0 \leq y_n/n + 1 \leq 1$ for any n to be large enough. As in the proof of (5.4), for any $\varepsilon > 0$, we can choose δ to be big enough such that $\sup_n \rho_n(z_2 > \delta) < \varepsilon$, which implies that (5.8) tends to 0 as $n \to \infty$. When $y_n < -M$, we also have

$$\begin{aligned} A_{n,2} f(x_n, y_n) &= \theta_n (n y_n + n^2) \int_{\mathbb{N}^2} f(x_n, y_n + \frac{z_2 - 1}{n}) \nu_n(dz) \\ &\quad + \int_{\mathbb{R}_+^2} f(x_n + z_1, y_n + z_2) \kappa_n(dz). \end{aligned}$$

We obtain as in the proof of (5.8) that $A_{n,2} f(x_n, y_n) \to 0$ as $n \to \infty$. In addition, $Af(x_n, y_n) \to 0$ as $n \to \infty$. Hence, (5.3) holds in case 3. □

Lemma 5.1 *Let $(X(\cdot), Y(\cdot))$ be a spectrally positive regular affine process with admissible parameters $(a, (\alpha_{ij}), (b_1, b_2), (\beta_{ij}), m, \mu)$. Suppose that $\beta_{22} \leq 0$ and $a \geq -\beta_{22}$. Moreover, if (α_{ij}) satisfies condition* (i) *or* (ii) *of Proposition 3.2, then there exists a sequence of catalytic DBI-processes with parameters $(n^2\theta_n, (\gamma_n, \gamma_n/n^2), \theta_n; h_n, g_n, f_n)$ that satisfies conditions (A), (D1,2) and (E).*

Proof. Suppose, for a moment, that $\beta_{22} < 0$. Let $c = -\beta_{22}$ and $d = a + \beta_{22}$. Since $a \geq -\beta_{22}$, we have $d \geq 0$. The following proof is divided into three steps.

Step 1: Set $a_n = 1 - f_n'(1-) = c/(c + 2d + 1)$, $\sigma_n = f_n''(1-) = 2d/(c + 2d + 1)$ and

$$f_n(\lambda_2) = \lambda_2 + \frac{c}{c + 2d + 1}(1 - \lambda_2) + \frac{d}{c + 2d + 1}(1 - \lambda_2)^2.$$

Step 2: Suppose that $m \neq 0$. Let $D_n = \{(z_1, z_2) \in \mathbb{R}_+^2 : z_1 > 1/\sqrt{n},\ z_2 > 1/\sqrt{n}\}$ and $v_n = \int_{D_n} (z_2 - a_n/n) \mu(dz)$. Then we define the sequences

$$\theta_{1,n} = (c + 2d + 1) + \frac{(c + 2d + 1) v_n}{cn} + m(D_n)/n^2$$

$$\begin{aligned} h_{1,n}(\lambda_1, \lambda_2) &= \frac{(c + 2d + 1) v_n}{cn\theta_{1,n}} + \frac{1}{n^2 \theta_{1,n}} \int_{D_n} e^{-n\langle \mathbf{1} - \lambda, z\rangle} m(dz) \\ &\quad + \frac{(c + 2d + 1)}{\theta_{1,n}} \Big[\frac{a_n n^2}{n^2 + 1} \lambda_2 + \Big(1 - \frac{a_n n^2}{n^2 + 1}\Big) \Big]. \end{aligned}$$

Suppose that $b_1 + |b_2| > 0$. If $b_1 \geq |b_2|$, we set

$$\begin{aligned} h_{2,n}(\lambda_1, \lambda_2) &= \frac{2d + 1}{c + 2d + 1} + \frac{c}{c + 2d + 1} \lambda_1 \lambda_2 \\ &\quad + \frac{b_2 c}{b_1 (c + 2d + 1)} \lambda_1 (\lambda_2 - 1) \lambda_2^{1_{\{b_2 > 0\}}} \end{aligned}$$

and $\theta_{2,n} = b_1(c + 2d + 1)/c$. If $|b_2| > b_1$, we set

$$h_{2,n}(\lambda_1, \lambda_2) = \begin{cases} 1 + \dfrac{b_1 c}{b_2 (c + 2d + 1)}(1 - \lambda_1), & \text{if } b_2 < 0; \\[2ex] \dfrac{2d + 1}{c + 2d + 1} + \dfrac{c}{c + 2d + 1} \lambda_2^2 + \dfrac{b_1 c}{b_2 (c + 2d + 1)} (\lambda_1 - 1) \lambda_2^2, & \text{if } b_2 > 0. \end{cases}$$

and $\theta_{2,n} = |b_2|(c+2d+1)/(cn)$. Then let $\theta_n = (\theta_{1,n} + \theta_{2,n})$ and $h_n = \theta_n^{-1}(\theta_{1,n}h_{1,n} + \theta_{2,n}h_{2,n})$. Then the sequence $\{F_n\}$ defined by (2.16) satisfies conditions (D1,2).

Step 3: By Proposition 3.2, there exists a sequence $\{R_n\}$ defined by (2.4) with (γ_n, g_n) such that $\{R_n\}$ satisfies conditions (A) and the limit function R has the representation (2.6).

We find the desired sequence of catalytic DBI-processes with parameters $(n^2\theta_n,\ (\gamma_n,\ \gamma_n/n^2),\ \theta_n;\ h_n,\ g_n,\ f_n)$. Now, if $c := -\beta_{22} = 0$, then we define the sequences $c_n = \sqrt{\frac{v_n+1}{n}}$. It is easy to check that $\lim_{n\to\infty} c_n = 0$ and the above proof still holds if c is replaced by c_n. $\square$

Proof of Theorem 2.3 (ii) By Lemma 5.1, we only need to consider the case that $\alpha_{12} < 0$. Let $r_1 = \alpha_{12}^2/k\alpha_{22}$ and $r_2 = |\alpha_{12}|/k$, where k is a positive integer such that $r_2 < 1$. Then define the bijection $\Gamma : \mathbb{R}_+^2 \ni (z_1, z_2) \longrightarrow (z_1/r_1, z_2/r_2) \in \mathbb{R}_+^2$. Consider another set of admissible parameters:

$$\tilde{a} = a/r_2^2; \quad \tilde{b}_1 = b_1/r_1,\ \tilde{b}_2 = b_2/r_2; \quad \tilde{\beta}_{11} = \beta_{11},\ \tilde{\beta}_{12} = 0,\ \tilde{\beta}_{21} = \frac{r_1\beta_{21}}{r_2},\ \tilde{\beta}_{22} = \beta_{22};$$

$$\tilde{\alpha}_{11} = \frac{k\alpha_{11}\alpha_{22}}{\alpha_{12}^2},\ \tilde{\alpha}_{12} = \tilde{\alpha}_{21} = -k,\ \tilde{\alpha}_{22} = k; \quad \tilde{m}(dz) = m \circ \Gamma^{-1}(dz),\ \tilde{\mu}(dz) = r_1\mu \circ \Gamma^{-1}(dz).$$

The above set of admissible parameters determines a unique regular affine process $(\tilde{X}(\cdot), \tilde{Y}(\cdot))$, which satisfies the condition of Lemma 5.1. Then there exists a sequence of catalytic DBI-processes $(\xi_n(\cdot), \eta_n(\cdot))$, such that $(X_n(\cdot), Y_n(\cdot))$ converges weakly to $(\tilde{X}(\cdot), \tilde{Y}(\cdot))$, where $(X_n(\cdot), Y_n(\cdot))$ is defined by (2.19). We also have $(r_1X_n(\cdot), r_2Y_n(\cdot))$ converges weakly to $(r_1\tilde{X}(\cdot), r_2\tilde{Y}(\cdot))$; see, e.g., [14]. It is easy to check that $(r_1\tilde{X}(\cdot), r_2\tilde{Y}(\cdot))$ and $(X(\cdot), Y(\cdot))$ have the same finite-dimensional distributions. $\square$

Proof of Theorem 2.4 This theorem can be proved with the same method as Theorem 2.3. $\square$

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