Published in: *Statistics and Probability Letters* **76** (2006), 4: 401–411.

Fluctuation Limit Theorems of Immigration Superprocesses with Small Branching[1]

Zenghu Li and Mei Zhang[2]

School of Mathematical Sciences, Beijing Normal University,
Beijing 100875, People's Republic of China

lizh@bnu.edu.cn and meizhang@bnu.edu.cn

Abstract

We establish fluctuation limit theorems of immigration superprocesses with small branching rates. The weak convergence of the processes on a Sobolev space is established, which improves the result of Gorostiza and Li (2000). The limiting processes are infinite-dimensional Ornstein-Uhlenbeck type processes.

Mathematics Subject Classifications (2000): Primary 60J80; Secondary 60G20, 60G10

Key words and phrases: immigration superprocess, fluctuation limit, tightness, weak convergence, Schwartz space, Sobolev space.

1 Introduction

Fluctuation limits of branching particle systems and superprocesses have been studied extensively. Since the branching particle systems are usually unstable, in many cases one uses time-dependent scalings which lead to time-inhomogeneous Ornstein-Uhlenbeck processes; see, e.g., Bojdecki and Gorostiza [3], Dawson et al [6] and the references therein. For subcritical branching systems with immigration, it is usually easy to find a stationary distribution. In the study of fluctuation limits of those systems, we can use a time-independent scaling, which lead to homogeneous Ornstein-Uhlenbeck processes. Fluctuation limits of this kind were studied in [14, 15, 18]. In [18] it was shown that a class of distribution-valued Ornstein-Uhlenbeck diffusions arise as fluctuation limits of immigration superprocesses by three different scalings (high density, small branching and large scale). The tightness was established there by checking Kolmogorov's criterion based on moment calculations. A high density fluctuation limit theorem for branching particle systems was proved in [15]. In [14] a fluctuation limit of a immigration superprocess with small branching rate was proved. However, the tightness of the convergent

[1]Supported by NSFC (No. 10121101)
[2]Corresponding author.

sequence was not established there, so the limit theorem there only asserts the convergence of the finite dimensional distributions. The main difficulty comes from the general branching mechanisms. In this note, we shall prove that the sequence considered in [14] is really tight in a typical situation. Indeed, for a large class of Feller underlying processes, we show that the sequence is tight in the space of càdlàg paths with values in a suitable Sobolev space. Instead of Kolmogorov's criterion, we shall use a criterion of tightness given in Either and Kurtz [10] based on the martingale characterization. A number of properties of Ornstein-Uhlenbeck processes on Hilbert spaces have been proved under the formulation of generalized Mehler semigroups introduced by Bogachev et al [2]; see, e.g., [7, 8, 12, 20] and the references therein. Our Theorem 4.1 puts the limiting Ornstein-Uhlenbeck process into this framework so that one can easily derive regularities and properties of the processes from the existing literature.

2 Immigration superprocesses

Let $C(\mathbb{R}^d)$ be the Banach space of bounded and continuous functions on $\mathbb{R}^d$ endowed with the supremum norm $\|\cdot\|$. Let $C^2(\mathbb{R}^d)$ denote the space of smooth functions on $\mathbb{R}^d$ with all partial derivatives up to the second order belonging to $C(\mathbb{R}^d)$. Let $(P_t^0)_{t\geq 0}$ be a Feller transition semigroup on $C(\mathbb{R}^d)$ with strong generator A_0. For concreteness we assume that $C^2(\mathbb{R}^d) \subset \mathscr{D}(A_0) \subset C(\mathbb{R}^d)$ and

$$\begin{aligned} A_0 f(x) &= \sum_{i,j=1}^d \alpha_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{j=1}^d \beta_j(x)\frac{\partial f}{\partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d}\Big[f(x+y) - f(x) - \frac{1}{1+|y|^2}\sum_{j=1}^d y_j \frac{\partial f}{\partial x_j}(x)\Big]\mu(x,dy) \end{aligned} \tag{2.1}$$

for $f \in C^2(\mathbb{R}^d)$, where $\alpha_{ij}(x)$ and $\beta_j(x)$ are continuous functions and $\mu(x,dy)$ is a Lévy kernel on $\mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d}\Big[\sum_{i,j=1}^d |\alpha_{ij}(x)| + \sum_{j=1}^d |\beta_j(x)| + \int_{\mathbb{R}^d\setminus\{0\}} (1\wedge |y|^2)\mu(x,dy)\Big] < \infty.$$

By a result of Courrège [4], the above representation is valid for a large class of Feller generators.

Let $M(\mathbb{R}^d)$ be the space of finite Borel measures on $\mathbb{R}^d$ equipped with the topology of weak convergence. Write $\mu(f) = \int f d\mu$ for $f \in C(\mathbb{R}^d)$ and $\mu \in M(\mathbb{R}^d)$. Let $C(\mathbb{R}^d)^+$ denote the subset of non-negative elements of $C(\mathbb{R}^d)$. Let ϕ be a continuous function on $\mathbb{R}^d \times [0,\infty)$ given by

$$\phi(x,z) = c(x)z^2 + \int_0^\infty (e^{-zu} - 1 + zu)n(x,du), \quad x\in\mathbb{R}^d, z\geq 0, \tag{2.2}$$

where $c(\cdot) \in C(\mathbb{R}^d)^+$ and $u^2 n(x,du)$ is a bounded kernel from $\mathbb{R}^d$ to $[0,\infty)$. We fix a strictly positive function $b(\cdot) \in C(\mathbb{R}^d)^+$ which is bounded away from zero. It is well-known that the evolution equation

$$V_t f(x) + \int_0^t ds \int_{\mathbb{R}^d}[\phi(y, V_s f(y)) + b(y)V_s f(y)]P_{t-s}^0(x,dy) = P_t^0 f(x), \quad t\geq 0, x\in\mathbb{R}^d, \tag{2.3}$$

defines a semigroup of non-linear operators $(V_t)_{t\geq 0}$ on $C(\mathbb{R}^d)^+$. Moreover, there is a transition semigroup $(Q_t)_{t\geq 0}$ on $M(\mathbb{R}^d)$ determined by

$$\int_{M(\mathbb{R}^d)} e^{-\nu(f)} Q_t(\mu, d\nu) = \exp\{-\mu(V_t f)\}, \quad f \in C(\mathbb{R}^d)^+. \tag{2.4}$$

A Markov process X with transition semigroup $(Q_t)_{t\geq 0}$ is called a *Dawson-Watanabe superprocess*; see, e.g., Dawson [5]. (Since $b(\cdot)$ is positive and bounded away from zero, the superprocess has a subcritical branching mechanism.) A number of basic regularities of Dawson-Watanabe superprocesses were proved in Fitzsimmons [11]. In particular, it follows from [11, Corollary 3.6] that the Dawson-Watanabe superprocess has a Hunt realization.

Let $(P_t)_{t\geq 0}$ be the semigroup generated by the operator $A := A_0 - b$. It is easy to show that if $\xi = (\Omega, \mathscr{F}, \mathscr{F}_t, \xi_t, \mathbf{P}_x^0)$ is a Hunt realization of $(P_t^0)_{t\geq 0}$, then

$$P_t f(x) = \mathbf{P}_x^0 f(\xi_t) \exp\bigg\{ -\int_0^t b(\xi_s) ds \bigg\}.$$

In terms of $(P_t)_{t\geq 0}$, we can rewrite (2.3) as

$$V_t f(x) + \int_0^t \int_{\mathbb{R}^d} \phi(y, V_s f(y)) P_{t-s}(x, dy) = P_t f(x), \qquad t \geq 0, x \in \mathbb{R}^d, \tag{2.5}$$

which is more convenient for the discussions in this work.

Given $m \in M(\mathbb{R}^d)$, we can define another transition semigroup $(Q_t^m)_{t\geq 0}$ on $M(\mathbb{R}^d)$ by

$$\int_{M(\mathbb{R}^d)} e^{-\nu(f)} Q_t^m(\mu, d\nu) = \exp\bigg\{ -\mu(V_t f) - \int_0^t m(V_s f) ds \bigg\}, \quad f \in C(\mathbb{R}^d)^+. \tag{2.6}$$

A Markov process in $M(\mathbb{R}^d)$ is called an *immigration superprocess with parameters* (A, ϕ, m) if its transition semigroup $(Q_t^m)_{t\geq 0}$. Intuitively, new particles immigrate to $\mathbb{R}^d$ according to a Poisson random field on the space-time space with intensity proportional to $m(dx)ds$; see, e.g., [16, 17]. It was shown in [16] that the immigration superprocess has regularities similar to the superprocess without immigration defined by (2.4) and (2.5). Indeed, following the arguments of [11] it can be proved that the immigration superprocess also has a Hunt realization $Y = (W, \mathscr{G}, \mathscr{G}_t, Y_t, \mathbf{Q}_\mu^m)$.

As in [9] and [11], it is not hard to show that the immigration superprocess has generator L defined by

$$\begin{aligned} LF(\mu) \;=\; & \int_{\mathbb{R}^d} AF'(\mu)(x)\mu(dx) + \int_{\mathbb{R}^d} F'(\mu, x) m(dx) + \int_{\mathbb{R}^d} c(x) F''(\mu, x)\mu(dx) \\ & + \int_{\mathbb{R}^d} \mu(dx) \int_0^\infty [F(\mu + u\delta_x) - F(\mu) - uF'(\mu, x)] n(x, du), \end{aligned} \tag{2.7}$$

where

$$F'(\mu, x) = \lim_{r\to 0} \frac{1}{r}[F(\mu + r\delta_x) - F(\mu)]$$

and $F''(\mu, x)$ is defined by the limit with $F(\mu)$ replaced by $F'(\mu, x)$. The domain $\mathscr{D}(L)$ of L contains all functions F of the form

$$F(\mu) = h(\mu(f_1), \cdots, \mu(f_m)) \tag{2.8}$$

where $h \in C^2(\mathbb{R}^m)$ and $\{f_j : j = 1, \cdots, m\} \subset \mathscr{D}(A)$. If F is given by (2.8), then

$$\begin{aligned} LF(\mu) \quad = \quad & \sum_{i=1}^{m} h_i'(\mu(f_1), \cdots, \mu(f_m)) \int_{\mathbb{R}^d} Af_i(x)\mu(dx) \\ & + \sum_{i=1}^{m} h_i'(\mu(f_1), \cdots, \mu(f_m)) \int_{\mathbb{R}^d} f_i(x)m(dx) \\ & + \sum_{i,j=1}^{m} h_{ij}''(\mu(f_1), \cdots, \mu(f_m)) \int_{\mathbb{R}^d} c(x)f_i(x)f_j(x)\mu(dx) \\ & + \int_{\mathbb{R}^d} \mu(dx) \int_0^\infty [h(\mu(f_1) + uf_1(x), \cdots, \mu(f_m) + uf_m(x)) \\ & - h(\mu(f_1), \cdots, \mu(f_m)) - u \sum_{i=1}^{m} h_i'(\mu(f_1), \cdots, \mu(f_m))f_i(x)]n(x, du). \qquad (2.9) \end{aligned}$$

Proposition 2.1 *For each $f \in \mathscr{D}(A)$, the process*

$$\langle Y_t, f\rangle - \langle Y_0, f\rangle - \int_0^t \langle Y_s, Af\rangle ds - \langle m, f\rangle t, \quad t \geq 0, \qquad (2.10)$$

is a martingale.

Proof. By (2.9) and the martingale characterization of Markov processes, we see that (2.10) is a local martingale. Since $\phi_z'(x, 0) = 0$, we may differentiating both sides of (2.5) and (2.6) to show that

$$\mathbf{Q}_\mu^m[\langle Y_t, f\rangle] = \mu(P_t f) + \int_0^t m(P_s f)ds \qquad (2.11)$$

first for $f \in C(\mathbb{R}^d)^+$ and then for all $f \in C(\mathbb{R}^d)$. Similarly we get

$$\begin{aligned} \mathbf{Q}_\mu^m[\langle Y_t, f\rangle^2] \quad = \quad & \Big[\mu(P_t f) + \int_0^t m(P_s f)ds\Big]^2 + \int_0^t \mu P_{t-s}[\phi_z''(\cdot, 0)(P_s f)^2]ds \\ & + \int_0^t du \int_0^u mP_{u-s}[\phi_z''(\cdot, 0)(P_s f)^2]ds \qquad (2.12) \end{aligned}$$

Therefore, $\mathbf{Q}_\mu^m[\langle Y_t, f\rangle^2]$ is a locally bounded function of $t \geq 0$. It follows that the process (2.10) has locally bounded second moment, so it is actually a martingale. □

From (2.6) it is easy to see that, as $t \to \infty$ the distribution of Y_t converges to the probability measure $Q^m(d\nu)$ on $M(\mathbb{R}^d)$ such that

$$\int_{M(\mathbb{R}^d)} e^{-\nu(f)} Q^m(d\nu) = \exp\Big\{ - \int_0^\infty m(V_s f)ds\Big\}, \qquad f \in C(\mathbb{R}^d)^+. \qquad (2.13)$$

Moreover, it is easy to show that

$$\gamma(f) = \int_0^\infty m(P_s f)ds = \int_{M(\mathbb{R}^d)} \nu(f)Q^m(d\nu), \qquad f \in C(\mathbb{R}^d)^+, \qquad (2.14)$$

defines a purely excessive measure $\gamma \in M(\mathbb{R}^d)$ for $(P_t)_{t\geq 0}$. We are interested in the asymptotic fluctuations of the immigration superprocess around the long-term average γ.

Proposition 2.2 *Let $Z_t = Y_t - \gamma$. Then for any $f \in C(\mathbb{R}^d)$ we have $\mathbf{Q}^m_\gamma[\langle Z_t, f\rangle] = 0$ and*

$$\mathbf{Q}^m_\gamma[\langle Z_t, f\rangle^2] = \int_0^t \gamma(\phi''_z(\cdot, 0)(P_s f)^2)ds, \tag{2.15}$$

where

$$\phi''_z(x, 0) = 2c(x) + \int_0^\infty u^2 n(x, du).$$

Moreover, if $f \in \mathscr{D}(A)$, the process

$$\langle Z_t, f\rangle - \int_0^t \langle Z_s, Af\rangle ds, \qquad t \geq 0, \tag{2.16}$$

is a martingale.

Proof. By (2.14) it is easy to show that

$$\gamma(P_t f) + \int_0^t m(P_s f)ds = \gamma(f) \tag{2.17}$$

for each $t \geq 0$. Then we have $\mathbf{Q}^m_\gamma[\langle Z_t, f\rangle] = 0$ from (2.11). From (2.11) and (2.12) we obtain

$$\begin{aligned}
\mathbf{Q}^m_\gamma[\langle Z_t, f\rangle^2] &= \int_0^t \gamma P_{t-s}[\phi''_z(\cdot, 0)(P_s f)^2]ds + \int_0^t ds \int_s^t m P_{u-s}[\phi''_z(\cdot, 0)(P_s f)^2]du \\
&= \int_0^t \gamma P_{t-s}[\phi''_z(\cdot, 0)(P_s f)^2]ds + \int_0^t ds \int_0^{t-s} m P_r[\phi''_z(\cdot, 0)(P_s f)^2]dr.
\end{aligned}$$

Then we get (2.15) by applying (2.17) with f replaced by $\phi''_z(\cdot, 0)(P_s f)^2$. Using (2.17) to Af, we have

$$\begin{aligned}
\langle \gamma, Af\rangle t &= \int_0^t \left[\int_0^s m P_r(Af)dr + \gamma P_s(Af)\right]ds \\
&= \int_0^t m(P_s f - f)ds + \int_0^t \gamma P_s(Af)ds \\
&= \int_0^t m(P_s f - f)ds + \gamma(P_t f - f) \\
&= -\langle m, f\rangle t.
\end{aligned}$$

It follows that

$$\langle Z_t, f\rangle - \int_0^t \langle Z_s, Af\rangle ds = \langle Y_t, f\rangle - \langle \gamma, f\rangle - \int_0^t \langle Y_s, Af\rangle ds - \langle m, f\rangle t.$$

Then the martingale property of (2.16) follows from Proposition 2.1. $\square$

3 Convergence in the Schwartz space

Let $\gamma \in M(\mathbb{R}^d)$ be a purely excessive measure of the transition semigroup $(P_t)_{t\geq 0}$. Then there is a finite entrance law $(\eta_s)_{s>0}$ such that

$$\gamma = \int_0^\infty \eta_s ds;$$

see, e.g., Getoor [13, p.43]. Since all finite entrance laws for the Feller semigroup $(P_t^0)_{t\geq 0}$ are closable, Li [16, Lemma 2.1] implies that all those laws for $(P_t)_{t\geq 0}$ are also closable. In particular, there is $m \in M(\mathbb{R}^d)$ such that

$$\gamma = \int_0^\infty m P_s ds.$$

For any integer $k \geq 1$ let $\phi_k(x,z) = \phi(x, z/k)$. Suppose that $\{Y_t^{(k)} : t \geq 0\}$ is a càdlàg immigration superprocess with parameters (A, ϕ_k, m) and $Y_0^{(k)} = \gamma$. Let $S_k^\gamma(\mathbb{R}^d)$ be the set of signed-measures μ on $\mathbb{R}^d$ such that $\mu + k\gamma \in M(\mathbb{R}^d)$. We define the $S_k^\gamma(\mathbb{R}^d)$-valued process $\{Z_t^{(k)}, t \geq 0\}$ by

$$Z_t^{(k)} = k[Y_t^{(k)} - \gamma], \qquad t \geq 0. \tag{3.1}$$

By Proposition 2.2 we have $\mathbf{E}[\langle Z_t^{(k)}, f\rangle] = 0$ for each $f \in C(\mathbb{R}^d)$. By the calculations in [14], $\{Z_t^{(k)} : t \geq 0\}$ is a Markov process with transition semigroup

$$\int_{S_k^\gamma(\mathbb{R}^d)} e^{-\nu(f)} R_t^{(k)}(\mu, d\nu) = \exp\bigg\{ - \mu(k^{-1} V_t^{(k)}(kf)) + \int_0^t \gamma(\phi(k^{-1} V_s^{(k)}(kf)))ds \bigg\}, \tag{3.2}$$

where $(V_t^{(k)})_{t\geq 0}$ is determined by

$$V_t^{(k)} f(x) + \int_0^t \int_{\mathbb{R}^d} \phi_k(y, V_s^{(k)} f(y)) P_{t-s}(x, dy) = P_t f(x). \tag{3.3}$$

Let $\mathscr{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing real functions on $\mathbb{R}^d$. That is, each $f \in \mathscr{S}(\mathbb{R}^d)$ is infinitely differentiable and for each non-negative integer k and each non-negative integer-valued vector $\alpha = (\alpha_1, \cdots, \alpha_d)$ we have

$$\lim_{|x|\to\infty} |x|^k |\partial^\alpha f(x)| = 0,$$

where

$$\partial^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_d^{\alpha_d}} f(x_1, \ldots, x_d)$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The topology of $\mathscr{S}(\mathbb{R}^d)$ is defined by the sequence of semi-norms

$$f \mapsto p_n(f) := \sup\{(1 + |x|^n)|\partial^\alpha f(x)| : x \in \mathbb{R}^d, |\alpha| \leq n\}, \qquad n = 0, 1, 2, \cdots.$$

Let $\mathscr{S}'(\mathbb{R}^d)$ denote the dual space of $\mathscr{S}(\mathbb{R}^d)$ equipped with the strong topology. Then both $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ are nuclear spaces; see, e.g., Schaefer [21, p.107]. We regard $S_k^\gamma(\mathbb{R}^d)$ as a subspace of $\mathscr{S}'(\mathbb{R}^d)$ in the usual sense. Thus $\{Z_t^{(k)} : t \geq 0\}$ is a processes taking values from $\mathscr{S}'(\mathbb{R}^d)$.

Theorem 3.1 *As $k \to \infty$, the finite dimensional distributions of $\{Z_t^{(k)} : t \geq 0\}$ converge to those of the $\mathscr{S}'(\mathbb{R}^d)$-valued Markov process $\{Z_t : t \geq 0\}$ with $Z_0 = 0$ and with semigroup $(R_t)_{t\geq 0}$ determined by*

$$\int_{\mathscr{S}'(\mathbb{R}^d)} e^{i\nu(f)} R_t(\mu, d\nu) = \exp\left\{ i\mu(P_t f) + \int_0^t \gamma(\phi(-iP_s f))ds \right\}, \quad f \in \mathscr{S}(\mathbb{R}^d), \tag{3.4}$$

where $\phi(-iP_s f)$ is given by (2.2) with z replaced by $-iP_s f(x)$.

The above theorem was established by Gorostiza and Li [14]. The main purpose of this section is to show the weak convergence of $\{Z_t^{(k)} : t \geq 0\}$ on the space $D([0,\infty), \mathscr{S}'(\mathbb{R}^d))$. To this end it suffices to prove the tightness of the sequence $\{Z_t^{(k)} : t \geq 0; k \geq 1\}$.

Lemma 3.1 *For $\eta > 0$, $T > 0$ and $f \in \mathscr{D}(A)$, there is a constant $C = C(T, \|f\|, \|Af\|)$ such that*

$$\sup_{k\geq 1} \mathbf{P}\Big\{ \sup_{0\leq t\leq T} |\langle Z_t^{(k)}, f\rangle| > \eta \Big\} \leq C/\eta. \tag{3.5}$$

Proof. Observe that

$$\begin{aligned} \mathbf{P}\Big\{ \sup_{0\leq t\leq T} |\langle Z_t^{(k)}, f\rangle| > \eta \Big\} \;&\leq\; \mathbf{P}\Big\{ \sup_{0\leq t\leq T} \Big| \langle Z_t^{(k)}, f\rangle - \int_0^t \langle Z_s^{(k)}, Af\rangle ds \Big| > \eta/2 \Big\} \\ &\quad + \mathbf{P}\Big\{ \int_0^T |\langle Z_s^{(k)}, Af\rangle| ds > \eta/2 \Big\}. \end{aligned} \tag{3.6}$$

By Proposition 2.2,

$$\langle Z_t^{(k)}, f\rangle - \int_0^t \langle Z_s^{(k)}, Af\rangle ds, \quad t \geq 0,$$

is a right continuous martingale. Then by Doob's martingale inequality, the first term on the right hand side of (3.6) is dominated by

$$\frac{2}{\eta}\mathbf{E}\Big| \langle Z_T^{(k)}, f\rangle - \int_0^T \langle Z_s^{(k)}, Af\rangle ds \Big| \leq \frac{2}{\eta}\Big[\mathbf{E}^{1/2}[\langle Z_T^{(k)}, f\rangle^2] + \int_0^T \mathbf{E}^{1/2}[\langle Z_s^{(k)}, Af\rangle^2] ds \Big]$$

By (2.15), we get

$$\mathbf{E}\{\langle Z_s^{(k)}, h\rangle^2\} = \int_0^s \gamma(\phi_z''(\cdot,0)|P_r h|^2) dr \leq \|\phi_z''(\cdot,0)\| \|h\|^2 \gamma(1) T \tag{3.7}$$

for any $0 \leq s \leq T$ and $h \in C(\mathbb{R}^d)$. Applying this estimate to f and Af we obtain

$$\begin{aligned} &\mathbf{P}\Big\{ \sup_{0\leq t\leq T} |\langle Z_t^{(k)}, f\rangle - \int_0^t \langle Z_s^{(k)}, Af\rangle ds| > \eta/2 \Big\} \\ &\qquad \leq \frac{2}{\eta}\Big[\|\phi_z''(\cdot,0)\| \|f\|^2 \gamma(1) T \Big]^{1/2} + \frac{2T}{\eta}\Big[\|\phi_z''(\cdot,0)\| \|Af\|^2 \gamma(1) T \Big]^{1/2}. \end{aligned}$$

By Chebyshev's inequality,

$$\begin{aligned}
\mathbf{P}\Big\{\int_0^T |\langle Z_s^{(k)}, Af\rangle| ds \geq \eta/2\Big\} &\leq \frac{2}{\eta}\int_0^T \mathbf{E}[|\langle Z_s^{(k)}, Af\rangle|]ds \\
&\leq \frac{2}{\eta}\int_0^T \mathbf{E}^{1/2}[\langle Z_s^{(k)}, Af\rangle^2]ds \\
&\leq \frac{2T}{\eta}\Big[\|\phi_z''(\cdot,0)\|\|Af\|^2\gamma(1)T\Big]^{1/2}.
\end{aligned}$$

Then we obtain (3.5). □

Lemma 3.2 *For any $f \in \mathscr{D}(A)$, the sequence $\{\langle Z_t^{(k)}, f\rangle : t \geq 0; k \geq 1\}$ is tight in $D([0,\infty), \mathbb{R})$.*

Proof. Let $\mathscr{L}_k$ denote the generator of $\{Z_t^{(k)}, t \geq 0\}$. Let $F(\mu) = h(\langle \mu, f\rangle)$ for $h \in C^2(\mathbb{R})$ and $\mu \in S_k^\gamma(\mathbb{R}^d)$. By (2.9) it is not hard to show that

$$\begin{aligned}
\mathscr{L}_k F(\mu) &= h'(\langle \mu, f\rangle)\langle \mu, Af\rangle + k^{-1}h''(\langle \mu, f\rangle)\langle \mu, cf^2\rangle + h''(\langle \mu, f\rangle)\langle \gamma, cf^2\rangle \\
&\quad + k^{-1}\int_{\mathbb{R}^d}\mu(dx)\int_0^\infty l(x,f,h,\mu)n(x,du) \\
&\quad + \int_{\mathbb{R}^d}\gamma(dx)\int_0^\infty l(x,f,h,\mu)n(x,du), \qquad (3.8)
\end{aligned}$$

where

$$l(x,f,h,\mu) = \int_0^\infty \Big[h(\langle \mu, f\rangle + uf(x)) - h(\langle \mu, f\rangle) - h'(\langle \mu, f\rangle)uf(x)\Big]n(x,du).$$

Since

$$F(Z_t^{(k)}) - F(Z_0^{(k)}) - \int_0^t \mathscr{L}_k F(Z_s^{(k)})ds, \quad t \geq 0,$$

is a martingale, according to Ethier and Kurtz [10, p.142 and p.145], to obtain the desired tightness it suffices to show

$$\sup_{k\geq 1}\int_0^T \mathbf{E}[|\mathscr{L}_k F(Z_s^{(k)})|^2]ds < \infty. \qquad (3.9)$$

for each $T > 0$. Recall that $Z_s^{(k)} = k[Y_s^{(k)} - \gamma]$. Then we have

$$\begin{aligned}
\mathscr{L}^{(k)}F(Z_s^{(k)}) &= h'(\langle Z_s^{(k)}, f\rangle)\langle Z_s^{(k)}, Af\rangle + h''(\langle Z_s^{(k)}, f\rangle)\langle Y_s^{(k)}, cf^2\rangle \\
&\quad + \int_{\mathbb{R}^d} l(x,f,h,Z_s^{(k)})Y_s^{(k)}(dx).
\end{aligned}$$

It follows that

$$|\mathscr{L}^{(k)}F(Z_s^{(k)})|^2 \leq C_1[\langle Z_s^{(k)}, Af\rangle^2 + \langle Y_s^{(k)}, cf^2\rangle^2 + \langle Y_s^{(k)}, l(f,h,Z_s^{(k)})\rangle^2] \qquad (3.10)$$

for some constant C_1 only depending on $\|h'\|$ and $\|h''\|$. By (3.7) we have

$$\mathbf{E}[\langle Z_s^{(k)}, Af\rangle^2] \leq \|\phi_z''(\cdot,0)\|\|Af\|^2\gamma(1)T.$$

From (2.12) it follows that

$$\mathbf{E}[\langle Y_t^{(k)}, g\rangle^2] \leq \|g\|^2[\gamma(1)^2 + m(1)^2T^2 + \|\phi_z''(\cdot,0)\|\gamma(1)T + \|\phi_z''(\cdot,0)\|m(1)T^2],$$

for any $g \in C(\mathbb{R}^d)$. By Taylor's expansion it is easy to find that

$$|l(x, f, h, Z_s^{(k)})| \leq C_2 \int_0^\infty u^2 n(x, du) \tag{3.11}$$

for a constant C_2 depending on $\|f\|$ and $\|h''\|$. Under our assumption, the right hand side of (3.11) is bounded in $x \in \mathbb{R}^d$. Thus (3.9) follows. $\square$

Theorem 3.2 *As $k \to \infty$, the sequence $\{Z_t^{(k)} : t \geq 0\}$ converges weakly to a process $\{Z_t : t \geq 0\}$ in $D([0,\infty), \mathscr{S}'(\mathbb{R}^d))$ with $Z_0 = 0$ and with transition semigroup $(R_t)_{t\geq 0}$ determined by (3.4).*

Proof. By Lemma 3.2 and Mitoma's result, the sequence $\{Z_t^{(k)} : t \geq 0; k \geq 1\}$ is tight in $D([0,\infty), \mathscr{S}'(\mathbb{R}^d))$. Then the result follows from Theorem 3.1. See, e.g., Walsh [22, pp.364-365]. $\square$

4 Convergence in Sobolev spaces

In this section, we give a formulation of the fluctuation limit theorem in a Sobolev space. This puts the limiting Ornstein-Uhlenbeck process into the framework of Bogachev et al [2] and Fuhrman and Röckner [12]; see also Dawson et al [8]. For any $f \in \mathscr{S}(\mathbb{R}^d)$ we define its Fourier transform $\hat{f}$ by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx,$$

where " $\cdot$ " denotes the inner product on $\mathbb{R}^d$. For $f \in \mathscr{S}'(\mathbb{R}^d)$, we define the Fourier transform $\hat{f}$ as a distribution by $\langle \hat{f}, g\rangle = \langle f, \hat{g}\rangle$ for $g \in \mathscr{S}(\mathbb{R}^d)$. For each $s \in \mathbb{R}$ we consider the Sobolev space

$$H^s(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : (1 + |x|^2)^s \hat{f}(x) \in L^2(\mathbb{R}^d)\}$$

which is endowed with the Hilbertian norm $\|\cdot\|_s$ defined by

$$\|f\|_s^2 = \int_{\mathbb{R}^d} (1 + |x|^2)^s |\hat{f}(x)|^2 dx. \tag{4.1}$$

It is well-known that each $H^s(\mathbb{R}^d)$ is a separable Hilbert space and its strong topological dual can be identified with $H^{-s}(\mathbb{R}^d)$. In particular, for any integer $n \geq 0$ we may also define $H^n(\mathbb{R}^d)$ by

$$H^n(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \partial^\alpha f \in L^2(\mathbb{R}^d) \text{ whenever } |\alpha| \leq n\}$$

with the norm $\|\cdot\|_{n,2}$ defined by

$$\|f\|_{n,2}^2 = \sum_{|\alpha|\leq n} \int_{\mathbb{R}^d} |\partial^\alpha f(x)|^2 dx,$$

which is equivalent to the norm $\|\cdot\|_n$ given by (4.1). Moreover, for any $s \leq t$ we have

$$\mathscr{S}'(\mathbb{R}^d) \supset H^s(\mathbb{R}^d) \supset H^t(\mathbb{R}^d) \supset \mathscr{S}(\mathbb{R}^d) \tag{4.2}$$

with continuous embeddings. See, e.g., Barros-Neto [1, Theorem 5.5]. Let $\{Z_t : t \geq 0\}$ and $\{Z_t^{(k)} : t \geq 0\}$ be as in the last section. Now we have

Theorem 4.1 *For $n > d+2$ the process $\{Z_t : t \geq 0\}$ has a realization in $D([0,\infty), H^{-n}(\mathbb{R}^d))$ and $\{Z_t^{(k)} : t \geq 0\}$ converges weakly to $\{Z_t : t \geq 0\}$ in $D([0,\infty), H^{-n}(\mathbb{R}^d))$.*

Proof. By Sobolev's embedding theorem, when $p > d/2$, any function $f \in H^p(\mathbb{R}^d)$ possess bounded and continuous derivatives in the classical sense. Then $H^{p+2}(\mathbb{R}^d) \subset C^2(\mathbb{R}^d) \subset \mathscr{D}(A)$ and (3.5) holds for $f \in H^{p+2}(\mathbb{R}^d)$. Since $\|\cdot\|_{p+2} < \|\cdot\|_n$ in the Hilbert-Schmidt sense for any $n > d/2+p+2$, the result follows from Theorem 3.1 and Lemma 3.2. See, e.g., Walsh [22, p.335 and p.365]. □

Acknowledgement. We thank Professor L.G. Gorostiza for some helpful comments on an earlier version of this note.

References

[1] Barros-Neto, J.: *An introduction to the theory of distributions.* Marcel Dekker, New York (1973). Chinese Translation (1981).

[2] Bogachev, V.I., Röckner, M. and Schmuland, B.: *Generalized Mehler semigroups and applications.* Probab. Theory Related Fields **105** (1996), 193-225.

[3] Bojdecki, T. and Gorostiza, L.G.: *Langevin equation for $\mathcal{S}'$-valued Gaussian processes and fluctuation limits of infinite particle systems.* Probab. Theory Related Fields **73** (1986), 227-244.

[4] Courrège, P.: *Sur la forme intégro-différentielle de opérateurs de C_K^∞ dans C satisfaisant au principe du maximum.* Sém. Théorie du Potentiel (1965/6), 38pp.

[5] Dawson, D.A.: *Measure-Valued Markov Processes.* In: Lect. Notes. Math. **1541**, 1-260, Springer-Verlag, Berlin (1993).

[6] Dawson, D.A., Fleischmann, K. and Gorostiza, L.G: *Stable hydrodynamic limit fluctuations of a critical branching particle system in a random medium.* Ann. Probab. **17** (1989), 1083-1117.

[7] Dawson, D.A. and Li, Z.H.: *Non-differentiable skew convolution semigroups and related Ornstein-Uhlenbeck processes.* Potent. Anal. **20** (2004), 285-302.

[8] Dawson, D.A., Li, Z.H., Schmuland, B. and Sun, W.: *Generalized Mehler semigroups and catalytic branching processes with immigration.* Potential Anal. **21** (2004), 75-97.

[9] El Karoui, N. and Roelly, S.: *Propriétés de martingales, explosion et representation de Lévy-Khintchine d'une classe de processus de branchement à valeurs mesures.* Stochastic Process. Appl. **38** (1991), 239-266.

[10] Ethier, S.N. and Kurtz, T.G.: *Markov Processes: Characterization and Convergence.* Wiley, New York (1986).

[11] Fitzsimmons, P.J.: *Construction and regularity of measure-valued Markov branching processes.* Israel J. Math. **64** (1988), 337-361.

[12] Fuhrman, M. and Röckner, M.: *Generalized Mehler semigroups: The non-Gaussian case.* Potential Anal. **12**

[13] Getoor, R.K.: *Excessive Measures.* Birkhäuser Boston, Inc., Boston, MA (1990).

[14] Gorostiza, L. G. and Li, Z.H.: *Fluctuation limits of measure-valued immigration processes with small branching.* In: "Modelos Estocásticos", Investigación **14** (1998), 261-268, Aportaciones Matemáticas, Sociedad Matemática Mexicana.

[15] Gorostiza, L.G. and Li, Z.H.: *High density fluctuations of immigration branching particle systems.* Stochastic Models (Ottawa, Ontario, 1998), CMS Conference Proceedings, Series **26** (2000), 159-171, Edited by Gorostiza, L.G. and Ivanoff, B.G., AMS, Providence, RI.

[16] Li, Z.H.: *Immigration structures associated with Dawson-Watanabe superprocesses.* Stochastic Process. Appl. **62** (1996), 73-86.

[17] Li, Z.H.: *Immigration processes associated with branching particle systems.* Adv. Appl. Probab. **30** (1998), 657-675.

[18] Li, Z.H.: *Measure-valued immigration diffusions and generalized Ornstein-Uhlenbeck diffusions.* Acta Mathematicae Applicatae Sinica **15** (1999), 310-320.

[19] Mitoma, I.: *Tightness of probabilities on $C([0,1],\mathscr{S})$ and $D([0,1],\mathscr{S}')$.* Ann. Probab. **11** (1983), 989-999.

[20] Röckner, M. and Wang, F.Y.: *Harnack and functional inequalities for generalized Mehler semigroups.* J. Funct. Anal. **203** (2003), 237-261.

[21] Schaefer, H.H.: *Topological Vector Spaces.* Springer-Verlag, New York (1980).

[22] Walsh, J.B.: *An Introduction to Stochastic Partial Differential Equations.* In: Lect. Notes Math. **1180** (1986), Springer-Verlag, Berlin.