

# A limit theorem for discrete Galton-Watson branching processes with immigration <sup>1</sup>

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**Abstract.** We provide a simple set of sufficient conditions for the weak convergence of discrete Galton-Watson branching processes with immigration to continuous time and continuous state branching processes with immigration.

**Mathematics Subject Classification (2000):** 60J80

**Keywords and phrases:** Markov chain, branching process with immigration, limit theorem, weak convergence, generating function.

**Abbreviated Title:** Branching processes with immigration

## 1 Introduction

Let  $(g, h)$  be a pair of probability generating functions. By a *discrete time and discrete state Galton-Watson branching process with immigration* (DBI-process) corresponding to  $(g, h)$  we mean a discrete-time Markov chain  $\{y(n) : n = 0, 1, 2, \dots\}$  with state space  $\mathbb{N} := \{0, 1, 2, \dots\}$  and one-step transition matrix  $P(i, j)$  defined by

$$\sum_{j=0}^{\infty} P(i, j)z^j = g(z)^i h(z), \quad i = 0, 1, 2, \dots, 0 \leq z \leq 1. \quad (1.1)$$

The intuitive meaning of the process is clear from (1.1). In particular, if  $h(z) \equiv 1$ , we simply call  $\{y(n) : n = 0, 1, 2, \dots\}$  a *discrete time and discrete state Galton-Watson branching process* (DB-process).

Kawazu and Watanabe (1971) studied systematically the limit theorems of DBI-processes. They also characterized completely the class of the limit processes as *continuous time and continuous state branching processes with immigration* (CBI-processes). Let us consider a special class of the CBI-processes introduced in Kawazu and Watanabe (1971). Suppose that  $R$  is a function on  $[0, \infty)$  defined by

$$R(\lambda) = \beta\lambda - \alpha\lambda^2 - \int_0^{\infty} \left( e^{-\lambda u} - 1 + \frac{\lambda u}{1 + u^2} \right) \mu(du), \quad (1.2)$$

where  $\beta \in \mathbb{R}$  and  $\alpha \geq 0$  are constants and  $(1 \wedge u^2)\mu(du)$  is a finite measure on  $(0, \infty)$ , and  $F$  is a function on  $[0, \infty)$  defined by

$$F(\lambda) = b\lambda + \int_0^{\infty} (1 - e^{-\lambda u}) m(du), \quad (1.3)$$

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<sup>1</sup> Supported by NCET and NSFC Grants (No. 10121101 and No. 10525103).

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where  $b \geq 0$  is a constant and  $(1 \wedge u)m(du)$  is a finite measure on  $(0, \infty)$ . A Markov process  $\{y(t) : t \geq 0\}$  with state space  $\mathbb{R}_+ := [0, \infty)$  is called a *CBI-process* if it has transition semigroup  $(P_t)_{t \geq 0}$  given by

$$\int_0^\infty e^{-\lambda y} P_t(x, dy) = \exp \left\{ -x\psi_t(\lambda) - \int_0^t F(\psi_s(\lambda)) ds \right\}, \quad \lambda \geq 0, \quad (1.4)$$

where  $\psi_t(\lambda)$  is the unique solution of

$$\frac{d\psi_t}{dt}(\lambda) = R(\psi_t(\lambda)), \quad \psi_0(\lambda) = \lambda. \quad (1.5)$$

Clearly, the transition semigroup  $(P_t)_{t \geq 0}$  defined by (1.4) is stochastically continuous. In particular, if  $F(\lambda) \equiv 0$ , we simply call  $\{y(t) : t \geq 0\}$  a *continuous time and continuous state branching process* (CB-process).

A CBI-process is said to be *conservative* if it does not explode, that is,  $\mathbf{P}_x\{y(t) < \infty\} = 1$  for every  $t \geq 0$  and  $x \in \mathbb{R}_+$  where  $\mathbf{P}_x$  denotes the conditional law given  $y(0) = x$ . By Kawazu and Watanabe (1971, Theorem 1.2), the process is conservative if and only if

$$\int_{0+} R^*(\lambda)^{-1} d\lambda = \infty,$$

where  $R^*(\lambda) = R(\lambda) \vee 0$ . (This is a correction to equation (1.21) of Kawazu and Watanabe (1971).)

Let  $\{b_k\}$  and  $\{c_k\}$  be sequences of positive numbers such that  $b_k \rightarrow \infty$  and  $c_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Let  $\{y_k(n) : n \geq 0\}$  be a sequence of DBI-processes corresponding to the parameters  $\{(g_k, h_k)\}$  and assume  $y_k(0) = c_k$ . Suppose that for all  $t \geq 0$  and  $\lambda \geq 0$  the limits

$$\lim_{k \rightarrow \infty} g_k^{[kt]}(e^{-\lambda/b_k})^{c_k} = \phi_1(t, \lambda) \quad \text{and} \quad \lim_{k \rightarrow \infty} \prod_{j=0}^{[kt]-1} h_k(g_k^j(e^{-\lambda/b_k})) = \phi_2(t, \lambda) \quad (1.6)$$

exist and the convergence is locally uniform in  $\lambda \geq 0$  for each fixed  $t \geq 0$ , where  $g_k^j$  denotes the  $j$ -order composition of  $g_k$  and  $[kt]$  denotes the integer part of  $kt$ . The following result was proved in Kawazu and Watanabe (1971, Theorem 2.1):

**Theorem 1.1** *Suppose that (1.6) holds and  $\phi_1(t, \lambda) < 1$  for some  $t > 0$  and  $\lambda > 0$ . Then  $\{y_k([kt])/b_k : t \geq 0\}$  converges in finite-dimensional distributions to a stochastically continuous and conservative CBI-process  $\{y(t) : t \geq 0\}$  with transition semigroup given by (1.4).*

Based on this theorem, Kawazu and Watanabe (1971) showed that, given each stochastically continuous and conservative CBI-process  $\{y(t) : t \geq 0\}$ , there is a sequence of positive numbers  $\{b_k\}$  with  $b_k \rightarrow \infty$  and a sequence of DBI-processes  $\{y_k(n) : n \geq 0\}$  such that  $\{y_k([kt])/b_k : t \geq 0\}$  converges in finite-dimensional distributions to  $\{y(t) : t \geq 0\}$ . Their results have become the basis of many studies of branching processes with immigration; see e.g. Pitman and Yor (1982) and Shiga and Watanabe (1973). On the other hand, since condition (1.6) involves complicated compositions of the probability generating functions, it is some times not so easy to verify. In view of the characterizations (1.1), (1.4) and (1.5) of the two classes of processes, one naturally expect some simple sufficient conditions for the convergence of the DBI-processes to the CBI-processes given in terms of the parameters  $(g, h)$  and  $(R, F)$ .

The purpose of this note is to provide a set of conditions of this type. For the convenience of proof, we shall discuss the convergence of  $\{y_k([\gamma_k t])/k : t \geq 0\}$  for some sequence of positive numbers  $\{\gamma_k\}$  with  $\gamma_k \rightarrow \infty$ , which is slightly different from the scaling of Kawazu and Watanabe (1971). Instead of the convergence of finite-dimensional distributions, we shall consider the weak convergence on the space of càdlàg functions  $D([0, \infty), \mathbb{R}_+)$ .

## 2 The limit theorem

In this section, we prove a limit theorem for DBI-processes on the space  $D([0, \infty), \mathbb{R}_+)$ . Let  $F$  be defined by (1.3). For simplicity we assume the function  $R$  is given by

$$R(\lambda) = \beta\lambda - \alpha\lambda^2 - \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\mu(du), \quad \lambda \geq 0, \quad (2.1)$$

where  $\beta \in \mathbb{R}$  and  $\alpha \geq 0$  are constants and  $(u \wedge u^2)\mu(du)$  is a finite measure on  $(0, \infty)$ . Suppose that  $\{y(t) : t \geq 0\}$  is a CBI-process corresponding to  $(R, F)$ . Let  $\{y_k(n) : n \geq 0\}$  be a sequence of DBI-processes corresponding to the parameters  $\{(g_k, h_k)\}$  and let  $\{\gamma_k\}$  be a sequence of positive numbers. For  $0 \leq \lambda \leq k$  set

$$F_k(\lambda) = \gamma_k[1 - h_k(1 - \lambda/k)] \quad (2.2)$$

and

$$R_k(\lambda) = k\gamma_k[(1 - \lambda/k) - g_k(1 - \lambda/k)]. \quad (2.3)$$

Let us consider the following set of conditions:

(2.A) As  $k \rightarrow \infty$ , we have  $\gamma_k \rightarrow \infty$  and  $\gamma_k/k \rightarrow$  some  $\gamma_0 \geq 0$ .

(2.B) As  $k \rightarrow \infty$ , the sequence  $\{F_k\}$  defined by (2.2) converges to a continuous function.

(2.C) The sequence  $\{R_k\}$  defined by (2.3) is uniformly Lipschitz on each bounded interval and converges to a continuous function as  $k \rightarrow \infty$ .

We remark that conditions (2.B) and (2.C) parallel the sufficient conditions for the convergence of continuous-time and discrete state branching processes with immigration, see e.g., Li (1992) for the discussions in the setting of measure-valued processes. Based the results of Li (1991), the following lemma can be proved by modifying the arguments of the proofs of Li (1992, Lemmas 3.4 and 4.1).

**Lemma 2.1** (i) Under conditions (2.B) and (2.C), the limit functions  $F$  and  $R$  of  $\{F_k\}$  and  $\{R_k\}$  have representations (1.3) and (2.1), respectively. (ii) For any  $(F, R)$  given by (1.3) and (2.1), there are sequences  $\{\gamma_k\}$  and  $\{(g_k, h_k)\}$  as above such that (2.A), (2.B) and (2.C) hold with  $F_k \rightarrow F$  and  $R_k \rightarrow R$ .

For  $\lambda \geq 0$  we set

$$S_k(\lambda) = k\gamma_k[(1 - \lambda/k) - g_k(e^{-\lambda/k})]. \quad (2.4)$$

**Lemma 2.2** Under conditions (2.A) and (2.C), let  $R = \lim_{k \rightarrow \infty} R_k$ . Then we have

$$\lim_{k \rightarrow \infty} S_k(\lambda) = R(\lambda) - \gamma_0\lambda^2/2 \quad \text{and} \quad \lim_{k \rightarrow \infty} \gamma_k[1 - g_k(e^{-\lambda/k})] = \gamma_0\lambda \quad (2.5)$$

uniformly on each bounded interval.

*Proof.* By mean-value theorem we have

$$S_k(\lambda) = R_k(\lambda) - k\gamma_k g'_k(\eta_k)(e^{-\lambda/k} - 1 + \lambda/k), \quad (2.6)$$

where  $1 - \lambda/k < \eta_k < e^{-\lambda/k}$  and  $g'_k$  denotes the derivative of  $g_k$ . Under condition (2.C), the sequence  $R'_k(\lambda) = \gamma_k[g'_k(1 - \lambda/k) - 1]$  is uniformly bounded on each bounded interval  $\lambda \in [0, l]$  for  $l \geq 0$ . Then  $g'_k(1 - \lambda/k) \rightarrow 1$  uniformly on each bounded interval. In particular, we have  $g'_k(\eta_k) \rightarrow 1$  and the first equality in (2.5) follows from (2.A) and (2.6). The second equality follows by a similar argument.  $\square$

**Theorem 2.1** *Suppose conditions (2.A), (2.B) and (2.C) hold with  $F = \lim_{k \rightarrow \infty} F_k$  and  $R = \lim_{k \rightarrow \infty} R_k$ . If  $y_k(0)/k$  converges in distribution to  $y(0)$ , then  $\{y_k([\gamma_k t])/k : t \geq 0\}$  converges in distribution on  $D([0, \infty), \mathbb{R}_+)$  to the CBI-process  $\{y(t) : t \geq 0\}$  corresponding to  $(R, F)$  with initial value  $y(0)$ .*

*Proof.* Let  $(P_t)_{t \geq 0}$  denote the transition semigroup of the CBI-process corresponding to  $(R, F)$ . For  $\lambda > 0$  and  $x \geq 0$  set  $e_\lambda(x) = e^{-\lambda x}$ . We denote by  $D_1$  the linear hull of  $\{e_\lambda : \lambda > 0\}$ . Then  $D_1$  is an algebra which strongly separates the points of  $\mathbb{R}_+$ . Let  $C_0(\mathbb{R}_+)$  be the space of continuous functions on  $\mathbb{R}_+$  vanishing at infinity. By the Stone-Weierstrass theorem,  $D_1$  is dense in  $C_0(\mathbb{R}_+)$  for the supremum norm; see, e.g., Hewitt and Stromberg (1965, pp.98-99). For  $\lambda > 0$  we set

$$Ae_\lambda(x) = -e^{-\lambda x} [xR(\lambda) + F(\lambda)], \quad x \in \mathbb{R}_+, \quad (2.7)$$

and extend the definition of  $A$  to  $D_1$  by linearity. Then  $A := \{(f, Af) : f \in D_1\}$  is a linear subspace of  $C_0(\mathbb{R}_+) \times C_0(\mathbb{R}_+)$ . Since  $D_1$  is invariant under  $(P_t)_{t \geq 0}$ , it is a core of  $A$ ; see, e.g., Ethier and Kurtz (1986, p.17). With those observations it is not hard to see that the semigroup  $(P_t)_{t \geq 0}$  is generated by the closure of  $A$ ; see, e.g., Ethier and Kurtz (1986, p.15 and p.17). Note that  $\{y_k(n)/k : n \geq 0\}$  is a Markov chain with state space  $E_k := \{0, 1/k, 2/k, \dots\}$  and one-step transition probability  $Q_k(x, dy)$  determined by

$$\int_{E_k} e^{-\lambda y} Q_k(x, dy) = g_k(e^{-\lambda/k})^{kx} h_k(e^{-\lambda/k}).$$

Then the (discrete) generator  $A_k$  of  $\{y_k([\gamma_k t])/k : t \geq 0\}$  is given by

$$\begin{aligned} A_k e_\lambda(x) &= \gamma_k \left[ g_k(e^{-\lambda/k})^{kx} h_k(e^{-\lambda/k}) - e^{-\lambda x} \right] \\ &= \gamma_k \left[ \exp\{xk\alpha_k(\lambda)(g_k(e^{-\lambda/k}) - 1)\} \exp\{\beta_k(\lambda)(h_k(e^{-\lambda/k}) - 1)\} - e^{-\lambda x} \right], \end{aligned}$$

where

$$\alpha_k(\lambda) = (g_k(e^{-\lambda/k}) - 1)^{-1} \log g_k(e^{-\lambda/k})$$

and  $\beta_k(\lambda)$  is defined by the same formula with  $g_k$  replaced by  $h_k$ . Under conditions (2.A), (2.B) and (2.C), it is easy to show that

$$\lim_{k \rightarrow \infty} (g_k(e^{-\lambda/k}) - 1) = \lim_{k \rightarrow \infty} (h_k(e^{-\lambda/k}) - 1) = 0$$

and

$$\lim_{k \rightarrow \infty} \alpha_k(\lambda) = \lim_{k \rightarrow \infty} \beta_k(\lambda) = 1.$$

Then we have

$$A_k e_\lambda(x) = -e^{-\lambda x} [x\alpha_k(\lambda)S_k(\lambda) + x\gamma_k(\alpha_k(\lambda) - 1)\lambda + H_k(\lambda)] + o(1), \quad (2.8)$$

where

$$H_k(\lambda) = \gamma_k\beta_k(\lambda)(1 - h_k(e^{-\lambda/k})).$$

By elementary calculations we find that

$$\alpha_k(\lambda) = 1 + \frac{1}{2}(1 - g_k(e^{-\lambda/k})) + o(1 - g_k(e^{-\lambda/k})),$$

and so  $\lim_{k \rightarrow \infty} \gamma_k(\alpha_k(\lambda) - 1) = \gamma_0\lambda/2$  by Lemma 2.2. It follows that

$$\lim_{k \rightarrow \infty} [\alpha_k(\lambda)S_k(\lambda) + \gamma_k(\alpha_k(\lambda) - 1)\lambda] = R(\lambda).$$

By the argument of the proof of Lemma 2.2 one can show that

$$\lim_{k \rightarrow \infty} H_k(\lambda) = \lim_{k \rightarrow \infty} F_k(\lambda) = F(\lambda).$$

In view of (2.7) and (2.8) we get

$$\lim_{k \rightarrow \infty} \sup_{x \in E_k} |A_k e_\lambda(x) - A e_\lambda(x)| = 0$$

for each  $\lambda > 0$ . This clearly implies that

$$\lim_{k \rightarrow \infty} \sup_{x \in E_k} |A_k f(x) - A f(x)| = 0$$

for each  $f \in D_1$ . By Ethier and Kurtz (1986, p.226 and pp.233-234) we find that  $\{y_k([\gamma_k t])/k : t \geq 0\}$  converges in distribution on  $D([0, \infty), \mathbb{R}_+)$  to the CBI-process corresponding to  $(R, F)$ .  $\square$

By Lemma 2.1 and Theorem 2.1, for any functions  $(R, F)$  given by (1.3) and (2.1), there is a sequence of positive numbers  $\{\gamma_k\}$  and a sequence of DBI-processes  $\{y_k(n) : n \geq 0\}$  such that  $\{y_k([\gamma_k t])/k : t \geq 0\}$  converges in distribution on  $D([0, \infty), \mathbb{R}_+)$  to the CBI-process corresponding to  $(R, F)$ .

### 3 Generalized Ray-Knight theorems

As an example of the applications of their limit theorems, Kawazu and Watanabe (1971) reproved the Ray-Knight theorems of diffusion characterizations of the Brownian local time. In this section, we generalize the results to the case of a Brownian motion with drift. We refer the reader to Le Gall and Le Jan (1998) for another adequate formulation of the Ray-Knight theorems for general Lévy processes.

Let  $A = \alpha d^2/dx^2 + \beta d/dx$  for given constants  $\alpha > 0$  and  $\beta \in \mathbb{R}$ . Then  $A$  generates a one-dimensional Brownian motion with drift  $(X_t, \mathcal{F}_t, \mathbf{P}_x)$ . The *local time* of  $\{X_t : t \geq 0\}$  is a continuous two-parameter process  $\{l(t, x) : t \geq 0, x \in \mathbb{R}\}$  such that the following property holds almost surely:

$$2 \int_B l(t, x) dx = \int_0^t 1_B(X_s) ds, \quad B \in \mathcal{B}(\mathbb{R}), \quad (3.1)$$

where  $\mathcal{B}(\mathbb{R})$  denote the Borel  $\sigma$ -algebra of  $\mathbb{R}$  and  $1_B$  denotes the indicator function of  $B$ . For fixed  $a \geq 0$  let

$$l^{-1}(u, a) = \inf\{t \geq 0 : l(t, a) = u\}. \quad (3.2)$$

**Theorem 3.1** *Under  $\mathbf{P}_0$  the process*

$$\xi_u(t) = l(l^{-1}(u, a), a + t), \quad t \geq 0 \quad (3.3)$$

*is a diffusion generated by*

$$x \frac{d^2}{dx^2} + \frac{\beta}{\alpha} x \frac{d}{dx}. \quad (3.4)$$

*Proof.* We follow the ideas of Kawazu and Watanabe (1971, Example 2.2). For  $c \in \mathbb{R}$  let  $\sigma_c = \inf\{t \geq 0 : X_t = c\}$ . Let  $\delta > 0$  and let  $u_\delta(x) = \mathbf{P}_x\{\sigma_\delta < \sigma_{-\delta}\} = 1 - \mathbf{P}_x\{\sigma_\delta > \sigma_{-\delta}\}$ . Then  $u_\delta(\cdot)$  satisfies

$$\alpha \frac{d^2}{dx^2} u_\delta(x) + \beta \frac{d}{dx} u_\delta(x) = 0, \quad |x| \leq \delta,$$

with  $u_\delta(\delta) = 1$  and  $u_\delta(-\delta) = 0$ . Solving this boundary value problem we find that

$$u_\delta(x) = \frac{\exp\{\beta\delta/\alpha\} - \exp\{-\beta x/\alpha\}}{\exp\{\beta\delta/\alpha\} - \exp\{-\beta\delta/\alpha\}}.$$

By a  $\delta$ -downcrossing at  $x \in \mathbb{R}$  before time  $T > 0$  we mean an interval  $[u, v] \subset [0, T)$  such that  $X_u = x + \delta$ ,  $X_v = x$  and  $x < X_t < x + \delta$  for all  $u < t < v$ . Let  $\eta_\delta$  denote the number of  $\delta$ -downcrossings at 0 before time  $\sigma_{-\delta}$ . By the property of independent increments of the Brownian motion with drift we have

$$\mathbf{E}_0[z^{\eta_\delta}] = \sum_{i=0}^{\infty} (1-p)(pz)^i = \frac{q}{1-pz},$$

where  $p = u_\delta(0)$ ,  $q = 1 - p$  and  $\mathbf{E}_0$  denotes the expectation under  $\mathbf{P}_0$ . Let  $x_i = a + i/k$  for  $i \geq 0$  and  $k \geq 1$  and let  $Z_k(i)$  denote the number of  $1/k$ -downcrossings at  $x_i$  before time  $l^{-1}(u, a)$ . It is easy to see that  $Z_k(i+1)$  is the sum of  $Z_k(i)$  independent copies of  $\eta_{1/k}$ . Thus  $\{Z_k(i) : i = 0, 1, \dots\}$  is a DB-process corresponding to the generating function

$$g_k(z) = \frac{q_k}{1 - p_k z},$$

where  $p_k = u_{1/k}(0)$  and  $q_k = 1 - p_k$ . By a standard result for local times of diffusion processes,

$$\lim_{k \rightarrow 0} Z_{1/k}([kt])/k = l(l^{-1}(u, a), a + t) = \xi_u(t);$$

see Itô and McKean (1965, p.48 and p.222). Then Theorem 2.1 implies that the limit  $\{\xi_u(t) : t \geq 0\}$  is a CB-process corresponding to

$$R(\lambda) = \lim_{k \rightarrow \infty} k^2[(1 - \lambda/k) - g_k(1 - \lambda/k)] = \frac{\beta}{\alpha} \lambda - \lambda^2.$$

This proves the desired result. □

Kawazu and Watanabe (1971, Theorem 2.3 and Example 2.2) proved the results of Theorem 3.1 in the special case  $\beta = 0$ . In that case the generating function  $g_k$  is actually independent of  $k \geq 1$ . In the general case, it seems difficult to check condition (1.6) for the sequence  $\{g_k\}$ . By similar arguments as the above we obtain the following

**Theorem 3.2** Under  $\mathbf{P}_0$  the process

$$\eta_u(t) = l(l^{-1}(u, a), a - t), \quad 0 \leq t \leq a \quad (3.5)$$

is a diffusion generated by

$$x \frac{d^2}{dx^2} + \frac{\beta}{\alpha} x \frac{d}{dx} + \frac{d}{dx}. \quad (3.6)$$

**Acknowledgement.** I would like to thank the referee for a number of helpful comments.

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