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Large and moderate deviations for occupation times of immigration superprocesses[1]

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Abstract. Large and moderate deviation principles are proved for the occupation time process of a subcritical branching superprocess with immigration.

Key words: Large deviation, moderate deviation, immigration superprocess, occupation time.

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1 Introduction

A number of large and moderate deviation principles (LDPs and MDPs) for superprocesses with and without immigration have been established in recent years. Particularly, Iscoe and Lee (1993) and Lee (1993) obtained LDPs for occupation times of super Brownian motions. Deuschel and Rosen (1998) proved an accurate LDP for the occupation times weighted by a testing function with zero average, improving the results of Lee and Remillard (1995). Schied (1996) proved LDPs of Freidlin-Wentzell type for rescaled super-Brownian motions, and Schied (1997) derived MDPs and used the result to establish a Strassen-type law of the iterated logarithm. Hong (2002, 2003) proved LDPs and MDPs for super-Brownian motion with randomly controlled immigration. LDPs and MDPs for a super-Brownian motion with uniform immigration were obtained in Zhang (2004a, b). Most of those results and their variants concentrate on super Brownian motions and related processes. On the other hand, Fleischmann and Kaj (1994) proved a LDP for rescaled superprocesses with a good convex rate functional on the measure state space. They considered a general underlying spatial motion and characterized the rate functional in terms of solutions of an explosive reaction-diffusion equation.

In this paper, we study the asymptotics of the occupation times of a subcritical branching superprocess with immigration. We shall consider a general underlying motion and prove a LDP and a MDP. The proofs of those results are easier than the corresponding results for other models.

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2 Immigration superprocesses

Let $b > 0$ be a fixed constant and $(P_t)_{t\geq 0}$ a conservative Borel right semigroup on a Lusin topological space E. Let $B(E)^+$ denote the set of bounded non-negative Borel functions on E. Then for each $f \in B(E)^+$, there is a unique locally bounded solution $V_t f$ to the evolution equation

$$V_t f(x) = P_t f(x) - \int_0^t P_{t-s}[(V_s f)^2 + bV_s f](x)ds, \qquad t \geq 0. \tag{2.1}$$

Let $M(E)$ be the space of finite Borel measures on E endowed with the topology of weak convergence. Write $\langle \mu, f\rangle = \int_E f d\mu$ for $f \in B(E)^+$ and $\mu \in M(E)$. For any $\lambda \in M(E)$,

$$\int_{M(E)} e^{-\langle \nu,f\rangle} Q_t^\lambda(\mu, d\nu) = \exp\bigg\{ - \langle \mu, V_t f\rangle - \int_0^t \langle \lambda, V_s f\rangle ds\bigg\}, \qquad f \in B(E)^+, \tag{2.2}$$

defines the transition semigroup $(Q_t^\lambda)_{t\geq 0}$ of a diffusion process in $M(E)$, which is the so-called *immigration superprocess*; see, e.g., [8, 11].

Let $X = (\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \boldsymbol{Q}_\mu^\lambda)$ be a diffusion realization of $(Q_t^\lambda)_{t\geq 0}$ and define the occupation time process $\{Y_t : t \geq 0\}$ by

$$\langle Y_t, f\rangle = \int_0^t \langle X_s, f\rangle ds, \qquad f \in B(E)^+. \tag{2.3}$$

A characterization of this process is given by

$$\boldsymbol{Q}_\mu^\lambda \exp\{-\langle Y_t, f\rangle\} = \exp\bigg\{ - \langle \mu, U_t f\rangle - \int_0^t \langle \lambda, U_s f\rangle ds\bigg\}, \qquad f \in B(E)^+, \tag{2.4}$$

where $U_t f$ is the solution of

$$U_t f(x) = \int_0^t P_s f(x)ds - \int_0^t P_{t-s}[(U_s f)^2 + bU_s f](x)ds, \qquad t \geq 0. \tag{2.5}$$

Observe that (2.5) is equivalent to

$$U_t f(x) = \int_0^t P_s^b f(x)ds - \int_0^t P_{t-s}^b[(U_s f)^2](x)ds, \qquad t \geq 0, \tag{2.6}$$

where $P_s^b = e^{-bs}P_s$. For notational convenience, we write

$$G^b f(x) = \int_0^\infty P_s^b f(x)ds \quad \text{and} \quad G_t^b f(x) = \int_0^t P_s^b f(x)ds. \tag{2.7}$$

In view of (2.4) we have the canonical representation

$$U_t f(x) = \langle l_t(x), f\rangle + \int_{M(E)^\circ} (1 - e^{-\langle \nu,f\rangle})L_t(x, d\nu), \qquad f \in B(E)^+, \tag{2.8}$$

where $l_t(x) \in M(E)$ and $(1 \wedge \langle \nu, 1\rangle)L_t(x, d\nu)$ is a finite measure on $M(E)^\circ := M(E) \setminus \{0\}$.

We first show the following central limit theorem.

Theorem 2.1 *Assume $\lambda \in M(E)$ is an invariant measure of $(P_t)_{t\geq 0}$. Fix $f \in B(E)^+$ and let*

$$S_T(f) = \frac{1}{\sqrt{T}}\bigg[\langle Y_T, f\rangle - \int_0^T \langle \lambda, G_s^b f\rangle ds\bigg]. \tag{2.9}$$

Then as $T \to \infty$, the distribution of $S_T(f)$ under $\boldsymbol{Q}_\mu^\lambda$ converges as $T \to \infty$ to the Gaussian distribution with mean zero and variance $2b^{-1}\langle \lambda, (G^b f)^2\rangle$.

Proof. Write $f_T = T^{-1/2} f$. By (2.4) and (2.6) it is not hard to show that

$$\begin{aligned}
\boldsymbol{Q}_\mu^\lambda \exp\{-S_T(f)\} &= \boldsymbol{Q}_\mu^\lambda \exp\bigg\{ - \langle Y_T, f_T\rangle + \int_0^T \langle \lambda, G_s^b f_T\rangle ds\bigg\} \\
&= \exp\bigg\{ - \langle \mu, U_T f_T\rangle - \int_0^T \langle \lambda, U_s f_T\rangle ds + \int_0^T \langle \lambda, G_s^b f_T\rangle ds\bigg\} \\
&= \exp\bigg\{ - \langle \mu, U_T f_T\rangle - \int_0^T ds \int_0^s \langle \lambda, P_{s-r}^b[(U_r f_T)^2]\rangle dr\bigg\}.
\end{aligned}$$

From (2.6) it is easy to see that

$$\langle \mu, U_T f_T\rangle \leq T^{-1/2}\langle \mu, G_T^b f\rangle \to 0$$

as $T \to \infty$. By similar estimates one finds that

$$\int_0^T ds \int_0^s \bigg\langle \lambda, P_{s-r}^b \bigg[\bigg(\int_0^r P_{r-s}^b[(U_s f_T)^2] ds\bigg)^2\bigg]\bigg\rangle dr \to 0$$

and

$$\int_0^T ds \int_0^s \bigg\langle \lambda, P_{s-r}^b \bigg[G_r^b f_T \int_0^r P_{r-s}^b[(U_s f_T)^2] ds\bigg]\bigg\rangle dr \to 0.$$

Then, using the $(P_t)_{t\geq 0}$-invariance of λ we have easily

$$\begin{aligned}
\int_0^T ds \int_0^s \langle \lambda, P_{s-r}^b[(U_r f_T)^2]\rangle dr &= \int_0^T ds \int_0^s \langle \lambda, P_{s-r}^b[(G_r^b f_T)^2]\rangle dr + o(1) \\
&\to b^{-1}\langle \lambda, (G^b f)^2\rangle
\end{aligned}$$

as $T \to \infty$. Combining the above gives

$$\lim_{T\to\infty} \boldsymbol{Q}_\mu^\lambda \exp\{-S_T(f)\} = \exp\{b^{-1}\langle \lambda, (G^b f)^2\rangle\},$$

so the theorem follows. □

3 Extension of the Laplace functional

In this section, we shall give an extension for the characterization of the Laplace transform of the immigration superprocess. This is realized by a power series expansion of the solution of (2.6) following a similar argument used in Hong [5, 6]. For $f \in B(E)^+$ and $\theta \in I\!R$ define

$$v(t, x; \theta f) = \langle l_t(x), \theta f\rangle + \int_{M(E)^\circ} (e^{\langle \nu, \theta f\rangle} - 1) L_t(x, d\nu), \qquad t \geq 0, x \in E \tag{3.1}$$

with values in $(-\infty, \infty]$. In view of (2.4) and (2.8) we have

$$\boldsymbol{Q}_\mu^\lambda \exp\{\theta\langle Y_t, f\rangle\} = \exp\left\{\langle \mu, v(t,\cdot;\theta f)\rangle + \int_0^t \langle \lambda, v(s,\cdot;\theta f)\rangle ds\right\}. \tag{3.2}$$

For any functions $g(t,\cdot)$ and $h(t,\cdot) \in B(E)^+$, we define the convolution

$$g(t,x) * h(t,x) = \int_0^t P_{t-s}^b [g(s,\cdot)h(s,\cdot)](x) ds. \tag{3.3}$$

Define the sequence of positive numbers $\{B_n : n \geq 1\}$ by $B_1 = B_2 = 1$ and

$$B_n = \sum_{k=1}^{n-1} B_k B_{n-k}. \tag{3.4}$$

Let $g^{*1}(t,x) = g(t,x)$ and

$$g(t,x)^{*n} = \sum_{k=1}^{n-1} g(t,x)^{*k} * g(t,x)^{*(n-k)}; \tag{3.5}$$

see [3, 14].

Lemma 3.1 *Fix $f \in B(E)$ and write $F(t,x) = G_t^b f(x)$. Then*

$$|F(t,x)^{*n}| \leq B_n b^{1-2n} \|f\|^n. \tag{3.6}$$

Proof. By the definition it is immediate that

$$|F(t,x)| \leq \int_0^t e^{-bs} ||f|| ds = b^{-1}\|f\|.$$

If (3.6) is true for all $k < n$, we have

$$\begin{aligned} |F(t,x)^{*n}| &\leq \sum_{k=1}^{n-1} \int_0^t e^{-b(t-s)} B_k B_{n-k} b^{2-2n} \|f\|^n ds \\ &\leq \sum_{k=1}^{n-1} B_k B_{n-k} b^{1-2n} \|f\|^n \\ &= B_n b^{1-2n} \|f\|^n. \end{aligned}$$

Then the result follows by induction. □

Lemma 3.2 *When $|\theta| < b^2/4\|f\|$, the equation*

$$u(t,x;\theta) = \theta \int_0^t P_r^b f(x) dr + \int_0^t P_s^b [u(t-s,\cdot;\theta)^2](x) ds, \tag{3.7}$$

admits an unique solution $u(t,x;\theta) = u_f(t,x;\theta)$. Moreover, $u_f(t,x;\theta)$ is analytic in θ and

$$|u_f(t,x;\theta)| \leq \frac{b}{2}[1 - (1 - 4b^{-2}\|f\||\theta|)^{1/2}] \leq 2b^{-1}\|f\|\theta. \tag{3.8}$$

Proof. In terms of the convolution defined by (3.3), we can rewrite (3.7) as

$$u(t,x;\theta) = \theta F(t,x) + u(t,x;\theta) * u(t,x;\theta). \tag{3.9}$$

As observed in [3, 14], a formal solution of (3.9) is given by the series

$$u(t,x;\theta) = \sum_{n=1}^{\infty} F(t,x)^{*n}\theta^n. \tag{3.10}$$

By Lemma 3.1 we have

$$\sum_{n=1}^{\infty} |F(t,x)^{*n}\theta^n| \leq \sum_{n=1}^{\infty} B_n b^{1-2n}\|f\|^n|\theta|^n.$$

It is elementary to see that

$$\sum_{n=1}^{\infty} B_n z^n = \frac{1}{2}[1-(1-4z)^{1/2}], \qquad |z| < 1/4.$$

Then (3.10) is absolutely convergence when $|\theta| < b^2/4\|f\|$. Consequently, the series really defines a function $u(t,x;\theta)$ which solves (3.9) and is analytic in θ. The estimates in (3.8) are immediate. □.

Lemma 3.3 *If $f(\cdot) \equiv 1$, for $|\theta| < b^2/4$ we have*

$$u_1(t,x;\theta) \equiv \frac{2\theta(1-e^{-\gamma t})}{(b+\gamma)-(b-\gamma)e^{-\gamma t}}, \qquad x \in E, \tag{3.11}$$

where $\gamma = \sqrt{b^2 - 4\theta}$.

Proof. From (3.10) we see that $u_1(t;\theta) := u_1(t,\cdot;\theta)$ is actually independent of $x \in E$. Then (3.7) implies that

$$\frac{d}{dt}u_1(t;\theta) = u_1(t;\theta)^2 - bu_1(t;\theta) + \theta. \tag{3.12}$$

Solving this differential equation gives (3.11). □

Theorem 3.1 *For $\theta < b^2/4\|f\|$, we have*

$$u_f(t,x;\theta) = v(t,x;\theta f), \qquad t \geq 0, x \in E. \tag{3.13}$$

Proof. From (2.6) and the representations (2.8) and (3.1) we see that $v(t,x;\theta f)$ satisfies (3.7). Then (3.13) holds for $-b^2/4\|f\| < \theta \leq 0$. By Lemma 3.2, $u_f(t,x;\theta)$ is analytic in $\theta \in (-b^2/4\|f\|, b^2/4\|f\|)$. In view of (3.2), we also have (3.13) for $0 \leq \theta < b^2/4\|f\|$ by the property of Laplace transforms; see, e.g., [15]. □

4 A large deviation principle

Assume $\lambda \in M(E)$ is an invariant measure of $(P_t)_{t\geq 0}$. We shall establish a long time large deviation principle for the occupation time of the immigration superprocess. Let $f \in B(E)^+$ be fixed and let $v(t, x; \theta f)$ and $u(t, x; \theta)$ be respectively given by (3.1) and (3.7).

Lemma 4.1 *For any $x \in E$ and $\theta \in I\!R$, the limit $u_f(x;\theta) := \lim_{t\to\infty} v(t, x; \theta f)$ exists in $(-\infty, \infty]$. Moreover, $u_f(x;\theta)$ is finite when $\theta < b^2/4\|f\|$.*

Proof. As a special case of (3.2) we have

$$\boldsymbol{Q}_\mu^0 \exp\left\{\theta \int_0^t \langle X_s, f\rangle ds\right\} = \exp\{\langle \mu, v(t, \cdot; \theta f)\rangle\}. \tag{4.1}$$

Then $v(t, x; \theta f)$ is monotonous in $t \geq 0$, so the limit $u_f(x;\theta) := \lim_{t\to\infty} v(t, x; \theta f)$ exists. By Theorem 3.1, for $\theta < b^2/4\|f\|$ we have

$$0 \leq |v(t, x; \theta f)| \equiv |u_f(t, x; \theta)| \leq u_1(t, x; \theta\|f\|) = \frac{2|\theta|\|f\|(1 - e^{-\gamma t})}{(b+\gamma) - (b-\gamma)e^{-\gamma t}},$$

where $\gamma = \sqrt{b^2 - 4\theta\|f\|}$. As $t \to \infty$, $u_1(t, x; \theta\|f\|)$ increases to $(b - \sqrt{b^2 - 4\theta\|f\|})/2$. Then the limit $u_f(x;\theta)$ is finite when $\theta < b^2/4\|f\|$. □

Lemma 4.2 *For any $x \in E$ and $\theta \in (-\infty, b^2/4\|f\|)$, we have $u_f(x;\theta) := \lim_{t\to\infty} u_f(t, x; \theta)$. Moreover, $u_f(x;\theta)$ is strictly convex in $\theta \in (-\infty, b^2/4\|f\|)$.*

Proof. By the proof of the last lemma we have

$$\boldsymbol{Q}_\mu^0 \exp\left\{\theta \int_0^\infty \langle X_s, f\rangle ds\right\} = \exp\{\langle \mu, u_f(x;\theta)\rangle\} \tag{4.2}$$

Clearly, the random variable $\int_0^\infty \langle X_s, f\rangle ds$ under $\boldsymbol{Q}_{\delta_x}^0$ has an infinitely divisible distribution. Then we have the canonical representation

$$u_f(x;\theta) = l(x)\theta + \int_0^\infty (e^{\theta u} + 1) L(x, du),$$

where $l(x) \geq 0$, and the Lévy measure $L(x, du)$ is nontrivial. (Otherwise, (4.2) defines a degenerate distribution.) It follows that

$$\frac{d^2}{d\theta^2} u_f(x;\theta) = \int_0^\infty u^2 e^{\theta u} L(x, du),$$

which is finite and strictly positive when $\theta < b^2/4\|f\|$. Thus $u_f(x;\theta)$ is strictly convex in θ. □

Now we have the following large deviation principle for the occupation time of the immigration superprocess.

Theorem 4.1 *Let $f \in B(E)^+$ be fixed and note $\delta = b^2/4\|f\|$. Then*

$$a := \lim_{\theta \to \delta -} \frac{d}{d\theta}\langle \lambda, u_f(\cdot;\theta)\rangle > \frac{d}{d\theta}\langle \lambda, u_f(\cdot;0)\rangle = b^{-1}\langle \lambda, f\rangle. \tag{4.3}$$

For any open set $U \subset (0,a)$ and any closed set $L \subset (0,a)$, we have

$$\liminf_{T\to\infty} T^{-1}\log \boldsymbol{Q}^{\lambda}_{\mu}\{T^{-1}\langle Y_T, f\rangle \in U\} \geq -\inf_{x\in U} I(x), \tag{4.4}$$

and

$$\limsup_{T\to\infty} T^{-1}\log \boldsymbol{Q}^{\lambda}_{\mu}\{T^{-1}\langle Y_T, f\rangle \in L\} \leq -\inf_{x\in L} I(x), \tag{4.5}$$

where

$$I(x) = \sup_{\theta<\delta}[x\theta - \langle \lambda, u_f(\cdot;\theta)\rangle], \qquad 0 \leq x < a. \tag{4.6}$$

Proof. By Lemma 4.1, $\langle \lambda, u_f(\cdot;\theta)\rangle$ is finite and strictly convex in $\theta \in (-\infty, \delta)$. By Theorem 3.1 we have $u_f(t,x;\theta) = v(t,x;\theta f)$. Recall that λ is an invariant measure of $(P_t)_{t\geq 0}$. Then we may differentiate both sides of a special form of (3.2) in θ to see that

$$\boldsymbol{Q}^0_{\lambda}\left[\int_0^t \langle X_s, f\rangle ds \exp\left\{\theta \int_0^t \langle X_s, f\rangle ds\right\}\right] = \frac{d}{d\theta}\langle \lambda, u_f(t,\cdot;\theta)\rangle.$$

The above value is bounded below by

$$\boldsymbol{Q}^0_{\lambda}\left[\int_0^t \langle X_s, f\rangle ds\right] = \frac{d}{d\theta}\langle \lambda, u_f(t,\cdot;0)\rangle = \int_0^t e^{-bs}\langle \lambda, f\rangle ds,$$

where the second equality follows by (3.7). Then (4.3) follows. For any $\theta \in I\!R$ we have

$$\begin{aligned}\Lambda(T,\theta) &:= T^{-1}\log \boldsymbol{Q}^{\lambda}_{\mu}\exp[\theta\langle Y_T, f\rangle] \\ &= T^{-1}\left[\langle \mu, v(T,\cdot;\theta f)\rangle + \int_0^T \langle \lambda, v(s,\cdot;\theta f)\rangle ds\right].\end{aligned}$$

It follows that $\lim_{T\to\infty}\Lambda(T,\theta) = \langle \lambda, u_f(\cdot;\theta)\rangle$. Observe that

$$\lim_{\theta\to-\infty}\frac{d}{d\theta}\langle \lambda, u_f(\cdot;\theta)\rangle = 0.$$

Then for any $x \in (0,a)$ there is some $\theta_x < \delta$ such that

$$\frac{d}{d\theta}\langle \lambda, u_f(t,\cdot;\theta_x)\rangle = x$$

and hence

$$I(x) := \sup_{\theta\in I\!R}[x\theta - \langle \lambda, u_f(\cdot;\theta)\rangle] = \sup_{\theta<\delta}[x\theta - \langle \lambda, u_f(\cdot;\theta)\rangle] = x\theta_x - \langle \lambda, u_f(t,\cdot;\theta_x)\rangle.$$

That is, I is well-defined in $(0,a)$ by (4.6). Then the result follows from the Gärtner-Ellis Theorem; see, e.g., [1, p. 44]. □

Remark If $f \equiv 1$, we have

$$u_1(\theta) := \lim_{t\to\infty} u_1(t;\cdot,\theta) \equiv \frac{1}{2}[b - \sqrt{b^2 - 4\theta}]. \tag{4.7}$$

Note that

$$\frac{d}{d\theta} u_1(\theta) = \frac{1}{\sqrt{b^2 - 4\theta}} \to \infty$$

as θ increases to $4^{-1}b^2$. Then the proof of Theorem 4.1 gives a full large deviation principle. Moreover, from (4.6) and (4.7) it is not hard to get that

$$I(x) = \frac{b^2}{4x}(x - b^{-1}\langle\lambda, 1\rangle)^2, \qquad x \geq 0. \tag{4.8}$$

5 A moderate deviation principle

Assume $\lambda \in M(E)$ is an invariant measure of $(P_t)_{t\geq 0}$. Let $c(T)$ be such that $c(T) \to \infty$ and $Tc(T)^{-1} \to \infty$ as $T \to \infty$. Fix $f \in B(E)^+$ and let

$$\boldsymbol{Z}_T(f) = \frac{1}{\sqrt{Tc(T)}}\left[\langle Y_T, f\rangle - \int_0^T \langle\lambda, G_s^b f\rangle ds\right]. \tag{5.1}$$

Then we have the following

Theorem 5.1 *For any open set $U \subset I\!R$ and closed set $L \subset I\!R$,*

$$\liminf_{T\to\infty} c(T)^{-1} \log \boldsymbol{Q}_\mu^\lambda\{\boldsymbol{Z}_T(f) \in U\} \geq -\inf_{x\in U} I(x), \tag{5.2}$$

and

$$\limsup_{T\to\infty} c(T)^{-1} \log \boldsymbol{Q}_\mu^\lambda\{\boldsymbol{Z}_T(f) \in L\} \leq -\inf_{x\in L} I(x), \tag{5.3}$$

where

$$I(x) = \frac{bx^2}{4\langle\lambda, (G^b f)^2\rangle}. \tag{5.4}$$

Proof. For $\theta \in I\!R$ let

$$\Lambda(T,\theta) = c(T)^{-1} \log \boldsymbol{Q}_\mu^\lambda \exp\{\theta c(T)\boldsymbol{Z}_T(f)\}. \tag{5.5}$$

We shall prove

$$\Lambda(T,\theta) \to \Lambda(\theta) := b^{-1}\langle\lambda, (G^b f)^2\rangle\theta^2. \tag{5.6}$$

as $T \to \infty$. It is easy to show that $I(x)$ is the Legendre transform of $\Lambda(\theta)$, that is,

$$I(x) = \sup_{\theta\in I\!R}[\theta x - \Lambda(\theta)].$$

Consequently, once (5.6) is proved, the theorem is an application of the Gärtner-Ellis Theorem; see, e.g., [1, p. 44]. To establish (5.6), let $l(T) = \sqrt{T/c(T)}$ and let $u_T(t,\cdot;\theta)$ be the solution of (3.7) with f replaced by $f_T := f/l(T)$. When $|\theta| < b^2 l(T)/4\|f\|$,

$$\Lambda(T,\theta) = c(T)^{-1}\langle \mu, u_T(T,\cdot;\theta)\rangle + c(T)^{-1}\int_0^T ds \int_0^s \langle \lambda, P^b_{s-r}[u_T(r,\cdot;\theta)^2]\rangle dr, \tag{5.7}$$

By Lemma 3.2 we get

$$u_T(t,x;\theta) \le 2b^{-1}\theta l(T)^{-1}\|f\|. \tag{5.8}$$

It then follows that

$$I := c(T)^{-1}\langle \mu, u_T(T,\cdot;\theta)\rangle \le 2b^{-1}\theta c(T)^{-1} l(T)^{-1}\langle \mu, 1\rangle \|f\| \to 0$$

as $T \to \infty$. On the other hand,

$$\begin{aligned}
I\!I &:= c(T)^{-1}\int_0^T ds \int_0^s \langle \lambda, P^b_{s-r}[u_T(r,\cdot;\theta)^2]\rangle dr \\
&= c(T)^{-1}\int_0^T ds \int_0^s \langle \lambda, e^{b(s-r)} P_{s-r} u_T(r,\cdot;\theta)^2\rangle dr \\
&= c(T)^{-1}\int_0^T dr \int_0^{T-r} e^{bs}\langle \lambda, u_T(r,\cdot;\theta)^2\rangle ds \\
&= b^{-1}c(T)^{-1}\int_0^T \langle \lambda, u_T(r,\cdot;\theta)^2\rangle dr - b^{-1}c(T)^{-1}\int_0^T e^{-b(T-r)}\langle \lambda, u_T(r,\cdot;\theta)^2\rangle dr.
\end{aligned}$$

Let $I\!I_1$ and $I\!I_2$ denote respectively the absolute values of the first and the second terms on the right hand side. By (5.8),

$$I\!I_2 \le 4b^{-3}\theta^2 c(T)^{-1} l(T)^{-2}\langle \lambda, 1\rangle ||f||^2 \int_0^T e^{-b(T-r)} dr \to 0$$

as $T \to \infty$. By l'Hospital's rule,

$$\begin{aligned}
I\!I_1' &:= b^{-1}c(T)^{-1}\int_0^T \left\langle \lambda, \theta^2\left[\int_0^r P^b_s f_T ds\right]^2\right\rangle dr \\
&= \theta^2 b^{-1} c(T)^{-1} l(T)^{-2}\int_0^T \langle \lambda, (G^b_r f)^2\rangle dr \\
&= \theta^2 b^{-1} T^{-1}\int_0^T \langle \lambda, (G^b_r f)^2\rangle dr \\
&\to \theta^2 b^{-1}\langle \lambda, (G^b f)^2\rangle
\end{aligned}$$

as $T \to \infty$. In view of (3.7),

$$\begin{aligned}
u_T(r,x;\theta)^2 &= \theta^2\left[\int_0^r P^b_s f_T(x) ds\right]^2 + \left[\int_0^r P^b_s[u_T(r-s,\cdot;\theta)^2](x) ds\right]^2 \\
&\quad + 2\theta \int_0^r P^b_s f_T(x) ds \int_0^r P^b_s[u_T(r-s,\cdot;\theta)^2](x) ds.
\end{aligned}$$

It follows that

$$
\begin{aligned}
|I\!I_1 - I\!I_1'| &\le b^{-1}c(T)^{-1}\int_0^T \left\langle \lambda, \left| u_T(r,\cdot;\theta)^2 - \theta^2\left(\int_0^r P_s^b f_T ds\right)^2 \right| \right\rangle dr \\
&\le b^{-1}c(T)^{-1}\int_0^T \left\langle \lambda, \left(\int_0^r P_s^b[u_T(r-s,\cdot;\theta)^2]ds\right)^2 \right\rangle dr \\
&\quad + 2b^{-1}\theta c(T)^{-1}\int_0^T \left\langle \lambda, \int_0^r P_s^b f_T(x)ds \int_0^r P_s^b[u_T(r-s,\cdot;\theta)^2](x)ds \right\rangle dr \\
&\le 16b^{-5}\theta^4 c(T)^{-1} l(T)^{-4}\langle\lambda,1\rangle\|f\|^4 \int_0^T \left(\int_0^r e^{-bs}ds\right)^2 dr \\
&\quad + 8b^{-4}\theta^3 c(T)^{-1} l(T)^{-3}\langle\lambda,1\rangle\|f\|^3 \int_0^T \left(\int_0^r e^{-bs}ds\right) dr \\
&\le 16b^{-5}\theta^4 T^{-2} c(T)\langle\lambda,1\rangle\|f\|^4 \int_0^T \left(\int_0^r e^{-bs}ds\right)^2 dr \\
&\quad + 8b^{-4}\theta^3 T^{-3/2} c(T)^{1/2}\langle\lambda,1\rangle\|f\|^3 \int_0^T \left(\int_0^r e^{-bs}ds\right) dr \\
&\to 0.
\end{aligned}
$$

Combining the above gives (5.6). □

The above theorem is frequently referred to as the moderate deviation principle. Roughly speaking, the central limit theorem proved in Section 2 corresponds to the extremal case $c(T) \equiv 1$, and the large deviation principle established in Section 4 corresponds to the case $c(T) \equiv T$. In this sense, the moderate deviation principle fills up the gap between the central limit theorem and the large deviation principle.

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