Published in: *Probability Theory and Related Fields* **130** (2004), 1: 1–17.

A Degenerate Stochastic Partial Differential Equation for Superprocesses with Singular Interaction

Zenghu Li[1]
Department of Mathematics, Beijing Normal University,
Beijing 100875, P.R. China
lizh@bnu.edu.cn

Hao Wang[2]
Department of Mathematics, University of Oregon,
Eugene OR 97403-1222, U.S.A.
haowang@math.uoregon.edu

Jie Xiong[3]
Department of Mathematics, University of Tennessee
Knoxville, TN 37996-1300, U.S.A.
jxiong@math.utk.edu
and
Mathematical and Statistical Sciences, University of Alberta
Edmonton, Alberta T6G 2G1, Canada.
jxiong@math.ualberta.ca

Abstract

The scaling limit for a class of interacting superprocesses and the associated singular, degenerate stochastic partial differential equation (SDSPDE) are investigated. It is proved that the scaling limit is a coalescing, purely-atomic-measure-valued process which is the unique strong solution of a reconstructed, associated SDSPDE.

AMS Subject Classifications: Primary 60G57, 60H15; secondary 60J80.

Key words and phrases: coalescing Brownian motion, scaling limit, purely atomic superprocess, interaction, stochastic partial differential equation, strong solution, pathwise uniqueness.

[1]Research supported by NNSF (No. 10131040 and No. 10121101).
[2]Research supported partially by the research grant of UO.
[2]Corresponding author.
[3]Research supported partially by NSA, NSERC, PIms, Lockheed Martin Naval Electronics and Surveillance Systems, Lockheed Martin Canada and VisionSmart through a MITACS center of excellence entitled "Prediction in Interacting Systems".

1 Introduction

Currently interactions and stochastic partial differential equations are two of the hot topics in the field of Superprocesses. We have seen that diverse interactions have been introduced into the models of superprocesses such as population interaction in the mutually catalytic model, interactive branching mechanism, inter-particle interaction, particle random medium interaction, mean-field interaction, and so on. (See the survey article [3] and the references therein). In Dawson et al. [5], a particle random medium interaction model with location dependent branching, which generalized the model introduced in Wang [14] [13], is introduced and the limiting superprocesses, which will be called superprocesses with dependent spatial motion and branching (SDSM's), are constructed and characterized. An open problem left from the recent work of Dawson et al. [6] and Wang [16] on this model has especially raised our interest in the investigation and consideration presented in the present paper. In order to describe the question clearly, we introduce necessary notations and the model first.

For fixed natural integers $k, m \geq 1$, let $C^k(\mathbb{R}^m)$ be the set of functions on $\mathbb{R}^m$ having continuous derivatives of order $\leq k$ and $C^k_\partial(\mathbb{R}^m)$ be the set of functions in $C^k(\mathbb{R}^m)$ which, together with their derivatives up to order k, can be extended continuously to $\bar{\mathbb{R}}^m := \mathbb{R}^m \cup \{\partial\}$, the one point compactification of $\mathbb{R}^m$. $C^k_0(\mathbb{R}^m)$ denotes the subset of $C^k_\partial(\mathbb{R}^m)$ of functions that, together with their derivatives up to order k, vanish at infinity. Let $M(\mathbb{R}^m)$ be the space of finite Borel measures on $\mathbb{R}^m$ equipped with the topology of weak convergence. We denote by $C_b(\mathbb{R}^m)$ the set of bounded continuous functions on $\mathbb{R}^m$, and by $C_0(\mathbb{R}^m)$ its subset of continuous functions vanishing at infinity. The subsets of non-negative elements of $C_b(\mathbb{R}^m)$ and $C_0(\mathbb{R}^m)$ are denoted by $C_b(\mathbb{R}^m)^+$ and $C_0(\mathbb{R}^m)^+$, respectively. $\mathcal{S}(\mathbb{R})$ stands for the space of all infinitely differentiable functions which, together with all their derivatives, are rapidly decreasing at infinity. Let $B(\mathbb{R})$ (resp. $C(\mathbb{R})$) be the collection of all Borel (resp. continuous) functions on $\mathbb{R}$. For $f \in B(\mathbb{R})$ and $\mu \in M(\mathbb{R})$, set $\langle f, \mu \rangle = \int_{\mathbb{R}} f d\mu$. Suppose that $\{W(x,t) : x \in \mathbb{R}, t \geq 0\}$ is a Brownian sheet (see Walsh [12]) and $\{B^i(t) : t \geq 0\}$,$i \in \mathbb{N}$, is a family of independent standard Brownian motions which are independent of $\{W(x,t) : x \in \mathbb{R}, t \geq 0\}$. For each natural number n, which serves as a control parameter for our finite branching particle systems, we consider a system of particles (initially, there are m^n_0 particles) which move, die and produce offspring in a random medium on $\mathbb{R}$.

The diffusive part of such a branching particle system has the form

$$dx^n_i(t) = c(x^n_i(t))dB^i(t) + \int_{\mathbb{R}} h(y - x^n_i(t))W(dy, dt), \quad t \geq 0, \tag{1.1}$$

where $c \in C_b(\mathbb{R})$ is a Lipschitz function and $h \in C^2_0(\mathbb{R})$ is a square-integrable function. By Lemma 3.1 of Dawson et al. [5], for any initial conditions $x^n_i(0) = x_i \in \mathbb{R}$, the stochastic equations (1.1) have unique strong solution $\{x^n_i(t) : t \geq 0\}$ and, for each integer $m \geq 1$, $\{(x^n_1(t), \cdots, x^n_m(t)) : t \geq 0\}$ is an m-dimensional diffusion process which is generated by the differential operator

$$G^m := \frac{1}{2}\sum_{i=1}^m a(x_i)\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i,j=1, i\neq j}^m \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.2}$$

In particular, $\{x^n_i(t) : t \geq 0\}$ is a one-dimensional diffusion process with generator $G :=$

$(a(x)/2)\Delta$, where Δ is the Laplacian operator,

$$\rho(x) := \int_{\mathbb{R}} h(y-x)h(y)dy, \tag{1.3}$$

and $a(x) := c^2(x) + \rho(0)$ for $x \in \mathbb{R}$. The function ρ is twice continuously differentiable with ρ' and ρ'' bounded since h is square-integrable and twice continuously differentiable with h' and h'' bounded. The quadratic variational process for the system given by (1.1) is

$$\langle x_i^n(t), x_j^n(t)\rangle = \int_0^t \rho(x_i^n(s) - x_j^n(s))ds + \delta_{\{i=j\}} \int_0^t c^2(x^i(s))\, ds, \tag{1.4}$$

where we set $\delta_{\{i=j\}} = 1$ or 0 according as $i = j$ or $i \neq j$, where $i, j \in \mathbb{N}$. Here $x_i^n(t)$ is the location of the i^{th} particle. We assume that each particle has mass $1/\theta^n$ and branches at rate $\gamma\theta^n$, where $\gamma \geq 0$ and $\theta \geq 2$ are fixed constants. We assume that when a particle $\frac{1}{\theta^n}\delta_x$, which has location at x, dies, it produces k particles with probability $p_k(x); x \in \mathbb{R}, k \in \mathbb{N} \cup \{0\}$. This means that the branching mechanism depends on the spatial location. The offspring distribution is assumed to satisfy:

$$p_1(x) = 0, \quad \sum_{k=0}^{\infty} kp_k(x) = 1, \quad \text{and} \quad m_2(x) := \sum_{k=0}^{\infty} k^2 p_k(x) < \infty \quad \text{for all } x \in \mathbb{R}. \tag{1.5}$$

The second condition indicates that we are solely interested in the critical case. After branching, the resulting set of particles evolve in the same way as their parents and they start off from the parent particle's branching site. Let m_t^n denote the total number of particles at time t . Denote the empirical measure process by

$$\mu_t^n(\cdot) := \frac{1}{\theta^n} \sum_{i=1}^{m_t^n} \delta_{x_i^n(t)}(\cdot). \tag{1.6}$$

In order to obtain measure-valued processes by use of an appropriate rescaling, we assume that there is a positive constant $\xi > 0$ such that $m_0^n/\theta^n \leq \xi$ for all $n \geq 0$ and that weak convergence of the initial laws $\mu_0^n \Rightarrow \mu$ holds, for some finite measure μ. As for the convergence from branching particle systems to a SDSM, the reader is referred to Wang [14] and Dawson et al. [5].

Let $E := M(\mathbb{R})$ be the Polish space of all finite Radon measures on $\mathbb{R}$ with the weak topology defined by

$$\mu^n \Rightarrow \mu \quad \text{if and only if} \quad \langle f, \mu^n\rangle \to \langle f, \mu\rangle \text{ for } \forall f \in C_b(\mathbb{R}) .$$

By Ito's formula and the conditional independence of motions and branching, we can obtain the following formal generators (usually called pregenerators) for the limiting measure-valued processes:

$$\mathcal{L}_{c,\sigma}F(\mu) := \mathcal{A}_c F(\mu) + \mathcal{B}_\sigma F(\mu), \tag{1.7}$$

where

$$\mathcal{B}_\sigma F(\mu) := \tfrac{1}{2} \int_{\mathbb{R}} \sigma(x) \frac{\delta^2 F(\mu)}{\delta\mu(x)^2} \mu(dx), \tag{1.8}$$

and

$$\mathcal{A}_c F(\mu) := \tfrac{1}{2}\int_{\mathbb{R}} a(x)(\frac{d^2}{dx^2})\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \tag{1.9}$$

$$+\tfrac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x-y)(\frac{d}{dx})(\frac{d}{dy})\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy)$$

for $F(\mu) \in \mathcal{D}(\mathcal{L}_{c,\sigma}) \subset C(E)$, where $\sigma(x) := \gamma(m_2(x)-1)$ for any $x \in \mathbb{R}$, the variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{h\downarrow 0}\frac{F(\mu+h\delta_x)-F(\mu)}{h}, \tag{1.10}$$

and $\mathcal{D}(\mathcal{L}_{c,\sigma})$ is the domain of the pregenerator $\mathcal{L}_{c,\sigma}$. Especially, we denote $\mathcal{L}_{0,\sigma} = \mathcal{A}_0 + \mathcal{B}_\sigma$ for $\mathcal{L}_{c,\sigma} = \mathcal{A}_c + \mathcal{B}_\sigma$ with $c(x) \equiv 0$. Let $B(\mathbb{R})^+$ be the space of all non-negative, bounded, measurable functions on $\mathbb{R}$. We cite one theorem proved in Dawson et al. [5].

Theorem 1.1 *Let $c \in C_b(\mathbb{R})$ be a Lipschitz function, $h \in C_0^2(\mathbb{R})$ be a square-integrable function on $\mathbb{R}$, and $\sigma(x) \in B(\mathbb{R})^+$. Then, for any $\mu \in E$, $(\mathcal{L}_{c,\sigma}, \delta_\mu)$-martingale problem (MP) has a unique solution which is denoted by X_t with sample paths in $D([0,\infty), M(\mathbb{R}))$. Then X_t is a diffusion process.*

Proof: For the proof of this theorem, the reader is referred to the section 5 of Dawson et al. [5]. $\square$

X_t is often called the high density limit of the branching particle systems we discussed.

Similar to Konno-Shiga's famous results for super-Brownian motion (See [9]), for this interactive model it was proved by Wang [13] that X_t is absolutely continuous. Also, Dawson et al [8] derived a stochastic partial differential equation (SPDE) for the density for the case of $c(\cdot) = \epsilon > 0$. An interesting case is due to Wang ([13], [15]) who proved that when $c(\cdot) \equiv 0$, X_t is a purely atomic measure valued process. In addition, Dawson et al [6] derived a degenerated SPDE

$$\begin{aligned} \langle\phi, X_t\rangle &= \langle\phi, X_{t_0}\rangle + \int_{t_0}^t\int_{\mathbb{R}} \left\langle h(y-\cdot)\phi', X_u\right\rangle W(dydu) \\ &\quad + \frac{1}{2}\rho(0)\int_{t_0}^t \left\langle\phi'', X_u\right\rangle du \\ &\quad + \sum_{i\in I(t_0)}\int_{t_0}^t \phi(x_i(u))\sqrt{\sigma(x_i(u))a_i(u)}dB_i(u), \qquad \phi\in\mathcal{S}(\mathbb{R}), t\geq t_0>0 \end{aligned} \tag{1.11}$$

for

$$X_t = \sum_{i\in I(t)} a_i(t)\delta_{x_i(t)}$$

and proved that the above degenerate SPDE has a pathwise or strong unique solution. On the other hand, Wang [16] recently proved that if $c(\cdot) \geq \epsilon > 0$, and h is a singular function (See Wang [16] for the precise definition) which can be roughly defined as "the square root of the Dirac delta function", the high density limit X_t is just the super-Brownian motion. Wang [16] also pointed out by example that if $c(\cdot) \equiv 0$, the conclusion is not clear when h is a singular function due to coalescence. This naturally raised a challenging question: *Can we identify the high density limit as well as its associated degenerate SPDE when* $c(\cdot) \equiv 0$ *and* h *is a singular function*?

We now outline the rough idea for approaching the problem. Since the "square root of the Dirac delta function" cannot be defined in the sense of distribution, we have to find a way such that it makes sense. The first idea is to seek a method to handle it as a sequential limit. The scaling limit argument automatically becomes a good candidate for us. From the related literature, we found that Dawson and Fleischmann [2] studied the clumping property of the classical super-Brownian motion (SBM) using the scaling limit. Mimicking this paper, we defined

$$X_t^K(B) = K^{-1} X_{Kt}(KB), \qquad \forall\ B \in \mathcal{B}(\mathbb{R}). \tag{1.12}$$

We found that the limit of X^K is the same as that of [2]. The external term is simply too weak to carry over at the end of the scaling. However, this does not give all that we want. When we carefully checked the proportions of scaling for different coefficients, we realized that we must adjust the scaling proportion of the random medium term. This adjustment also matches the real world situation. In fact, h and W characterize the outside force applying to the whole system. It should use a much larger scale compared to the motion of each individual in the system. Imagine our own movement and that of the earth! Therefore, we replaced the original $h(x)$ by $\sqrt{K}h(x)$ and applied the scaling (1.12). We denote the resulted process by Z_t^K. Assume that the scaled initial measures converge. We then prove that the limit Z_t of Z_t^K exists and is characterized as follows: At time $t_0 > 0$, Z_{t_0} is a Poisson random measure with intensity $t_0^{-1}\mu_0$. The particles move according to coalescing Brownian motion, whose mechanism is determined by the external term ρ, until its mass, governed by independent Feller's branching diffusions, reaches 0. Note that when $h = 0$, the particles do not move and we get the same result as Dawson and Fleischmann [2]. From this procedure, we derive a singular, degenerated SPDE for the limit process, where the motion dynamic is driven by a sequence of coalescing Brownian motions. Nevertheless, the singular, degenerated SPDE does not have strong uniqueness due to coalescence. After we replace the coalescing Brownian motions by killing Brownian motions, the strong uniqueness of the singular, degenerated SPDE is recovered and we get our expected results.

A similar problem is studied by Dawson et al. [7] from a different point of view. Since the paper is not yet published, we briefly list here the contents of that paper. The main purpose of the paper is to give a proof of the observation of Dawson et al. [5]. Section 2 gives characterizations for a coalescing Brownian motion flow and shows that the flow is actually the scaling limit of the interacting Brownian flow that serves as the carrier of the purely atomic SDSM in the excursion representation given in Dawson and Li [4]. Section 3 constructs the limiting superprocess in terms of one-dimensional excursions using the coalescing Brownian flow as a carrier. Section 4 derives the scaling limit of the SDSM from that of the interacting Brownian flow and the excursion representations. The major differences between [7] and our paper are as follows: First, we consider the usual scaling (1.12) and compare with the results of Dawson and Fleischmann for classical SBM. In [7], the scaling $K^{-2}X_{Kt}(KB)$ is used. Secondly, our scaling limit is for a general superprocess over a stochastic flow, namely, we derive the degenerate limit from random fields. In [7], it started from the degenerated process (i.e. $c = 0$) so that the scaling limit is essentially for finite-dimensional processes. Thirdly, our proof makes use of a new representation theorem which provides an easy approach to Wang's (cf. [15], [13]) result for the non-singular case. Finally, [7] has discussed the excursion representation of the limiting superprocess. In the current paper we consider an SPDE for the superprocess which is degenerated as well as singular. These two topics distinguish the two papers' emphases.

A related model was studied by Skoulakis and Adler [11], and its properties were investigated by Xiong [18], [17].

This article is organized as follows: In section 2, we prove the weak convergence of X^K and characterize the limit by a martingale problem. In section 3, we discuss the weak convergence of Z^K with h replaced by $\sqrt{K}h$ and prove that the limit martingale problem coincides with that studied by Dawson et al [7] which arises from a system of coalescing Brownian motions. In section 4, we show the nonuniqueness for the solution to the degenerated SPDE which is a natural extension of (1.11) for the singular case. Finally, We modify the driving Brownian motions and prove strong uniqueness for the modified SPDE.

2 Weak convergence under clumping scaling

To accommodate a larger class of processes, we consider tempered measures. Let

$$\phi_\lambda(x) = \int_{|y|<1} dy e^{-\lambda|x-y|} \exp\left(-\frac{1}{1-y^2}\right) \Bigg/ \int_{|y|<1} dy \exp\left(-\frac{1}{1-y^2}\right).$$

Note that to each $\lambda \in \mathbb{R}$ and $m \geq 0$ there are positive constants $\underline{c}_{\lambda,m}$ and $\bar{c}_{\lambda,m}$ such that

$$\underline{c}_{\lambda,m}\, \phi_\lambda(x) \leq \left|\frac{d^m}{dx^m}\, \phi_\lambda\,(x)\right| \leq \bar{c}_{\lambda,m}\, \phi_\lambda(x), \qquad \forall\ x \in \mathbb{R},$$

(cf. (2.1) of Mitoma [10]). We define $\mathcal{M}_{tem}(\mathbb{R})$ to be the collection of all measures μ such that

$$\langle \phi_\lambda, \mu \rangle < \infty, \qquad \forall\ \lambda > 0.$$

Let $C_{rap}(\mathbb{R})$ be the collection of all functions f such that for all $\lambda > 0$, there exists c_λ such that $|f(x)| \leq c_\lambda \phi_\lambda(x)$ for all $x \in \mathbb{R}$.

Lemma 2.1 *Assume that a, σ and ρ are bounded and*

$$\sup_K \left\langle \phi_\lambda, \mu^K \right\rangle < \infty, \qquad \forall\ \lambda > 0$$

where μ^K is define by the same fashion as in (1.12). Then for any $\alpha \geq 2$, λ, $T > 0$, we have

$$\sup_K \mathbb{E} \sup_{t \leq T} \left\langle \phi_\lambda, X_t^K \right\rangle^\alpha < \infty.$$

Proof: Denote $a_K(x) = K^{-1}a(Kx)$. σ_K and ρ_K are defined similarly. Note that X^K satisfies the following martingale problem: $\forall\ \phi \in C_{rap}(\mathbb{R})$,

$$M_t^K(\phi) \equiv \left\langle \phi, X_t^K \right\rangle - \left\langle \phi, \mu^K \right\rangle - \frac{1}{2}\int_0^t \left\langle a_K \phi'', X_u^K \right\rangle du \tag{2.1}$$

is a martingale with quadratic variation process

$$\langle M^K(\phi)\rangle_t = \int_0^t \langle K\sigma_K\phi^2, X_u^K\rangle\, du + \int_0^t du \int_{\mathbb{R}^2} \rho_K(y-z)\phi'(y)\phi'(z)X_u^K(dy)X_u^K(dz). \tag{2.2}$$

By Burkholder's inequality, it is easy to see that

$$\begin{aligned}\mathbb{E}\sup_{t\le s}\langle \phi_\lambda, X_t^K\rangle^\alpha &\le c + c\mathbb{E}\left(\int_0^s \langle \phi_\lambda, X_t^K\rangle\, dt\right)^\alpha \\ &\quad + c\mathbb{E}\left(\int_0^s \langle \phi_\lambda^2, X_t^K\rangle\, dt\right)^{\alpha/2} + c\mathbb{E}\left(\int_0^s \langle \phi_\lambda, X_t^K\rangle^2\, dt\right)^{\alpha/2}.\end{aligned}$$

Since $\phi_\lambda^2 \le \phi_\lambda$, using $|x| \le 1 + x^2$ and Hölder's inequality, we can continue with

$$\le c + c\int_0^s \mathbb{E}\langle \phi_\lambda, X_t^K\rangle^\alpha\, dt.$$

The conclusion of the lemma then follows from Gronwall's inequality. □

Theorem 2.1 *Under the conditions of Lemma 2.1, $\{X^K : K \ge 1\}$ is tight in $C(\mathbb{R}_+, \mathcal{M}_{tem}(\mathbb{R}))$.*

Proof: It is well known that we only need to prove the tightness of $\{\langle \phi, X^K\rangle : K \ge 1\}$ in $C(\mathbb{R}_+, \mathbb{R})$ for each fixed $\phi \in C_{rap}(\mathbb{R})$. Note that by the martingale problem (2.1,2.2) and Lemma 2.1, we have

$$\mathbb{E}\left|\langle \phi, X_t^K\rangle - \langle \phi, X_s^K\rangle\right|^\alpha \le c|t-s|^{\alpha/2}.$$

Take $\alpha > 2$; the tightness then follows from Kolmogorov's criterion. □

Theorem 2.2 *Suppose that $\sigma(\infty) = \lim_{|x|\to\infty}\sigma(x)$ and $\mu^\infty = \lim_{K\to\infty}\mu^K$ exist and 0 is not an atom of μ^∞. Under the conditions of Lemma 2.1, X^K converges in law to the unique solution of the following martingale problem: $\forall\ \phi \in C_{rap}(\mathbb{R})$,*

$$M_t^\infty(\phi) \equiv \langle \phi, X_t^\infty\rangle - \langle \phi, \mu^\infty\rangle \tag{2.3}$$

is a martingale with quadratic variation process

$$\langle M^\infty(\phi)\rangle_t = \int_0^t \langle \sigma(\infty)\phi^2, X_u^\infty\rangle\, du. \tag{2.4}$$

Proof: Note that for any $\epsilon > 0$ fixed, $K\sigma_K(y)$ converges to $\sigma(\infty)$ as $K \to \infty$ uniformly for $y \in S_\epsilon^c$ where $S_\epsilon = (-\epsilon, \epsilon)$. On the other hand, if we choose $f_\epsilon \in C_b(\mathbb{R})$ such that $1_{S_\epsilon} \le f_\epsilon \le 1_{S_{2\epsilon}}$, then

$$\begin{aligned}\limsup_{K\to\infty}\mathbb{E}\langle \phi^2 1_{S_\epsilon}, X_u^K\rangle &\le \limsup_{K\to\infty}\int \mu^K(dx)\int f_\epsilon(y)\phi^2(y)p_u^K(x,dy) \\ &= \int \mu^\infty(dx) f_\epsilon(x)\phi^2(x),\end{aligned}$$

where $p_u^K(x,dy)$ is the transition probability of the Markov process generated by $\mathcal{L}_K\phi = \frac{1}{2}a_K\phi''$. Let $\epsilon \downarrow 0$; we have

$$\int \mu^\infty(dx) f_\epsilon(x)\phi^2(x) \to 0.$$

By (2.1), (2.2), it is then easy to see that every limit point of X^K solves the martingale problem (2.3), (2.4). The uniqueness of this martingale problem follows from [2]. The conclusion of the theorem then follows easily. □

3 Weak convergence under strong interaction

In this section, we consider strong interaction, namely, replace h by $\sqrt{K}h$ and then apply clumping scaling discussed in the previous section. For any $\phi \in C_{rap}(\mathbb{R})$, we have that

$$U_t^K(\phi) \equiv \langle \phi, Z_t^K \rangle - \langle \phi, \mu^K \rangle - \frac{1}{2} \int_0^t \langle (K^{-1}c^2(Kx) + \rho(0))\phi'', Z_u^K \rangle \, du$$

is a martingale with quadratic variation process

$$\langle U^K(\phi) \rangle_t = \int_0^t \langle K\sigma_K \phi^2, Z_u^K \rangle \, du + \int_0^t du \int_{\mathbb{R}^2} \rho(K(y-z))\phi'(y)\phi'(z) Z_u^K(dy) Z_u^K(dz).$$

We shall prove that Z^K is a tight sequence and characterize the limit Z. Note that Z^K is a measure-valued process with density and, as we will show, Z is a purely atomic measure-valued process. Therefore, it is not easy to derive the limit martingale problem of type (3.8) below from the martingale problem for Z^K studied in Dawson et al. [5]. With a complicated argument as that in Xiong and Zhou [19], we believe that it can be proved that Z satisfies the martingale problem (3.6), (3.7). However, that martingale problem is not well-posed. To determine the distribution of Z uniquely, we start with the dual relation between Z^K and (Y^K, M) and prove the convergence of the latter; then the distribution of Z is determined by the limit of (Y^K, M). Finally, we construct a process which is clearly a Markov process, since it is the unique solution to the martingale problem (3.8) and has the same distribution as Z.

First we need the following lemma.

Lemma 3.1 *Suppose that* $\lim_{|x|\to\infty} \rho(x) = 0$ *and* η_K *is governed by the following SDE:*

$$d\eta_K(t) = \sqrt{\rho(0) - \rho(K\eta_K(t))} dB_t.$$

Then $\eta_K \to \eta$ *which is a Brownian motion with* 0 *as an absorbing boundary.*

Proof: It is easy to see that η_K is a tight sequence of diffusions on $[0, \infty)$, each with 0 as an absorbing boundary. Let

$$\mathcal{D}_0 = \{f \in C^2([0,\infty)) : f''(0) = 0\}.$$

Then for any $f \in \mathcal{D}_0$,

$$f(\eta_K(t)) - \int_0^t \frac{1}{2}(\rho(0) - \rho(K\eta_K(s)))f''(\eta_K(s))ds$$

is a martingale. Let η be a limit point. Then for any $f \in \mathcal{D}_0$,

$$f(\eta(t)) - \int_0^t \frac{1}{2}\rho(0) f''(\eta(s))ds$$

is a martingale. This proves the conclusion of the lemma. □

To characterize the limit of Z^K, we need the concept of the coalescing Brownian motion (CBM) which was first introduced by R. Arratia [1], where the coalescing Brownian motion is constructed from a system of discrete random walks. An interesting thing here is that we can construct the coalescing Brownian motion simply from a given Brownian sheet.

Definition 3.1 $(x_1(t), \cdots, x_m(t))$ *is a CBM if the components move as independent Brownian motions until any pair, say* $x_i(t)$ *and* $x_j(t)$, $(i < j)$, *meet. Starting from the meeting time,* $x_j(t)$ *assumes the values of* $x_i(t)$, $x_i(t)$ *disappears, and the system continues to evolve in the same fashion.*

Theorem 3.1 *Suppose* $\lim_{|x|\to\infty} \rho(x) = 0$ *and the conditions of Theorem 2.2 hold. Then,* Z^K *is tight and its limit finite marginal distribution is determined by the following duality relation: for all* $f \in C_{rap}(\mathbb{R}^m)$,

$$\mathbb{E}\left\langle f, (Z_t)^m \right\rangle = \mathbb{E}_{m,f}\left[\left\langle Y_t, \mu^{M_t}\right\rangle \exp\left\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\right\}\right] \tag{3.1}$$

and recursive relation: for $t_1 < t_2 < \cdots < t_{j+1}$,

$$\begin{aligned} &\mathbb{E}\left(\Pi_{i=1}^{j+1}\left\langle f_i, (Z_{t_i})^{m_i}\right\rangle\right) \qquad (3.2)\\ &= \mathbb{E}\Bigg(\mathbb{E}^Y_{m_{j+1},f_{j+1}}\left[\left\langle Y_{t_{j+1}-t_j}, (Z_{t_j})^{M_{t_{j+1}-t_j}}\right\rangle \exp\left\{\frac{1}{2}\int_0^{t_{j+1}-t_j} M_s(M_s-1)ds\right\}\right]\\ &\quad \times \Pi_{i=1}^{j}\left\langle f_i, (Z_{t_i})^{m_i}\right\rangle\Bigg)\end{aligned}$$

where M_t *is Kingman's coalescent process starting at* m *with jumping time* $0 = \tau_0 < \tau_1 < \cdots < \tau_m = \infty$, *and where* Y_t, *starting at* f, *is a function-valued process defined by*

$$Y_t = P^{M_{\tau_k}}_{t-\tau_k}\Gamma_k \cdots P^{M_{\tau_1}}_{\tau_2-\tau_1}\Gamma_1 P^{M_{\tau_0}}_{\tau_1} Y_0, \quad \forall\ t \in [\tau_k, \tau_{k+1}), 0 \le k < m \tag{3.3}$$

where P^m_t *is the semigroup of the* m*-dimensional coalescing Brownian motion,* Γ_k *is taking one of the* Φ_{ij} *randomly and*

$$\Phi_{ij} f(x_1, \cdots, x_{m-1}) = \sigma(\infty) f(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}), \tag{3.4}$$

where x_{m-1} *is in the places of the* i*th and* j*th variables.*

Proof: The tightness follows from an argument similar to that in section 2. By [5], we have

$$\mathbb{E}\left\langle f, (Z^K_t)^m \right\rangle = \mathbb{E}_{m,f}\left[\left\langle Y^K_t, \mu^{M_t}\right\rangle \exp\left\{\frac{1}{2}\int_0^t M_s(M_s-1)ds\right\}\right],$$

where Y^K starting from f is defined similarly as in (3.3)-(3.4) with Φ_{ij}, $\sigma(\infty)$ and P^m_t replaced by

$$\Phi^K_{ij} f(x_1, \cdots, x_{m-1}) = \sigma(Kx_{m-1}) f(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}),$$

$\sigma(K\cdot)$, and $P^{m,K}_t$, respectively. $P^{m,K}_t$ is the semigroup with generator

$$G^{m,K} = \frac{1}{2}\sum_{i=1}^m K^{-1}c^2(Kx_i)\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{1\le i,j\le m}\rho(K(x_i-x_j))\frac{\partial^2}{\partial x_i \partial x_j}.$$

Let

$$G^m = \frac{1}{2}\sum_{1\le i,j\le m} \rho(0) 1_{\{x_i=x_j\}} \frac{\partial^2}{\partial x_i \partial x_j}.$$

We define

$$\mathcal{D}_1 = \{f \in C_b^2(\mathbb{R}^m) : \frac{\partial^2 f}{\partial x_i \partial x_j} = 0 \text{ if } x_i = x_j, \text{for some } i \ne j\}.$$

Let $x^K(t)$ be the process generated by $G^{m,K}$. It is easy to see that x^K is tight as an m-dimensional process. Let x be a limit point. For $f \in \mathcal{D}_1$, we have $G^{m,K} f \to G^m f$ and hence

$$f(x(t)) - f(x(0)) - \int_0^t G^m f(x(s)) ds$$

is a martingale. From this, it is easy to see that $x(t)$ is an m-dimensional Brownian motion before it reaches the set $\{x : \exists\, i < j, x_i = x_j\}$. To study its behavior after it reaches that set, we consider $x_i^K(t) - x_j^K(t)$. Note that

$$\frac{d}{dt}\left\langle x_i^K - x_j^K \right\rangle_t = K^{-1} c^2(K x_i^K(t)) + K^{-1} c^2(K x_j^K(t)) + (\rho(0) - \rho(K(x_i^K(t) - x_j^K(t))).$$

The first and the second terms converge to 0 uniformly. As in Lemma 3.1, it is then easy to show that $x_i(t) - x_j(t)$ is a Brownian motion with 0 as absorbing boundary. This implies that $x(t)$ is the coalescing Brownian motion. Hence, $P_t^{m,K}$ converges to P_t^m, and hence Y^K converges to Y.

As $f \in C_{rap}(\mathbb{R}^m)$, we have $f(x_1, \cdots, x_m) \le c_\lambda \phi_\lambda(x_1) \cdots \phi_\lambda(x_m) \equiv c_\lambda \phi_\lambda(x)$. It is easy to verify that $P_t^{m,K}\phi(x) \le c_{\lambda,t}^m \phi_\lambda(x)$. Therefore, $\left\langle Y_t^K, \mu^{M_t}\right\rangle \le c_{\lambda,t}^{M_t} \left\langle \phi_\lambda, \mu\right\rangle^{M_t}$. (3.1) follows from the dominated convergence theorem. The same argument as in [5] shows that (3.1) determines the distribution of Z_t uniquely. Thus, for any fixed m and n, we have that $\left\langle f, (Z_t^K)^m\right\rangle^n$ is integrable uniformly in K since

$$\mathbb{E}\left(\left\langle f, (Z_t^K)^m\right\rangle^{2n}\right) = \mathbb{E}\left\langle f^{\otimes 2n}, (Z_t^K)^{2mn}\right\rangle.$$

Finally we prove (3.2). Note that

$$\begin{aligned}
\mathbb{E}\left(\Pi_{i=1}^{j+1}\left\langle f_i, (Z_{t_i}^K)^{m_i}\right\rangle\right) &= \mathbb{E}\left(\mathbb{E}\left(\left\langle f_{j+1}, (Z_{t_{j+1}}^K)^{m_{j+1}}\right\rangle | \mathcal{F}_{t_j}^K\right)\Pi_{i=1}^j \left\langle f_i, (Z_{t_i}^K)^{m_i}\right\rangle\right) \\
&= \mathbb{E}\Bigg(\mathbb{E}_{m_{j+1}, f_{j+1}}^Y\Bigg[\left\langle Y_{t_{j+1}-t_j}^K, (Z_{t_j}^K)^{M_{t_{j+1}-t_j}}\right\rangle \qquad (3.5)\\
&\qquad \times \exp\left\{\frac{1}{2}\int_0^{t_{j+1}-t_j} M_s(M_s-1) ds\right\}\Bigg]\Pi_{i=1}^j \left\langle f_i, (Z_{t_i}^K)^{m_i}\right\rangle\Bigg) \\
&\equiv \mathbb{E}\left(\Pi_{i=1}^j \left\langle f_i^K, (Z_{t_i}^K)^{m_i}\right\rangle\right),
\end{aligned}$$

where $f_i^K = f_i$ for $i < j$ and

$$f_j^K = f_j \mathbb{E}_{m_{j+1}, f_{j+1}}^Y\left(Y_{t_{j+1}-t_j}^K \exp\left\{\frac{1}{2}\int_0^{t_{j+1}-t_j} M_s(M_s-1) ds\right\}\right).$$

Note that $f_j^K \le c_\lambda \phi_\lambda$. Letting $K \to \infty$ in (3.5), we have (3.2). $\square$

Next we construct Z_t from another point of view. Let

$$\left\langle f, \tilde{Z}_t^n \right\rangle = \frac{1}{n} \sum_{i=1}^{n} \xi_i(t) f(x_i(t)),$$

where $\{\xi_i\}$ are independent Feller's branching diffusions with branching rate $\sigma(\infty)$ and $(x_i(t), \cdots, x_n(t))$ is the n-dimensional coalescent Brownian motion with diffusion coefficient $\rho(0)$. Here we consider the limit of $\tilde{Z}^n$ by adapting the method of Xiong and Zhou [19].

Theorem 3.2 *Under the conditions of Theorem 3.1, $\tilde{Z}^n$ is tight and its limit $\tilde{Z}$ solves the following martingale problem:*

$$U_t^\infty(\phi) \equiv \left\langle \phi, \tilde{Z}_t \right\rangle - \langle \phi, \mu^\infty \rangle - \frac{1}{2}\rho(0) \int_0^t \left\langle \phi'', \tilde{Z}_u \right\rangle du \tag{3.6}$$

is a martingale with quadratic variation process

$$\langle U^\infty(\phi) \rangle_t = \int_0^t \left\langle \sigma(\infty)\phi^2, \tilde{Z}_u \right\rangle du + \int_0^t du \int_\Delta \rho(0)\phi'(y)\phi'(z)\tilde{Z}_u(dy)\tilde{Z}_u(dz), \tag{3.7}$$

where $\Delta = \{(x, x) : \ x \in \mathbb{R}\}$.

Proof: Applying Itô's formula, it is easy to show that

$$U_t^n(\phi) \equiv \left\langle \phi, \tilde{Z}_t^n \right\rangle - \langle \phi, \mu^\infty \rangle - \frac{1}{2}\rho(0) \int_0^t \left\langle \phi'', \tilde{Z}_u^n \right\rangle du$$

is a martingale with quadratic variation process

$$\begin{aligned} \langle U^n(\phi) \rangle_t &= \frac{1}{n} \sum_{i=1}^{n} \int_0^t \sigma(\infty)\xi_i(s)\phi(x_i(s))^2 ds \\ &\quad + \frac{\rho(0)}{n^2} \sum_{i,j=1}^{n} \int_{\tau_{ij}\wedge t}^{t} \xi_i(s)\xi_j(s)\phi'(x_i(s))\phi'(x_j(s)) ds \\ &= \int_0^t \left\langle \sigma(\infty)\phi^2, \tilde{Z}_u^n \right\rangle du + \int_0^t du \int_\Delta \rho(0)\phi'(y)\phi'(z)\tilde{Z}_u^n(dy)\tilde{Z}_u^n(dz), \end{aligned}$$

where τ_{ij} is the first time $x_i(t)$ and $x_j(t)$ meet. It is easy to prove the tightness of $\{(\tilde{Z}^n, \langle U^n(\phi) \rangle)\}$. Denote the limit by $(\tilde{Z}, \Lambda)$. By arguments similar to those in [19], we see that

$$\Lambda(t) = \int_0^t \left\langle \sigma(\infty)\phi^2, \tilde{Z}_u \right\rangle du + \int_0^t du \int_\Delta \rho(0)\phi'(y)\phi'(z)\tilde{Z}_u(dy)\tilde{Z}_u(dz)$$

and hence $\tilde{Z}$ solves the martingale problem (3.6)-(3.7). □

Remark 3.3 *The solution to the martingale problem (3.6)-(3.7) is not unique.*

For example, if we replace $(x_1(t), \cdots, x_n(t))$ by n-dimensional ordinary Brownian motion, the limit will provide another example.

The ordinary SBM with diffusion coefficient $\rho(0)$ and branching rate $\sigma(\infty)$ is a third example of the solution to the martingale problem (3.6)-(3.7).

Finally, we prove Z and $\tilde{Z}$ have the same distribution.

Theorem 3.4 *Under the conditions of Theorem 3.1, $Z = \tilde{Z}$ in distribution. Therefore, Z_t is a Markov process.*

Proof: We only need to prove (3.1) holds with Z replaced by $\tilde{Z}$. For $f \in C_b^2(\mathbb{R}^m)$, let $F_{m,f}(\mu) = \langle f, \mu^m \rangle$. Define

$$\mathcal{L}F_{m,f}(\mu) = F_{m,G^m f}(\mu) + \frac{1}{2} \sum_{1 \le i \ne j \le m} F_{m-1,\Phi_{ij} f}(\mu).$$

By Itô's formula, it is easy to see that $\tilde{Z}_t^n$ solves the following martingale problem: $\forall\, f \in C_b^2(\mathbb{R}^m)$,

$$F_{m,f}(\tilde{Z}_t^n) - F_{m,f}(\tilde{Z}_0^n) - \int_0^t \mathcal{L}F_{m,f}(\tilde{Z}_s^n) ds$$

is a martingale. Let $n \to \infty$, we see that $\tilde{Z}_t$ satisfies: $\forall\, f \in C_b^2(\mathbb{R}^m)$,

$$F_{m,f}(\tilde{Z}_t) - F_{m,f}(\mu) - \int_0^t \mathcal{L}F_{m,f}(\tilde{Z}_s) ds \tag{3.8}$$

is a martingale. Mimicking the proof of Theorem 2.1 in [5], we see that (3.1) holds with Z_t replaced by $\tilde{Z}_t$. Since $\tilde{Z}_t$ is a Markov process, it is easy to verify that (3.2) holds with Z_t replaced by $\tilde{Z}_t$. Therefore, Z and $\tilde{Z}$ have the same distribution and Z is a Markov process. $\Box$

4 SPDE

In this section, we first show that Z_t is of purely atomic type. Then we characterize Z_t as the unique strong solution to an SPDE.

Since we are only interested in establishing the property for Z_t, we may and will assume that $c = 0$ and $\sigma = \sigma(\infty)$ are constants. Let h_n converges to the "square root of the delta function" so that $\rho_n(x)$ converges to 0 when $x \ne 0$ and $\rho_n(0) \to \rho(0) > 0$.

Fix h_n as in [5], we first reprove a theorem of Wang [13] (see also [15]) by a new representation of Z_t^n. This representation in a more general setup will involve Perkins' historical calculus and will be developed in another paper. Throughout this section, we assume that the conditions of Theorem 3.1 remain in force.

Theorem 4.1 *Let $X_t^n(x, W)$ be the strong solution of*

$$X_t^n = x + \int_0^t \int h_n(y - X_s^n) W(dsdy).$$

Let ζ_t be the superprocess with branching rate σ and spatial motion-free. Define

$$Z_t^n(\cdot) = \zeta_t\{x : X_t^n(x, W) \in \cdot\}.$$

Then

$$U_t^n(\phi) = \langle \phi, Z_t^n \rangle - \langle \phi, Z_0^n \rangle - \int_0^t \left\langle \frac{1}{2}\rho_n(0)\phi'', Z_u^n \right\rangle du$$

is a martingale with quadratic variation process

$$\langle U^n(\phi)\rangle_t = \int_0^t \left\langle \sigma\phi^2, Z_u^n\right\rangle du + \int_0^t du \int_{\mathbb{R}^2} \rho_n(y-z)\phi'(y)\phi'(z)Z_u^n(dy)Z_u^n(dz).$$

Proof: Applying Itô's formula, we have

$$\begin{aligned}
d\left\langle \phi, Z_t^n\right\rangle &= d\left\langle \phi(X_t^n(\cdot,W)),\zeta_t\right\rangle \\
&= \langle d\phi(X_t^n(\cdot,W)),\zeta_t\rangle + \langle \phi(X_t^n(\cdot,W)), d\zeta_t\rangle \\
&= \frac{\rho_n(0)}{2}\left\langle \phi'', Z_t^n\right\rangle dt + \int \left\langle \phi'(X_t^n(\cdot,W))h_n(y-\cdot),\zeta_t\right\rangle W(dtdy) \\
&\quad + \left\langle \phi(X_t^n(\cdot,W)), d\zeta_t\right\rangle.
\end{aligned}$$

Hence, $U_t^n(\phi)$ is a martingale with quadratic variation process

$$\begin{aligned}
\langle U^n(\phi)\rangle_t &= \int_0^t \int \left\langle \phi'(X_s^n(\cdot,W))h_n(y-\cdot),\zeta_s\right\rangle^2 dsdy + \int_0^t \left\langle \sigma\phi^2(X_s^n(\cdot,W)),\zeta_s\right\rangle ds \\
&= \int_0^t du\int_{\mathbb{R}^2}\rho_n(y-z)\phi'(y)\phi'(z)Z_u^n(dy)Z_u^n(dz) + \int_0^t \left\langle \sigma\phi^2, Z_u^n\right\rangle du.
\end{aligned}$$

$\square$

Theorem 4.2 *For all $t>0$, Z_t is of purely-atomic type.*

Proof: It is well-known that $\zeta_t = \sum_{i\in I(t)} \xi_t^i \delta_{x^i}$ is of purely-atomic type (cf. [2] and [13]). Hence $Z_t^n = \sum_{i\in I(t)} \xi_t^i \delta_{X_t^n(x^i,W)}$. Taking a limit, it is clear that Z_t is of purely-atomic type. $\square$

Mimicking [6], we consider the following SPDE

$$\begin{aligned}
\langle \phi, \mu_t\rangle &= \langle \phi, \mu_{t_0}\rangle + \sum_{i\in I(t_0)} \sqrt{\rho(0)}\int_{t_0}^t \phi'(x_i(u))\xi_i(u)dW_i(u) \\
&\quad + \frac{1}{2}\rho(0)\int_{t_0}^t \left\langle \phi'', \mu_u\right\rangle du \\
&\quad + \sum_{i\in I(t_0)} \int_{t_0}^t \phi(x_i(u))\sqrt{\sigma(\infty)\xi_i(u)}dB_i(u),
\end{aligned} \tag{4.1}$$

where $(W_1, W_2, \cdots)$ are coalescing Brownian motions independent of $\{B_i:\ i\ge 1\}$.

If $x_i(t) = \sqrt{\rho(0)}W_i(t)$ and $\xi_i(t)$ is the Feller's branching diffusion generated by B_i, it is easy to see that

$$\mu_t = \sum \xi_i(t)\delta_{x_i(t)} \tag{4.2}$$

satisfies (4.1). It is easy to verify that the solution of (4.1) satisfies the martingale problem (3.8). Therefore, weak uniqueness holds for the solution to (4.1). The next theorem shows the non-strong-uniqueness of the solution.

Theorem 4.3 *Let $t_0=0$ and $\mu_0 = \sum_{i\in I}\xi_i(0)\delta_{x_i(0)}$, where I is a countable set. Then, the pathwise uniqueness for the SPDE (4.1) does not hold.*

Proof: Let

$$\tau_1 = \inf\{t: \ \exists i \neq j, W_i(t) = W_j(t)\}.$$

Define τ_2 similarly. Suppose W_i and W_j $(i < j)$ meet at time τ_1. For $t \leq \tau_1$, we define $\tilde{\xi}_k(t) = \xi_k(t)$ for all k. For $0 < \tau_1 < t \leq \tau_2$, if $k \neq i,\ j$ we define $\tilde{\xi}_k(t) = \xi_k(t)$; otherwise, we define that $\tilde{\xi}_j(t) = 0$ and let $\tilde{\xi}_i(t)$ be the unique solution of the following SDE

$$\tilde{\xi}_i(t) = \xi_i(\tau_1) + \xi_j(\tau_1) + \sqrt{\sigma(\infty)} \int_{\tau_1}^t \sqrt{\tilde{\xi}_i(u)} dB_i(u).$$

Then

$$\tilde{\mu}_t = \sum \tilde{\xi}_i(t) \delta_{x_i(t)} \tag{4.3}$$

is another solution to (4.1). Therefore, the pathwise uniqueness for the SPDE (4.1) does not hold. □

To derive an SPDE with strong uniqueness, we need to modify the Brownian driving system in (4.1).

Definition 4.1 $\tilde{W} = \{\tilde{W}_k\}$ *is a system of killing Brownian motions (KBM) if each starts with an independent Brownian motion until a pair of them meet; at that time, the process with higher index will be killed and the indexes of the other higher indexed processes is lowered by 1. The system continues to evolve in this fashion.*

$\tilde{B} = \{\tilde{B}_k\}$ *is a system of adjoint (to* $\tilde{W}$*) killing Brownian motions (AKBM) if each starts with an independent Brownian motion. A member will be killed when the corresponding member in* $\tilde{W}$ *is killed.*

Now we modify (4.1) and consider

$$\begin{aligned} \langle \phi, \mu_t \rangle &= \langle \phi, \mu_{t_0} \rangle + \sum_{i \in I(t_0)} \sqrt{\rho(0)} \int_{t_0}^t \phi'(x_i(u)) \xi_i(u) d\tilde{W}_i(u) \\ &\quad + \frac{1}{2} \rho(0) \int_{t_0}^t \left\langle \phi'', \mu_u \right\rangle du \\ &\quad + \sum_{i \in I(t_0)} \int_{t_0}^t \phi(x_i(u)) \sqrt{\sigma(\infty) \xi_i(u)} d\tilde{B}_i(u). \end{aligned} \tag{4.4}$$

If we construct $\tilde{W}$ and $\tilde{B}$ in an obvious way, it is then clear that $\tilde{\mu}_t$ defined by (4.3) is a solution to (4.4).

Theorem 4.4 *Let* $t_0 = 0$ *and* $\mu_0 = \sum_{i \in I} \xi_i(0) \delta_{x_i(0)}$, *where* I *is a countable set. Then, the SPDE (4.4) has a pathwise unique solution.*

Proof: Let μ_t be a solution and

$$\tilde{\tau}_1 = \inf\{t: \ \exists i \neq j, \tilde{W}_i(t) = \tilde{W}_j(t)\}.$$

Now we prove the uniqueness of the solution before time $\tilde{\tau}_1$ by adapting the technique of [6] to the present setup. Let

$$\epsilon_0 = \inf\{|x_i(0) - x_j(0)| : i \neq j \in I\}$$

and

$$\eta_1 = \inf\{t \in [0, \tau_1) : |x_i(t) - x_i(0)| \geq \frac{1}{3}\epsilon_0 \text{ for some } i \in I\}.$$

Then η_1 is a stopping time. Take ϕ such that its support is within $\frac{2}{3}\epsilon_0$ of $x_i(0)$. Then

$$\begin{aligned}
\xi_i(t)\phi(x_i(t)) &= \xi_i(0)\phi(x_i(0)) + \sqrt{\rho(0)}\int_0^t \phi'(x_i(u))\xi_i(u)d\tilde{W}_i(u) \\
&\quad + \frac{1}{2}\rho(0)\int_0^t \xi_i(u)\phi''(x_i(u))du + \sqrt{\sigma(\infty)}\int_0^t \phi(x_i(u))\sqrt{\xi_i(u)}d\tilde{B}_i(u).
\end{aligned}$$

Also, take $\phi(x) = 1$ for x within $\frac{1}{3}\epsilon_0$ of $x_i(0)$. Then for $t \leq \eta_1$

$$\xi_i(t) = \xi_i(0) + \sqrt{\sigma(\infty)}\int_0^t \sqrt{\xi_i(u)}d\tilde{B}_i(u).$$

Applying Itô's formula, we then have

$$d\phi(x_i(t)) = \sqrt{\rho(0)}\phi'(x_i(t))d\tilde{W}_i(t) + \frac{1}{2}\rho(0)\phi''(x_i(t))dt.$$

This implies that $x_i(t) = \sqrt{\rho(0)}\tilde{W}_i(t)$. By the definition of η_1, we have $\{x_i(\eta_1) : i \in I\}$ are all distinct; hence we may start from η_1 and define η_2 accordingly. As in [6], we then can prove that η_n converges to τ_1 and get the uniqueness for $t \leq \tau_1$. Continuing this procedure, we get the uniqueness for all t.

The argument for countable I is the same as that at the end of the proof of Theorem 4.1 in [6]. □

Acknowledgment The authors would like to express their gratitude to Professor Steven N. Evans and the anonymous referee for their helpful suggestions. In particular, the authors thank Professor Kenneth A. Ross for his comments and suggestions.

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