Published in: *Journal of Theoretical Probability* **17** (2004), 3: 673–692.

Superprocesses with Coalescing Brownian Spatial Motion as Large-Scale Limits

Donald A. Dawson,[1] Zenghu Li,[2] and Xiaowen Zhou[3]

Abstract. A superprocess with coalescing spatial motion is constructed in terms of one-dimensional excursions. Based on this construction, it is proved that the superprocess is purely atomic and arises as scaling limit of a special form of the superprocess with dependent spatial motion studied in Dawson *et al.* (2001) and Wang (1997, 1998).

Mathematics Subject Classifications (2000): Primary 60J80; Secondary 60G57

Key words: superprocess, coalescing spatial motion, excursions, scaling limit, Poisson random measure.

1 Introduction

Large scale limits of interacting particle systems and measure-valued processes have been studied by many authors; see, e.g., Bojdecki and Gorostiza (1986), Cox *et al.* (2000), Dawson (1977), Dawson and Fleischmann (1988), Durrett and Perkins (1999), Hara and Slade (2000a,b). In particular, Dawson and Fleischmann (1988) investigated the scaling limit of a class of critical space-time branching models, giving a precise description of the growth of large clumps at spatially rare sites in low dimensions. They showed that a space-time-mass scaling limit exists and is a measure-valued branching process without migration. The clumps are located at Poissonian points and their sizes evolve according to continuous-state branching processes. Durrett and Perkins (1999) proved that suitably rescaled contact processes converge to super-Brownian motion in two or more dimensions. Cox *et al.* (2000) proved convergence of some rescaled voter models to super-Brownian motion. Hara and Slade (2000a,b) studied the convergence of rescaled percolation clusters to integrated super-Brownian excursions. Those results provide interesting connections between superprocesses and interacting particle systems.

[1]School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Canada K1S 5B6. E-mail: ddawson@math.carleton.ca

[2]Department of Mathematics, Beijing Normal University, Beijing 100875, People's Republic of China. E-mail: lizh@email.bnu.edu.cn

[3]Department of Mathematics and Statistics, Concordia University, 7141 Sherbrooke Street West, Montreal, Canada, H4B 1R6. E-mail: zhou@alcor.concordia.ca

A class of superprocesses with dependent spatial motion (SDSM) over the real line $\mathbb{R}$ were introduced and constructed in Wang (1997, 1998). The construction was then generalized in Dawson *et al.* (2001). In the model, the spatial motion is defined by a system of differential equations driven by a family of independent Brownian motions, the individual noises, and a time-space white noise, the common noise. If the coefficient of the individual noises are uniformly bounded away from zero, the SDSM is absolutely continuous and its density satisfies a stochastic differential equation (SPDE) that generalizes the Konno-Shiga equation satisfied by super Brownian motion over $\mathbb{R}$; see Dawson *et al.* (2000) and Konno and Shiga (1988). When the individual noises vanish, the SDSM is purely atomic; see Wang (1997, 2002). A construction of the purely atomic SDSM in terms of one-dimensional excursions was given in Dawson and Li (2003), where an immigration diffusion process was also constructed as the pathwise unique solution of a stochastic equation. An SPDE for the purely atomic SDSM was derived recently in Dawson *et al.* (2003). It was proved in Dawson *et al.* (2001) that a suitably rescaled absolutely continuous SDSM converges to the usual super Brownian motion. This describes another situation where the super Brownian motion arises universally. For the purely atomic SDSM, it was mentioned in the introduction of Dawson *et al.* (2001) that the same rescaled limit would lead to a superprocess with coalescing spatial motion (SCSM), a continuous state version of the coalescing-branching particle system. This seems to be a new phenomenon in scaling limits of interacting particle systems and superprocesses. The statement was not proved in Dawson *et al.* (2001) since the construction and characterization of the SCSM remained open at that time.

The main purpose of this paper is to give a proof of the observation of Dawson *et al.* (2001). As a preliminary, we give in Section 2 some characterizations for a coalescing Brownian flow in terms of martingale problems and show that the flow is actually the scaling limit of the interacting Brownian flow that serves as the carrier of the purely atomic SDSM in the excursion representation given in Dawson and Li (2003).

In Section 3, we construct the SCSM from the coalescing Brownian flow and one-dimensional excursions following the idea of Dawson and Li (2003). It has been known for a long time that a superprocess without spatial motion reduces to a Poisson system of point masses that evolve according to Feller branching diffusions without interaction; see Shiga (1990). The SCSM adds a coalescing Brownian flow which carries the point masses. Any masses join together when their carriers coalesce.

In Section 4, we derive the scaling limit theorem of the SDSM from that of the interacting Brownian flow and the excursion representations. This result shows that excursion representations play important roles not only in the construction of the superprocesses but also in the study of some of their properties.

2 Interacting Brownian flows

An m-dimensional continuous process $\{(y_1(t), \cdots, y_m(t)) : t \geq 0\}$ is called an *m-system of coalescing Brownian motions* (m-SCBM) with speed $\rho > 0$ if each $\{y_i(t) : t \geq 0\}$ is a Brownian motion with speed $\rho > 0$ and, for $i \neq j$, $\{|y_i(t) - y_j(t)| : t \geq 0\}$ is a Brownian motion with speed 2ρ stopped at the origin. Clearly, $\{(y_1(t), \cdots, y_m(t)) : t \geq 0\}$ is an m-SCBM with speed $\rho > 0$ if and only if

$$\langle y_i, y_j \rangle(t) = \rho \cdot (t - t \wedge \tau_{ij}), \qquad 1 \leq i, j \leq m, \tag{2.1}$$

where $\tau_{ij} = \inf\{t \geq 0 : y_i(t) = y_j(t)\}$.

To give a martingale characterization of the m-SCBM, we need to choose a convenient core of its generator. For any permutation $(i_1, i_2, \cdots, i_m)$ of $(1, 2, \cdots, m)$ let

$$\mathbb{R}^m_{i_1 i_2 \cdots i_m} = \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m : x_{i_1} < x_{i_2} < \cdots < x_{i_m}\}. \tag{2.2}$$

Let $\mathscr{D}_0^{(1)} = C^2(\mathbb{R})$ and for $m \geq 2$ let $\mathscr{D}_0^{(m)}$ be the set of functions $f \in C(\mathbb{R}^m)$ such that

(2.A) f is twice continuously differentiable in each $\mathbb{R}^m_{i_1 i_2 \cdots i_m}$ with bounded partial derivatives up to the second degree;

(2.B) all partial derivatives of f up to the second degree can be extended to the closure of each $\mathbb{R}^m_{i_1 i_2 \cdots i_m}$ as uniformly continuous functions with

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \cdots, x_m) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_1, \cdots, x_m) = 0 \tag{2.3}$$

for $1 \leq i < j \leq m$ and $x_i = x_j$. (We simply write $\frac{\partial^2 f}{\partial x_i \partial x_j}$ for the continuous extension of the derivative.)

For any $1 \leq i < j \leq m$, define the operator $p_{ij}^{(m)} : C(\mathbb{R}^m) \to C(\mathbb{R}^{m-1})$ by

$$p_{ij} f(x_1, \cdots, x_{m-1}) = f(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}),$$

where x_{m-1} occurs at the places of the ith and the jth variables on the right hand side. Let $\mathscr{D}^{(m)}$ be the totality of functions $f \in C(\mathbb{R}^m)$ such that $p_{i_k j_k}^{(m-k)} \cdots p_{i_0 j_0}^{(m)} f \in \mathscr{D}_0^{(m-k-1)}$ for all $1 \leq i_l < j_l \leq m - l$ and $1 \leq l \leq m-1$. For $f \in \mathscr{D}^{(m)}$, let

$$G_0^{(m)} f(x_1, \cdots, x_m) = \frac{1}{2}\rho \Delta_m f(x_1, \cdots, x_m), \quad (x_1, \cdots, x_m) \in \mathbb{R}^m, \tag{2.4}$$

where Δ_m denotes the m-dimensional Laplace operator.

A continuous process $\{(x_1(t), \cdots, x_m(t)) : t \geq 0\}$ is called a solution of the $(G_0^{(m)}, \mathscr{D}^{(m)})$-*martingale problem* if

$$f(x_1(t), \cdots, x_m(t)) - f(x_1(0), \cdots, x_m(0)) - \int_0^t G_0^{(m)} f(x_1(s), \cdots, x_m(s)) ds$$

is a martingale for each $f \in \mathscr{D}^{(m)}$.

Theorem 2.1 *The distribution $\boldsymbol{P}_{(b_1,b_2)}$ on $C([0,\infty), \mathbb{R}^2)$ of the 2-SCBM with speed $\rho > 0$ and initial state (b_1, b_2) is the unique probability measure on $C([0,\infty), \mathbb{R}^2)$ such that $\boldsymbol{P}_{(b_1,b_2)}\{(w_1(0), w_2(0)) = (b_1, b_2)\} = 1$ and $\{(w_1(t), w_2(t)) : t \geq 0\}$ under $\boldsymbol{P}_{(b_1,b_2)}$ solves the $(G_0^{(2)}, \mathscr{D}^{(2)})$-martingale problem.*

Proof. We first show that the 2-SCBM solves the $(G_0^{(2)}, \mathscr{D}^{(2)})$-martingale problem. Let $f \in \mathscr{D}^{(2)}$. If $b_1 = b_2$, then $\boldsymbol{P}_{(b_1,b_2)}\{w_1(t) = w_2(t) \text{ for all } t \geq 0\} = 1$. By Itô's formula we get

$$\begin{aligned} & f(w_1(t), w_2(t)) - f(w_1(0), w_2(0)) \\ = & \sum_{i=1}^2 \int_0^t f_i'(w_1(s), w_2(s)) dw_i(s) + \frac{1}{2}\rho \sum_{i,j=1}^2 \int_0^t f_{ij}''(w_1(s), w_2(s)) d\langle w_i, w_j\rangle(s) \\ = & \sum_{i=1}^2 \int_0^t f_i'(w_1(s), w_2(s)) dw_i(s) + \frac{1}{2}\rho \sum_{i=1}^2 \int_0^t f_{ii}''(w_1(s), w_2(s)) ds, \end{aligned} \tag{2.5}$$

where we have used the assumption $f''_{12}(x,x) = f''_{21}(x,x) = 0$ for the last equality. If $b_1 \neq b_2$, we have

$$\begin{aligned} & f(w_1(t\wedge\tau), w_2(t\wedge\tau)) - f(w_1(0), w_2(0)) \\ = & \sum_{i=1}^{2}\int_0^{t\wedge\tau} f'_i(w_1(s), w_2(s))dw_i(s) + \frac{1}{2}\rho\sum_{i=1}^{2}\int_0^{t\wedge\tau} f''_{ii}(w_1(s), w_2(s))ds, \end{aligned} \tag{2.6}$$

where $\tau = \inf\{t \geq 0 : w_1(t) = w_2(t)\}$. Summing up (2.5) and (2.6) we see that $\{(w_1(t), w_2(t)) : t \geq 0\}$ under $\boldsymbol{P}_{(b_1,b_2)}$ is a solution of the $(G_0^{(2)}, \mathscr{D}^{(2)})$-martingale problem. Conversely, suppose that $\boldsymbol{P}_{(b_1,b_2)}$ is a probability measure on $C([0,\infty), \mathbb{R}^2)$ such that $\boldsymbol{P}_{(b_1,b_2)}\{(w_1(0), w_2(0)) = (b_1, b_2)\} = 1$ and $\{(w_1(t), w_2(t)) : t \geq 0\}$ under $\boldsymbol{P}_{(b_1,b_2)}$ solves the $(G_0^{(2)}, \mathscr{D}^{(2)})$-martingale problem. For $f \in C^2(\mathbb{R})$ we apply the martingale problem to the function $(y_1, y_2) \mapsto f(y_1)$ to see that

$$f(w_1(t)) - f(y_1) = \text{mart.} + \frac{1}{2}\rho\int_0^t f''(w_1(s))ds.$$

Therefore, $\{w_1(t) : t \geq 0\}$ under $\boldsymbol{P}_{(b_1,b_2)}$ is a Brownian motion with speed ρ. Similarly, $\{w_2(t) : t \geq 0\}$ under $\boldsymbol{P}_{(y_1,y_2)}$ is also a Brownian motion with speed ρ. On the other hand, for $f \in C_0^2([0,\infty))$ satisfying $f''(0) = 0$ we find by applying the martingale problem to the function $(y_1, y_2) \mapsto f(|y_1 - y_2|)$ that

$$f(|w_1(t) - w_2(t)|) - f(|y_1 - y_2|) = \text{mart.} + \rho\int_0^t f''(|w_1(s) - w_2(s)|)ds. \tag{2.7}$$

For $n \geq 1$ let $f_n \in C_0^2([0,\infty))$ be such that $f_n(x) = x$ for $0 \leq x \leq n$. Applying (2.7) to the sequence $\{f_n\}$ we see that $\{|w_1(t) - w_2(t)| : t \geq 0\}$ under $\boldsymbol{P}_{(b_1,b_2)}$ is a non-negative local martingale so it must be absorbed at zero. By Itô's formula,

$$|w_1(t) - w_2(t)|^2 - |y_1 - y_2|^2 = \text{local mart.} + \int_0^t d\langle|w_1 - w_2|\rangle(s).$$

For $n \geq 1$ let $h_n \in C_0^2([0,\infty))$ be such that $h''_n(0) = 0$ and $h_n(x) = x^2$ for $n^{-1} \leq x \leq n$. Since $\{|w_1(t) - w_2(t)| : t \geq 0\}$ is absorbed at zero, applying (2.7) to $\{h_n\}$ we see that

$$|w_1(t) - w_2(t)|^2 - |y_1 - y_2|^2 = \text{local mart.} + 2\rho(t\wedge\tau),$$

where $\tau = \inf\{t \geq 0 : w_1(t) = w_2(t)\}$. It follows that $\langle|w_1 - w_2|\rangle(t) = 2\rho \cdot (t\wedge\tau)$ and hence $\boldsymbol{P}_{(b_1,b_2)}$ is the distribution of the 2-SCBM. $\square$

Theorem 2.2 *The distribution $\boldsymbol{P}_{(b_1,\cdots,b_m)}$ on $C([0,\infty), \mathbb{R}^m)$ of the m-SCBM with speed $\rho > 0$ and initial state $(b_1, \cdots, b_m)$ is the unique probability measure on $C([0,\infty), \mathbb{R}^m)$ such that $\boldsymbol{P}_{(b_1,\cdots,b_m)}\{(w_1(0), \cdots, w_m(0)) = (b_1, \cdots, b_m)\} = 1$ and $\{(w_1(t), \cdots, w_m(t)) : t \geq 0\}$ under $\boldsymbol{P}_{(b_1,\cdots,b_m)}$ solves the $(G_0^{(m)}, \mathscr{D}^{(m)})$-martingale problem.*

Proof. By considering the m-SCBM piece by piece between the coalescing times as in the proof of the last theorem, one can show that it is indeed a solution of the $(G_0^{(m)}, \mathscr{D}^{(m)})$-martingale problem. To see the uniqueness, observe that for any $1 \le i < j \le m$ and $f \in \mathscr{D}^{(2)}$, the function $(y_1, \cdots, y_m) \mapsto f(y_i, y_j)$ belongs to $\mathscr{D}^{(m)}$. It follows that if $\{(w_1(t), \cdots, w_m(t)) : t \ge 0\}$ under $\boldsymbol{P}_{(b_1,\cdots,b_m)}$ is a solution of the $(G_0^{(m)}, \mathscr{D}^{(m)})$-martingale problem, then the pair $\{(w_i(t), w_j(t)) : t \ge 0\}$ is a solution of the $(G_0^{(2)}, \mathscr{D}^{(2)})$-martingale problem. By Theorem 2.1, $\{(w_i(t), w_j(t)) : t \ge 0\}$ under $\boldsymbol{P}_{(b_1,\cdots,b_m)}$ is a 2-SCSM and hence $\langle w_i, w_j\rangle(t) = \rho \cdot (t - t \wedge \tau_{ij})$, where $\tau_{ij} = \inf\{t \ge 0 : w_i(t) = w_j(t)\}$. Then $\{(w_1(t), \cdots, w_m(t)) : t \ge 0\}$ is an m-SCBM. $\square$

We may embed the m-SCBM into an inhomogeneous Markov process with state space $W := C([0,\infty), \mathbb{R})$. To this end, let $W^{\mathbb{R}}$ denote the totality of W-valued paths $\{w(a,\cdot) : a \in \mathbb{R}\}$, which contains all possible paths of the Markov process to be defined. For any $\{b_1, \cdots, b_m\} \subset \mathbb{R}$, let $F_{b_1,\cdots,b_m}$ denote the distribution on W^m of the m-SCBM $\{(y_1(t), \cdots, y_m(t)) : t \ge 0\}$ with speed $\rho > 0$ and initial state $(b_1, \cdots, b_m)$. It is easy to see that $\{F_{b_1,\cdots,b_m} : b_1, \cdots, b_m \in \mathbb{R}\}$ is a consistent family. By Kolmogorov's extension theorem, there is a unique probability measure $\boldsymbol{P}^{cb}$ on $W^{\mathbb{R}}$ which has finite dimensional distributions $\{F_{b_1,\cdots,b_m} : b_1, \cdots, b_m \in \mathbb{R}\}$. A two parameter process $\{y(a,t) : a \in \mathbb{R}, t \ge 0\}$ is called a *coalescing Brownian flow* if the path-valued process $\{y(a,\cdot) : a \in \mathbb{R}\}$ has distribution $\boldsymbol{P}^{cb}$ on $W^{\mathbb{R}}$. Indeed, $\{y(a,\cdot) : a \in \mathbb{R}\}$ is an inhomogeneous Markov process. For $a \in \mathbb{R}$ let W_a denote the set of paths $w \in W$ with $w(0) = a$. For any $a \in \mathbb{R}$ let $\{B_a(t) : t \ge 0\}$ be a Brownian motion with speed $\rho > 0$ and initial state $B_a(0) = a$ and let $Q_a(\cdot)$ denote the distribution of $\{B_a(t) : t \ge 0\}$ on W_a. For $a \le b \in \mathbb{R}$ and $w_a \in W_a$ let $Q_{a,b}(w_a, \cdot)$ denote the distribution on W_b of the process $\{\xi_b(t) : t \ge 0\}$ defined by

$$\xi_b(t) = \begin{cases} B_b(t) & \text{if } t \le \tau_{ab}, \\ w_a(t) & \text{if } t > \tau_{ab}, \end{cases}$$

where $\tau_{ab} = \inf\{t \ge 0 : B_b(t) = w_a(t)\}$. Then $(Q_{a,b})_{a \le b}$ is a Markov transition semigroup with state spaces $\{W_a : a \in \mathbb{R}\}$ and $(Q_a)_{a\in\mathbb{R}}$ is an entrance law for $(Q_{a,b})_{a\le b}$. It is not hard to see that a coalescing Brownian flow $\{y(a,\cdot) : a \in \mathbb{R}\}$ is a Markov process with transition semigroup $(Q_{a,b})_{a\le b}$ and one-dimensional distributions $(Q_a)_{a\in\mathbb{R}}$. See Dynkin (1978, p.724) for discussions of inhomogeneous Markov processes determined by entrance laws. A more general coalescing Brownian flow is defined and studied in Harris (1984), where interaction is allowed between the particles before they coalesce. We refer the reader to Evans and Pitman (1998) and the references therein for some recent work on related models.

Now we consider an interacting Brownian flow driven by a time-space white noise. Let $h \in C(\mathbb{R})$ be square-integrable and continuously differentiable with square-integrable derivative h'. Suppose we are given on some standard probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ a time-space white noise $W(ds, dy)$ on $[0,\infty) \times \mathbb{R}$ based on the Lebesgue measure; see, e.g., Walsh (1986). By Dawson *et al.* (2001) and Wang (1997, 1998), for each $a \in \mathbb{R}$ the equation

$$x(t) = a + \int_0^t \int_{\mathbb{R}} h(y - x(s)) W(ds, dy), \quad t \ge 0, \tag{2.8}$$

has a unique solution $\{x(a,t) : t \ge 0\}$. We call $\{x(a,t) : t \ge 0; a \in \mathbb{R}\}$ an *interacting Brownian flow* driven by the time-space white noise. It is not hard to check that for any $(a_1, \cdots, a_m) \in \mathbb{R}^m$, the solutions $\{(x(a_1,t), \cdots, x(a_m,t)) : t \ge 0\}$ of (2.8) constitute an m-dimensional diffusion process generated by the differential operator

$$G^{(m)} := \frac{1}{2} \sum_{i,j=1}^m \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{2.9}$$

where

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R}. \tag{2.10}$$

In particular, each $\{x(a_i, t) : t \geq 0\}$ is a one-dimensional Brownian motion with quadratic variation process $\rho(0)t$, so we call $\{(x(a_1, t), \cdots, x(a_n, t)) : t \geq 0\}$ an *m-system of interacting Brownian motions* (m-SIBM).

Given $\theta > 0$ and $f \in C(\mathbb{R})$, let $f_\theta(x) = f(\theta x)$. Replacing $h(\cdot)$ in (2.8) by $\sqrt{\theta}h_\theta(\cdot)$ we obtain the function $\rho_\theta(\cdot)$, so the latter can also serve as the interaction parameter of an interacting Brownian motion. The following theorem shows that the coalescing Brownian flow arises in some sense as the scaling limit of the interacting Brownian flow driven by the time-space white noise.

Theorem 2.3 *Suppose that $\rho(x) \to 0$ as $|x| \to \infty$. For each $\theta \geq 1$, let $\{(x_1^\theta(t), \cdots, x_m^\theta(t)) : t \geq 0\}$ be an m-SIBM with interaction parameter $\rho_\theta(\cdot)$ and initial state $(a_1^\theta, \cdots, a_m^\theta)$. If $a_i^\theta \to b_i$ as $\theta \to \infty$, then the law of $\{(x_1^\theta(t), \cdots, x_m^\theta(t)) : t \geq 0\}$ on $C([0,\infty), \mathbb{R}^m)$ converges to that of the m-SCBM with speed $\rho(0)$ starting from $(b_1, \cdots, b_m)$.*

Proof. The result could be proved using Theorem 2.2. The following proof directly based on the definition of the SIBM seems more readable. Since each $x_i^\theta(t)$ is a Brownian motion with speed $\rho(0)$, we get by Doob's martingale inequality that

$$\boldsymbol{P}\bigg\{\sup_{0\leq t\leq T} |x_i^\theta(t)| > \eta\bigg\} \leq \boldsymbol{P}\{x_i^\theta(T)^2\}/\eta^2 = 2(|a_i^\theta|^2 + \rho(0)T)/\eta^2,$$

where we also use $\boldsymbol{P}$ to denote the expectation; see, e.g., Ikeda and Watanabe (1989, p.34). Then for each $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset \mathbb{R}^m$ such that

$$\sup_{\theta\geq 1} \boldsymbol{P}\{(x_1^\theta(t), \cdots, x_m^\theta(t)) \in K_\varepsilon \text{ for } 0 \leq t \leq T\} \leq \varepsilon, \tag{2.11}$$

that is, the family $\{(x_1^\theta(\cdot), \cdots, x_m^\theta(\cdot)) : \theta \geq 1\}$ satisfies the compact containment condition of Ethier and Kurtz (1986, p.142). Let

$$G_\theta^{(m)} := \frac{1}{2}\sum_{i,j=1}^m \rho_\theta(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}.$$

Let $C_\kappa(\mathbb{R}^m)$ denote the set of continuous functions on $\mathbb{R}$ with compact supports and let $H = C_\kappa(\mathbb{R}^m) \cap C^2(\mathbb{R}^m)$. Then for $f \in H$,

$$f(x_1^\theta(t), \cdots, x_m^\theta(t)) - f(a_1^\theta, \cdots, a_m^\theta) - \int_0^t G_\theta^{(m)} f(x_1^\theta(s), \cdots, x_m^\theta(s))ds \tag{2.12}$$

is a martingale. Observe that $\sup_{\theta\geq 1} \|G_\theta^{(m)} f\| < \infty$, so for each $T > 0$ we have

$$\sup_{\theta\geq 1} \boldsymbol{E}\bigg[\bigg(\int_0^T |G_\theta^{(m)} f(x_1^\theta(s), \cdots, x_m^\theta(s))|^2 ds\bigg)^{1/2}\bigg] < \infty.$$

By Ethier and Kurtz (1986, p.145), $\{f(x_1^\theta(\cdot),\cdots,x_m^\theta(\cdot)):\theta\geq 1\}$ is a tight family in $C([0,\infty),\mathbb{R})$, which is a closed subspace of $D([0,\infty),\mathbb{R})$. Since H is dense in $C_\kappa(\mathbb{R}^m)$ in the topology of uniform convergence on compact sets, by Ethier and Kurtz (1986, p.142), $\{(x_1^\theta(\cdot),\cdots,x_m^\theta(\cdot)):\theta\geq 1\}$ is tight in $C([0,\infty),\mathbb{R}^m)$. Let $\boldsymbol{P}_0$ be the limit distribution on $C([0,\infty),\mathbb{R}^m)$ of any convergent subsequence $(x_1^{\theta_k}(t),\cdots,x_m^{\theta_k}(t))$ with $\theta_k\to\infty$. Since each $x_i^\theta(t)$ is a Brownian motion with speed $\rho(0)$, so is $w_i(t)$ under $\boldsymbol{P}_0$. As in Wang (1998, p.756), one may see that $\{x_i^\theta(t)-x_j^\theta(t):t\geq 0\}$ is a diffusion process for which the origin is an unaccessible trap. It follows that $\boldsymbol{P}\{x_i^\theta(t)=x_j^\theta(t)$ for all $t\geq 0\}=1$ if $a_i^\theta=a_j^\theta$ and $\boldsymbol{P}\{x_i^\theta(t)\neq x_j^\theta(t)$ for all $t\geq 0\}=1$ if $a_i^\theta\neq a_j^\theta$. In view of (2.12), for any $f\in C_0^2([0,\infty))$ with $f''(0)=0$,

$$\begin{aligned} f(|x_j^\theta(t)-x_i^\theta(t)|)-f(|a_j^\theta-a_i^\theta|)\\ -\int_0^t[\rho(0)-\rho_\theta(x_j^\theta(s)-x_i^\theta(s))]f''(|x_j^\theta(s)-x_i^\theta(s)|)ds \end{aligned}\tag{2.13}$$

is a martingale. Since $f''(0)=0$ and $\rho(x)\to 0$ as $|x|\to\infty$, we have $[\rho(0)-\rho_\theta(\cdot)]f''(|\cdot|)\to\rho(0)f''(|\cdot|)$ uniformly as $\theta\to\infty$. Letting $\theta\to\infty$ in (2.13) along $\{\theta_k\}$ we see

$$f(|w_j(t)-w_i(t)|)-f(|b_j-b_i|)-\int_0^t\rho(0)f''(|w_j(s)-w_i(s)|)ds$$

under $\boldsymbol{P}_0$ is a martingale. As in the proof of Theorem 2.1, $\{|w_j(t)-w_i(t)|:t\geq 0\}$ under $\boldsymbol{P}_0$ must be a non-negative local martingale having quadratic variation process $2\rho(0)(t\wedge\tau_{ij})$ with $\tau_{ij}=\inf\{t\geq 0:w_i(t)=w_j(t)\}$. Thus $\boldsymbol{P}_0$ is the law of the m-SCBM Brownian motion starting from $(b_1,\cdots,b_m)$ with speed $\rho(0)$. □

3 Superprocesses with coalescing spatial motion

In this section, we give some constructions for the SCSM. Let $\rho>0$ be a constant. Suppose that $\sigma\in C(\mathbb{R})^+$ and there is a constant $\epsilon>0$ such that $\inf_x\sigma(x)\geq\epsilon$. The formal generator of the SCSM is given by

$$\begin{aligned}\mathscr{L}F(\mu) &= \frac{1}{2}\int_{\mathbb{R}}\sigma\frac{\delta^2F(\mu)}{\delta\mu(x)^2}\mu(dx)+\frac{1}{2}\rho\int_{\mathbb{R}}\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx)\\ &\quad+\frac{1}{2}\int_{\varDelta}\frac{d^2}{dxdy}\frac{\delta^2F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy),\end{aligned}\tag{3.1}$$

where $\varDelta=\{(x,x):x\in\mathbb{R}\}$. Note that the first two terms on the right hand side simply give the generator of a usual super Brownian motion, where the first term describes the branching and the second term gives the spatial motion. The last term shows that interactions in the spatial motion only occur between 'particles' located at the same positions. Those descriptions are justified by the constructions to be given.

We first consider a purely atomic initial state with a finite number of atoms. In the sequel, a martingale diffusion $\{\xi(t):t\geq 0\}$ is called a *standard Feller branching diffusion* if it has quadratic variation $\xi(t)dt$. Let $\{(y_1(t),\cdots,y_n(t)):t\geq 0\}$ be an n-SCBM with speed ρ and initial state $(b_1,\cdots,b_n)\in\mathbb{R}^n$. Let $\{(\xi_1(t),\cdots,\xi_n(t)):t\geq 0\}$ be a system of independent standard Feller branching diffusions with initial state $(\xi_1,\cdots,\xi_n)\in\mathbb{R}_+^n$. We assume that $\{(y_1(t),\cdots,y_n(t)):$

$t \geq 0\}$ and $\{(\xi_1(t), \cdots, \xi_n(t)) : t \geq 0\}$ are defined on a standard complete probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ and are independent of each other. Set

$$\psi_i^\sigma(t) = \int_0^t \sigma(y_i(s))ds \tag{3.2}$$

and $\xi_i^\sigma(t) = \xi_i(\psi_i^\sigma(t))$. Then

$$X_t = \sum_{i=1}^n \xi_i^\sigma(t)\delta_{y_i(t)}, \quad t \geq 0, \tag{3.3}$$

defines a continuous $M(\mathbb{R})$-valued process. Intuitively, this process consists of n particles carried by the n-SCBM $\{(y_1(t), \cdots, y_n(t)) : t \geq 0\}$. The mass of the ith particle is given by $\{\xi_i^\sigma(t) : t \geq 0\}$, which is obtained from a standard Feller branching diffusion by a time change depending on the position of the ith carrier. Thus we have here a spatially dependent branching mechanism.

Let $\mathscr{G}_t$ be the σ-algebra generated by the family of $\boldsymbol{P}$-null sets in $\mathscr{F}$ and the family of random variables

$$\{(y_1(s), \cdots, y_n(s), \xi_1^\sigma(s), \cdots, \xi_n^\sigma(s)) : 0 \leq s \leq t\}. \tag{3.4}$$

Then we have

Theorem 3.1 *The process $\{X_t : t \geq 0\}$ defined by (3.3) is a diffusion process relative to the filtration $(\mathscr{G}_t)_{t\geq 0}$ with state space $M_a(\mathbb{R})$, the set of purely atomic measures on $\mathbb{R}$.*

Proof. Let $\mu = \sum_{i=1}^n \xi_i \delta_{a_i}$. By symmetry, the distribution $Q_t(\mu, \cdot)$ of X_t on $M_a(\mathbb{R})$ only depends on $t \geq 0$ and μ. Clearly, under $\boldsymbol{P}\{\cdot\,|\mathscr{G}_r\}$ the process $\{(x_1(r+t), \cdots, x_n(r+t)) : t \geq 0\}$ is an n-SCBM and $\{(\xi_1(\psi_i^\sigma(r)+t), \cdots, \xi_n(\psi_i^\sigma(r)+t)) : t \geq 0\}$ is a system of independent standard Feller branching diffusions. Moreover, the two systems are conditionally independent of each other. Then X_{r+t} under $\boldsymbol{P}\{\cdot\,|\mathscr{G}_r\}$ has distribution $Q_t(X_r, \cdot)$. The Feller property of the $Q_t(\mu, \cdot)$ follows from those of $(x_1(t), \cdots, x_n(t))$ and $(\xi_1(t), \cdots, \xi_n(t))$. Then the strong Markov property holds by the continuity of $\{X_t : t \geq 0\}$. $\square$

Theorem 3.2 *If $\{X_t : t \geq 0\}$ is given by (3.3), then for each $\phi \in C^2(\mathbb{R})$,*

$$M_t(\phi) = \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - \frac{1}{2}\rho \int_0^t \langle \phi'', X_s \rangle ds, \quad t \geq 0, \tag{3.5}$$

is a continuous martingale relative to $(\mathscr{G}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, X_s \rangle ds + \int_0^t ds \int_\Delta \phi'(x)\phi'(y)X_s(dx)X_s(dy), \tag{3.6}$$

where $\Delta = \{(x, x) : x \in \mathbb{R}\}$.

Proof. As in the proof of Dawson and Li (2003, Theorem 3.3), $\{(\xi_i^\sigma(t) : t \geq 0\}$ is a continuous martingale with quadratic variation $\sigma(y_i(t))dt$ and $\langle \xi_i^\sigma, \xi_j^\sigma \rangle(t) \equiv 0$ if $i \neq j$. By Itô's formula,

$$\begin{aligned} \xi_i^\sigma(t)\phi(y_i(t)) \quad &= \quad \xi_i^\sigma(0)\phi(y_i(0)) + \int_0^t \phi(y_i(s))d\xi_i^\sigma(s) + \int_0^t \xi_i^\sigma(s)\phi'(y_i(s))dy_i(s) \\ &\quad + \frac{1}{2}\rho \int_0^t \xi_i^\sigma(s)\phi''(y_i(s))ds. \end{aligned} \tag{3.7}$$

Taking the summation we get

$$\langle \phi, X_t \rangle - \langle \phi, X_0 \rangle = M_t(\phi) + \frac{1}{2}\rho \int_0^t \langle \phi'', X_s \rangle ds, \quad t \geq 0,$$

where

$$M_t(\phi) := \sum_{i=1}^n \int_0^t \phi(y_i(s)) d\xi_i^\sigma(s) + \sum_{i=1}^n \int_0^t \xi_i^\sigma(s)\phi'(y_i(s))dy_i(s),$$

is a continuous martingale relative to $(\mathscr{G}_t)_{t\geq 0}$ with quadratic variation process

$$\begin{aligned} \langle M(\phi)\rangle_t &= \sum_{i=1}^n \int_0^t \sigma(y_i(s))\xi_i^\sigma(s)\phi(y_i(s))^2 ds + \sum_{i,j=1}^n \int_{\tau_{ij}}^t \xi_i^\sigma(s)\xi_j^\sigma(s)\phi'(y_i(s))\phi'(y_j(s))ds \\ &= \int_0^t \langle \sigma\phi^2, X_s\rangle ds + \int_0^t ds \int_{\varDelta} \phi'(x)\phi'(y) X_s^2(dx, dy), \end{aligned}$$

where $\tau_{ij} = \inf\{s \geq 0 : y_i(s) = y_j(s)\}$. This gives the desired result. □

We can give another martingale characterization of the process as follows. Let $\mathscr{D}(\mathscr{L})$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in \mathscr{D}^{(m)}$. Observe that

$$\mathscr{L} F_{m,f}(\mu) = F_{m, G_0^{(m)} f}(\mu) + \frac{1}{2} \sum_{i,j=1, i\neq j}^m F_{m-1, \varPhi_{ij} f}(\mu), \tag{3.8}$$

where $G_0^{(m)}$ is the generator of the m-SCBM with speed ρ and $\varPhi_{ij}$ denotes the operator from $C(\mathbb{R}^m)$ to $C(\mathbb{R}^{m-1})$ defined by

$$\varPhi_{ij} f(x_1, \cdots, x_{m-1}) = \sigma(x_{m-1}) f(x_1, \cdots, x_{m-1}, \cdots, x_{m-1}, \cdots, x_{m-2}), \tag{3.9}$$

where x_{m-1} takes the places of the ith and the jth variables of f on the right hand side.

Theorem 3.3 *Let $\{X_t : t \geq 0\}$ be defined by (3.3). Then $\boldsymbol{E}\{\langle 1, X_t\rangle^m\}$ is locally bounded in $t \geq 0$ for each $m \geq 1$ and $\{X_t : t \geq 0\}$ solves the $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$-martingale problem, that is, for each $F_{m,f} \in \mathscr{D}(\mathscr{L})$,*

$$F_{m,f}(X_t) - F_{m,f}(X_0) - \int_0^t \mathscr{L} F_{m,f}(X_s) ds \tag{3.10}$$

is a martingale.

Proof. Based on Theorem 3.2, it is not hard to show that $\boldsymbol{E}\{\langle 1, X_t\rangle^m\}$ is locally bounded in $t \geq 0$ for each $m \geq 1$. Since $\{(\xi_i^\sigma(t) : t \geq 0\}$ is a continuous martingale with quadratic variation $\sigma(y_i(t))dt$ and $\langle \xi_i^\sigma, \xi_j^\sigma\rangle(t) \equiv 0$ if $i \neq j$, for $m \geq 1$ and $f \in \mathscr{D}^{(m)}$ we have

$$\langle f, X_t^m \rangle = \sum_{i_1, \cdots, i_m = 1}^n \xi_{i_1}^\sigma(t) \cdots \xi_{i_m}^\sigma(t) f(y_{i_1}(t), \cdots, y_{i_m}(t))$$

$$
\begin{aligned}
&= \sum_{i_1,\cdots,i_m=1}^{n} \xi_{i_1}^\sigma(0)\cdots\xi_{i_m}^\sigma(0) f(y_{i_1}(0),\cdots,y_{i_m}(0)) + \text{mart.} \\
&\quad + \frac{1}{2}\rho \sum_{i_1,\cdots,i_m=1}^{n}\sum_{j=1}^{m}\int_0^t \xi_{i_1}^\sigma(s)\cdots\xi_{i_m}^\sigma(s) f''_{jj}(y_{i_1}(s),\cdots,y_{i_m}(s))ds \\
&\quad + \frac{1}{2}\sum_{\alpha,\beta=1}^{m}\sum_{\{\text{cond.}\}}\int_0^t \sigma(y_{i_\alpha}(s))\xi_{i_1}^\sigma(s)\cdots\xi_{i_m}^\sigma(s)\xi_{i_\alpha}^\sigma(s)^{-1} f(y_{i_1}(s),\cdots,y_{i_m}(s))ds \\
&= \langle f, X_0^m\rangle + \text{mart.} + \frac{1}{2}\rho\int_0^t \langle \Delta f, X_s^m\rangle ds + \frac{1}{2}\sum_{\alpha,\beta=1}^{m}\int_0^t \langle \Phi_{\alpha\beta} f, X_s^{m-1}\rangle ds,
\end{aligned}
$$

where $\{\text{cond.}\} = \{$ for all $1 \le i_1,\cdots,i_m \le n$ with $i_\alpha = i_\beta\}$ and we used the fact $f''_{ij}(x_1,\cdots,x_m) = 0$ for $x_i = x_j$ in the second equality. By (3.8) we see that $\{X_t : t \ge 0\}$ solves the $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$-martingale problem. □

The distribution of $\{X_t : t \ge 0\}$ can be characterized in terms of a dual process as follows. Let $\{M_t : t \ge 0\}$ be a nonnegative integer-valued cádlág Markov process with transition intensities $\{q_{i,j}\}$ such that $q_{i,i-1} = -q_{i,i} = i(i-1)/2$ and $q_{i,j} = 0$ for all other pairs (i,j). That is, $\{M_t : t \ge 0\}$ only has downward jumps which occur at rate $M_t(M_t - 1)/2$. Such a Markov process is known as Kingman's coalescent process. Let $\tau_0 = 0$ and $\tau_{M_0} = \infty$. For $1 \le k \le M_0 - 1$ let τ_k denote the kth jump time of $\{M_t : t \ge 0\}$. Let $\{\Gamma_k : 1 \le k \le M_0 - 1\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \ge 0\}$ and satisfy

$$
\boldsymbol{P}\{\Gamma_k = \Phi_{ij} | M(\tau_k^-) = l\} = \frac{1}{l(l-1)}, \qquad 1 \le i \ne j \le l, \tag{3.11}
$$

where Φ_{ij} is defined by (3.9). Let $\boldsymbol{C}$ denote the topological union of $\{C(\mathbb{R}^m) : m = 1, 2, \cdots\}$ endowed with pointwise convergence on each $C(\mathbb{R}^m)$. Let $(P_t^{(m)})_{t\ge 0}$ denote the transition semigroup of the m-SCBM. Then

$$
Y_t = P_{t-\tau_k}^{(M_{\tau_k})}\Gamma_k P_{\tau_k-\tau_{k-1}}^{(M_{\tau_{k-1}})}\Gamma_{k-1}\cdots P_{\tau_2-\tau_1}^{(M_{\tau_1})}\Gamma_1 P_{\tau_1}^{(M_0)} Y_0, \quad \tau_k \le t < \tau_{k+1}, 0 \le k \le M_0 - 1, \tag{3.12}
$$

defines a Markov process $\{Y_t : t \ge 0\}$ taking values from $\boldsymbol{C}$. The process evolves in the time interval $[0, \tau_1)$ according to the linear semigroup $(P_t^{(M_0)})_{t\ge 0}$ and then it makes a jump given by Γ_1 at time τ_1. After that, it evolves in interval $[\tau_1, \tau_2)$ according to $(P_t^{(M_{\tau_1})})_{t\ge 0}$ and then it makes another jump given by Γ_2 at time τ_2, and so on. Clearly, $\{(M_t, Y_t) : t \ge 0\}$ is also a Markov process. Let $\boldsymbol{E}_{m,f}^\sigma$ denote the expectation related to $\{(M_t, Y_t) : t \ge 0\}$ given $M_0 = m$ and $Y_0 = f \in C(\mathbb{R}^m)$.

Theorem 3.4 *If $\{X_t : t \ge 0\}$ is a continuous $M(\mathbb{R})$-valued process such that $\boldsymbol{E}\{\langle 1, X_t\rangle^m\}$ is locally bounded in $t \ge 0$ for each $m \ge 1$ and $\{X_t : t \ge 0\}$ solves the $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$-martingale problem with $X_0 = \mu$, then the distribution of X_t is uniquely determined by*

$$
\boldsymbol{E}\langle f, X_t^m\rangle = \boldsymbol{E}_{m,f}^\sigma\left[\langle Y_t, \mu^{M_t}\rangle \exp\left\{\frac{1}{2}\int_0^t M_s(M_s - 1)ds\right\}\right], \tag{3.13}
$$

where $t \ge 0$, $f \in C(\mathbb{R}^m)$ and $m \ge 1$.

Proof. It suffices to prove the equation for $Y_0 = f \in \mathscr{D}^{(m)}$. In this case, we have a.s. $Y_t \in \mathscr{D}(\mathscr{L})$ for all $t \geq 0$. Set $F_\mu(m,f) = F_{m,f}(\mu) = \langle f, \mu^m \rangle$. By the construction (3.12), it is not hard to see that $\{(M_t, Y_t) : t \geq 0\}$ has generator $\mathscr{L}^*$ given by

$$\mathscr{L}^* F_\mu(m,f) = F_\mu(m, G^{(m)} f) + \frac{1}{2} \sum_{i,j=1, i \neq j}^{m} [F_\mu(m-1, \Phi_{ij} f) - F_\mu(m,f)]. \tag{3.14}$$

In view of (3.8) and (3.14) we have

$$\mathscr{L} F_{m,f}(\mu) = \mathscr{L}^* F_\mu(m,f) + \frac{1}{2} m(m-1) F_\mu(m,f). \tag{3.15}$$

The right hand side corresponds to a Feynman-Kac formula for the process $\{(M_t, Y_t) : t \geq 0\}$. Guided by this relation, it is not hard to get

$$\boldsymbol{E}\left[F_{m,f}(X_t)\right] = \boldsymbol{E}^\sigma_{m,f}\left[F_\mu(M_t, Y_t) \exp\left\{\frac{1}{2}\int_0^t M_s(M_s - 1) ds\right\}\right],$$

which is just (3.13). This formula gives in particular all the moments of $\langle f_1, X_t \rangle$ for $f_1 \in C(\mathbb{R})$ and hence determines uniquely the distribution of X_t. We omit the details since they are almost identical with the proofs of Dawson *et al.* (2001, Theorems 2.1 and 2.2). □

By Theorems 3.3 and 3.4, the process $\{X_t : t \geq 0\}$ constructed by (3.3) is a diffusion process. Let $Q_t(\mu, d\nu)$ denote the distribution of X_t on $M(\mathbb{R})$ given $X_0 = \mu \in M_a(\mathbb{R})$. The above theorem asserts that

$$\int_{M(\mathbb{R})} \langle f, \nu^m \rangle Q_t(\mu, d\nu) = \boldsymbol{E}^\sigma_{m,f}\left[\langle Y_t, \mu^{M_t} \rangle \exp\left\{\frac{1}{2}\int_0^t M_s(M_s - 1) ds\right\}\right] \tag{3.16}$$

for $t \geq 0$, $m \geq 1$ and $f \in C(\mathbb{R}^m)$. As in the proof of Dawson *et al.* (2001, Theorem 5.1) one can extend $Q_t(\mu, d\nu)$ to a Feller transition semigroup on $M(\mathbb{R})$. A Markov process on $M(\mathbb{R})$ with transition semigroup $(Q_t)_{t \geq 0}$ given by (3.16) is called a *superprocess with coalescing spatial motion* (SCSM) with speed ρ and branching rate $\sigma(\cdot)$.

A construction of the SCSM with a general initial state $\mu \in M(\mathbb{R})$ is given as follows. Let $W = C([0,\infty), \mathbb{R}^+)$ and let $\tau_0(w) = \inf\{s > 0 : w(s) = 0\}$ for $w \in W$. Let W_0 be the set of paths $w \in W$ such that $w(0) = w(t) = 0$ for $t \geq \tau_0(w)$. We endow W and W_0 with the topology of locally uniform convergence. Let $(q_t)_{t \geq 0}$ denote the transition semigroup of the standard Feller branching diffusion. For $t > 0$ and $y > 0$ let $\kappa_t(dy) = 4t^{-2} e^{-2y/t} dy$. Then $(\kappa_t)_{t > 0}$ is an entrance law for the restriction of $(q_t)_{t \geq 0}$ to $(0, \infty)$. Let $\boldsymbol{Q}_\kappa$ denote the corresponding excursion law, which is the unique σ-finite measure on W_0 satisfying

$$\boldsymbol{Q}_\kappa\{w(t_1) \in dy_1, \cdots, w(t_n) \in dy_n\} = \kappa_{t_1}(dy_1) q_{t_2 - t_1}(y_1, dy_2) \cdots q_{t_n - t_{n-1}}(y_{n-1}, dy_n)$$

for $0 < t_1 < t_2 < \cdots < t_n$ and $y_1, y_2, \cdots, y_n \in (0, \infty)$; see, e.g., Pitman and Yor (1982) or Dawson and Li (2003, p.41) for details. Suppose that $\{y(a,t) : a \in \mathbb{R}, t \geq 0\}$ is a coalescing Brownian flow and $N(dx, dw)$ is a Poisson random measure on $\mathbb{R} \times W_0$ with intensity $\mu(dx)\boldsymbol{Q}_\kappa(dw)$. Assume that $\{y(a,t)\}$ and $\{N(dx,dw)\}$ are defined on a standard probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ and are independent of each other. As in Dawson and Li (2003), we can enumerate the atoms of $N(dx,dw)$ into a sequence $\mathrm{supp}(N) = \{(a_i, w_i) : i = 1, 2, \cdots\}$ such that a.s. $\tau_0(w_{i+1}) < \tau_0(w_i)$ for all $i \geq 1$ and $\tau_0(w_i) \to 0$ as $i \to \infty$. Let

$$\psi_i^\sigma(t) = \int_0^t \sigma(y(a_i, s))ds \tag{3.17}$$

and $w_i^\sigma(t) = w_i(\psi_i^\sigma(t))$. For $t \geq 0$ let $\mathscr{G}_t$ be the σ-algebra generated by the family

$$\{y(a,s) : 0 \leq s \leq t, a \in \mathbb{R}\} \quad \text{and} \quad \{w_i^\sigma(s) : 0 \leq s \leq t, i = 1, 2, \cdots\}. \tag{3.18}$$

Theorem 3.5 *Let $X_0 = \mu$ and let*

$$X_t = \sum_{i=1}^\infty w_i^\sigma(t)\delta_{y(a_i,t)}, \quad t > 0. \tag{3.19}$$

Then $\{X_t : t \geq 0\}$ is a SCSM relative to $(\mathscr{G}_t)_{t\geq 0}$.

Proof. For $r > 0$ let $\mathrm{supp}_r^\sigma(N) = \{(x_i, w_i) \in \mathrm{supp}(N) : w_i^\sigma(r) > 0\}$ and $m^\sigma(r) = \#\{\mathrm{supp}_r^\sigma(N)\}$. As in Dawson and Li (2003, Lemmas 3.3 and 3.4), we have a.s. $m^\sigma(r) < \infty$ and there is a permutation $\{w_{i_j} : j = 1, \cdots, m^\sigma(r)\}$ of $\mathrm{supp}_r^\sigma(N)$ so that $\{w_{i_j}(t) : t \geq r; j = 1, \cdots, m(r)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathscr{G}_r\}$ are independent σ-branching diffusions which are independent of $\{x(a,t) : a \in \mathbb{R}, t \geq r\}$. By Theorem 3.1, $\{X_t : t \geq r\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathscr{G}_r\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. It follows that $\{X_t : t > 0\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. We shall prove that the random measure X_t has distribution $Q_t(\mu, \cdot)$ for $t > 0$ so that the desired result follows from the uniqueness of distribution of the SDSM. By Theorem 3.1 we can also show that

$$X_t^{(r)} := \sum_{j=1}^{m(\epsilon r/\beta)} w_{k_j}(\epsilon r/\beta + \psi_{k_j}^\sigma(t))\delta_{x_{k_j}(t)}, \quad t \geq 0, \tag{3.20}$$

under the non-conditional probability $\boldsymbol{P}$ is a SDSM with initial state

$$X_0^{(r)} = \sum_{j=1}^{m(\epsilon r/\beta)} w_{k_j}(\epsilon r/\beta)\delta_{a_{k_j}}.$$

By Shiga (1990, Theorem 3.6), $\{X_0^{(r)} : r > 0\}$ is a measure-valued branching diffusion without migration and $X_0^{(r)} \to \mu$ a.s. as $r \to 0$. By the Feller property of $(Q_t)_{t\geq 0}$, the distribution of $X_t^{(r)}$ converges to $Q_t(\mu, \cdot)$ as $r \to 0$. Since $\phi_{k_j}(t) \geq \epsilon t/\beta$, we can rewrite (3.20) as

$$X_t^{(r)} := \sum_{j=1}^{m(\epsilon t/\beta)} w_{k_j}(\epsilon r/\beta + \phi_{k_j}(t))\delta_{x_{k_j}(t)}.$$

Then for fixed $t > 0$ we have $X_t^{(r)} \to X_t$ a.s. as $r \to 0$ and hence X_t has distribution $Q_t(\mu, \cdot)$. $\square$

The excursion representation (3.19) allows us to construct the SCSM directly without consideration of high density limits of the corresponding coalescing-branching particle systems. This representation also provides a useful tool for the study of the SCSM. In particular, by (3.19) and the proof of Theorem 3.4, for each $r > 0$ the process $\{X_{r+t} : t \geq 0\}$ consists of only a finite number of atoms. By this observation and the fact $X_r \to \mu$ a.s. as $r \to 0$ implied by the statements of Theorem 3.4, it is easy to see that Theorems 3.2 and 3.3 also hold for a general initial state $\mu \in M(\mathbb{R})$. Another application of (3.19) is the proof of the scaling limit theorem in the next section.

4 A limit theorem of rescaled superprocesses

In this section, we show that the SCSM arises naturally as scaling limit of the SDSM studied in Dawson *et al.* (2001) and Wang (1997, 1998). In particular, the result confirms an observation of Dawson *et al.* (2001) on the scaling limit of the purely atomic SDSM.

Suppose that $h \in C(\mathbb{R})$ is a square-integrable function with continuous square-integrable derivative h'. Let $\rho(\cdot)$ be defined as in section 2. Suppose that $\sigma \in C(\mathbb{R})^+$ and $\inf_x \sigma(x) \geq \epsilon$ for some constant $\epsilon > 0$. We define the operator $\mathscr{L}$ by

$$\begin{aligned}\mathscr{L}F(\mu) &= \frac{1}{2}\rho(0)\int_{\mathbb{R}} \frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} \rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \sigma\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx). \end{aligned} \tag{4.1}$$

Let $\mathscr{D}(\mathscr{L})$ denote the collection of functions on $M(\mathbb{R})$ of the form $F_{n,f}(\nu) := \int f d\nu^n$ with $f \in C^2(\mathbb{R}^n)$ and functions of the form

$$F_{f,\{\phi_i\}}(\nu) := f(\langle\phi_1, \nu\rangle, \cdots, \langle\phi_n, \nu\rangle) \tag{4.2}$$

with $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$. An $M(\mathbb{R})$-valued diffusion process is called a *superprocess with dependent spatial motion* (SDSM) if it solves the $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$-martingale problem. The existence of solution of the $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$-martingale problem was proved in Dawson *et al.* (2001, Theorem 5.2) and its uniqueness follows from Dawson *et al.* (2001, Theorem 2.2); see also Wang (1997, 1998).

Suppose that $\sigma(x) \to \sigma_\partial$ and $\rho(x) \to 0$ as $|x| \to \infty$. Given $\theta > 0$, we defined the operator K_θ on $M(\mathbb{R})$ by $K_\theta\mu(B) = \mu(\{\theta x : x \in B\})$. Let $\{X_t^{(\theta)} : t \geq 0\}$ be a SDSM with parameters (ρ, σ) and deterministic initial state $X_0^{(\theta)} = \mu^{(\theta)} \in M(\mathbb{R})$. Let $X_t^\theta = \theta^{-2}K_\theta X_{\theta^2 t}^{(\theta)}$ and assume $\mu_\theta := \theta^{-2}K_\theta\mu^{(\theta)} \to \mu$ as $\theta \to \infty$. By Dawson *et al.* (2001, Lemma 6.1), $\{X_t^\theta : t \geq 0\}$ is a SDSM with parameters $(\rho_\theta, \sigma_\theta)$.

Lemma 4.1 *Under the above assumptions, $\{X_t^\theta : t \geq 0; \theta \geq 1\}$ is tight in $C([0, \infty), M(\bar{\mathbb{R}}))$.*

Proof. By Dawson and Li (2003, Theorem 3.2), $\{\langle 1, X_t^\theta\rangle : t \geq 0\}$ is a continuous positive martingale. Then we have

$$\boldsymbol{P}\Big\{\sup_{t\geq 0}\langle 1, X_t^\theta\rangle > \eta\Big\} \leq \frac{\langle 1, \mu_\theta\rangle}{\eta}$$

for any $\eta > 0$. That is, $\{X_t^\theta : t \geq 0; \theta \geq 1\}$ satisfy the compact containment condition of Ethier and Kurtz (1986, p.142). Let $\mathscr{L}_\theta$ denote the generator of $\{X_t^\theta : t \geq 0\}$ and let $F = F_{f,\{\phi_i\}}$ be given by (4.2) with $f \in C_0^2(\mathbb{R}^n)$ and with each $\phi_i \in C_\partial^2(\mathbb{R})$ bounded away from zero. Then

$$F(X_t^\theta) - F(X_0^\theta) - \int_0^t \mathscr{L}_\theta F(X_s^\theta) ds, \qquad t \geq 0,$$

is a martingale and the desired tightness follows from the result of Ethier and Kurtz (1986, p.145). $\square$

Let us adopt a useful representation of the SDSM in terms of excursions similar to the one discussed in section 3. Suppose we have on some standard probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ a time-space white noise $W(ds, dy)$ on $[0, \infty) \times \mathbb{R}$ based on the Lebesgue measure and a Poisson random measure $N_\theta(dx, dw)$ on $\mathbb{R} \times W_0$ with intensity $\mu_\theta(dx)\boldsymbol{Q}_\kappa(dw)$, where $\boldsymbol{Q}_\kappa$ denotes the excursion law of the standard Feller branching diffusion. Assume that $\{W(ds, dy)\}$ and $\{N_\theta(dx, dw)\}$ are independent. We enumerate the atoms of $N_\theta(dx, dw)$ into a sequence $\mathrm{supp}(N_\theta) = \{(a_i, w_i) : i = 1, 2, \cdots\}$ so that a.s. $\tau_0(w_{i+1}) < \tau_0(w_i)$ and $\tau_0(w_i) \to 0$ as $i \to \infty$. Let $\{x^\theta(a_i, t) : t \geq 0\}$ be the solution of (2.8) with a_i replacing a and $\sqrt{\theta} h_\theta(\cdot)$ replacing $h(\cdot)$. Let

$$\psi_i^\theta(t) = \int_0^t \sigma_\theta(x^\theta(a_i, s)) ds \tag{4.3}$$

and $w_i^\theta(t) = w_i(\psi_i^\theta(t))$. By Dawson and Li (2003, Theorem 3.4), the process $\{Y_t^\theta : t \geq 0\}$ defined by $Y_0^\theta = \mu_\theta$ and

$$Y_t^\theta = \sum_{i=1}^\infty w_i^\theta(t) \delta_{x^\theta(a_i, t)}, \quad t > 0. \tag{4.4}$$

has the same distribution on $C([0, \infty), M(\mathbb{R}))$ as $\{X_t^\theta : t \geq 0\}$. The following theorem confirms an observation given in the introduction of Dawson *et al.* (2001).

Theorem 4.1 *The distribution of $\{X_t^\theta : t \geq 0\}$ on $C([0, \infty), M(\mathbb{R}))$ converges as $\theta \to \infty$ to that of a SCSM with speed $\rho(0)$, constant branching rate σ_∂ and initial state μ.*

Proof. For any $r > 0$, let $\boldsymbol{Q}_\kappa^r$ denote the restriction of $\boldsymbol{Q}_\kappa$ to $W_r := \{w \in W_0 : \tau_0(w) > r\}$. Then we have $\boldsymbol{Q}_\kappa(W_r) = \boldsymbol{Q}_\kappa^r(W_r) = 2/r$; see, e.g., Dawson and Li (2003). Since $\inf_x \sigma \geq \epsilon$, we have $\psi_i^\theta(t) \geq \epsilon t$. Then $w_i^\theta(t) = 0$ for all $t \geq r$ if $w_i(\epsilon r) = 0$. Thus we only need to consider the restriction of N_θ to $W_{\epsilon r}$ for the construction of the process $\{Y_t^\theta : t \geq r\}$. To avoid triviality we assume $\langle 1, \mu \rangle > 0$. Suppose we have on a probability space the following:

(i) a family of Poisson random variables η_θ with parameter $\langle 1, \mu_\theta \rangle \langle 1, \boldsymbol{Q}_\kappa^{\epsilon r} \rangle$ such that $\eta_\theta \to \eta$ a.s. as $\theta \to \infty$, where η is a Poisson random variable with parameter $\langle 1, \mu \rangle^{-1} \langle 1, \boldsymbol{Q}_\kappa^{\epsilon r} \rangle$.

(ii) sequences of i.i.d. real random variables $\{a_{\theta,1}, a_{\theta,2}, \cdots\}$ with distributions $\langle 1, \mu_\theta \rangle^{-1} \mu_\theta(dx)$ such that $a_{\theta,i} \to a_i$ a.s. as $\theta \to \infty$, where $\{a_1, a_2, \cdots\}$ are i.i.d. real random variables with distribution $\langle 1, \mu \rangle^{-1} \mu(dx)$.

(iii) a sequence of i.i.d. random variables $\{\xi_1, \xi_2, \cdots\}$ in $W_{\epsilon r}$ with distribution $\langle 1, \boldsymbol{Q}_\kappa^{\epsilon r} \rangle^{-1} \boldsymbol{Q}_\kappa^{\epsilon r}(dw)$.

Under those assumptions, it is not hard to see that $\sum_{i=1}^{\eta_\theta}\delta_{(a_{\theta,i},\xi_i)}$ and $\sum_{i=1}^{\eta}\delta_{(a_i,\xi_i)}$ are Poisson random measures with intensities $\mu_\theta(dx)\boldsymbol{Q}_\kappa^{\epsilon r}(dw)$ and $\mu(dx)\boldsymbol{Q}_\kappa^{\epsilon r}(dw)$ respectively. Let $\{x^\theta(a_{\theta,i},t): t\geq 0\}$ be the solution of (2.8) with $a_{\theta,i}$ replacing a and $\sqrt{\theta}h_\theta(\cdot)$ replacing $h(\cdot)$. Let

$$\psi_{\theta,i}(t)=\int_0^t\sigma_\theta(x^\theta(a_{\theta,i},s))ds \tag{4.5}$$

and $\xi_{\theta,i}(t)=\xi_i(\psi_{\theta,i}(t))$. In view of (4.4), the process

$$Z_t^\theta:=\sum_{i=1}^{\eta_\theta}\xi_{\theta,i}(t)\delta_{x^\theta(a_{\theta,i},t)},\quad t\geq r, \tag{4.6}$$

has the same distribution on $C([r,\infty),M(\mathbb{R}))$ as $\{Y_t^\theta: t\geq r\}$ and $\{X_t^\theta: t\geq r\}$. By Theorem 2.3 it is easy to show that $\{Z_t^\theta: t\geq r\}$ converges in distribution to

$$X_t:=\sum_{i=1}^{\eta}\xi_i(\sigma_\partial t)\delta_{x(a_i,t)},\quad t\geq r, \tag{4.7}$$

where $\{x(a_i,t)\}$ is a system of coalescing Brownian motions. By Theorem 3.5, $\{X_t: t\geq r\}$ has the same distribution on $C([r,\infty),M(\mathbb{R}))$ as the SCSM described in the theorem. Then the above arguments show that the distribution of $\{X_t^\theta: t\geq r\}$ on $C([r,\infty),M(\mathbb{R}))$ converges as $\theta\to\infty$ to that of the SCSM. The convergence is certainly true if we consider the distributions on $C([r,\infty),M(\bar{\mathbb{R}}))$. By Lemma 4.1 it is easy to conclude that the distribution of $\{X_t^\theta: t\geq 0;\theta\geq 1\}$ on $C([0,\infty),M(\bar{\mathbb{R}}))$ converges to that of the SCSM. Since all the distributions are supported by $C([0,\infty),M(\mathbb{R}))$, the desired result follows. $\square$

Acknowledgements. We thank S.N. Evans and T.G. Kurtz for enlightening comments on absorbing and coalescing Brownian motions. We are indebted to a referee for a list of comments and suggestions which helped us in improving the presentation of the results. We are also grateful to H. Wang and J. Xiong for helpful discussions on the subject. Dawson and Zhou were supported by NSERC Grants, and Li was supported by NSFC Grants.

References

[1] Bojdecki, T. and Gorostiza, L.G.: *Langevin equation for $\mathscr{S}'$-valued Gaussian processes and fluctuation limits of infinite particle systems.* Probab. Theory Related Fields **73** (1986), 227-244.

[2] Cox, J.T.; Durrett, R. and Perkins, E.A.: *Rescaled voter models converge to super-Brownian motion.* Ann. Probab. **28** (2000), 185-234.

[3] Dawson, D.A.: *The critical measure diffusion process.* Z. Wahrsch. Verw. Gebiete **40** (1977), 125-145.

[4] Dawson, D.A. and Fleischmann, K.: *Strong clumping of critical space-time branching models in subcritical dimensions.* Stochastic Process. Appl. **30** (1988), 193-208.

[5] Dawson, D.A. and Li, Z.H.: *Construction of immigration superprocesses with dependent spatial motion from one-dimensional excursions.* Probab. Theory Related Fields **127** (2003), 37-61.

[6] Dawson, D.A.; Li, Z.H. and Wang, H.: *Superprocesses with dependent spatial motion and general branching densities*. Elect. J. Probab. **6** (2001), Paper No. 25, 1-33.

[7] Dawson, D.A.; Li, Z.H. and Wang, H.: *A degenerate stochastic partial differential equation for the purely atomic superprocess with dependent spatial motion*. Infin. Dimen. Anal., Quant. Probab. Related Topics **6** (2003), 597-607.

[8] Dawson, D.A.; Vaillancourt, J. and Wang, H.: *Stochastic partial differential equations for a class of measure-valued branching diffusions in a random medium*. Ann. Inst. H. Poincaré, Probabilités and Statistiques **36** (2000), 167-180.

[9] Durrett, R. and Perkins, E. A.: *Rescaled contact processes converge to super-Brownian motion in two or more dimensions*. Probab. Theory Related Fields **114** (1999), 309-399.

[10] Dynkin, E.B.: *Sufficient statistics and extreme points*. Ann. Probab. **6** (1978), 705-730.

[11] Evans, S.N. and Pitman, J: *Construction of Markovian coalescents*. Ann. Inst. H. Poincaré Probab. Statist. **34** (1998), 339-383.

[12] Ethier, S.N. and Kurtz, T.G.: *Markov Processes: Characterization and Convergence*. Wiley, New York (1986).

[13] Hara, T. and Slade, G.: *The scaling limit of the incipient infinite cluster in high-dimensional percolation I: Critical exponents*. J. Statist. Phys. **99** (2000), 1075-1168.

[14] Hara, T. and Slade, G.: *The scaling limit of the incipient infinite cluster in high-dimensional percolation II: Integrated super-Brownian excursion*. J. Math. Phys. **41** (2000), 1244-1293.

[15] Harris, T.E.: *Coalescing and noncoalescing stochastic flows in R^1*. Stochastic Process. Appl. **17** (1984), 187-210.

[16] Ikeda, N. and Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes*. North-Holland/Kodansha, Amsterdam/Tokyo (1989).

[17] Konno, N. and Shiga, T.: *Stochastic partial differential equations for some measure-valued diffusions*. Probab. Theory Related Fields **79** (1988), 201-225.

[18] Pitman, J. and Yor, M.: *A decomposition of Bessel bridges*, Z. Wahrsch. verw. Geb. **59** (1982), 425-457.

[19] Shiga, T.: *A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes*. J. Math. Kyoto Univ. **30** (1990), 245-279.

[20] Walsh, J.B.: *An Introduction to Stochastic Partial Differential Equations*. In: Lect. Notes Math. **1180**, 265-439, Springer-Verlag (1986).

[21] Wang, H.: *State classification for a class of measure-valued branching diffusions in a Brownian medium*. Probab. Theory Related Fields **109** (1997), 39-55.

[22] Wang, H.: *A class of measure-valued branching diffusions in a random medium*. Stochastic Anal. Appl. **16** (1998), 753-786.

[23] Wang, H.: *State classification for a class of interacting superprocesses with location dependent branching*. Elect. Commun. Probab. **7** (2002), Paper No. 16, 157-167.