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Generalized Mehler Semigroups and Ornstein-Uhlenbeck Processes Arising from Superprocesses over the Real Line [1]

Zenghu Li and Zikun Wang
Department of Mathematics, Beijing Normal University,
Beijing 100875, People's Republic of China
lizh@email.bnu.edu.cn

Abstract

We study the fluctuation limits of a class of superprocesses with dependent spatial motion on the real line, which give rise to some new Ornstein-Uhlenbeck processes with values of Schwartz distributions.

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1 Introduction

To put the investigation into perspectives, we first give a brief review of some recent progresses in the study of generalized Mehler semigroups and Ornstein-Uhlenbeck processes. Suppose that $(S,+)$ is a Hausdorff topological semigroup and $(Q_t)_{t\geq 0}$ is a transition semigroup on S satisfying

$$Q_t(x_1+x_2,\cdot) = Q_t(x_1,\cdot) * Q_t(x_2,\cdot), \quad t\geq 0, x_1,x_2\in S, \tag{1.1}$$

where "$*$" denotes the convolution operation. A family of probability measures $(\mu_t)_{t\geq 0}$ on S is called a *skew convolution semigroup* (SC-semigroup) associated with $(Q_t)_{t\geq 0}$ if it satisfies

$$\mu_{r+t} = (\mu_r Q_t) * \mu_t, \quad r,t\geq 0. \tag{1.2}$$

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This equation is of interest since it is satisfied if and only if

$$Q_t^\mu(x,\cdot) := Q_t(x,\cdot) * \mu_t(\cdot), \quad t \geq 0, x \in S, \tag{1.3}$$

defines another transition semigroup $(Q_t^\mu)_{t\geq 0}$ on S. In the special case where S is the space of all finite measures on some given measurable space, $(Q_t)_{t\geq 0}$ corresponds to a measure-valued branching process and $(Q_t^\mu)_{t\geq 0}$ corresponds to an immigration process. In this setting, it was proved in [24] that SC-processes are in one-to-one correspondence with infinitely divisible probability entrance laws for the semigroup $(Q_t)_{t\geq 0}$; see also [27]. In [25], a characterization of such laws for the so-called Dawson-Watanabe superprocesses is given. With different formulations, measure-valued immigration processes corresponding to closable infinitely divisible probability entrance laws have been studied by a number of authors; see e.g. [6, 7, 12, 17, 21, 23, 34, 35].

The second well-studied case is where $S = H$ is a real separable Hilbert space and $Q_t(x,\cdot) \equiv \delta_{T_t x}$ for a semigroup of bounded linear operators $(T_t)_{t\geq 0}$ on H. In this case, we can rewrite equation (1.2) as

$$\mu_{r+t} = (\mu_r \circ T_t^{-1}) * \mu_t, \quad r, t \geq 0, \tag{1.4}$$

and the transition semigroup $(Q_t^\mu)_{t\geq 0}$ is given by

$$Q_t^\mu f(x) := \int_H f(T_t x + y)\mu_t(dy), \quad x \in H, f \in B(H), \tag{1.5}$$

where $B(H)$ denotes the totality of bounded Borel measurable functions on H. The semigroup $(Q_t^\mu)_{t\geq 0}$ defined by (1.5) is called a *generalized Mehler semigroup* associated with $(T_t)_{t\geq 0}$, which corresponds to a generalized Ornstein-Uhlenbeck process (OU-process). This definition of the generalized OU-process was given in [1]. The similarity between this formulation and that of the immigration process was first noticed by L.G. Gorostiza (1999, personal communication); see also [5, 33]. In the setting of cylindrical probability measures, it was proved in [1] that an SC-semigroup $(\mu_t)_{t\geq 0}$ is uniquely determined by an infinitely divisible probability measure on H if the function $t \mapsto \hat{\mu}_t(a)$ is differentiable at $t = 0$ for all $a \in H$, where $\hat{\mu}_t$ denotes the characteristic functional of μ_t. Constructions of Gaussian and non-Gaussian generalized OU-processes with differentiable SC-semigroups were given respectively in [1, 18]. A characterization for general non-differentiable SC-semigroups was given in [9], where it was also observed that the corresponding OU-processes may have no right continuous realizations; see also [30, 33] for the study of non-differentiable SC-semigroups. A general construction of such OU-processes was given in [8]. Some powerful inequalities for differentiable generalized Mehler semigroups were proved recently by [32].

Another rich class of generalized OU-processes can also be formulated by (1.4) and (1.5) if we replace H by the space of Schwartz distributions $\mathcal{S}'(I\!R^d)$. Some of the $\mathcal{S}'(I\!R^d)$-valued OU-processes arise as high density fluctuation limits of measure-valued branching processes with or without immigration; see e.g. [2, 3, 8, 14, 15, 16, 20, 26, 36]. As pointed out in [36, p.308], for sufficiently regular semigroup $(T_t)_{t\geq 0}$, the $\mathcal{S}'(I\!R)$-valued generalized OU-process usually has an $L^2(I\!R)$-valued version. Some of those processes can also be regarded as multi-parameter OU-processes and defined by stochastic integrals; see [36, 39, 40]. It was observed in [8, 9] that OU-processes in $L^2(I\!R)$ or $L^2(0,\infty)$ corresponding to non-differentiable skew convolution semigroups may arise as fluctuation limits of superprocesses with measure-valued branching catalysts. In general, the $\mathcal{S}'(I\!R^d)$-valued OU-processes really live in the space of distributions when $d \geq 2$, and it is neither convenient nor natural to treat them as processes in Hilbert spaces. Therefore, generalized Mehler semigroups and OU-processes on the space $\mathcal{S}'(I\!R^d)$ with $d \geq 2$ need to be studied separately. Indeed, those distribution-valued OU-processes involve interesting mathematical structures. For example, in [4, 5] the self-intersection local times of the $\mathcal{S}'(I\!R^d)$-valued generalized OU-processes were studied.

This work arose from the curiosity in whether or not an $\mathcal{S}'(I\!R)$-valued generalized OU-process associated with the Brownian semigroup always has a well-defined $L^2(I\!R)$-valued version. As a testing example, we study the fluctuation limits of the superprocess with dependent spatial motion over the real line recently constructed in [10, 37, 38]. The model is described as follows. Let $M(I\!R)$ be the space of finite Borel measures on $I\!R$ endowed with the weak convergence topology. Let $C(I\!R)$ be the set of bounded continuous functions on $I\!R$. Let $c \geq 0$ be a non-negative constant and $\sigma(\cdot)$ a bounded non-negative Borel function on $I\!R$. Given a square-integrable function $h \in C(I\!R)$, let

$$\rho(x) = \int_{I\!R} h(y-x)h(y)dy, \quad x \in I\!R, \tag{1.6}$$

and $a = c^2 + \rho(0)$. We assume in addition that h is continuously differentiable with square-integrable derivative h'. Then ρ is twice continuously differentiable with bounded derivatives ρ' and ρ''. Based on the results of [10, 37, 38], we shall prove that there is an $M(I\!R)$-valued diffusion process $\{X_t : t \geq 0\}$ such that, for each $\phi \in C^2(I\!R)$,

$$M_t(\phi) = \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - \frac{1}{2}\, a \int_0^t \langle \phi'', X_s \rangle ds, \quad t \geq 0, \tag{1.7}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \sigma \int_0^t \langle \phi^2, X_s \rangle ds + \int_0^t ds \int_{I\!R} \langle h(z-\cdot)\phi', X_s \rangle^2 dz. \tag{1.8}$$

We call $\{X_t : t \geq 0\}$ a *superprocess with dependent spatial motion* (SDSM) with parameters (a, ρ, σ), where a represents the speed of the underlying motion, ρ represents the interaction of migration between the "particles" and σ represents the branching density. Clearly, the SDSM reduces to a usual super Brownian motion with independent spatial motion when $\rho(\cdot) \equiv 0$; see e.g. [6]. The SDSM is also related with McKean-Vlasov type interacting diffusion systems and superprocess arising from stochastic flows; see e.g. [22, 28].

In the study of high density fluctuation limits of measure-valued branching processes with independent spatial motion, techniques of Laplace functionals play an important role; see e.g. [9, 15, 16, 20]. Since the Laplace functionals are not neatly represented for the SDSM, we have to find some replacements. Our approach is to embed the fluctuation processes into a family of continuous martingales and observe those martingales as time changed Brownian motions. By a weak law of large numbers, we show that the quadratic variation processes of the martingales converge to a deterministic increasing process, from which we get the central limit theorem of the finite-dimensional distributions. The tightness of the fluctuation processes is proved using a criterion of [13]. Our fluctuation limit theorems lead to a class of $\mathcal{S}'(I\!R)$-valued generalized OU-processes. Those processes have differentiable skew convolution semigroups, but we cannot define the function-valued versions for some of them in the natural way. Therefore, it is not convenient to deal with their function-valued versions even they do exist. This phenomenon seems new and shows that the study of generalized Mehler semigroups and OU-processes on the space $\mathcal{S}'(I\!R^d)$ for all dimension numbers $d \geq 1$ is of interest. The complete description of all generalized Mehler semigroups on the space $\mathcal{S}'(I\!R^d)$ is still a challenging open problem.

Notation: Let $C^n(I\!R^m)$ denote the set of continuous functions on $I\!R^m$ with bounded continuous partial derivatives up to the n-th order. Let $\phi_p(x) = (1+x^2)^{-p}$ for $x \in I\!R$ and $p > 0$. Let $C_p(I\!R^m)$ be the set of functions $f \in C(I\!R^m)$ with $\|f/\phi_p^{\otimes m}\| < \infty$, where $\phi_p^{\otimes m}(x_1, \cdots, x_m) = \phi_p(x_1)\cdots\phi_p(x_m)$. Let $C_p^2(I\!R)$ denote the set of twice continuously differentiable functions $f \in C_p(I\!R)$ with $\|f'/\phi_p\|+\|f''/\phi_p\| < \infty$. In particular, we have $\phi_p \in C_p^2(I\!R)$. We use the superscript "+" to denote the subsets of non-negative elements of the function spaces, e.g. $C^2(I\!R)^+$. We also write $C(E)$ for the totality of all bounded continuous functions on a general topological space E. For a function f and measure μ, we write $\langle f, \mu\rangle$ for $\int f d\mu$ if the integral is meaningful. Let $M_p(I\!R)$ be the set of all σ-finite Borel measures on $I\!R$ satisfying $\langle \mu, \phi_p\rangle < \infty$. We define a topology on $M_p(I\!R)$ by the convention that $\mu_n \to \mu$ if and only if $\langle \mu_n, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(I\!R)$. Let $\mathcal{S}(I\!R)$ denote the Schwartz space on $I\!R$; see e.g. [19, p.305], and let $\mathcal{S}'(I\!R)$ denote the dual space of $\mathcal{S}(I\!R)$. We also write $\langle\cdot,\cdot\rangle$ for the duality on $(\mathcal{S}'(I\!R), \mathcal{S}(I\!R))$. Let $(T_t)_{t\geq 0}$ denote the transition semigroup of a standard Brownian motion.

Because of the presence of the derivative ϕ' in the variation process (2.2), it is not obvious how extend the definition of $\{M_t(\phi) : t \geq 0\}$ to a general function $\phi \in C(I\!R)$. However, following the method of [36], we can still define the stochastic integral

$$\int_0^t \int_{I\!R} \phi(s,x) M(ds,dx), \qquad t \geq 0, \tag{1.9}$$

if both $\phi(s,x)$ and $\phi'_x(s,x)$ are belong to $C([0,\infty)\times I\!R)$.

In section 2 we recall some basic characterizations of the SDSM. The convergence of finite dimensional distributions is established in section 3. In section 4 we discuss tightness and weak convergence on the path space. Two extreme cases are discussed in section 5.

2 Characterizations of the SDSM

We here recall the existence and some characterizations of the SDSM given in [10]. For a function F on $M(I\!R)$ let

$$\frac{\delta F(\mu)}{\delta\mu(x)} = \lim_{r\to 0^+} \frac{1}{r}[F(\mu + r\delta_x) - F(\mu)], \qquad x \in I\!R, \tag{2.1}$$

if the limit exists. Let $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ be defined in the same way with F replaced by $\delta F/\delta\mu(y)$ on the right hand side. Define the operator $\mathcal{L}$ by

$$\begin{aligned}
\mathcal{L}F(\mu) \;=\; & \frac{a}{2}\int_{I\!R} \frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\
& + \frac{1}{2}\int_{I\!R^2} \rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy) \\
& + \frac{1}{2}\int_{I\!R} \sigma(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(dx),
\end{aligned} \tag{2.2}$$

which acts on a class of continuous functions on $M(I\!R)$. The domain of the generator $\mathcal{L}$ defined by (2.2) includes all functions of the form $F_{m,f}(\mu) := \langle f, \mu^m\rangle$ with $f \in C^2(I\!R^m)$ and all functions of the form $F_{f,\phi}(\mu) := f(\langle\phi, \mu\rangle)$ with $f \in C^2(I\!R)$ and $\phi \in C^2(I\!R)$. Let $\mathcal{D}(\mathcal{L})$ denote the collection of all those functions, which is a subset of the domain of $\mathcal{L}$. By Theorems 2.2 and 5.2 in [10] we have

Theorem 2.1 *There is an $M(I\!R)$-valued diffusion process $(X_t, \mathcal{G}_t, \boldsymbol{Q}_\mu)$ with transition semigroup $(Q_t)_{\geq 0}$ generated by the closure of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$.*

The diffusion process given by the above theorem is the so-called SDSM. A useful martingale characterization of the SDSM is given in the following

Theorem 2.2 *A continuous $M(I\!R)$-valued process $\{X_t : t \geq 0\}$ is a diffusion process with semigroup $(Q_t)_{\geq 0}$ if and only if for each $\phi \in C^2(I\!R)$,*

$$M_t(\phi) := \langle\phi, X_t\rangle - \langle\phi, X_0\rangle - \frac{a}{2}\int_0^t \langle\phi'', X_s\rangle ds, \qquad t \geq 0, \tag{2.3}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle\sigma\phi^2, X_s\rangle ds + \int_0^t ds \int_{I\!R} \langle h(z - \cdot)\phi', X_s\rangle^2 dz. \tag{2.4}$$

Proof. Suppose that $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem, that is,

$$F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s)ds, \quad t \geq 0, \tag{2.5}$$

is a continuous martingale for every $F \in \mathcal{D}(\mathcal{L})$. Comparing the martingales related to the functions $\mu \mapsto \langle\phi, \mu\rangle$ and $\mu \mapsto \langle\phi, \mu\rangle^2$ and using Itô's formula we see that (2.3) is a continuous martingale with quadratic variation process (2.4). Conversely, suppose that $\boldsymbol{Q}_\mu$ is a probability measure on $C([0, \infty), M(I\!R))$ under which (2.3) is a continuous martingale with quadratic variation process (2.4) for each $\phi \in C^2(I\!R)$. If

$$F_{f,\{\phi_i\}}(\nu) := f(\langle\phi_1, \nu\rangle, \cdots, \langle\phi_n, \nu\rangle)$$

for $f \in C^2(I\!R^n)$ and $\{\phi_i\} \subset C^2(I\!R)$, we have

$$\begin{aligned}
\mathcal{L}F_{f,\{\phi_i\}}(\nu) \;=\; & \frac{1}{2}\rho(0)\sum_{i=1}^n f_i'(\langle\phi_1, \nu\rangle, \cdots, \langle\phi_n, \nu\rangle)\langle\phi_i'', \nu\rangle \\
& + \frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle\phi_1, \nu\rangle, \cdots, \langle\phi_n, \nu\rangle)\int_{I\!R^2} \rho(x - y)\phi_i'(x)\phi_j'(y)\mu(dx)\mu(dy) \\
& + \frac{1}{2}\sigma\sum_{i,j=1}^n f_{ij}''(\langle\phi_1, \nu\rangle, \cdots, \langle\phi_n, \nu\rangle)\langle\phi_i\phi_j, \nu\rangle.
\end{aligned}$$

By Itô's formula we see that (2.5) is a continuous martingale if $F = F_{f,\{\phi_i\}}$. Then the theorem follows by an approximation of an arbitrary $F \in \mathcal{D}(\mathcal{L})$. □

The following theorem can be proved by similar arguments as Lemma 4.6 of [10].

Theorem 2.3 *For $t \geq 0$ and $\phi \in C^1(I\!R)$ we have a.s.*

$$\langle \phi, X_t \rangle = \langle T_{at}\phi, X_0 \rangle + \int_0^t \int_{I\!R} T_{a(t-s)}\phi(x) M(ds, dx). \tag{2.6}$$

To study the fluctuation limits of the SDSM, we need the following moment estimate.

Lemma 2.1 *Let $\beta(p) = \|\phi_p'/\phi_p\| + \|\phi_p''/\phi_p\|$ and let $\alpha = \beta(p)^2(c^2 + \|\rho\|)/2$. Then for any $f \in C_p(I\!R^m)$ and any integer $m \geq 1$,*

$$\int_{M(I\!R)} \langle f, \nu^m \rangle Q_t(\mu, d\nu) \leq m^{2m} e^{\alpha m^2 t}(1 + \|\sigma\|^m)\|f/\phi_p^{\otimes m}\| \sum_{k=1}^m \langle \phi_p, \mu \rangle^k, \tag{2.7}$$

where $\|\cdot\|$ denotes the supremum norm.

Proof. Let $(P_t^m)_{t \geq 0}$ be the transition semigroup on $I\!R^m$ generated by the differential operator

$$G^m := \frac{a}{2}\sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i,j=1, i \neq j}^m \rho(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.8}$$

It is easy to check that $|G^m \phi_p^{\otimes m}(x)| \leq \alpha m^2 \phi_p^{\otimes m}(x)$, and hence

$$\frac{d}{dt} P_t^{\otimes m} \phi_p^{\otimes m} = P_t^{\otimes m} G^m \phi_p^{\otimes m} \leq \alpha m^2 \phi_p^{\otimes m}.$$

By a comparison theorem we get $P_t^m \phi_p^{\otimes m} \leq e^{\alpha m^2 t}\phi_p^{\otimes m}$ for all $t \geq 0$. As in the proof of Lemma 2.1 of [10] we have

$$\begin{aligned}\int_{M(I\!R)} \langle f, \nu^m \rangle Q_t(\mu, d\nu) &\leq \|f/\phi_p^{\otimes m}\| \int_{M(I\!R)} \langle \phi_p^{\otimes m}, \nu^m \rangle Q_t(\mu, d\nu) \\ &\leq e^{\alpha m^2 t}\|f/\phi_p^{\otimes m}\| \sum_{k=0}^{m-1} 2^{-k} m^k (m-1)^k \|\sigma\|^k \langle \phi_p, \mu \rangle^{m-k},\end{aligned}$$

from which the desired estimate follows. □

3 A fluctuation limit theorem

We fix the constant $p > 1/2$. For each $\theta \geq 1$, let $\rho_\theta(\cdot)$ be defined by (1.1) with $h(\cdot)$ replaced by $h(\cdot)/\sqrt{\theta}$ and let $a_\theta = c^2 + \rho(0)/\theta$. Let $\mathcal{L}_\theta$ be defined by (2.2) with a and $\rho(\cdot)$ replaced respectively by a_θ and $\rho_\theta(\cdot)$. Let $(\Omega, \mathcal{A}, \boldsymbol{P})$ be a complete probability space on which an SDSM $\{X_t^{(\theta)} : t \geq 0\}$ is defined with generator $\mathcal{L}_\theta$ and initial state $X_0^{(\theta)} = \mu_\theta \in M(I\!R)$. We define the $\mathcal{S}'(I\!R)$-valued process $\{Z_t^{(\theta)} : t \geq 0\}$ by

$$\langle \phi, Z_t^{(\theta)} \rangle = (\langle \phi, X_t^{(\theta)} \rangle - \theta\langle \phi, \lambda \rangle)/\sqrt{\theta}, \qquad \phi \in \mathcal{S}(I\!R). \tag{3.1}$$

The following lemma establishes a weak law of large numbers for $\{X_t^{(\theta)} : t \geq 0\}$.

Lemma 3.1 *Suppose that $\mu_\theta/\theta \to \mu \in M_p(I\!R)$ as $\theta \to \infty$. Then for $t \geq 0$ and $\phi \in C_p^2(I\!R)$, we have $\langle \phi/\theta, X_t^{(\theta)}\rangle \to \langle T_t^c\phi, \mu\rangle$ in $L^2(\Omega, \boldsymbol{P})$ as $\theta \to \infty$, where $T_t^c = T_{c^2t}$.*

Proof. Let $M^{(\theta)}(ds, dx)$ denote the stochastic integral with respect to the martingale measure determined by (2.3) with $\{X_t : t \geq 0\}$ replaced by $\{X_t^{(\theta)} : t \geq 0\}$. Then for fixed $t \geq 0$ and $\phi \in C_p^2(I\!R)$,

$$M_{t,u}^{(\theta)}(\phi) := \int_0^{t\wedge u}\int_{I\!R} T_{a_\theta(t-s)}\phi(x) M^{(\theta)}(ds, dx), \qquad u \geq 0 \tag{3.2}$$

is a continuous martingale with quadratic variation process

$$\begin{aligned}\langle M_t^{(\theta)}(\phi)\rangle_u &= \int_0^{t\wedge u}\langle \sigma(T_{a_\theta(t-s)}\phi)^2, X_s^{(\theta)}\rangle ds \\ &\quad + \int_0^{t\wedge u} ds\int_{I\!R}\langle h(z-\cdot)T_{a_\theta(t-s)}(\phi'), X_s^{(\theta)}/\sqrt{\theta}\rangle^2 dz.\end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned}\boldsymbol{E}\{\langle M_t^{(\theta)}(\phi/\theta)\rangle_t\} &= \frac{1}{\theta}\int_0^t \boldsymbol{E}\{\langle \sigma[T_{a_\theta(t-s)}\phi]^2/\theta, X_s^{(\theta)}\rangle\} ds \\ &\quad + \frac{1}{\theta}\int_0^t ds\int_{I\!R}\boldsymbol{E}\{\langle h(z-\cdot)T_{a_\theta(t-s)}(\phi')/\theta, X_s^{(\theta)}\rangle^2\} dz.\end{aligned} \tag{3.4}$$

Observe that $\phi_p''(x) \leq (4p+6)\phi_p(x)$. Thus there is a constant $C_t \geq 0$ such that

$$|T_{a_\theta(t-s)}\phi_p(x)| \leq C_t\phi_p(x), \quad x \in I\!R, \theta \geq 1, 0 \leq s \leq t.$$

Then we get by Lemma 2.1 that

$$\begin{aligned}\boldsymbol{E}\{\langle \sigma[T_{a_\theta(t-s)}\phi]^2/\theta, X_s^{(\theta)}\rangle\} &\leq C_t\|\sigma\phi^2/\phi_p\|\boldsymbol{E}\{\langle \phi_p/\theta, X_s^{(\theta)}\rangle\} \\ &\leq C_t\|\sigma\phi^2/\phi_p\|e^{\alpha t}(1+\|\sigma\|)\langle \phi_p/\theta, \mu_\theta\rangle.\end{aligned} \tag{3.5}$$

By Schwarz' inequality and a similar procedure as the above one finds that

$$\begin{aligned}&\int_{I\!R}\boldsymbol{E}\{\langle h(z-\cdot)T_{a_\theta(t-s)}(\phi')/\theta, X_s^{(\theta)}\rangle^2\} dz \\ \leq\ &\int_{I\!R}\boldsymbol{E}\{\langle T_{a_\theta(t-s)}(\phi')/\theta, X_s^{(\theta)}\rangle\langle h(z-\cdot)^2 T_{a_\theta(t-s)}(\phi')/\theta, X_s^{(\theta)}\rangle\} dz \\ \leq\ &\rho(0)C_t^2\|\phi'/\phi_p\|^2\boldsymbol{E}\{\langle \phi_p/\theta, X_s^{(\theta)}\rangle^2\} \\ \leq\ &\rho(0)C_t^2\|\phi'/\phi_p\|^2 e^{4\alpha t}(1+\|\sigma\|^2)(\langle \phi_p/\theta, \mu_\theta\rangle + \langle \phi_p/\theta, \mu_\theta\rangle^2).\end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we see that

$$\boldsymbol{E}\{(\langle \phi/\theta, X_t^{(\theta)}\rangle - \langle T_{a_\theta t}\phi/\theta, \mu_\theta\rangle)^2\} = \boldsymbol{E}\{\langle M_t^{(\theta)}(\phi/\theta)\rangle_t\} \to 0$$

uniformly on each finite interval of $t \geq 0$ as $\theta \to \infty$. Observe also that

$$\begin{aligned}&|\langle T_{a_\theta t}\phi/\theta, \mu_\theta\rangle - \langle T_t^c\phi, \mu\rangle| \\ \leq\ &\int_{c^2t}^{a_\theta t}\langle T_s|\phi''|/\theta, \mu_\theta\rangle ds + |\langle T_t^c\phi/\theta, \mu_\theta\rangle - \langle T_t^c\phi, \mu\rangle| \\ \leq\ &\|\phi''/\phi_p\|\int_{c^2t}^{(c^2+\rho(0)/\theta)t}\langle T_s\phi_p/\theta, \mu_\theta\rangle ds + |\langle T_t^c\phi/\theta, \mu_\theta\rangle - \langle T_t^c\phi, \mu\rangle|.\end{aligned}$$

Clearly, the right hand side also goes to zero as $\theta \to \infty$. Consequently,

$$\boldsymbol{E}\{(\langle \phi/\theta, X_t^{(\theta)}\rangle - \langle T_t^c \phi/\theta, \mu\rangle)^2\} \to 0$$

uniformly on each finite interval of $t \geq 0$ as $\theta \to \infty$. □

Theorem 3.1 *Suppose that $\sigma \in C^2(I\!R)^+$ and $h \in C^2(I\!R)$. If $\zeta_\theta := (\mu_\theta - \theta\lambda)/\sqrt{\theta} \to \zeta \in \mathcal{S}'(I\!R)$ as $\theta \to \infty$, then the distribution of $Z_t^{(\theta)}$ converges to a probability measure $\tilde{Q}_t(\zeta, \cdot)$ on $\mathcal{S}'(I\!R)$ determined by*

$$\begin{aligned}\int_{\mathcal{S}'(I\!R)} e^{i\langle \phi, \nu\rangle} \tilde{Q}_t(\zeta, d\nu) &= \exp\Big\{ i\langle T_t^c \phi, \zeta\rangle - \frac{1}{2}\int_0^t \langle \sigma (T_s^c \phi)^2, \lambda\rangle ds \\ &\quad - \frac{1}{2}\int_0^t ds \int_{I\!R} \langle h'(z-\cdot) T_s^c \phi, \lambda\rangle^2 dz \Big\}. \end{aligned} \tag{3.7}$$

Proof. We use the notation introduced in the proof of Lemma 3.1. By (2.6) and (3.2) we have

$$M_{t,t}^{(\theta)}(\phi/\sqrt{\theta}) = (\langle \phi, X_t^{(\theta)}\rangle - \langle T_{a_\theta t}\phi, \mu_\theta\rangle)/\sqrt{\theta}, \quad t \geq 0.$$

It follows that

$$\langle \phi, Z_t^{(\theta)}\rangle = M_{t,t}^{(\theta)}(\phi/\sqrt{\theta}) + (\langle T_{a_\theta t}\phi, \mu_\theta\rangle - \langle T_t^c \phi, \mu_\theta\rangle)/\sqrt{\theta} + \langle T_t^c \phi, \zeta_\theta\rangle. \tag{3.8}$$

That is, the main part of $\langle \phi, Z_t^{(\theta)}\rangle$ can be embedded into the continuous martingale (3.2). By (3.3),

$$\begin{aligned}\langle M_t^{(\theta)}(\phi/\sqrt{\theta})\rangle_u &= \int_0^{t\wedge u} \langle \sigma (T_{a_\theta(t-s)}\phi)^2, X_s^{(\theta)}/\theta\rangle ds \\ &\quad + \int_0^{t\wedge u} ds \int_{I\!R} \langle h(z-\cdot) T_{a_\theta(t-s)}(\phi'), X_s^{(\theta)}/\theta\rangle^2 dz. \end{aligned} \tag{3.9}$$

Under the assumptions, we have $\mu_\theta/\theta \to \lambda$. Thus by Lemma 3.1,

$$\langle M_t^{(\theta)}(\phi/\sqrt{\theta})\rangle_u \to \int_0^{t\wedge u} \langle \sigma (T_{t-s}^c \phi)^2, \lambda\rangle ds + \int_0^{t\wedge u} ds \int_{I\!R} \langle h(z-\cdot) T_{t-s}^c(\phi'), \lambda\rangle^2 dz \tag{3.10}$$

in $L^2(\Omega, \boldsymbol{P})$ as $\theta \to \infty$. By a representation of continuous martingales, there is a standard Brownian motion $\{B_{t,\phi}^{(\theta)}(u) : u \geq 0\}$ defined on an extension of $(\Omega, \mathcal{A}, \boldsymbol{P})$ such that $M_{t,u}^{(\theta)}(\phi/\sqrt{\theta}) = B_{t,\phi}^{(\theta)}(\langle M_t^{(\theta)}(\phi/\sqrt{\theta})\rangle_u)$ for all $u \geq 0$; see e.g. [31, p.171]. Thus (3.10) implies that the distribution of $M_{t,t}^{(\theta)}(\phi/\sqrt{\theta})$ converges to the Gaussian distribution with mean zero and variance

$$\begin{aligned}&\int_0^t \langle \sigma (T_{t-s}^c \phi)^2, \lambda\rangle ds + \int_0^t ds \int_{I\!R} \langle h(z-\cdot) T_{t-s}^c(\phi'), \lambda\rangle^2 dz \\ =\ &\int_0^t \langle \sigma (T_{t-s}^c \phi)^2, \lambda\rangle ds + \int_0^t ds \int_{I\!R} \langle h(z-\cdot) (T_{t-s}^c \phi)', \lambda\rangle^2 dz \\ =\ &\int_0^t \langle \sigma (T_{t-s}^c \phi)^2, \lambda\rangle ds + \int_0^t ds \int_{I\!R} \langle h'(z-\cdot) T_{t-s}^c \phi, \lambda\rangle^2 dz. \end{aligned} \tag{3.11}$$

On the other hand, observe that

$$\begin{aligned}\frac{1}{\sqrt{\theta}}\,|\langle T_{a_\theta t}\phi,\mu_\theta\rangle-\langle T^c_t\phi,\mu_\theta\rangle| &\leq \frac{1}{\sqrt{\theta}}\int_{c^2t}^{a_\theta t}\langle T_s|\phi''|,\mu_\theta\rangle ds\\ &\leq \frac{1}{\sqrt{\theta}}\|\phi''/\phi_p\|\int_{c^2t}^{(c^2+\rho(0)/\theta)t}\langle T_s\phi_p,\mu_\theta\rangle ds. \end{aligned}\tag{3.12}$$

The right hand side clearly goes to zero as $\theta\to\infty$. In view of (3.8), the distribution of $\langle\phi,Z^{(\theta)}_t\rangle$ converges to the Gaussian distribution with mean $\langle T^c_t\phi,\zeta\rangle$ and variance (3.11), giving the desired result. □

It is easy to see that (3.7) defines a transition semigroup $(\tilde{Q}_t)_{t\geq0}$ on $\mathcal{S}'(I\!R)$. Clearly, Theorem 3.1 implies that the finite dimensional distributions of $\{Z^{(\theta)}_t:t\geq0\}$ converge as $\theta\to\infty$ to those of an $\mathcal{S}'(I\!R)$-valued Markov process $\{Z_t:t\geq0\}$ with transition semigroup $(\tilde{Q}_t)_{t\geq0}$. Note that $N_t=\tilde{Q}_t(0,\cdot)$ satisfies

$$N_{r+t}=(T^c_tN_r)*N_t,\quad r\geq0,t\geq0,\tag{3.13}$$

where $T^c_tN_r$ denotes the image of N_r under the adjoint operator of T^c_t on $\mathcal{S}'(I\!R)$. That is, $(\tilde{Q}_t)_{t\geq0}$ is a generalized Mehler semigroup associated with $(T^c_t)_{t\geq0}$. In view of (3.7), the characteristic functional of N_t is differentiable for any testing function $\phi\in\mathcal{S}(I\!R)$.

4 Weak convergence and generalized OU-diffusions

In this section, we prove that the process $\{Z^{(\theta)}_t:t\geq0\}$ defined by (3.1) converges weakly on the space $C([0,\infty),\mathcal{S}'(I\!R))$. Therefore, the limiting generalized OU-process $\{Z_t:t\geq0\}$ has a diffusion realization.

Lemma 4.1 *Suppose that $\zeta_\theta:=(\mu_\theta-\theta\lambda)/\sqrt{\theta}\to\zeta\in\mathcal{S}'(I\!R)$ as $\theta\to\infty$. Then $\{Z^{(\theta)}_t:t\geq0;\theta\geq1\}$ is a tight family in $C([0,\infty),\mathcal{S}'(I\!R))$.*

Proof. We use the notation introduced in the proof of Lemma 3.1. By (3.9),

$$\begin{aligned}\boldsymbol{E}\{M^{(\theta)}_{t,t}(\phi/\sqrt{\theta})^2\} &= \int_0^t\boldsymbol{E}\{\langle\sigma(T_{a_\theta(t-s)}\phi)^2/\theta,X^{(\theta)}_s\rangle\}ds\\ &\quad+\int_0^t ds\int_{I\!R}\boldsymbol{E}\{\langle h(z-\cdot)T_{a_\theta(t-s)}(\phi'/\theta),X^{(\theta)}_s\rangle^2\}dz.\end{aligned}$$

In view of (3.5) and (3.6), this value is bounded above by a locally bounded function $C_1(\phi,t)$ of $t\geq0$. From (3.8) we see that $\boldsymbol{E}\{\langle\phi,Z^{(\theta)}_t\rangle^2\}$ is bounded above by a locally bounded function $C_2(\phi,t)$ of $t\geq0$. Similarly,

$$\begin{aligned}\boldsymbol{E}\{M^{(\theta)}_t(\phi/\sqrt{\theta})^2\} &= \boldsymbol{E}\{\langle M^{(\theta)}(\phi/\sqrt{\theta})\rangle_t\}\\ &= \int_0^t\langle\sigma\phi^2/\theta,\mu_\theta\rangle ds+\int_0^t ds\int_{I\!R}\boldsymbol{E}\{\langle h(z-\cdot)\phi'/\theta,X^{(\theta)}_s\rangle^2\}dz\end{aligned}$$

is bounded above by a locally bounded function $C_3(\phi,t)$ of $t\geq0$. For each $\phi\in\mathcal{S}(I\!R)$ we have $\phi'\in\mathcal{S}(I\!R)$ and hence $\langle\phi'',\lambda\rangle=0$. Then we get from (2.3) that

$$\langle\phi,Z^{(\theta)}_t\rangle-\langle\phi,\zeta_\theta\rangle=M^{(\theta)}_t(\phi/\sqrt{\theta})+\frac{a_\theta}{2}\int_0^t\langle\phi'',Z^{(\theta)}_s\rangle ds,\qquad t\geq0.\tag{4.1}$$

For $u > 0$ and $\eta > 0$ we have

$$\begin{aligned}
&\boldsymbol{P}\Big\{\sup_{0\le t\le u}|\langle\phi, Z_t^{(\theta)}\rangle - \langle\phi,\zeta_\theta\rangle| \ge \eta\Big\} \\
\le\;& \frac{1}{\eta^2}\boldsymbol{E}\Big\{\sup_{0\le t\le u}|\langle\phi, Z_t^{(\theta)}\rangle - \langle\phi,\zeta_\theta\rangle|^2\Big\} \\
\le\;& \frac{2}{\eta^2}\boldsymbol{E}\Big\{\sup_{0\le t\le u}|M_t^{(\theta)}(\phi/\sqrt{\theta})|^2\Big\} + \frac{a_\theta^2 u}{2\eta^2}\int_0^u \boldsymbol{E}\{\langle\phi'', Z_s^{(\theta)}\rangle^2\}ds \\
\le\;& \frac{8}{\eta^2}\boldsymbol{E}\Big\{\langle M^{(\theta)}(\phi/\sqrt{\theta})\rangle_u\Big\} + \frac{a_\theta^2 u}{2\eta^2}\int_0^u \boldsymbol{E}\{\langle\phi'', Z_s^{(\theta)}\rangle^2\}ds.
\end{aligned}$$

The right hand side goes to zero as $\eta \to \infty$. Then $\{Z_t^{(\theta)}(\phi) : t \ge 0\}$ satisfy the compact containment condition of [13, p.142]. For $f \in C^2(I\!R)$ we consider the function $F(\mu) := f((\langle\phi,\mu\rangle - \theta\langle\phi,\lambda\rangle)/\sqrt{\theta})$ on $M(I\!R)$. Let

$$\begin{aligned}
\mathcal{L}^{(\theta)}F(\mu) \;=\;& \frac{1}{2}f'((\langle\phi,\mu\rangle - \theta\langle\phi,\lambda\rangle)/\sqrt{\theta})(\langle\phi'',\mu\rangle - \theta\langle\phi'',\lambda\rangle)/\sqrt{\theta} \\
&+\frac{1}{2\theta^2}f''((\langle\phi,\mu\rangle - \theta\langle\phi,\lambda\rangle)/\sqrt{\theta})\int_{I\!R^2}\rho(y-x)\phi'(x)\phi'(y)\mu(dx)\mu(dy) \\
&+\frac{1}{2\theta}f''((\langle\phi,\mu\rangle - \theta\langle\phi,\lambda\rangle)/\sqrt{\theta})\langle\sigma\phi^2,\mu\rangle.
\end{aligned}$$

Then

$$f(\langle\phi, Z_t^{(\theta)}\rangle) - f(\langle\phi,\zeta_\theta\rangle) - \int_0^t \mathcal{L}^{(\theta)}F(X_s^{(\theta)})ds, \quad t \ge 0, \tag{4.2}$$

is a martingale. By Lemma 2.1 and the proof of Theorem 3.1, it is not hard to find that $\boldsymbol{E}\{\mathcal{L}^{(\theta)}F(X_s^{(\theta)})^2\}$ is a locally bounded function of $s \ge 0$. By [13, pp.142-145], the family $\{\langle\phi, Z_t^{(\theta)}\rangle : t \ge 0; \theta \ge 1\}$ is tight in $C([0,\infty), I\!R)$, which is a closed subset of $D([0,\infty), I\!R)$. The tightness of $\{Z_t^{(\theta)} : t \ge 0; \theta \ge 1\}$ then follows by a theorem of [29]. □

By Lemma 4.1 and the observations at the end of the last section we get the following weak convergence on the path space, which also gives the existence of a diffusion realization of the limiting generalized OU-process.

Theorem 4.1 *Assume in addition that $\sigma \in C^2(I\!R)^+$ and $h \in C^2(I\!R)$. If $\zeta_\theta := (\mu_\theta - \theta\lambda)/\sqrt{\theta} \to \zeta \in \mathcal{S}'(I\!R)$ as $\theta \to \infty$, then the distribution of $\{Z_t^{(\theta)} : t \ge 0\}$ on $C([0,\infty), \mathcal{S}'(I\!R))$ converges as $\theta \to \infty$ to that of a generalized OU-diffusion process $\{Z_t : t \ge 0\}$ with initial value $Z_0 = \zeta$ and transition semigroup $(\tilde{Q}_t)_{t\ge0}$ given by (3.7).*

5 Two extreme cases

Suppose that $\sigma \in C^2(I\!R)^+$ and $h \in C^2(I\!R) \cap L^2(I\!R,\lambda)$ with $h' \in L^2(I\!R,\lambda)$. It is not hard to check that $(\tilde{Q}_t)_{t\ge0}$ has generator $\mathcal{J}$ given by

$$\mathcal{J}F(\mu) \;=\; \frac{1}{2}c^2\langle\Delta\delta F(\mu)/\delta\mu(\cdot), \mu\rangle$$

$$+\frac{1}{2}\int_{\mathbb{R}^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2F(\mu)}{\delta\mu(x)\delta\mu(y)}dxdy$$
$$+\frac{1}{2}\int_{\mathbb{R}}\sigma(x)\frac{\delta^2F(\mu)}{\delta\mu(x)^2}dx, \tag{5.1}$$

where $\delta F(\mu)/\delta\mu(x)$ is defined as in (2.1). Let us consider two extreme cases separately.

Example 5.1 Assume that $h \equiv 0$ and $\sigma \in L^1(\mathbb{R},\lambda)$ is non-trivial. In this case, the corresponding $\mathcal{S}'(\mathbb{R})$-valued generalized OU-diffusion process $\{Z_t : t \geq 0\}$ satisfies the following *Langevin equation*:

$$\langle\phi, Z_t\rangle = \langle\phi, Z_0\rangle + \langle\phi, W_t\rangle + \frac{1}{2}c^2\int_0^t\langle\phi'', Z_s\rangle ds, \qquad t\geq 0, \tag{5.2}$$

where $\{W_t : t \geq 0\}$ is an $\mathcal{S}'(\mathbb{R})$-valued Wiener process such that

$$\langle W(\phi)\rangle_t = t\langle\sigma\phi^2, \lambda\rangle; \tag{5.3}$$

see [2, 15, 16, 26]. Let $W(ds,dx)$ denote the stochastic integral determined by the process $\{W_t : t\geq 0\}$. Then for $t \geq 0$ and $\phi \in \mathcal{S}(\mathbb{R})$ we have a.s.

$$\langle\phi, Z_t\rangle = \langle T_t^c\phi, Z_0\rangle + \int_0^t\int_{\mathbb{R}} T_{t-s}^c\phi(x)W(ds,dx). \tag{5.4}$$

Let $g_t^c(x,y)$ denote the density of $T_t^c(x,dy)$ for $t>0$. It is easy to check that

$$\int_{\mathbb{R}}dy\int_0^t\langle\sigma g_{t-s}^c(\cdot,y)^2,\lambda\rangle ds \leq \langle\sigma,\lambda\rangle\int_0^t\frac{1}{\sqrt{2\pi(t-s)}}ds<\infty. \tag{5.5}$$

In view of (5.4) and (5.5), if $Z_0 \in L^2(\mathbb{R},\lambda)$,

$$Z_t(y) := \langle g_t^c(\cdot,y), Z_0\rangle + \int_0^t\int_{\mathbb{R}} g_{t-s}^c(x,y)W(ds,dx), \quad t\geq 0, x\in\mathbb{R}, \tag{5.6}$$

defines an $L^2(\mathbb{R},\lambda)$-valued version of the generalized OU-process $\{Z_t : t\geq 0\}$.

Example 5.2 Assume that $\sigma\equiv 0$ and h is non-trivial. In this case, (5.2) is valid if we replace (5.3) by

$$\langle W(\phi)\rangle_t = t\int_{\mathbb{R}}\langle h(z-\cdot)\phi',\lambda\rangle^2dz = t\int_{\mathbb{R}}\langle h'(z-\cdot)\phi,\lambda\rangle^2dz. \tag{5.7}$$

But, now (5.6) is not well-defined since

$$\begin{aligned}
& \int_{\mathbb{R}}dy\int_{\mathbb{R}}\langle h'(z-\cdot)g_{t-s}^c(\cdot,y),\lambda\rangle^2dz \\
= \; & \int_{\mathbb{R}}dy\int_{\mathbb{R}}dz\int_{\mathbb{R}}h'(z-x_1)dx_1\int_{\mathbb{R}}h'(z-x_2)g_{t-s}^c(x_1,y)g_{t-s}^c(x_2,y)dx_2 \\
= \; & \int_{\mathbb{R}}dz\int_{\mathbb{R}}h'(z-x_1)dx_1\int_{\mathbb{R}}h'(z-x_2)g_{2(t-s)}^c(x_1,x_2)dx_2 \\
= \; & -\int_{\mathbb{R}}dx_1\int_{\mathbb{R}}\rho''(x_2-x_1)g_{2(t-s)}^c(x_2-x_1)dx_2 \\
= \; & -\int_{\mathbb{R}}T_{2(t-s)}^c\rho''(0)dx_1 \\
= \; & \infty.
\end{aligned}$$

This is very different from the situation observed in [36] and shows that it is not convenient to deal with the function-valued version of $\{Z_t : t > 0\}$ even it does exist. Actually, we expect that the process $\{Z_t : t > 0\}$ only lives in a Sobolev space of negative index. Therefore, it is a non-trivial problem to look into the existence of local times or intersection local times of the process following [4, 5].

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