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Measure-valued Diffusions and Stochastic Equations with Poisson Process [1]

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1 Introduction

Suppose that we are given a locally compact metric space E. Let $C(E)$ denote the set of bounded continuous functions on E, and $C_0(E)$ its subset of continuous functions vanishing at infinity. The subsets of non-negative elements of $C(E)$ and $C_0(E)$ are denoted respectively by $C^+(E)$ and $C_0^+(E)$. Let $(P_t)_{t\geq 0}$ be a strongly continuous conservative Feller semigroup on $C_0(E)$ with generator $(A, \mathcal{D}(A))$, where $\mathcal{D}_0(A) \subset C_0(E)$, and let $\mathcal{D}(A) = \mathcal{D}_0(A) \cup \{1\}$. Suppose in addition that $b(\cdot) \in C(E)$ and $c(\cdot) \in C^+(E)$ have continuous extensions to $\bar{E}$, the one point compactification of E, and that $c(\cdot)$ is bounded away from zero.

Let $M(E)$ be the space of finite Borel measures on E equipped with the topology of weak convergence. Let $W = C([0,\infty), M(E))$ be the space of all continuous paths $w : [0,\infty) \to M(E)$. Let $\tau_0(w) = \inf\{s > 0 : w(s) = 0\}$ for $w \in W$ and let W_0 be the set of paths $w \in W$ satisfying $w(0) = w(t) = 0$ for all $t \geq \tau_0(w)$. We fix a metric on $M(E)$ which is compatible with its topology and endow W and W_0 with the topology of uniform convergence. Then for each $\mu \in M(E)$ there is a unique Borel probability measure $\boldsymbol{Q}_\mu$ on W such that for $f \in \mathcal{D}(A)$,

$$M_t(f) = w_t(f) - \mu(f) - \int_0^t w_s(Af - bf)ds, \quad t \geq 0, \tag{1.1}$$

under $\boldsymbol{Q}_\mu$ is a martingale with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t w_s(cf^2)ds, \quad t \geq 0, \tag{1.2}$$

where $\mu(f) = \int f d\mu$. The system $\{\boldsymbol{Q}_\mu : \mu \in M(E)\}$ defines a measure-valued diffusion, which is the well-known Dawson-Watanabe superprocess. In the sequel, we shall simply refer to it as a (A, b, c)-*superprocess*. We refer the reader to Dawson [1] and the references

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therein for the construction and basic properties of the process. A modification of the above model is to replace (1.1) by

$$M_t(f) = w_t(f) - \mu(f) - \int_0^t w_s(Af - bf)ds - \int_0^t V(w_s, f)ds, \quad t \geq 0, \tag{1.3}$$

by using a kernel $V(\mu, dx)$ from $M(E)$ to E, which can be regarded as a (A, b, c)-*superprocess with interactive immigration.* Some interesting special cases of this modification have been studied in the literature. Using a Cameron-Martin-Girsanov formula, Dawson [1, pp.172-173] treated the special case where $b(\cdot) \equiv 0$ and

$$V(\mu, dx) = r(\mu, x)\mu(dx), \quad \mu \in M(E), x \in E,$$

for a continuous function $r(\cdot, \cdot)$ on $M(E) \times E$ and obtained a superprocess with non-linear birth-death rate. The conditioned superprocess constructed by Evans and Perkins [5] and Roelly-Coppoletta and Rouault [16] corresponds to the case

$$V(\mu, dx) = \mu(1)^{-1}\mu(dx), \quad \mu \in M(E) \setminus \{0\}, x \in E.$$

An interesting representation of the conditioned superprocess was given by Evans [4] in terms of an "immortal particle" that moves around according to the underlying process and throws off pieces of mass into the space.

Let m be a σ-finite Borel measure on E and let $q(\cdot, \cdot)$ be a non-negative Borel function on $M(E) \times E$. We have another particular form of (1.3) given by

$$M_t(f) = w_t(f) - \mu(f) - \int_0^t w_s(Af - bf)ds - \int_0^t m(q(w_s, \cdot)f)ds, \quad t \geq 0, \tag{1.4}$$

where $q(\cdot, \cdot)$ can be interpreted as an interactive immigration rate relative to the reference measure m. The process defined by (1.4) and (1.2) is of interest since it includes as special cases (at least formally) the superprocess with non-linear birth-death rate and the conditioned superprocess as they are a.s. absolutely continuous with respect to the reference measure m, both of which has arisen considerable research interest. If $q(\nu, x) = q(x)$ only depends on $x \in E$, the martingale problem has a unique solution and defines a superprocess with independent immigration; see e.g. Konno and Shiga [8] and Li and Shiga [12]. In the general case, a solution of the martingale problem could be constructed by an approximation by particle systems, but the uniqueness of solution seems hard. This is similar to the superprocess with mean field interaction studied by Méléard and Roelly [13, 14] for which the uniqueness still remains open. Instead of the martingale problem, Shiga [17] suggested another approach to the interactive immigration superprocess, who gave the formulation of a stochastic integral equation involving a superprocess and a system of independent Poisson processes on the space of excursions of one-dimensional branching diffusions. For the particular case where $A \equiv 0$ and $\mu \mapsto m(q(\mu, \cdot))$ is bounded and Lipschitz relative to the total variation metric, Shiga [17] constructed a solution of the integral equation and showed that his solution also solves the martingale problem

(1.4) and (1.2). He proved that the pathwise uniqueness of solution for the stochastic integral equation holds so his solution is a diffusion process. This is a very interesting result since the uniqueness of solution of (1.4) and (1.2) is not known. A generalization of his result was given in the recent work by Dawson and Li [2], where some superprocesses with dependent spatial motion and interactive immigration were constructed from one-dimensional excursions carried by stochastic flows.

The main purpose of this paper is to establish the results of Shiga [17] when the spatial migration mechanism A is non-trivial. Since in this case the mass is mixed, it is not clear how to construct the process from one-dimensional excursions as in [17]. Fortunately, the techniques developed by Li and Shiga [12] can be combined with those of Shiga [17] to solve the difficulty. The main idea of our approach is to formulate a stochastic equation with a Poisson process on the space of *measure-valued* excursions. Let $\{X_t : t \geq 0\}$ be an (A, b, c)-superprocess with deterministic initial state $X_0 = \mu$ and $N(ds, dx, du, dw)$ a Poisson random measure on $[0, \infty) \times E \times [0, \infty) \times W_0$ with intensity $dsm(dx)du\boldsymbol{Q}^x(dw)$, where $\boldsymbol{Q}^x$ is an excursion law of the (A, b, c)-superprocess carried by excursions growing up at $x \in E$. We assume $\{X_t : t \geq 0\}$ and $N(ds, dx, du, dw)$ are defined on a standard probability space and are independent of each other. We shall prove that the stochastic equation

$$Y_t = X_t + \int_0^t \int_E \int_0^{q(Y_s,x)} \int_{W_0} w(t-s) N(ds, dx, du, dw), \quad t \geq 0, \tag{1.5}$$

has a pathwise unique continuous solution $\{Y_t : t \geq 0\}$ and its distribution on W solves the martingale problem given by (1.2) and (1.4); see Theorem 4.1. The pathwise uniqueness implies the strong Markov property of $\{Y_t : t \geq 0\}$, so our result gives a partial solution of the open problem on the Markov property of the superprocess with mean field interaction; see Méléard and Roelly [14, p.103].

In particular, when $E = \{a\}$ is a singleton, equation (1.5) gives a decomposition of the one-dimensional diffusion process $\{y(t) : t \geq 0\}$ defined by

$$dy(t) = \sqrt{cy(t)}dB(t) + \beta(y(t))y(t)dt + \gamma(y(t))dt, \quad t \geq 0, \tag{1.6}$$

where $c > 0$ is a constant, $\beta(\cdot)$ is a bounded Lipschitz function on $[0, \infty)$ and $\gamma(\cdot)$ is a non-negative locally Lipschitz function on $[0, \infty)$ satisfying the linear growth condition. In the special case where $\beta(\cdot)$ and $\gamma(\cdot)$ are constant, Pitman and Yor [15] gave a construction of $\{y(t) : t \geq 0\}$ by picking up excursions by a Poisson point process, which served as a preliminary to their well-known results on decomposition of Bessel bridges. See also Le Gall and Yor [9].

In section 2 we recall some basic facts on the (A, b, c)-superprocess and its immigration processes with deterministic immigration rates. In section 3, we discuss construction of immigration processes with predictable immigration rates. The stochastic equation with a Poisson process of excursions is studied in section 4.

2 Deterministic immigration rate

In this section, we summarize some basic facts on the (A,b,c)-superprocess and its immigration processes with deterministic immigration rates. Let $(Q_t)_{t\geq 0}$ denote the transition semigroup of the (A,b,c)-superprocess, which is determined by

$$\int_{M(E)} e^{-\nu(f)} Q_t(\mu, d\nu) = \exp\{-\mu(V_t f)\}, \quad f \in C^+(E), \mu \in M(E), \tag{2.1}$$

where $V_t f$ is the unique positive solution of the evolution equation

$$V_t f(x) + \frac{1}{2}\int_0^t ds \int_E c(y) V_s f(y)^2 P^b_{t-s}(x, dy) = P^b_t f(x), \quad t \geq 0, x \in E, \tag{2.2}$$

where $(P^b_t)_{t\geq 0}$ denotes the semigroup of kernels on E generated by $A^b := A - b$. By [1, pp.195-196], there is a family of finite measures $L_t(x, d\nu)$ on $M(E)^\circ := M(E) \setminus \{0\}$ such that

$$\int_{M(E)^\circ} (1 - e^{-\nu(f)}) L_t(x, d\nu) = V_t f(x), \quad t > 0, x \in E, f \in C^+(E). \tag{2.3}$$

Let $(Q^\circ_t)_{t\geq 0}$ be the restriction of $(Q_t)_{t\geq 0}$ to $M(E)^\circ$. It is easy to check that $(L_t(x,\cdot))_{t>0}$ is an *entrance law* for $(Q^\circ_t)_{t\geq 0}$, that is $L_r(x,\cdot)Q^\circ_t = L_{r+t}(x,\cdot)$ for $r > 0$ and $t > 0$. Then there is a unique σ-finite Borel measure $\boldsymbol{Q}^x$ on $(W_0, \mathcal{B}(W_0))$ such that

$$\begin{aligned} & \boldsymbol{Q}^x(w(t_1) \in d\nu_1, \cdots, w(t_n) \in d\nu_n) \\ = \ & L_{t_1}(x, d\nu_1) Q^\circ_{t_2 - t_1}(\nu_1, d\nu_2) \cdots Q^\circ_{t_n - t_{n-1}}(\nu_{n-1}, d\nu_n) \end{aligned} \tag{2.4}$$

for $0 < t_1 < t_2 < \cdots < t_n$ and $\nu_1, \nu_2, \cdots, \nu_n \in M(E)^\circ$. Indeed, $\boldsymbol{Q}^x$ is carried by the paths $w \in W_0$ such that $w_t(1)^{-1} w_t \to \delta_x$ as $t \to 0$; see [11] and [12]. Moreover, it is easy to obtain that

$$\boldsymbol{Q}^x\{w(t)(f)\} = P^b_t f(x), \quad t > 0, x \in E, f \in C^+(E). \tag{2.5}$$

Let $\mathcal{B}_t(W_0)$ be the σ-algebra on W_0 generated by $\{w(s) : 0 \leq s \leq t\}$. Roughly speaking, $(W_0, \mathcal{B}_t(W_0), w(t))$ under $\boldsymbol{Q}^x$ is a Markov process with semigroup $(Q^\circ_t)_{t>0}$ and one-dimensional distributions $(L_t(x,\cdot))_{t>0}$. The measure $\boldsymbol{Q}^x$ is known as an *excursion law* of $(Q_t)_{t\geq 0}$.

Now we fix a σ-finite reference measure m on E and suppose that $q(\cdot,\cdot)$ is a non-negative Borel function on $[0,\infty) \times E$ such that $m(q(t,\cdot))$ is a locally bounded function of $t \geq 0$. Then

$$\int_0^\infty e^{-\nu(f)} Q^q_{r,t}(\mu, d\nu) = \exp\Big\{ - \mu(V_{t-r} f) - \int_r^t m(q(s,\cdot) V_{t-s} f) ds \Big\} \tag{2.6}$$

defines an inhomogeneous transition semigroup $(Q^q_{r,t})_{t\geq r}$. A diffusion process with transition semigroup $(Q^q_{r,t})_{t\geq r}$ can be constructed as follows. Let $\{X_t : t \geq 0\}$ be an (A,b,c)-superprocess with deterministic initial state $X_0 = \mu$ and $N(ds, dx, du, dw)$ a Poisson random measure on $[0,\infty)\times E\times[0,\infty)\times W_0$ with intensity $dsm(dx)du\boldsymbol{Q}^x(dw)$. We assume $\{X_t : t\geq 0\}$ and $N(ds,dx,du,dw)$ are defined on a standard probability space $(\Omega,\mathcal{A},\boldsymbol{P})$ and are independent of each other. For $t\geq 0$, let $\mathcal{G}_t$ be the σ-algebra generated by the $\boldsymbol{P}$-null sets in $\mathcal{A}$ and the random variables

$$\{X_s, N(J\times A) : J\in\mathcal{B}([0,s]\times E\times[0,\infty)), A\in\mathcal{B}_{t-s}(W_0), 0\leq s\leq t\}. \tag{2.7}$$

We define the $M(E)$-valued process $\{Y_t : t\geq 0\}$ by

$$Y_t = X_t + \int_0^t\int_E\int_0^{q(s,x)}\int_{W_0} w(t-s)N(ds,dx,du,dw), \quad t\geq 0, \tag{2.8}$$

where the integration area refers to

$$\{(s,x,u,w) : 0<s\leq t, x\in E, 0<u\leq q(s,x), w\in W_0\}.$$

(We shall make the same convention in the sequel.)

Theorem 2.1 *The process $\{Y_t : t\geq 0\}$ defined by (2.8) is an inhomogeneous diffusion process relative to $(\mathcal{G}_t)_{t\geq 0}$ with transition semigroup $(Q^q_{r,t})_{t\geq r}$. Moreover, for each $f\in\mathcal{D}(A)$,*

$$M_t(f) = Y_t(f) - Y_0(f) - \int_0^t Y_s(Af-bf)ds - \int_0^t m(q(s,\cdot)f)ds, \quad t\geq 0, \tag{2.9}$$

is a martingale relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t Y_s(cf^2)ds, \quad t\geq 0. \tag{2.10}$$

Proof. Let $N_1(ds,dx,du,dw)$ denote the restriction of $N(ds,dx,du,dw)$ to

$$\{(s,x,u,w) : s>0, x\in E, 0<u\leq q(s,x), w\in W_0\}$$

and let $N_1(ds,dw)$ be the image of $N_1(ds,dx,du,dw)$ under the map $(s,x,u,w)\mapsto(s,w)$. Then $N_1(ds,dw)$ is a Poisson random measure on $[0,\infty)\times W_0$ with intensity $ds\boldsymbol{Q}^\kappa_s(dw)$, where

$$\boldsymbol{Q}^\kappa_s(dw) = \int_E q(s,x)\boldsymbol{Q}^x(dw)m(dx), \quad w\in W_0.$$

Then the first assertion follows by an obvious modification of the arguments of [12, Theorem 1.3] and [17, Theorem 3.6]; see also [10, Theorem 3.2]. The martingale characterization (2.9) and (2.10) can be proved by a calculation of the generator of $(Q^q_{r,t})_{t\geq r}$. □

The construction (2.8) gives clear interpretations for reference measure m and immigration rate $q(\cdot,\cdot)$ in the phenomenon. Since (2.9) is linear in $f\in\mathcal{D}(A)$, it defines a martingale measure $M(ds,dx)$ with quadratic variation measure $c(x)Y_s(dx)ds$ in the sense of Walsh [18]. By a standard argument one gets the following

Theorem 2.2 *For each $t\geq 0$ and $f\in C(E)$ we have a.s.*

$$Y_t(f) = Y_0(P^b_t f) + \int_0^t\int_E P^b_{t-s}f(x)M(ds,dx) + \int_0^t m(q(s,\cdot)P^b_{t-s}f)ds. \tag{2.11}$$

3 Predictable immigration rate

In this section, we fix a σ-finite reference measure m on E. Let $(\Omega, \mathcal{A}, \boldsymbol{P})$ be a standard probability space and $N(ds, dx, du, dw)$ and $\{X_t : t \geq 0\}$ be as in the last section. Let $\mathcal{G}_t$ be the σ-algebra on Ω generated by the $\boldsymbol{P}$-null sets in $\mathcal{A}$ and the random variables in (2.7). Let $\mathcal{P}$ be the σ-algebra on $[0, \infty) \times E \times \Omega$ generated by functions of the form

$$g(s, x, \omega) = \eta_0(x, \omega)1_{\{r_0\}}(s) + \sum_{i=0}^{\infty} \eta_i(x, \omega)1_{(r_i, r_{i+1}]}(s), \tag{3.1}$$

where $0 = r_0 < r_1 < r_2 < \ldots$ and $\eta_i(\cdot, \cdot)$ is $\mathcal{B}(E) \times \mathcal{G}_{r_i}$-measurable. We say a function on $[0, \infty) \times E \times \Omega$ is *predictable* if it is $\mathcal{P}$-measurable.

Theorem 3.1 *Suppose that $q(\cdot, \cdot, \cdot)$ is a non-negative predictable function on $[0, \infty) \times E \times \Omega$ such that $\boldsymbol{E}\{m(q(t, \cdot))^2\}$ is locally bounded in $t \geq 0$. Then the $M(E)$-valued process*

$$Y_t = X_t + \int_0^t \int_E \int_0^{q(s,x)} \int_{W_0} w(t - s)N(ds, dx, du, dw), \quad t \geq 0, \tag{3.2}$$

has a continuous modification. Moreover, for this modification and each $f \in \mathcal{D}(A)$,

$$M_t(f) = Y_t(f) - Y_0(f) - \int_0^t Y_s(Af - bf)ds - \int_0^t m(q(s, \cdot)f)ds, \quad t \geq 0, \tag{3.3}$$

is a martingale relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t Y_s(cf^2)ds, \quad t \geq 0. \tag{3.4}$$

Let $M(ds, dx)$ denote the stochastic integral with respect to the martingale measure with quadratic variation measure $c(x)Y_s(dx)ds$ defined by (3.3) and (3.4). Then we have

Theorem 3.2 *For each $t \geq 0$ and $f \in C(E)$ we have a.s.*

$$Y_t(f) = Y_0(P_t^b f) + \int_0^t \int_E P_{t-s}^b f(x)M(ds, dx) + \int_0^t m(q(s, \cdot)P_{t-s}^b f)ds. \tag{3.5}$$

The process $\{Y_t : t \geq 0\}$ constructed by (3.2) can be regarded as an (A, b, c)-superprocess allowing immigration with immigration rate given by the predictable function $q(\cdot, \cdot, \cdot)$. To give the proof of the above theorems we need a set of lemmas.

Lemma 3.1 *The results of Theorems 3.1 and 3.2 hold if $q(\cdot, \cdot, \cdot)$ is of the form (3.1).*

Proof. Observe that $\eta_i(x)$ is a deterministic function on E under the regular conditional probability $\boldsymbol{P}\{\cdot|\mathcal{G}_{r_i}\}$. Since $\mathcal{G}_{r_i}$ and the restriction of $N(ds, dx, du, dw)$ to $(r_i, \infty) \times E \times [0, \infty) \times W_0$ are independent, this restriction under $\boldsymbol{P}\{\cdot|\mathcal{G}_{r_i}\}$ is still a Poisson random measure with intensity $dsm(dx)du\boldsymbol{Q}^x(dw)$. Note that $\{X_t : t \geq 0\}$ is also an a.s. continuous (A, b, c)-superprocess under $\boldsymbol{P}\{\cdot|\mathcal{G}_0\}$. Then we conclude by Theorem 2.1 that $\{Y_t : 0 \leq t \leq \ r_1\}$ under $\boldsymbol{P}\{\cdot|\mathcal{G}_0\}$ is an a.s. continuous (A, b, c)-superprocess allowing immigration with immigration rate $\eta_0(\cdot)$. Let

$$Y_t^{(0)} = X_t + \int_0^{r_1} \int_E \int_0^{\eta_0(x)} \int_{W_0} w(t-s) N(ds, dx, du, dw), \quad t \geq r_1.$$

By Theorem 2.1, $\{Y_t^{(0)} : t \geq r_1\}$ under $\boldsymbol{P}\{\cdot|\mathcal{G}_0\}$ is an a.s. continuous (A, b, c)-superprocess. Of course, $\{Y_t^{(0)} : t \geq r_1\}$ is still an a.s. continuous (A, b, c)-superprocess under $\boldsymbol{P}\{\cdot|\mathcal{G}_{r_1}\}$. It is not difficult to see that

$$Y_t = Y_t^{(0)} + \int_{r_1}^{t} \int_E \int_0^{\eta_1(x)} \int_{W_0} w(t-s) N(ds, dx, du, dw), \quad r_1 \leq t \leq r_2.$$

By Theorem 2.1 again, $\{Y_t : r_1 \leq t \leq r_2\}$ under $\boldsymbol{P}\{\cdot|\mathcal{G}_{r_1}\}$ is an a.s. continuous (A, b, c)-superprocess allowing immigration with immigration rate $\eta_1(\cdot)$. Using the above argument inductively we can see that $\{Y_t : r_i \leq t \leq r_{i+1}\}$ under $\boldsymbol{P}\{\cdot|\mathcal{G}_{r_i}\}$ is an a.s. continuous (A, b, c)-superprocess allowing immigration with immigration rate $\eta_i(\cdot)$. By Theorem 2.1, $\{Y_t : t \geq 0\}$ has a continuous modification. The martingale characterizations of Theorems 3.1 and 3.2 follow from those of the immigration process with deterministic immigration rate. $\square$

Lemma 3.2 *Suppose that there is a non-negative deterministic function $q_1(\cdot) \in L^1(E, m)$ such that $q(t, x, \omega) \leq q_1(x)$ for a.a. $(t, x, \omega) \in [0, \infty) \times E \times \Omega$. Let $\{g_n\}$ be a sequence of non-negative predictable functions of the form (3.1) such that $g_n(t, x, \omega) \leq q_1(x)$ and $g_n(t, x, \omega) \to q(t, x, \omega)$ for a.a. $(t, x, \omega) \in [0, \infty) \times E \times \Omega$. Let $\{Y_t^{(n)} : t \geq 0\}$ be defined by (3.2) in terms of $g_n(\cdot, \cdot, \cdot)$. Then there is an $M(E)$-valued process $\{Y_t : t \geq 0\}$ such that $\lim_{n\to\infty} \boldsymbol{E}\{\|Y_t^{(n)} - Y_t\|\} = 0$ uniformly on each finite interval of $t \geq 0$, where $\|\cdot\|$ denotes the total variation metric.*

Proof. Since the result of Theorem 3.2 holds for $\{Y_t^{(n)} : t \geq 0\}$, we have

$$\boldsymbol{E}\{Y_t^{(n)}(f)\} = \mu(P_t^b f) + \int_0^t \boldsymbol{E}\{m(g_n(s, \cdot) P_{t-s}^b f)\} ds, \quad f \in C(E). \tag{3.6}$$

Observe that for any $k \geq n \geq 1$, both $g_n \vee g_k$ and $g_n \wedge g_k$ are predictable functions of the form (3.1). Let

$$Y_t^{(n,k)} = X_t + \int_0^t \int_E \int_0^{g_n(s,x) \vee g_k(s,x)} \int_{W_0} w(t-s) N(ds, dx, du, dw)$$

and

$$Z_t^{(n,k)} = X_t + \int_0^t \int_E \int_0^{g_n(s,x)\wedge g_k(s,x)} \int_{W_0} w(t-s) N(ds, dx, du, dw).$$

Since $\|Y_t^{(n)} - Y_t^{(k)}\| \leq Y_t^{(n,k)}(1) - Z_t^{(n,k)}(1)$, we may apply (3.6) to $\{Y_t^{(n,k)} : t \geq 0\}$ and $\{Z_t^{(n,k)} : t \geq 0\}$ so that

$$\boldsymbol{E}\{\|Y_t^{(n)} - Y_t^{(k)}\|\} \leq \int_0^t e^{\|b\|(t-s)} \boldsymbol{E}\{m(|g_n(s,\cdot) - g_k(s,\cdot)|)\} ds.$$

By dominated convergence, the right hand side goes to zero uniformly on each finite interval of $t \geq 0$ as $n \to \infty$. Then there is an $M(E)$-valued process $\{Y_t : t \geq 0\}$ such that

$$\boldsymbol{E}\{Y_t(f)\} = \mu(P_t^b f) + \int_0^t \boldsymbol{E}\{m(q(s,\cdot)P_{t-s}^b f)\} ds, \quad f \in C(E), \tag{3.7}$$

and $\lim_{n\to\infty} \boldsymbol{E}\{\|Y_t^{(n)} - Y_t\|\} = 0$ uniformly on each finite interval of $t \geq 0$. □

Lemma 3.3 *Suppose that the condition of Lemma 3.2 holds. Then the process $\{Y_t : t \geq 0\}$ obtained there is independent of the choice of $\{g_n\}$ in the sense that if $\{Z_t : t \geq 0\}$ obtained from another sequence with the same properties, then $Y_t = Z_t$ a.s. for each $t \geq 0$. Moreover, (3.2) holds a.s. for each $t \geq 0$.*

Proof. Let $\{q_n\}$ be another sequence having the properties of $\{g_n\}$. Then $\{g_n \vee q_n\}$ and $\{g_n \wedge q_n\}$ have the same properties. Let $\{Y_t' : t \geq 0\}$ and $\{Y_t'' : t \geq 0\}$ be the processes obtained respectively from $\{g_n \vee q_n\}$ and $\{g_n \wedge q_n\}$. Clearly, $Y_t'' \leq Y_t \leq Y_t'$ a.s. for each $t \geq 0$. But, $\boldsymbol{E}\{Y_t'(1)\} = \boldsymbol{E}\{Y_t''(1)\} = \boldsymbol{E}\{Y_t(1)\}$ by (3.7), so we have $Y_t'' = Y_t' = Y_t$ a.s. for each $t \geq 0$. Thus $\{Y_t : t \geq 0\}$ is independent of the choice of $\{g_n\}$. To show (3.2), let Z_t denote the value of its right hand side. We first assume in addition there is a strictly positive deterministic functions $q_2(\cdot) \in L^1(E, m)$ such that $q_2(x) \leq q(t, x, \omega)$ for all $(t, x, \omega) \in [0, \infty) \times E \times \Omega$. For $k \geq 1$, let $\{Y_{k,t} : t \geq 0\}$ and $\{Z_{k,t} : t \geq 0\}$ be the process obtained by Lemma 3.2 from the non-negative predictable functions $q(t, x, \omega) + q_2(x)/k$ and $q(t, x, \omega) - q_2(x)/k$, respectively. Since

$$q(t, x, \omega) - q_2(x)/k < q(t, x, \omega) < q(t, x, \omega) + q_2(x)/k,$$

we have $Z_{k,t} \leq Z_t$, $Y_t \leq Y_{k,t}$ a.s. for each $t \geq 0$. But, by (3.7) it is easy to show that

$$\boldsymbol{E}\{Y_{k,t}(1) - Z_{k,t}(1)\} \leq 2te^{\|b\|t} m(q_2)/k,$$

so we must have $Y_t = Z_t$ a.s. for each $t \geq 0$. In the general case, we may apply the above reasoning to $q(t, x, \omega) + q_2(x)/k$ and $\{Y_{k,t} : t \geq 0\}$ to get

$$Y_{k,t} = X_t + \int_0^t \int_E \int_0^{q(s,x)+q_2(x)/k} \int_{W_0} w(t-s) N(ds, dx, du, dw)$$

and

$$\boldsymbol{E}\{Y_{k,t}(f)\} = \mu(P_t^b f) + \int_0^t \boldsymbol{E}\{m([q(s,\cdot) + q_2/k]P_{t-s}^b f)\}ds.$$

Clearly, $Y_{k,t}$ decreases to Z_t as $k \to \infty$. As in the proof of Lemma 3.2, it is easy to show that $\lim_{k\to\infty} \boldsymbol{E}\{\|Y_{k,t} - Y_t\|\} = 0$ uniformly on each finite interval of $t \geq 0$, so the desired results hold. □

Lemma 3.4 *Under the assumptions of Theorem 3.1, choose a strictly positive function $q_1(\cdot) \in L^1(E, m)$ and let $q_n(t, x, \omega) = q(t, x, \omega) \wedge (nq_1(x))$. Let $\{Z_t^{(n)} : t \geq 0\}$ be defined by (3.2) in terms of $q_n(\cdot, \cdot, \cdot)$. Then we have $\lim_{n\to\infty} \boldsymbol{E}\{\|Z_t^{(n)} - Y_t\|\} = 0$ uniformly on each finite interval of $t \geq 0$, where $\{Y_t : t \geq 0\}$ is defined by (3.2).*

Proof. As in the proof of Lemma 3.2 one can show that there is an $M(E)$-valued process $\{Z_t : t \geq 0\}$ such that $\lim_{n\to\infty} \boldsymbol{E}\{\|Z_t^{(n)} - Z_t\|\} = 0$ uniformly on each finite interval of $t \geq 0$. As in the proof of Lemma 3.3 we have $Y_t = Z_t$ a.s. for each $t \geq 0$. □

Lemma 3.5 *The results of Theorems 3.1 and 3.2 hold if E is compact.*

Proof. We first assume the condition of Lemma 3.2 holds. Let $\{Y_t^{(n)} : t \geq 0\}$ be the approximating sequence given by Lemma 3.2 and define $\{M_t^{(n)} : t \geq 0\}$ by (3.3) in terms of $\{Y_t^{(n)} : t \geq 0\}$ and $g_n(\cdot, \cdot, \cdot)$. By (3.6) we have

$$\int_0^t \boldsymbol{E}\{Y^{(n)}(1)\}ds \leq \int_0^t e^{\|b\|s}[\mu(1) + sm(q_1)]ds.$$

Then for $T > 0$ and $\varepsilon > 0$, there is $\eta > 0$ such that

$$\boldsymbol{P}\Big\{\int_0^T Y^{(n)}(|b|)ds > \eta/2\Big\} \leq 2\eta^{-1}\|b\| \int_0^T e^{\|b\|s}[\mu(1) + sm(q_1)]ds \leq \varepsilon.$$

Moreover, from (3.4) we obtain

$$\boldsymbol{E}\{M_T^{(n)}(1)^2\} = \int_0^T \boldsymbol{E}\{Y_s^{(n)}(c)\}ds \leq \|c\| \int_0^T e^{\|b\|s}[\mu(1) + sm(q_1)]ds. \tag{3.8}$$

In view of the martingale characterization (3.3) and (3.4) for $\{Y_t^{(n)} : t \geq 0\}$, choosing $\eta > 2\mu(1) + 2m(q_1)T$ we have

$$\begin{aligned} &\boldsymbol{P}\Big\{\sup_{0\leq t\leq T} Y_t^{(n)}(1) > \eta\Big\} \\ \leq\ &\varepsilon + \boldsymbol{P}\Big\{\sup_{0\leq t\leq T} Y_t^{(n)}(1) > \eta, \int_0^T Y_s^{(n)}(|b|)ds \leq \eta/2\Big\} \end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon + \boldsymbol{P}\Big\{ \sup_{0\le t\le T} \Big[\mu(1) + M_t^{(n)}(1) + \int_0^t m(g_n(s,\cdot))ds\Big] > \eta/2\Big\} \\
&\le \varepsilon + \boldsymbol{P}\Big\{ \sup_{0\le t\le T} M_t^{(n)}(1) > \eta/2 - \mu(1) - m(q_1)T\Big\} \\
&\le 4(\eta/2 - \mu(1) - m(q_1)T)^{-2}\boldsymbol{E}\left\{M_T^{(n)}(1)^2\right\} \\
&\le 4(\eta/2 - \mu(1) - m(q_1)T)^{-2}\|c\| \int_0^T e^{\|b\|s}[\mu(1) + sm(q_1)]ds
\end{aligned}
$$

by a martingale inequality; see e.g. [6, p.34]. Consequently,

$$
\lim_{\eta\to\infty} \sup_{n\ge 1} \boldsymbol{P}\Big\{ \sup_{0\le t\le T} Y_t^{(n)}(1) > \eta\Big\} = 0.
$$

Thus $\{Y_t^{(n)} : t \ge 0\}$ viewed as processes in $C([0,\infty), M(E))$ satisfy the compact containment condition of [3, p.142]. (Note that $C([0,\infty), M(E))$ is a closed subspace of $D([0,\infty), M(E))$.) By Itô's formula, for $G \in C^2(I\!R^m)$ and $\{f_1, \cdots, f_m\} \subset \mathcal{D}(A)$,

$$
\begin{aligned}
&G(Y_t^{(n)}(f_1), \cdots, Y_t^{(n)}(f_m)) - G(Y_0^{(n)}(f_1), \cdots, Y_0^{(n)}(f_m)) \\
&\quad - \sum_{i=1}^m \int_0^t G_i'(Y_s^{(n)}(f_1), \cdots, Y_s^{(n)}(f_m))[m(g_n(s,\cdot)f_i) + Y_s^{(n)}(Af_i - bf_i)]ds \\
&\quad - \frac{1}{2}\sum_{i,j=1}^m \int_0^t G_{ij}''(Y_s^{(n)}(f_1), \cdots, Y_s^{(n)}(f_m))Y_s^{(n)}(cf_i^2)ds
\end{aligned}
$$

is a continuous martingale. From (3.8) and the martingale characterizations of Lemma 3.1 we see that $\boldsymbol{E}\{Y_t^{(n)}(1)^2\}$ is dominated by a locally bounded positive function independent of $n \ge 1$. By [3, pp.142–145] we conclude that $\{Y_t^{(n)} : t \ge 0\}$ is a tight sequence in $C([0,\infty), M(E))$. Consequently, $\{Y_t : t \ge 0\}$ has a continuous modification and $\{Y_t^{(n)} : t \ge 0\}$ converges a.s. to this modification in the topology of $C([0,\infty), M(E))$. Note also that

$$
\int_0^t m(g_n(s,\cdot)f_i)ds \to \int_0^t m(q(s,\cdot)f_i)ds, \quad t \ge 0,
$$

in the topology of $C([0,\infty), I\!R)$. Then the martingale characterization (3.3) and (3.4) for $\{Y_t : t \ge 0\}$ follows from Lemma 3.1 and [7, p.342]. If the condition of Lemma 3.2 does not hold, we may consider the additional approximating sequence $\{Z_t^{(n)} : t \ge 0\}$ given by Lemma 3.4. Then a modification of the above arguments shows that $\{Z_t^{(n)} : t \ge 0\}$ is a tight sequence, so we also have (3.3) and (3.4). The equality (3.5) follows in the same way as Theorem 2.2. □

Proof of Theorems 3.1 and 3.2. Note that $(P_t)_{t\ge 0}$ can be extended to a Feller transition semigroup $(\bar{P}_t)_{t\ge 0}$ on $\bar{E}$, the one point compactification of E. Since m can be viewed as a σ-finite measure on $\bar{E}$ and since $b(\cdot)$ and $c(\cdot)$ have continuous extensions $\bar{b}(\cdot)$ and $\bar{c}(\cdot)$ on $\bar{E}$, we can also regard $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ as objects associated with $(\bar{P}_t)_{t\geq 0}$. Applying Lemma 3.5 in this way we see that $\{Y_t : t \geq 0\}$ has a $M(\bar{E})$-valued continuous modification $\{\bar{Y}_t : t \geq 0\}$ which satisfies the corresponding martingale characterization (3.3) and (3.4). Then the two theorems will follow from Lemma 3.5 once it is proved that

$$\boldsymbol{P}\{\bar{Y}_t(\{\partial\}) = 0 \text{ for all } t \in [0,T]\} = 1, \quad T > 0. \tag{3.9}$$

Observe that for any $\bar{f} \in C(\bar{E})$,

$$\begin{aligned} M_t^T(\bar{f}) &:= \bar{Y}_t(\bar{P}_{T-t}^b \bar{f}) - \bar{Y}_0(\bar{P}_T^b \bar{f}) - \int_0^t m(\bar{P}_{T-s}^b \bar{f} q(s,\cdot))ds \\ &= \int_0^t \int_{\bar{E}} \bar{P}_{T-s}^b \bar{f}(x) \bar{M}(ds,dx) \end{aligned}$$

is a continuous martingale in $t \in [0,T]$ with quadratic variation process

$$\langle M^T(\bar{f})\rangle_t = \int_0^t \bar{Y}_s(\bar{c}(\bar{P}_{T-s}^b \bar{f})^2)ds,$$

where $(\bar{P}_t^b)_{t\geq 0}$ is defined from $(\bar{P}_t)_{t\geq 0}$ and $\bar{b}$. By a martingale inequality we have

$$\begin{aligned} &\boldsymbol{P}\Big\{\sup_{0\leq t\leq T}\Big|\bar{Y}_t(\bar{P}_{T-t}^b \bar{f}) - \bar{Y}_0(\bar{P}_T^b \bar{f}) - \int_0^t m(\bar{P}_{T-s}^b \bar{f} q(s,\cdot))ds\Big|^2\Big\} \\ \leq\ & 4\int_0^T \boldsymbol{E}\{\bar{Y}_s(\bar{c}(\bar{P}_{T-s}^b \bar{f})^2)\}ds. \end{aligned}$$

Choose a sequence $\{\bar{f}_k\} \subset C(\bar{E})$ such that $\bar{f}_k \to 1_{\{\partial\}}$ boundedly as $k \to \infty$. Since each $\bar{Y}_s$ is a.s. supported by E, replacing $\bar{f}$ by $\bar{f}_k$ in the above and letting $k \to \infty$ we obtain (3.9). $\square$

4 A stochastic equation with Poisson process

We fix a σ-finite reference measure m on E. Let $(\Omega, \mathcal{A}, \boldsymbol{P})$ be a standard probability space on which $N(ds,dx,du,dw)$ and $\{X_t : t \geq 0\}$ are given as in section 2. Let $\mathcal{G}_t$ be the σ-algebra on Ω generated by the $\boldsymbol{P}$-null sets in $\mathcal{A}$ and the random variables in (2.7). Suppose that $q(\cdot,\cdot)$ is a Borel function on $M(E) \times E$ such that there is a constant K such that

$$m(q(\nu,\cdot)) \leq K(1 + \|\nu\|), \quad \nu \in M(E), \tag{4.1}$$

and for each $R > 0$ there is a constant $K_R > 0$ such that

$$m(|q(\nu,\cdot) - q(\gamma,\cdot)|) \leq K_R\|\nu - \gamma\| \tag{4.2}$$

for ν and $\gamma \in M(E)$ satisfying $\nu(1) \leq R$ and $\gamma(1) \leq R$. We consider the stochastic integral equation:

$$Y_t = X_t + \int_0^t \int_E \int_0^{q(Y_s,x)} \int_{W_0} w(t-s) N(ds,dx,du,dw), \quad t \geq 0. \tag{4.3}$$

By a (strong) *solution* of (4.3) we mean a continuous $M(E)$-valued process $\{Y_t : t \geq 0\}$ which is adapted to the filtration $(\mathcal{G}_t)_{t\geq 0}$ and satisfies (4.3) with probability one. A solution of this equation can be regarded as an immigration (A,b,c)-superprocess with interactive immigration rate given by $q(\cdot,\cdot)$.

Lemma 4.1 *Let $R \geq 0$ and let $q_1(\cdot,\cdot)$ and $q_2(\cdot,\cdot)$ be Borel functions on $M(E) \times E$ satisfying $q_1(\nu,\cdot) \equiv q_2(\nu,\cdot) \equiv q(\nu,\cdot)$ for $\nu(1) \leq R$. Suppose that $\{Y_t^{(1)} : t \geq 0\}$ and $\{Y_t^{(2)} : t \geq 0\}$ are solution of (4.3) with $q(\cdot,\cdot)$ replaced by $q_1(\cdot,\cdot)$ and $q_2(\cdot,\cdot)$ respectively. Let $\tau = \inf\{t \geq 0 : Y_t^{(1)}(1) \geq R$ or $Y_t^{(2)}(1) \geq R\}$. Then $\{Y_{t\wedge\tau}^{(1)} : t \geq 0\}$ and $\{Y_{t\wedge\tau}^{(2)} : t \geq 0\}$ are indistinguishable.*

Proof. Since each $\{Y_t^{(i)} : t \geq 0\}$ is continuous, $q(Y_t^{(i)},x)I_{\{t\leq\tau\}}$ is predictable. Note also that $m(q(Y_t^{(i)},\cdot)I_{\{t\leq\tau\}})$ is bounded. Let

$$Y_t^* = \int_0^{t\wedge\tau} \int_E \int_0^{q(Y_s^{(1)},x)\vee q(Y_s^{(2)},x)} \int_{W_0} w(t-s) N(ds,dx,du,dw)$$

and

$$Z_t^* = \int_0^{t\wedge\tau} \int_E \int_0^{q(Y_s^{(1)},x)\wedge q(Y_s^{(2)},x)} \int_{W_0} w(t-s) N(ds,dx,du,dw).$$

Applying Theorem 3.1 to the predictable function

$$(s,x,\omega) \mapsto q(Y_s^{(1)},x) \vee q(Y_s^{(2)},x) I_{\{s\leq\tau\}}$$

we see that

$$M_t^*(1) = Y_t^*(1) + \int_0^t Y_s^*(b)ds - \int_0^t m(q(Y_s^{(1)},\cdot) \vee q(Y_s^{(2)},\cdot))I_{\{s\leq\tau\}}ds$$

is a continuous martingale. By Doob's stopping theorem,

$$\boldsymbol{E}\{Y_{t\wedge\tau}^*(1)\} = \int_0^t \boldsymbol{E}\{m(q(Y_s^{(1)},\cdot) \vee q(Y_s^{(2)},\cdot))I_{\{s\leq\tau\}}\}ds - \int_0^t \boldsymbol{E}\{Y_s^*(b)I_{\{s\leq\tau\}}\}ds.$$

Similarly, we have

$$\boldsymbol{E}\{Z_{t\wedge\tau}^*(1)\} = \int_0^t \boldsymbol{E}\{m(q(Y_s^{(1)},\cdot) \wedge q(Y_s^{(2)},\cdot))I_{\{s\leq\tau\}}\}ds - \int_0^t \boldsymbol{E}\{Z_s^*(b)I_{\{s\leq\tau\}}\}ds.$$

By (4.2) we obtain

$$\begin{aligned}
\boldsymbol{E}\{[Y^*_{t\wedge\tau}(1) - Z^*_{t\wedge\tau}(1)]\} &= \int_0^t \boldsymbol{E}\{m(|q(Y_s^{(1)},\cdot) - q(Y_s^{(2)},\cdot)|I_{\{s\le\tau\}})\}ds \\
&\quad + \int_0^t \boldsymbol{E}\{[Y^*_s(b) - Z^*_s(b)]I_{\{s\le\tau\}}\}ds \\
&\le K_R \int_0^t \boldsymbol{E}\{\|Y_s^{(1)} - Y_s^{(2)}\| I_{\{s\le\tau\}}\}ds \\
&\quad + \|b\| \int_0^t \boldsymbol{E}\{[Y^*_s(1) - Z^*_s(1)]I_{\{s\le\tau\}}\}ds \\
&\le K_R \int_0^t \boldsymbol{E}\{\|Y_{s\wedge\tau}^{(1)} - Y_{s\wedge\tau}^{(2)}\|\}ds \\
&\quad + \|b\| \int_0^t \boldsymbol{E}\{[Y^*_{s\wedge\tau}(1) - Z^*_{s\wedge\tau}(1)]\}ds \\
&\le (K_R + \|b\|) \int_0^t \boldsymbol{E}\{[Y^*_{s\wedge\tau}(1) - Z^*_{s\wedge\tau}(1)]\}ds.
\end{aligned}$$

Observe that $\|Y_{t\wedge\tau}^{(1)} - Y_{t\wedge\tau}^{(2)}\| \le Y^*_{t\wedge\tau}(1) - Z^*_{t\wedge\tau}(1)$. Then Gronwall's inequality yields

$$\boldsymbol{E}\{\|Y_{t\wedge\tau}^{(1)} - Y_{t\wedge\tau}^{(2)}\|\} \le \boldsymbol{E}\{[Y^*_{t\wedge\tau}(1) - Z^*_{t\wedge\tau}(1)]\} = 0$$

for all $t \ge 0$. Since $\{Y_{t\wedge\tau}^{(1)} : t \ge 0\}$ and $\{Y_{t\wedge\tau}^{(2)} : t \ge 0\}$ are continuous, they are indistinguishable. $\square$

Lemma 4.2 *There is at most one solution of (4.3).*

Proof. Suppose $\{Y_t : t \ge 0\}$ and $\{Y'_t : t \ge 0\}$ are two solutions of (4.3). Let $\tau_n = \inf\{t \ge 0 : Y_t(1) \ge n \text{ or } Y'_t(1) \ge n\}$. By Lemma 4.1, $\{Y_{t\wedge\tau_n} : t \ge 0\}$ and $\{Y'_{t\wedge\tau_n} : t \ge 0\}$ are indistinguishable for each $n \ge 1$. Thus

$$\tau_n = \inf\{t \ge 0 : Y_t(1) \ge n\} = \inf\{t \ge 0 : Y'_t(1) \ge n\}.$$

By continuity of paths, $\tau_n \uparrow \infty$ a.s. as $n \to \infty$ and hence $\{Y_t : t \ge 0\}$ and $\{Y'_t : t \ge 0\}$ are indistinguishable, that is, (4.3) has at most one solution. $\square$

Lemma 4.3 *Suppose there is a constant $L \ge 0$ such that $m(q(\nu,\cdot)) \le L$ for all $\nu \in M(E)$ and (4.2) holds for all ν and $\gamma \in M(E)$ with K_R replaced by L. Then there is a solution $\{Y_t : t \ge 0\}$ of (4.3). Moreover, for this solution and each $f \in \mathcal{D}(A)$,*

$$M_t(f) = Y_t(f) - Y_0(f) - \int_0^t Y_s(Af - bf)ds - \int_0^t m(q(Y_s,\cdot)f)ds, \quad t \ge 0,$$

is a continuous martingale relative to the filtration $(\mathcal{G}_t)_{t\ge0}$ with quadratic variation process

$$\langle M(f)\rangle_t = \int_0^t Y_s(cf^2)ds, \quad t \ge 0.$$

Proof. Since $\{X_t : t \geq 0\}$ is a.s. continuous, the function $(s, x, \omega) \mapsto q(X_s(\omega), x)$ is predictable. We define an approximating sequence $\{Y_t^{(n)} : t \geq 0\}$ inductively by $Y_t^{(0)} = X_t$ and

$$Y_t^{(n)} = X_t + \int_0^t \int_E \int_0^{q(Y_s^{(n-1)},x)} \int_{W_0} w(t-s) N(ds, dx, du, dw)$$

for $n \geq 1$. Let

$$Y_t^{*(n)} = \int_0^t \int_E \int_0^{q(Y_s^{(n-1)},x) \vee q(Y_s^{(n-2)},x)} \int_{W_0} w(t-s) N(ds, dx, du, dw)$$

and

$$Z_t^{*(n)} = \int_0^t \int_E \int_0^{q(Y_s^{(n-1)},x) \wedge q(Y_s^{(n-2)},x)} \int_{W_0} w(t-s) N(ds, dx, du, dw).$$

From Theorems 3.1 and 3.2 it follows that

$$\boldsymbol{E}\{Y_t^{*(n)}(1)\} = \int_0^t \boldsymbol{E}\{m([q(Y_s^{(n-1)}, \cdot) \vee q(Y_s^{(n-2)}, \cdot)]P_{t-s}^b 1)\} ds$$

and

$$\boldsymbol{E}\{Z_t^{*(n)}(1)\} = \int_0^t \boldsymbol{E}\{m([q(Y_s^{(n-1)}, \cdot) \wedge q(Y_s^{(n-2)}, \cdot)]P_{t-s}^b 1)\} ds.$$

Then we use (4.2) and the fact $\|Y_t^{(n)} - Y_t^{(n-1)}\| \leq Y_t^{*(n)}(1) - Z_t^{*(n)}(1)$ to see

$$\begin{aligned} \boldsymbol{E}\{\|Y_t^{(n)} - Y_t^{(n-1)}\|\} &\leq \int_0^t \boldsymbol{E}\{m(|q(Y_s^{(n-1)}, \cdot) - q(Y_s^{(n-2)}, \cdot)|P_{t-s}^b 1)\} ds \\ &\leq Le^{\|b\|T} \int_0^t \boldsymbol{E}\{\|Y_s^{(n-1)} - Y_s^{(n-2)}\|\} ds, \end{aligned}$$

and

$$\boldsymbol{E}\{\|Y_t^{(1)} - Y_t^{(0)}\|\} = \int_0^t \boldsymbol{E}\{m(q(X_s, \cdot)P_{t-s}^b 1)\} ds \leq LTe^{\|b\|T}$$

for $0 \leq t \leq T$. By a standard argument, one shows that there is an $M(E)$-valued process $\{Y_t : t \geq 0\}$ such that $\lim_{n\to\infty} \boldsymbol{E}\{\|Y_t^{(n)} - Y_t\|\} = 0$ uniformly on each finite interval of $t \geq 0$. Let

$$Y_t' = X_t + \int_0^t \int_E \int_0^{q(Y_s,x)} \int_{W_0} w(t-s) N(ds, dx, du, dw).$$

By a calculation similar as the above we get

$$\boldsymbol{E}\{\|Y_t^{(n)} - Y_t'\|\} \leq Le^{\|b\|T} \int_0^t \boldsymbol{E}\{\|Y_s^{(n-1)} - Y_s\|\} ds$$

for $0 \leq t \leq T$. Then we also have $\lim_{n\to\infty} \boldsymbol{E}\{\|Y_t^{(n)} - Y_t'\|\} = 0$ uniformly on each finite interval of $t \geq 0$, so that a.s. $Y_t' = Y_t$ and (4.3) is satisfied. By Theorem 3.1, $\{Y_t : t \geq 0\}$ has a continuous modification and we have the martingale characterization. □

Lemma 4.4 *For each $n \geq 1$ define a continuously differentiable function $a_n(\cdot)$ on $[0, \infty)$ such that*

$$a_n(z) = \begin{cases} 1 & \text{if } z \leq n-1, \\ n/z & \text{if } z \geq n+1, \end{cases}$$

and $0 \geq a_n'(z) \geq -1/z$ for all $z \geq 0$. Then $q_n(\nu, x) := q(a_n(\nu(1))\nu, x)$ satisfies the conditions of Lemma 4.3.

Proof. Observe that the assumptions on $a(\cdot)$ imply that $a_n(z)z \leq n$ for every $z \geq 0$. By (4.1) and the definition of $q_n(\cdot, \cdot)$ we have

$$m(q_n(\nu, \cdot)) \leq K(1 + a_n(\nu(1))\nu(1)) \leq K(1+n).$$

On the other hand, for ν and $\gamma \in M(E)$ let $\eta = \nu + \gamma$ and let g_ν and g_γ denote respectively the densities of ν and γ with respect to η. Without loss of generality, we may assume $\nu(1) \leq \gamma(1)$. By the mean-value theorem we have that

$$\nu(1)|a_n(\nu(1)) - a_n(\gamma(1))| \leq \nu(1)|a_n'(z)||\nu(1) - \gamma(1)| \leq \|\nu - \gamma\|,$$

where $\nu(1) \leq z \leq \gamma(1)$. It follows that

$$\begin{aligned} |m(q_n(\nu, \cdot) - q_n(\gamma, \cdot))| &= |m(q(a_n(\nu(1))\nu, \cdot) - q(a_n(\gamma(1))\gamma, \cdot))| \\ &\leq K_n \|a_n(\nu(1))\nu - a_n(\gamma(1))\gamma\| \\ &\leq K_n \eta(|a_n(\nu(1))g_\nu - a_n(\gamma(1))g_\gamma|) \\ &\leq K_n[|a_n(\nu(1)) - a_n(\gamma(1))|\eta(g_\nu) + a_n(\gamma(1))\eta(|g_\nu - g_\gamma|)] \\ &\leq K_n[|a_n(\nu(1)) - a_n(\gamma(1))|\nu(1) + \|\nu - \gamma\|] \\ &\leq 2K_n\|\nu - \gamma\|. \end{aligned}$$

That is, $q_n(\cdot, \cdot)$ satisfies the conditions of Lemma 4.3. □

The following theorem generalizes the result of [17, Corollary 5.5]:

Theorem 4.1 *Under the conditions (4.1) and (4.2), there is a unique solution $\{Y_t : t \geq 0\}$ of (4.3). Moreover, $\{Y_t : t \geq 0\}$ is a measure-valued diffusion and for each $f \in \mathcal{D}(A)$,*

$$M_t(f) = Y_t(f) - Y_0(f) - \int_0^t Y_s(Af - bf)ds - \int_0^t m(q(Y_s, \cdot)f)ds, \quad t \geq 0, \tag{4.4}$$

is a continuous martingale relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$ *with quadratic variation process*

$$\langle M(f)\rangle_t = \int_0^t Y_s(cf^2)ds, \quad t \geq 0. \tag{4.5}$$

Proof. The uniqueness of (4.3) holds by Lemma 4.2. For the proof of existence, we first construct an approximating sequence. For each integer $n \geq 1$ let $q_n(\cdot,\cdot)$ be defined as in Lemma 4.4. By Lemma 4.3 there is an unique continuous solution $\{Y_t^{(n)} : t \geq 0\}$ of (4.3) with $q(\cdot,\cdot)$ replaced by $q_n(\cdot,\cdot)$. Then, by Lemma 4.1, for $k \geq n$, we have a.s. $Y_{t\wedge\tau_n}^{(n)} = Y_{t\wedge\tau_n}^{(k)}$ for each $t \geq 0$, where

$$\tau_n = \inf\{t \geq 0 : Y_t^{(n)}(1) \geq n\} = \inf\{t \geq 0 : Y_t^{(k)}(1) \geq n\}.$$

Since $\{Y_{t\wedge\tau_n}^{(n)} : t \geq 0\}$ and $\{Y_{t\wedge\tau_n}^{(k)} : t \geq 0\}$ have continuous paths, they are indistinguishable. Using Theorem 3.2, condition (4.1) and noticing that $q_n(Y_t^{(n)},\cdot) = q(Y_t^{(n)},\cdot)$ for $s \in [0, t\wedge\tau_n]$ we get

$$\begin{aligned} \boldsymbol{E}\{Y_{t\wedge\tau_n}^{(n)}(1))\} &\leq e^{\|b\|t}\mu(1) + \int_0^t e^{\|b\|(t-s)}\boldsymbol{E}\{m(q(Y_{s\wedge\tau_n}^{(n)},\cdot))\}ds \\ &\leq e^{\|b\|t}(\mu(1) + Kt) + Ke^{\|b\|t}\int_0^t \boldsymbol{E}\{Y_{s\wedge\tau_n}^{(n)}(1)\}ds. \end{aligned}$$

By Gronwall's inequality there is a locally bounded function $C(\cdot)$ on $[0,\infty)$ independent of $n \geq 1$ such that

$$\boldsymbol{E}\{Y_{t\wedge\tau_n}^{(n)}(1)\} \leq C(t). \tag{4.6}$$

By the definition of τ_n we have $n\boldsymbol{P}\{0 < \tau_n < t\} \leq C(t)$, and so

$$\boldsymbol{P}\{\tau_n \leq t\} = \boldsymbol{P}(\tau_n = 0) + \boldsymbol{P}(0 < \tau_n < t) \leq 1_{[n,\infty)}(\mu(1)) + n^{-1}C(t),$$

which goes to zero as $n \to \infty$. But $\{\tau_n\}$ is an increasing sequence, so we conclude that a.s. $\tau_n \uparrow \infty$ as $n \to \infty$. Thus there is a continuous process $\{Y_t : t \geq 0\}$ such that a.s. $Y_t^{(n)} = Y_t$ for all $t \in [0,\tau_n]$. Clearly, $\{Y_t : t \geq 0\}$ satisfies (4.3) with probability one. By (4.6) and Fatou's lemma, $\boldsymbol{E}\{Y_t(1)\} \leq C(t)$. The martingale characterization (4.4) and (4.5) follows by Lemma 4.3. The strong Markov property can be proved as [17, Theorem 4.4]. □

Suppose that $c > 0$ is a constant, $\beta(\cdot)$ is a bounded Lipschitz function on $[0,\infty)$ and $\gamma(\cdot)$ is a non-negative locally Lipschitz function on $[0,\infty)$ satisfying the linear growth condition. The stochastic differential equation

$$dy(t) = \sqrt{cy(t)}dB(t) + \beta(y(t))y(t)dt + \gamma(y(t))dt, \quad t \geq 0, \tag{4.7}$$

defines diffusion process $\{y(t) : t \geq 0\}$, which may be called a continuous state branching diffusion with interactive growth and immigration. Setting

$$b = -\inf_z \beta(z) \text{ and } q(z) = \beta(z)z + bz + \gamma(z), \quad z \geq 0,$$

we can rewrite (4.7) as

$$dy(t) = \sqrt{cy(t)}dB(t) - by(t)dt + q(y(t))dt, \quad t \geq 0. \tag{4.8}$$

The last equation may be regarded as the special case of the martingale problem (4.4) and (4.5) with $E = \{a\}$ being a singleton. Thus equation (4.3) gives a decomposition of the paths of $\{y(t) : t \geq 0\}$ into excursions of the diffusion process $\{x(t) : t \geq 0\}$ defined by

$$dx(t) = \sqrt{cx(t)}dB(t) - bx(t)dt, \quad t \geq 0. \tag{4.9}$$

This generalizes a result of [15], who considered the case where $\beta(\cdot)$ and $\gamma(\cdot)$ are constants and hence the right hand side of (4.3) is independent of $\{y(t) : t \geq 0\}$. See also [9].

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