

Immigration Superprocesses with Dependent Spatial Motion and Non-critical Branching

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Abstract

A class of immigration superprocess with dependent spatial motion is constructed by a passage to the limit from a sequence of superprocesses with positive jumps. A non-critical branching is then obtained by using a Girsanov transform of Dawson's type, which also gives a state-dependent spatial drift.

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1 Introduction

Let $B(\mathbb{R})$ be the totality of all bounded Borel functions on \mathbb{R} and let $C(\mathbb{R})$ denote its subset comprising of continuous functions. Let $M(\mathbb{R})$ denote the space of finite Borel measures on \mathbb{R} endowed with the topology of weak convergence. We write $\langle f, \mu \rangle$ for $\int f d\mu$ and for a function F on $M(\mathbb{R})$ let

$$\frac{\delta F(\mu)}{\delta \mu(x)} = \lim_{r \rightarrow 0^+} \frac{1}{r} [F(\mu + r\delta_x) - F(\mu)], \quad x \in \mathbb{R},$$

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if the limit exists. Let $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ be defined in the same way with F replaced by $(\delta F/\delta\mu(y))$ on the right hand side. Suppose that h is a continuously differentiable function on \mathbb{R} such that both h and h' are square-integrable. Then the function

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R}, \quad (1.1)$$

is twice continuously differentiable with bounded derivatives ρ' and ρ'' . Suppose that $c \in C(\mathbb{R})$ is Lipschitz and $\sigma \in B(\mathbb{R})^+$. We may define an operator \mathcal{L} by

$$\begin{aligned} \mathcal{L}F(\mu) &= \frac{1}{2} \int_{\mathbb{R}} a(x) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) \\ &+ \frac{1}{2} \int_{\mathbb{R}^2} \rho(x-y) \frac{d^2}{dx dy} \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} \mu(dx)\mu(dy) \\ &+ \frac{1}{2} \int_{\mathbb{R}} \sigma(x) \frac{\delta^2 F(\mu)}{\delta\mu(x)^2} \mu(dx), \end{aligned} \quad (1.2)$$

which acts on a class of functions on $M(\mathbb{R})$ to be specified. A Markov process with generator \mathcal{L} was constructed in Dawson *et al* [2], generalizing the construction of Wang [9, 10]. The process generated by \mathcal{L} is naturally called an *superprocess with dependent spatial motion* (SDSM) with parameters (a, ρ, σ) , where $a(\cdot)$ represents the rate of the underlying motion, $\rho(\cdot)$ represents the interaction between the ‘‘particles’’ and $\sigma(\cdot)$ represents the branching density. We shall also call the process simply a (a, ρ, σ) -*superprocess*. We refer the reader to [2, 9, 10] for detailed descriptions of the model. Given $\lambda \in M(\mathbb{R})$, we may define another operator \mathcal{J} by

$$\mathcal{J}F(\mu) = \mathcal{L}F(\mu) + \int_{\mathbb{R}} \frac{\delta F(\mu)}{\delta\mu(x)} \lambda(dx). \quad (1.3)$$

A Markov process generated by \mathcal{J} can be called an *SDSM with immigration* with parameters $(a, \rho, \sigma, \lambda)$ or simply a $(a, \rho, \sigma, \lambda)$ -*superprocess*, where λ represents the immigration rate.

In this work, we give a construction of the $(a, \rho, \sigma, \lambda)$ -superprocess by a passage to the limit from a sequence of SDSM’s with positive jumps. From the $(a, \rho, \sigma, \lambda)$ -superprocess we shall use Girsanov transform of Dawson’s type to derive an $M(\mathbb{R})$ -valued diffusion process with generator

$$\begin{aligned} \mathcal{J}^b F(\mu) &= \mathcal{J}F(\mu) - \int_{\mathbb{R}} b(x) \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) \\ &- \int_{\mathbb{R}^2} \rho(x-y)b'(y) \frac{d}{dx} \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx)\mu(dy), \end{aligned} \quad (1.4)$$

where $b \in C^1(\mathbb{R})$. Note that the generator \mathcal{J}^b not only involves a non-critical branching given by the second term on the right hand side, it also involves a state-dependent drift in the spatial motion represented by the last term. This is different from the classical case where the Girsanov transform does not effect the spatial motion; see Dawson [1].

2 Function-valued dual processes

As in Dawson *et al* [2], we shall define a function-valued dual process and investigate its connection to the solution of the martingale problem for the immigration SDSM. For $\mu \in M(\mathbb{R})$

and a subset $\mathcal{D}(\mathcal{J})$ of the domain of \mathcal{J} , we say an $M(\mathbb{R})$ -valued càdlàg process $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem if

$$F(X_t) - F(X_0) - \int_0^t \mathcal{J}F(X_s) ds, \quad t \geq 0, \quad (2.1)$$

is a martingale for each $F \in \mathcal{D}(\mathcal{J})$. Let G^m denote the generator of the interacting particle system introduced in [2], and let $(P_t^m)_{t \geq 0}$ denote the transition semigroup generated by the operator G^m . Observe that, if $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ for $f \in C_{\partial}^2(\mathbb{R}^m)$, then

$$\mathcal{J}F_{m,f}(\mu) = F_{m,G^m f}(\mu) + \frac{1}{2} \sum_{i,j=1, i \neq j}^m F_{m-1, \Phi_{ij} f}(\mu) + \sum_{i=1}^m F_{m-1, \Psi_i f}(\mu), \quad (2.2)$$

with $\Phi_{ij} f \in C_{\partial}^2(\mathbb{R}^{m-1})$ defined by

$$\Phi_{ij} f(x_1, \dots, x_{m-1}) = \sigma(x_{m-1}) f(x_1, \dots, x_{m-1}, \dots, x_{m-1}, \dots, x_{m-2}), \quad (2.3)$$

where x_{m-1} is in the places of the i th and the j th variables of f on the right hand side, and $\Psi_i f \in C_{\partial}^2(\mathbb{R}^{m-1})$ defined by

$$\Psi_i f(x_1, \dots, x_{m-1}) = \int_{\mathbb{R}} f(x_1, \dots, x_{i-1}, x, x_i, \dots, x_{m-1}) \lambda(dx), \quad x_j \in \mathbb{R}, \quad (2.4)$$

where $x \in \mathbb{R}$ is the i th variable of f on the right hand side.

Let $\{M_t : t \geq 0\}$ be a nonnegative integer-valued càdlàg Markov process with transition intensities $\{q_{i,j}\}$ such that $q_{i,i-1} = -q_{i,i} = i(i+1)/2$ and $q_{i,j} = 0$ for all other pairs (i, j) . Let $\tau_0 = 0$ and $\tau_{M_0+1} = \infty$, and let $\{\tau_k : 1 \leq k \leq M_0\}$ be the sequence of jump times of $\{M_t : t \geq 0\}$. Let $\{\Gamma_k : 1 \leq k \leq M_0\}$ be a sequence of random operators which are conditionally independent given $\{M_t : t \geq 0\}$ and satisfy

$$\mathbf{P}\{\Gamma_k = \Phi_{i,j} | M(\tau_k^-) = l\} = \frac{1}{l(l+1)}, \quad 1 \leq i \neq j \leq l, \quad (2.5)$$

and

$$\mathbf{P}\{\Gamma_k = \Psi_i | M(\tau_k^-) = l\} = \frac{2}{l(l+1)}, \quad 1 \leq i \leq l. \quad (2.6)$$

Let \mathbf{B} denote the topological union of $\{B(\mathbb{R}^m) : m = 1, 2, \dots\}$ endowed with pointwise convergence on each $B(\mathbb{R}^m)$. Then

$$Y_t = P_{t-\tau_k}^{M_{\tau_k}} \Gamma_k P_{\tau_k-\tau_{k-1}}^{M_{\tau_{k-1}}} \Gamma_{k-1} \dots P_{\tau_2-\tau_1}^{M_{\tau_1}} \Gamma_1 P_{\tau_1}^{M_0} Y_0, \quad \tau_k \leq t < \tau_{k+1}, 0 \leq k \leq M_0, \quad (2.7)$$

defines a Markov process $\{Y_t : t \geq 0\}$ taking values from \mathbf{B} . Clearly, $\{(M_t, Y_t) : t \geq 0\}$ is also a Markov process. To simplify the presentation, we shall let $\mathbf{E}_{m,f}^{\sigma}$ denote the expectation given $M_0 = m$ and $Y_0 = f \in C(\mathbb{R}^m)$, just as we are working with a canonical realization of $\{(M_t, Y_t) : t \geq 0\}$.

Theorem 2.1 Let $\mathcal{D}(\mathcal{J})$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C_{\delta}^2(\mathbb{R}^m)$. Suppose that $\{X_t : t \geq 0\}$ is a continuous $M(\mathbb{R})$ -valued process and that $\mathbf{E}\{\langle 1, X_t \rangle^m\}$ is locally bounded in $t \geq 0$ for each $m \geq 1$. If $\{X_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem with $X_0 = \mu$, then

$$\mathbf{E}\langle f, X_t^m \rangle = \mathbf{E}_{m,f}^{\sigma} \left[\langle Y_t, \mu^{M_t} \rangle \exp \left\{ \frac{1}{2} \int_0^t M_s(M_s + 1) ds \right\} \right] \quad (2.8)$$

for any $t \geq 0$, $f \in B(\mathbb{R}^m)$ and integer $m \geq 1$. Consequently, the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem has at most one solution possessing locally bounded moments of all degrees.

Proof. The general equality follows by bounded pointwise approximation once it is proved for $f \in C_{\delta}^2(\mathbb{R}^m)$. Set $F_{\mu}(m, f) = F_{m,f}(\mu) = \langle f, \mu^m \rangle$. From the construction (2.7), it is not hard to see that $\{(M_t, Y_t) : t \geq 0\}$ has generator \mathcal{L}^* given by

$$\begin{aligned} \mathcal{L}^* F_{\mu}(m, f) &= F_{\mu}(m, G^m f) \\ &+ \frac{1}{2} \sum_{i,j=1, i \neq j}^m [F_{\mu}(m-1, \Phi_{ij} f) - F_{\mu}(m, f)] \\ &+ \sum_{i=1}^m [F_{\mu}(m-1, \Psi_i f) - F_{\mu}(m, f)]. \end{aligned}$$

In view of (2.2) we have

$$\mathcal{L} F_{m,f}(\mu) = \mathcal{L}^* F_{\mu}(m, f) + \frac{1}{2} m(m+1) F_{\mu}(m, f). \quad (2.9)$$

Guided by (2.9) one can prove (2.8) using similar calculations as in [2]. To show the last assertion of the theorem, we may first consider the special case $\sigma(x) \equiv \sigma_0$ for a constant σ_0 . In this case, (2.1) implies that $\{\langle 1, X_t \rangle : t \geq 0\}$ is a one-dimensional diffusion with generator $2^{-1} \sigma_0 x d^2/dx^2 + \langle 1, \lambda \rangle d/dx$. As in [5, pp.236-237] one sees that

$$\mathbf{E} \exp\{z \langle 1, X_t \rangle\} = [1 - \sigma_0 z t / 2]^{-2 \langle 1, \lambda \rangle / \sigma_0} \exp \left\{ \frac{\langle 1, \mu \rangle z}{1 - \sigma_0 z t / 2} \right\}, \quad t \geq 0, |z| < 2 / \sigma_0 t.$$

The remaining arguments are similar to those in the proof of Theorem 2.2 in [2]. \square

3 SDSM with discrete immigration

Suppose that $(P_t)_{t \geq 0}$ is a Feller transition semigroup on some metric space E which has a Hunt process realization ξ . Suppose that $K(x, dy)$ is a bounded kernel on E . We assume that $K(x, \cdot)$ depends on $x \in E$ continuously. Let $\beta(x) = K(x, E)$. Let $K_0(x, dy) = \beta(x)^{-1} K(x, dy)$ if $\beta(x) > 0$ and $K_0(x, dy) = \delta_x(dy)$ if $\beta(x) = 0$. By the concatenation argument described in Sharpe [7, p.82] it is not hard to construct a Markov process η with the following properties:

(3A) The process evolves in E according to the law given by the transition probabilities of ξ until the random time τ_1 with $\mathbf{P}\{\tau_1 > t\} = \exp\{-\int_0^t \beta(\eta_s) ds\}$.

(3B) At time τ_1 the particle jumps from $\eta_{\tau_1^-}$ to a new place in E according to the probability distribution $K(\eta_{\tau_1^-}, dy)$, and then moves randomly according to the transition probabilities of ξ again until the random time $\tau_1 + \tau_2$ with $\mathbf{P}\{\tau_2 > t\} = \exp\{-\int_{\tau_1}^{\tau_1+t} \beta(\eta_s) ds\}$; and so on.

Lemma 3.1 Suppose that ξ has generator $(A, \mathcal{D}(A))$, where $\mathcal{D}(A) \subset C(E)$. Then η has generator $(B, \mathcal{D}(B))$, where $\mathcal{D}(B) = \mathcal{D}(A)$ and

$$Bf(x) = Af(x) + \int_E [f(y) - f(x)]K(x, dy), \quad x \in E, f \in \mathcal{D}(B). \quad (3.1)$$

Moreover, the transition semigroup of η is Feller.

Proof. Let $(Q_t)_{t \geq 0}$ denote the transition semigroup of η . The properties (3A) and (3B) imply that

$$\begin{aligned} Q_t f(x) &= \mathbf{P}_x \left[f(\xi_t) \exp \left\{ - \int_0^t \beta(\xi_s) ds \right\} \right] \\ &\quad + \mathbf{P}_x \left[\int_0^t \exp \left\{ - \int_0^s \beta(\xi_u) du \right\} \beta(\xi_s) ds \int_E Q_{t-s} f(y) K_0(\xi_s, dy) \right] \\ &= \mathbf{P}_x \left[f(\xi_t) \exp \left\{ - \int_0^t \beta(\xi_s) ds \right\} \right] \\ &\quad + \mathbf{P}_x \left[\int_0^t \exp \left\{ - \int_0^s \beta(\xi_u) du \right\} K(\xi_s, Q_{t-s} f) ds \right] \\ &= \mathbf{P}_x \left[f(\xi_t) \exp \left\{ - \int_0^t \beta(\xi_s) ds \right\} \right] \\ &\quad + \int_0^t \mathbf{P}_x \left[\exp \left\{ - \int_0^s \beta(\xi_u) du \right\} K(\xi_s, Q_{t-s} f) \right] ds. \end{aligned} \quad (3.2)$$

This equation follows as we think about the behavior of the particle. It either moves according to ξ without jumping until time t , or it first jumps at some time $s \in (0, t]$. The first event happens with probability $\exp\{-\int_0^t \beta(\xi_s) ds\}$ and the second happens with probability $\exp\{-\int_0^t \beta(\xi_u) du\} \beta(\xi_s) ds$, giving the two terms of on the right hand side. For $f \in \mathcal{D}(A)$, we get from (3.2) that

$$\begin{aligned} Bf(x) &= \lim_{t \downarrow 0} t^{-1} \mathbf{P}_x \left[f(\xi_t) \exp \left\{ - \int_0^t \beta(\xi_s) ds \right\} - f(x) \right] \\ &\quad + \lim_{t \downarrow 0} t^{-1} \int_0^t \mathbf{P}_x \left[\exp \left\{ - \int_0^s \beta(\xi_u) du \right\} K(\xi_s, Q_{t-s} f) \right] ds \\ &= \lim_{t \downarrow 0} t^{-1} \mathbf{P}_x [f(\xi_t) - f(x)] + \lim_{t \downarrow 0} t^{-1} \mathbf{P}_x \left[f(\xi_t) \exp \left\{ - \int_0^t \beta(\xi_s) ds \right\} - f(\xi_t) \right] \\ &\quad + \lim_{t \downarrow 0} t^{-1} \int_0^t \mathbf{P}_x \left[\exp \left\{ - \int_0^s \beta(\xi_u) du \right\} K(\xi_s, Q_{t-s} f) \right] ds \\ &= Af(x) - \beta(x)f(x) + K(x, f) \\ &= Af(x) + \int_E [f(y) - f(x)]K(x, dy). \end{aligned}$$

Since $(A, \mathcal{D}(A))$ generates a Feller transition semigroup, so does $(B, \mathcal{D}(B))$; see e.g. [4, p.37]. \square

For a fixed non-trivial measure $\lambda \in M(\mathbb{R})$ we consider a random variable ζ in \mathbb{R} with distribution $\lambda(1)^{-1}\lambda$. For $\mu \in M(\mathbb{R})$, let $K(\mu, d\nu)$ denote the distribution of the random measure

$$X := \mu + \theta^{-1} \delta_\zeta.$$

Observe that

$$\int_{M(\mathbb{R})} [F(\nu) - F(\mu)] K(\mu, d\nu) = \lambda(1)^{-1} \int_{\mathbb{R}} [F(\mu + \theta^{-1}\delta_y) - F(\mu)] \lambda(dy). \quad (3.3)$$

For $\theta > 0$ we can define the generator \mathcal{J}_θ by

$$\mathcal{J}_\theta F(\mu) = \mathcal{L}F(\mu) + \theta \int_{\mathbb{R}} [F(\mu + \theta^{-1}\delta_x) - F(\mu)] \lambda(dx). \quad (3.4)$$

By the result in [2], \mathcal{L} generates a Feller semigroup on $M(\mathbb{R})$, then so does \mathcal{J}_θ by Lemma 3.1. We shall call the process generated by \mathcal{J}_θ a *SDSM with discrete immigration* with parameters $(a, \rho, \sigma, \lambda)$ and unit mass $1/\theta$. Intuitively, the immigrants come to \mathbb{R} by cliques with mass $1/\theta$ with time-space configuration given by a Poisson random measure with intensity $\theta ds \lambda(dx)$. A more general immigration model for superprocesses with independent spatial motions has been considered in Li [6].

4 SDSM with continuous immigration

In this section, we construct a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem by using an approximation by the SDSM with discrete immigration. Observe that, if

$$F_{f, \{\phi_i\}}(\mu) = f(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle), \quad \mu \in M(\mathbb{R}), \quad (4.1)$$

for $f \in C_0^2(\mathbb{R}^n)$ and $\phi_i \in C_\partial^2(\mathbb{R})$, then

$$\begin{aligned} \mathcal{J}F_{f, \{\phi_i\}}(\mu) &= \frac{1}{2} \sum_{i=1}^n f_i'(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \langle a\phi_i'', \mu \rangle \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n f_{ij}''(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \int_{\mathbb{R}^2} \rho(x-y) \phi_i'(x) \phi_j'(y) \mu(dx) \mu(dy) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n f_{ij}''(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \langle \sigma \phi_i \phi_j, \mu \rangle \\ &\quad + \sum_{i=1}^n f_i'(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \langle \phi_i, \lambda \rangle. \end{aligned} \quad (4.2)$$

Let $\{\theta_k\}$ be any non-negative sequence such that $\theta_k \rightarrow \infty$ as $k \rightarrow \infty$. For $k \geq 1$, let $\{X_t^{(k)} : t \geq 0\}$ be a càdlàg SDSM with discrete immigration with parameters (a, ρ, σ, m) , unit $1/\theta_k$ and initial state $X_0^{(k)} = \mu_k \in M_{\theta_k}(\mathbb{R})$. Let $M(\hat{\mathbb{R}})$ be the space of finite Borel measure on $\hat{\mathbb{R}}$ endowed with the weak convergence topology.

Lemma 4.1 *If the sequence $\{\langle 1, \mu_k \rangle\}$ is bounded, then $\{X_t^{(k)} : t \geq 0\}$ form a tight sequence in $D([0, \infty), M(\hat{\mathbb{R}}))$.*

Proof. Let $H(\nu) = \langle 1, \nu \rangle$. By (3.4), it is not hard to see that $\mathcal{J}_{\theta_k} H(\nu) = \langle 1, \lambda \rangle$. It follows that

$$\mathbf{E}_{\mu_k} \{\langle 1, X_t^{(k)} \rangle\} = \langle 1, \mu_k \rangle + \langle 1, \lambda \rangle t, \quad t \geq 0.$$

Then $\{\langle 1, X_t^{(k)} \rangle - \langle 1, \lambda \rangle t : t \geq 0\}$ is a martingale. By a martingale inequality, for $u > 0$ and $\eta > \langle 1, \lambda \rangle u$ we have

$$\begin{aligned} \mathbf{P} \left\{ \sup_{0 \leq t \leq u} \langle 1, X_t^{(k)} \rangle > 2\eta \right\} &\leq \mathbf{P} \left\{ \sup_{0 \leq t \leq u} |\langle 1, X_t^{(k)} \rangle - \langle 1, \lambda \rangle t| > \eta \right\} \\ &\leq 3\eta^{-1} \sup_{0 \leq t \leq u} \mathbf{E}_{\mu_k} \{ |\langle 1, X_t^{(k)} \rangle - \langle 1, \lambda \rangle t| \} \\ &\leq 3\eta^{-1} (\langle 1, \mu_k \rangle + 2\langle 1, \lambda \rangle u); \end{aligned}$$

see e.g. [3, p.66]. That is, $\{X_t^{(k)} : t \geq 0\}$ satisfies the compact containment condition of [4, p.142]. Let \mathcal{J}_k denote the generator of $\{X_t^{(k)} : t \geq 0\}$ and let $F_{f, \{\phi_i\}}$ be given by (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and with each $\phi_i \in C_\partial^2(\mathbb{R})$ bounded away from zero. Then

$$F_{f, \{\phi_i\}}(X_t^{(k)}) - F_{f, \{\phi_i\}}(X_0^{(k)}) - \int_0^t \mathcal{J}_k F_{f, \{\phi_i\}}(X_s^{(k)}) ds, \quad t \geq 0,$$

is a martingale and the desired tightness follows from the result of [4, p.145]. \square

Now suppose that all functions in $C_\partial(\mathbb{R})$ are extended to $\hat{\mathbb{R}}$ by continuity. If $\sigma \in C_\partial(\mathbb{R})^+$, the right hands of (4.1) and (4.2) may be regarded as functions on $M(\hat{\mathbb{R}})$. Let $\hat{\mathcal{J}}F(\mu)$ be defined by the right hand side of (4.2) as a function on $M(\hat{\mathbb{R}})$. Let $\mathcal{D}(\hat{\mathcal{J}})$ be the totality of all functions of the form (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and with each $\phi_i \in C_\partial^2(\mathbb{R})$ bounded away from zero. Suppose that $\mu_k \rightarrow \mu \in M(\hat{\mathbb{R}})$ as $k \rightarrow \infty$ and let \mathbf{Q}_μ be any limit point of the distributions of $\{X_t^{(k)} : t \geq 0\}$. As in the proof of Lemma 4.2 in [2], we may see that \mathbf{Q}_μ is supported by $C([0, \infty), M(\hat{\mathbb{R}}))$ and

$$F_{f, \{\phi_i\}}(w_t) - F_{f, \{\phi_i\}}(w_0) - \int_0^t \hat{\mathcal{J}}F_{f, \{\phi_i\}}(w_s) ds, \quad t \geq 0, \quad (4.3)$$

is a martingale for each $F_{f, \{\phi_i\}} \in \mathcal{D}(\hat{\mathcal{J}})$, where $\{w_t : t \geq 0\}$ denotes the coordinate process of $C([0, \infty), M(\hat{\mathbb{R}}))$.

Lemma 4.2 *Let \mathbf{Q}_μ be given as the above. Then for $n \geq 1$, $t \geq 0$ and $\mu \in M(\mathbb{R})$ we have*

$$\mathbf{Q}_\mu \{ \langle 1, w_t \rangle^n \} \leq \langle 1, \mu \rangle^n + n[(n-1)\|\sigma\|/2 + \langle 1, \lambda \rangle] \int_0^t \mathbf{Q}_\mu \{ \langle 1, w_s \rangle^{n-1} \} ds.$$

Consequently, $\mathbf{Q}_\mu \{ \langle 1, w_t \rangle^n \}$ is a locally bounded function of $t \geq 0$. Let $\mathcal{D}(\hat{\mathcal{J}})$ be the union of all functions of the form (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and $\phi_i \in C_\partial^2(\mathbb{R})$ and all functions of the form $F_{m, f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C_\partial^2(\mathbb{R}^m)$. Then (4.3) under \mathbf{Q}_μ is a martingale for each $F \in \mathcal{D}(\hat{\mathcal{J}})$.

Proof. For any $k \geq 1$, take $f_k \in C_0^2(\mathbb{R})$ such that $f_k(z) = z^n$ for $0 \leq z \leq k$ and $f_k''(z) \leq n(n-1)z^{n-2}$ for all $z \geq 0$. Let $F_k(\mu) = f_k(\langle 1, \mu \rangle)$. It is easy to see that

$$\hat{\mathcal{J}}F_k(\mu) \leq n[(n-1)\|\sigma\|/2 + \langle 1, \lambda \rangle] \langle 1, \mu \rangle^{n-1}.$$

Since

$$F_k(X_t) - F_k(X_0) - \int_0^t \hat{\mathcal{J}}F_k(\langle 1, X_s \rangle) ds, \quad t \geq 0,$$

is a martingale, we get

$$\begin{aligned} \mathbf{Q}_\mu f_k(\langle 1, X_t \rangle) &\leq f_k(\langle 1, \mu \rangle) + n[(n-1)\|\sigma\|/2 + \langle 1, \lambda \rangle] \int_0^t \mathbf{Q}_\mu(\langle 1, X_s \rangle^{n-1}) ds \\ &\leq \langle 1, \mu \rangle^n + n[(n-1)\|\sigma\|/2 + \langle 1, \lambda \rangle] \int_0^t \mathbf{Q}_\mu(\langle 1, X_s \rangle^{n-1}) ds. \end{aligned}$$

Then the desired estimate follows by Fatou's Lemma. The last assertion is immediate. \square

By the martingale problem (4.3) and the last lemma, it is easy to find that for each $\phi \in C_{\partial}^2(\mathbb{R})$,

$$M_t(\phi) := \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \langle \phi, m \rangle t - \frac{1}{2} \int_0^t \langle a\phi'', w_s \rangle ds, \quad t \geq 0, \quad (4.4)$$

is a \mathbf{Q}_μ -martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, w_s \rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z - \cdot)\phi', w_s \rangle^2 dz. \quad (4.5)$$

For a continuous branching density function $\sigma \in C_{\partial}(\mathbb{R})^+$, the existence of a SDSM with immigration is given by the following

Theorem 4.1 *Let $\mathcal{D}(\mathcal{J})$ be the union of all functions of the form (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and $\phi_i \in C^2(\mathbb{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C^2(\mathbb{R}^m)$. Let $\{w_t : t \geq 0\}$ denote the coordinate process of $C([0, \infty), M(\mathbb{R}))$. Then for each $\mu \in M(\mathbb{R})$ there is a unique probability measure \mathbf{Q}_μ on $C([0, \infty), M(\mathbb{R}))$ such that $\mathbf{Q}_\mu\{w_0 = \mu\} = 1$, the moments $\mathbf{Q}_\mu\{\langle 1, w_t \rangle^m\}$ are locally bounded and $\{w_t : t \geq 0\}$ under \mathbf{Q}_μ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem.*

Proof. Let \mathbf{Q}_μ be as in Lemma 4.2. By Theorem 2.1, the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem has at most one solution possessing locally bounded moments of all degrees. Then the desired result follows once it is proved that

$$\mathbf{Q}_\mu\{w_t(\{\partial\}) = 0 \text{ for all } t \in [0, u]\} = 1, \quad u > 0. \quad (4.6)$$

Let $M(ds, dx)$ denote the stochastic integral relative to the martingale measure defined by (4.4) and (4.5). As in [2], we have

$$\langle \phi, w_t \rangle = \langle \hat{P}_t \phi, \mu \rangle + \int_0^t \langle \hat{P}_{t-s} \phi, \lambda \rangle ds + \int_0^t \int_{\hat{\mathbb{R}}} \hat{P}_{t-s} \phi(x) M(ds, dx)$$

for $t \geq 0$ and $\phi \in C_{\partial}^2(\mathbb{R})$. For any fixed $u > 0$, we have that

$$\begin{aligned} M_t^u(\phi) &:= \langle \hat{P}_{u-t} \phi, w_t \rangle - \langle \hat{P}_u \phi, \mu \rangle - \int_0^t \lambda(\hat{P}_{u-s} \phi) ds \\ &= \int_0^t \int_{\hat{\mathbb{R}}} \hat{P}_{u-s} \phi M(ds, dx), \quad t \in [0, u], \end{aligned}$$

is a continuous martingale with quadratic variation process

$$\begin{aligned}\langle M^u(\phi) \rangle_t &= \int_0^t \langle \sigma(\hat{P}_{u-s}\phi)^2, w_s \rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z - \cdot) \hat{P}_{u-s}(\phi'), w_s \rangle^2 dz \\ &= \int_0^t \langle \sigma(\hat{P}_{u-s}\phi)^2, w_s \rangle ds + \int_0^t ds \int_{\hat{\mathbb{R}}} \langle h(z - \cdot) (\hat{P}_{u-s}\phi)', w_s \rangle^2 dz.\end{aligned}$$

By a martingale inequality we have

$$\begin{aligned}& \mathbf{Q}_\mu \left\{ \sup_{0 \leq t \leq u} \left| \langle \hat{P}_{u-t}\phi, w_t \rangle - \langle \hat{P}_u\phi, \mu \rangle - \int_0^t \lambda(\hat{P}_{u-s}\phi) ds \right|^2 \right\} \\ & \leq 4 \int_0^u \mathbf{Q}_\mu \{ \langle \sigma(\hat{P}_{u-s}\phi)^2, w_s \rangle \} ds + 4 \int_0^u ds \int_{\hat{\mathbb{R}}} \mathbf{Q}_\mu \{ \langle h(z - \cdot) \hat{P}_{u-s}(\phi'), w_s \rangle^2 \} dz \\ & \leq 4 \int_0^u \langle \sigma(\hat{P}_{u-s}\phi)^2, \mu \hat{P}_s \rangle ds + 4 \int_{\hat{\mathbb{R}}} h(z)^2 dz \int_0^u \mathbf{Q}_\mu \{ \langle 1, w_s \rangle \langle \hat{P}_{u-s}(\phi')^2, w_s \rangle \} ds \\ & \leq 4 \int_0^u \langle \sigma(\hat{P}_{u-s}\phi)^2, \mu \hat{P}_s \rangle ds + 4 \|\phi'\|^2 \int_{\hat{\mathbb{R}}} h(z)^2 dz \int_0^u \mathbf{Q}_\mu \{ \langle 1, w_s \rangle^2 \} ds.\end{aligned}$$

Choose a sequence $\{\phi_k\} \subset C_{\partial}^2(\mathbb{R})$ such that $\phi_k(\cdot) \rightarrow 1_{\{\partial\}}(\cdot)$ boundedly and $\|\phi_k'\| \rightarrow 0$ as $k \rightarrow \infty$. Replacing ϕ by ϕ_k in the above and letting $k \rightarrow \infty$ we obtain (4.6). \square

For a general $\sigma \in B(\mathbb{R})^+$, we may choose a bounded sequence of functions $\{\sigma_k\} \subset C_{\partial}(\mathbb{R})^+$ such that $\sigma_k \rightarrow \sigma$ pointwise out of a Lebesgue null set. Suppose that $\{\mu_k\} \subset M(\mathbb{R})$ and $\mu_k \rightarrow \mu \in M(\mathbb{R})$ as $k \rightarrow \infty$. For each $k \geq 1$, let $\{X_t^{(k)} : t \geq 0\}$ be an immigration SDSM with parameters (a, ρ, σ_k, m) and initial state $\mu_k \in M(\mathbb{R})$ and let \mathbf{Q}_k denote the distribution of $\{X_t^{(k)} : t \geq 0\}$ on $C([0, \infty), M(\mathbb{R}))$. By the arguments in the proofs of Theorems 5.1 and 5.2 in [2] we get

Theorem 4.2 *As $k \rightarrow \infty$, the sequence \mathbf{Q}_k converges to a probability \mathbf{Q}_μ on $C([0, \infty), M(\mathbb{R}))$. Let $\mathcal{D}(\mathcal{J})$ be as in Theorem 4.1 for the more general $\sigma \in B(\mathbb{R})^+$. Then \mathbf{Q}_μ is the unique probability measure on $C([0, \infty), M(\mathbb{R}))$ such that $\mathbf{Q}_\mu\{w_0 = \mu\} = 1$ and $\{w_t : t \geq 0\}$ under \mathbf{Q}_μ solves the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem. Consequently, $\{w_t : t \geq 0\}$ under \mathbf{Q}_μ is a diffusion process with transition semigroup $(Q_t)_{t \geq 0}$ defined by*

$$\int_{M(\mathbb{R})} \langle f, \nu^m \rangle Q_t(\mu, d\nu) = \mathbf{E}_{m,f}^\sigma \left[\langle Y_t, \mu^{M_t} \rangle \exp \left\{ \frac{1}{2} \int_0^t M_s(M_s + 1) ds \right\} \right]. \quad (4.7)$$

This gives the existence of the SDSM with continuous immigration for a bounded measurable branching density $\sigma \in B(\mathbb{R})^+$. Clearly, we have that for each $\phi \in C^2(\mathbb{R})$,

$$M_t(\phi) := \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \langle \phi, m \rangle t - \frac{1}{2} \int_0^t \langle a\phi'', w_s \rangle ds, \quad t \geq 0, \quad (4.8)$$

is a \mathbf{Q}_μ -martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)\phi', w_s \rangle^2 dz. \quad (4.9)$$

Conversely, if \mathbf{Q}_μ is a probability measure on $C([0, \infty), M(\mathbb{R}))$ such that (4.8) is a martingale with quadratic variation process (4.9), by Itô's formula one can show that \mathbf{Q}_μ is the solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem, which is unique by Theorem 2.1. Then (4.8) and (4.9) give an alternate characterization of the immigration SDSM.

5 Non-critical branching mechanism

Let \mathbf{Q}_μ denote the distribution on $C([0, \infty), M(\mathbb{R}))$ of an $(a, \rho, \sigma, \lambda)$ -superprocess with initial state $\mu \in M(\mathbb{R})$. Let $M(ds, dx)$ denote the martingale measure defined by (4.8) and (4.9). Then for any $b \in C^1(\mathbb{R})$ the stochastic integral

$$M_t(b) := \int_0^t b(x)M(ds, dx), \quad t \geq 0, \quad (5.1)$$

is well-defined and

$$\langle M(b) \rangle_t = \int_0^t \langle \sigma b^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)b', w_s \rangle^2 dz. \quad (5.2)$$

We consider the exponential martingale

$$Z_t(b) := \exp \left\{ -M_t(b) - \frac{1}{2} \langle M(b) \rangle_t \right\}, \quad t \geq 0. \quad (5.3)$$

Fix a constant $T > 0$ and let $\mathbf{Q}_\mu^b(dw) = Z_T(w, b)\mathbf{Q}_\mu(dw)$. By Girsanov's theorem,

$$\begin{aligned} N_t(\phi) &:= \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \langle \phi, m \rangle t - \frac{1}{2} \int_0^t \langle a\phi'', w_s \rangle ds - \int_0^t \langle \sigma b\phi, w_s \rangle ds \\ &\quad + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)b', w_s \rangle \langle h(z - \cdot)\phi', w_s \rangle dz, \quad 0 \leq t \leq T, \end{aligned} \quad (5.4)$$

is a \mathbf{Q}_μ^b -martingale with quadratic variation process

$$\langle N(\phi) \rangle_t = \int_0^t \langle \sigma \phi^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)\phi', w_s \rangle^2 dz, \quad 0 \leq t \leq T. \quad (5.5)$$

As usual, the coordinate process $\{w_t : 0 \leq t \leq T\}$ under \mathbf{Q}_μ^b is a diffusion process; see e.g. [5, pp.190-197]. We call the new process a $(a, \rho, \sigma, b, \lambda)$ -superprocess. Intuitively, the term $\int_0^t \langle \sigma b\phi, w_s \rangle ds$ in (5.4) represents a linear growth with growth rate $\sigma(\cdot)b(\cdot)$. Girsanov transformations of this type were introduced by Dawson [1] to get non-critical superprocesses for critical ones. Note that we have on the right hand side of (5.5) the term

$$\int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)b', w_s \rangle \langle h(z - \cdot)\phi', w_s \rangle dz, \quad (5.6)$$

which may be interpreted as a spatial drift with state-dependent coefficient

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h(z - y)b'(y)h(z - \cdot)w_s(dy)dz = \int_{\mathbb{R}} b'(y)\rho(y - \cdot)w_s(dy). \quad (5.7)$$

This is different from the classical case where the Girsanov transform does not effect the spatial motion; see [1]. Let $\mathcal{D}(\mathcal{J}^b)$ be the union of all functions of the form (4.1) with $f \in C_0^2(\mathbb{R}^n)$ and $\phi_i \in C^2(\mathbb{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C^2(\mathbb{R}^m)$.

Theorem 5.1 *The $(a, \rho, \sigma, b, \lambda)$ -superprocess solves the $(\mathcal{J}^b, \mathcal{D}(\mathcal{J}^b))$ -martingale problem.*

Proof. If $F_{f,\{\phi_i\}}$ is given by (4.1), we have

$$\begin{aligned}
\mathcal{J}^b F_{f,\{\phi_i\}}(\mu) &= \frac{1}{2} \sum_{i=1}^n f'_i(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \langle a\phi''_i - 2b\phi, \mu \rangle \\
&+ \frac{1}{2} \sum_{i,j=1}^n f''_{ij}(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \int_{\mathbb{R}^2} \rho(x-y) \phi'_i(x) \phi'_j(y) \mu(dx) \mu(dy) \\
&+ \frac{1}{2} \sum_{i,j=1}^n f''_{ij}(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \langle \sigma \phi_i \phi_j, \mu \rangle \\
&- \sum_{i=1}^n f'_i(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \int_{\mathbb{R}^2} \rho(x-y) b'(y) \phi'_i(x) \mu(dx) \mu(dy) \\
&+ \sum_{i=1}^n f'_i(\langle \phi_1, \mu \rangle, \dots, \langle \phi_n, \mu \rangle) \langle \phi_i, \lambda \rangle.
\end{aligned} \tag{5.8}$$

Based on (5.4) and (5.5), it is easy to check by Itô's formula that

$$F_{f,\{\phi_i\}}(w_t) - F_{f,\{\phi_i\}}(w_0) - \int_0^t \mathcal{J}^b F_{f,\{\phi_i\}}(w_s) ds, \quad 0 \geq t \leq T, \tag{5.9}$$

is a martingale under \mathbf{Q}_μ^b . Then the theorem follows by an approximation of an arbitrary $F \in \mathcal{D}(\mathcal{L})$. \square

If $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ for $f \in C_\partial^2(\mathbb{R}^m)$, then

$$\begin{aligned}
\mathcal{J} F_{m,f}(\mu) &= F_{m,G_b^m} f(\mu) + \frac{1}{2} \sum_{i,j=1, i \neq j}^m F_{m-1, \Phi_{ij}} f(\mu) \\
&+ \sum_{i=1}^m F_{m-1, \Psi_i} f(\mu) + \sum_{i=1}^m F_{m+1, \Gamma_i} f(\mu),
\end{aligned} \tag{5.10}$$

where

$$G_b^m f(x_1, \dots, x_m) = G^m f(x_1, \dots, x_m) - \sum_{i=1}^m b(x_i) f(x_1, \dots, x_m), \tag{5.11}$$

and

$$\Gamma_i f(x_1, \dots, x_m, x_{m+1}) = -\rho(x_{m+1} - x_i) b'(x_{m+1}) f'_i(x_1, \dots, x_m). \tag{5.12}$$

In view of this expression of the generator, one may construct a dual process which gives expressions for the moments of the $(a, \rho, \sigma, b, \lambda)$ -superprocess.

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