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Generalized Mehler Semigroups and Catalytic Branching Processes with Immigration[1]

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Abstract. Skew convolution semigroups play an important role in the study of generalized Mehler semigroups and Ornstein-Uhlenbeck processes. We give a characterization for a general skew convolution semigroup on real separable Hilbert space whose characteristic functional is not necessarily differentiable at the initial time. A connection between this subject and catalytic branching superprocesses is established through fluctuation limits, providing a rich class of non-differentiable skew convolution semigroups. Path regularity of the corresponding generalized Ornstein-Uhlenbeck processes in different topologies is also discussed.

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1 Introduction

Suppose that H is a real separable Hilbert space. Given a Borel probability measure ν on H, let $\hat{\nu}$ denote its characteristic functional. It is known that if ν is infinitely divisible, then $\hat{\nu}(a) \neq 0$ for all $a \in H$ and there is a unique continuous function $\log \hat{\nu}$ on H such that $\log \hat{\nu}(0) = 0$ and $\hat{\nu}(a) = \exp\{\log \hat{\nu}(a)\}$; see e.g. Linde [22, p.20 and p.58]. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup of linear operators on H with dual $(T_t^*)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$ a family of probability measures on H. The family $(\mu_t)_{t\geq 0}$ is called a *skew convolution semigroup* (SC-semigroup) associated with $(T_t)_{t\geq 0}$ if the following equation is satisfied:

$$\mu_{r+t} = (T_t\mu_r) * \mu_t, \qquad r, t \geq 0, \tag{1.1}$$

where "$*$" denotes the convolution operation. It is easy to check that (1.1) holds if and only if we can define a Markov transition semigroup $(Q_t^\mu)_{t\geq 0}$ on H by

$$Q_t^\mu f(x) := \int_H f(T_t x + y)\mu_t(dy), \quad x \in H, f \in B(H), \tag{1.2}$$

where $B(H)$ denotes the totality of bounded Borel measurable functions on H. In this case, $(Q_t^\mu)_{t\geq 0}$ is called a *generalized Mehler semigroup*, which corresponds to a generalized Ornstein-Uhlenbeck process (OU-process) with state space H. This formulation of OU-processes was given by Bogachev *et al* [3] as a generalization of the classical Mehler formula; see e.g. Malliavin [23, p.17 and p.25]. One motivation to study such OU-processes is that they constitute a large class of explicit examples of processes on infinite-dimensional spaces with rich mathematical structures. They arise in the study of Langevin type equations with generalized drift involving the generator of $(T_t)_{t\geq 0}$. We refer the reader to Bogachev *et al* [3], Fuhrman and Röckner [12], and van Neerven [25] for discussions from a theoretical viewpoint. See also Bogachev and Röckner [2], Fuhrman [11], and van Neerven [24] for some earlier related work. In the setting of cylindrical probability measures, Bogachev *et al* [3, Lemma 2.6] proved that, if the function $t \mapsto \hat{\mu}_t(a)$ is absolutely continuous on $[0, \infty)$ and differentiable at $t = 0$ for all $a \in H$, then (1.1) is equivalent to

$$\hat{\mu}_t(a) = \exp\bigg\{ -\int_0^t \lambda(T_s^* a)ds \bigg\}, \quad t \geq 0, a \in H, \tag{1.3}$$

where $\lambda(a) = -(d/dt)\hat{\mu}_t(a)|_{t=0}$ is a negative-definite functional on H. A necessary and sufficient condition for a Gaussian SC-semigroup to be differentiable was given in van Neerven [25]. These results give characterizations for interesting special classes of SC-semigroups defined by (1.1) and have stimulated the present work.

Skew convolution semigroups have also played an important role in the study of immigration structures associated with branching processes. Let E be a Lusin topological space, i.e., a homeomorph of a Borel subset of a compact metric space, with Borel σ-algebra $\mathscr{B}(E)$. We denote by $B(E)^+$ the set of bounded non-negative Borel functions on E. Let $M(E)$ be the totality of finite measures on $(E, \mathscr{B}(E))$ endowed with the topology of weak convergence and $(Q_t)_{t\geq 0}$ the transition semigroup of a *measure-valued branching process* (superprocess) X with state space $M(E)$. A family $(N_t)_{t\geq 0}$ of probability measures on $M(E)$ is called a *SC-semigroup* associated with $(Q_t)_{t\geq 0}$ if it satisfies

$$N_{r+t} = (N_r Q_t) * N_t, \qquad r, t \geq 0. \tag{1.4}$$

We use the same terminology for solutions of (1.1) and those of (1.4) since (1.1) is actually a special form of (1.4) when they are put in a slightly more general setting, say, when H and $M(E)$ are replaced by a topological semigroup. This similarity between the two equations was first noticed by L.G. Gorostiza (1999, personal communication); see also Bojdecki and Gorostiza [1] and Schmuland and Sun [27]. It is not hard to show that (1.4) holds if and only if

$$Q_t^N(\nu,\cdot) := Q_t(\nu,\cdot) * N_t, \qquad t \geq 0, \nu \in M(E) \tag{1.5}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. A Markov process Y in $M(E)$ is called an *immigration process* associated with X if it has transition semigroup $(Q_t^N)_{t\geq 0}$. The intuitive meaning of the immigration process is clear from (1.5), that is, $Q_t(\nu,\cdot)$ is the distribution of descendants of the people distributed as $\nu \in M(E)$ at time zero and N_t is the distribution of descendants of the people immigrating to E during the time interval $(0,t]$. By Li [17, Theorem 2] or [21, Theorem 3.2], the family $(N_t)_{t\geq 0}$ satisfies (1.4) if and only if there is an infinitely divisible probability entrance law $(K_s)_{s>0}$ for $(Q_t)_{t\geq 0}$ such that

$$\log \int_{M(E)} e^{-\nu(f)} N_t(d\nu) = \int_0^t \left[\log \int_{M(E)} e^{-\nu(f)} K_s(d\nu)\right] ds, \quad t \geq 0, f \in B(E)^+, \tag{1.6}$$

where $\nu(f) = \int_E f d\nu$; see also Li [19, 21] for some generalizations of this result. Then there is a 1-1 correspondence between SC-semigroups and a set of infinitely divisible probability entrance laws. Some representations of the infinitely divisible probability entrance laws and path regularity of the corresponding immigration processes were studied in Li [18]. The connection between immigration processes and generalized OU-processes was studied in Gorostiza and Li [14, 15] and Li [20]. In view of (1.6), the function

$$t \mapsto \log \int_{M(E)} e^{-\nu(f)} N_t(d\nu) \tag{1.7}$$

is always absolutely continuous on $[0,\infty)$, and it is differentiable at $t=0$ for all continuous $f \in B(E)^+$ if and nearly only if $(K_s)_{s>0}$ is closable by an infinitely divisible probability measure K_0 on $M(E)$. By the similarity of (1.1) and (1.4), one might expect similar results for the solutions of (1.1). However, the Hilbert space situation is much more complicated as Schmuland and Sun [27] showed that the linear part of $t \mapsto \log \hat{\mu}_t(a)$ can be discontinuous. Therefore, we can only discuss characterizations for the solutions of (1.1) under reasonable regularity conditions on the linear part of $t \mapsto \log \hat{\mu}_t(a)$.

This work is also related to the catalytic branching superprocess introduced by Dawson and Fleischmann [5, 6]. Let us consider the special case where the underlying motion is an absorbing barrier Brownian motion (ABM) in a domain D. Let $(P_t)_{t\geq 0}$ denote the transition semigroup of the ABM. Let $\eta \in M(D)$ and let $\phi(\cdot,\cdot)$ be a function on $D\times[0,\infty)$ of a certain form to be specified. A *catalytic branching superprocess* in $M(D)$ has transition semigroup $(Q_t)_{t\geq 0}$ determined by

$$\int_{M(D)} e^{-\nu(f)} Q_t(\mu, d\nu) = \exp\left\{-\mu(V_t f)\right\}, \qquad f \in B(D)^+, \tag{1.8}$$

where $(V_t)_{t\geq 0}$ is a semigroup of non-linear operators on $B(D)^+$ defined by

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_D \phi(y, V_s f(y)) p_{t-s}(x,y)\eta(dy), \quad t \geq 0, x \in D, \tag{1.9}$$

with $p_t(x, y)$ being the density of $P_t(x, dy)$. This process describes the catalytic reaction of a large number of infinitesimal particles moving in D according to the transition law of the ABM and splitting according to the branching mechanism given by $\phi(\cdot, \cdot)$. The measure $\eta(dx)$ represents the distribution of a catalyst in D which causes the splitting. More detailed descriptions of the model will been given in Section 3.

In this paper, we give a representation for the general SC-semigroup $(\mu_t)_{t\geq 0}$ defined by (1.1) whose characteristic functional is not necessarily differentiable at $t = 0$. This result extends the interesting characterizations given in [3] and [25]. The general representation is of interest since it includes some SC-semigroups arising in applications which are not included in (1.3). We provide a rich class of SC-semigroups of this type in the case where $H = L^2(0, \infty)$ and $(T_t)_{t\geq 0}$ is the transition semigroup of the ABM. Indeed, the corresponding generalized OU-processes arise naturally as fluctuation limits of catalytic branching superprocesses with immigration. An important feature of these OU-processes is that they usually do not have right continuous realizations, which is similar to the situation of immigration processes studied in [18, 19, 21]. Nevertheless, we show that some of these OU-processes are in fact quite regular if we regard them as processes with values of signed-measures. The study of generalized Mehler semigroups on Hilbert spaces and that of catalytic branching processes have evolved independently of each other with different motivations, techniques, and so on. The fluctuation limits establish a connection between the two subjects.

The remainder of this paper is organized as follows. In Section 2 we give the characterization for general SC-semigroups. Fluctuation limits of immigration processes are studied in Section 3, which lead to generalized OU-processes with distribution values. Under stronger assumptions, it is proved in Section 4 that some of these OU-processes actually live in the Hilbert space $L^2(D)$ of functions. Regularity properties of the processes in the space of signed-measures are discussed in Section 5.

2 Characterization of SC-semigroups

In this section, we give a general representation of the SC-semigroups defined by (1.1). It was proved in Schmuland and Sun [27] that, if $(\mu_t)_{t\geq 0}$ is a solution of (1.1), then each μ_t is an infinitely divisible probability measure. Let

$$K(x, a) := e^{i\langle x,a\rangle} - 1 - i\langle x, a\rangle\chi_{[0,1]}(\|x\|), \qquad x, a \in H.$$

By Linde [22, p.75 and p.84], the characteristic functional of μ_t on H is given by

$$\hat{\mu}_t(a) = \exp\left\{i\langle b_t, a\rangle - \frac{1}{2}\langle R_t a, a\rangle + \int_{H^\circ} K(x, a)M_t(dx)\right\}, \qquad a \in H, \tag{2.1}$$

where $b_t \in H$, R_t is a symmetric, positive-definite, nuclear operator on H, and M_t is a σ-finite measure (Lévy measure) on $H^\circ := H \setminus \{0\}$ satisfying

$$\int_{H^\circ} (1 \wedge \|x\|^2)M_t(dx) < \infty. \tag{2.2}$$

Thus, μ_t is uniquely determined by the triple (b_t, R_t, M_t) and is uniquely decomposed into the convolution of three infinitely divisible probabilities $\mu_t = \mu_t^c * \mu_t^g * \mu_t^j$ with

$$\hat{\mu}_t^c(a) = \exp\{i\langle b_t, a\rangle\}, \qquad \hat{\mu}_t^g(a) = \exp\left\{ - \frac{1}{2}\langle R_t a, a\rangle\right\}, \tag{2.3}$$

and

$$\hat{\mu}_t^j(a) = \exp\left\{ \int_{H^\circ} K(x,a) M_t(dx) \right\} \tag{2.4}$$

for $a \in H$. We call μ_t^c the *constant* (or *linear*) *part*, μ_t^g the *Gaussian part*, and μ_t^j the *jump part* of μ_t. By the uniqueness of the decomposition (2.1) it is not hard to show that (1.1) holds if and only if we have

$$R_{r+t} = T_t R_r T_t^* + R_t, \qquad M_{r+t} = (T_t M_r)|_{H^\circ} + M_t, \tag{2.5}$$

and

$$b_{r+t} = b_t + T_t b_r + \int_{H^\circ} \left(\chi_{[0,1]}(\|T_t x\|) - \chi_{[0,1]}(\|x\|)\right) T_t x \, M_r(dx) \tag{2.6}$$

for all $r, t \geq 0$.

Theorem 2.1 *If $(\mu_t)_{t\geq 0}$ is an SC-semigroup with decomposition (2.1), then we can write*

$$\langle R_t a, a\rangle = \int_0^t \langle U_s a, a\rangle ds, \qquad t \geq 0, a \in H, \tag{2.7}$$

where $(U_s)_{s>0}$ is a family of nuclear operators on H satisfying $U_{s+t} = T_t U_s T_t^*$ for all $s, t > 0$ and

$$\int_0^t \operatorname{Tr} U_s ds < \infty, \qquad t \geq 0.$$

The basic idea of the proof of this theorem is similar to that of [17, Theorem 2], but the argument in the present case is more involved. We first prove two lemmas.

Lemma 2.1 *Under the conditions of Theorem 2.1, the function $t \mapsto \langle R_t a, b\rangle$ is absolutely continuous in $t \geq 0$ for all $a, b \in H$.*

Proof. If $(\mu_t)_{t\geq 0}$ is an SC-semigroup, so is $(\mu_t^g)_{t\geq 0}$ by the first equation in (2.5). Then we have

$$\int_H \|x\|^2 \mu_{r+t}^g(dx) = \int_H \|T_t x\|^2 \mu_r^g(dx) + \int_H \|x\|^2 \mu_t^g(dx), \qquad r, t \geq 0. \tag{2.8}$$

It follows that

$$g(t) := \int_H \|x\|^2 \mu_t^g(dx), \qquad t \geq 0 \tag{2.9}$$

is a non-decreasing function. Since $(T_t)_{t\geq 0}$ is strongly continuous, there are constants $c \geq 1$ and $b \geq 0$ such that $\|T_t\| \leq ce^{bt}$. We claim that, for $0 < r_1 < t_1 < \cdots < r_n < t_n \leq l$,

$$\sum_{j=1}^n [g(t_j) - g(r_j)] \leq c^2 e^{2bl} g(\sigma_n), \tag{2.10}$$

where $\sigma_n = \sum_{j=1}^n (t_j - r_j)$. When $n = 1$, this follows from (2.8). Now assume that (2.10) holds for $n-1$. Applying (2.8) twice,

$$\begin{aligned}
\sum_{j=1}^n [g(t_j) - g(r_j)] &\leq [g(t_n) - g(r_n)] + c^2 e^{2bl} \int_H \|x\|^2 \mu^g_{\sigma_{n-1}}(dx) \\
&= \int_H \|T_{r_n} x\|^2 \mu^g_{t_n - r_n}(dx) + c^2 e^{2bl} \int_H \|x\|^2 \mu^g_{\sigma_{n-1}}(dx) \\
&\leq c^2 e^{2bl} \int_H \|T_{\sigma_{n-1}} x\|^2 \mu^g_{t_n - r_n}(dx) + c^2 e^{2bl} \int_H \|x\|^2 \mu^g_{\sigma_{n-1}}(dx) \\
&= c^2 e^{2bl} \int_H \|x\|^2 \mu^g_{\sigma_n}(dx),
\end{aligned}$$

which gives (2.10). Letting $r \to 0$ and $t \to 0$ in (2.8) and using the fact that g is a non-decreasing function one sees that $g(t) \to 0$ as $t \to 0$. By this and (2.10), g is absolutely continuous in $t \geq 0$. From (2.5) we see that $\langle R_t a, a \rangle$ is a non-decreasing function of $t \geq 0$ for any $a \in H$. For $t \geq r \geq 0$, (2.5) yields

$$\begin{aligned}
\langle R_t a, a \rangle - \langle R_r a, a \rangle &= \langle R_{t-r} T_r^* a, T_r^* a \rangle = \int_H \langle x, T_r^* a \rangle^2 \mu^g_{t-r}(dx) \\
&\leq \|a\|^2 \int_H \|T_r x\|^2 \mu^g_{t-r}(dx) = \|a\|^2 [g(t) - g(r)].
\end{aligned}$$

Then $\langle R_t a, a \rangle$ is absolutely continuous in $t \geq 0$. Polarization shows that $\langle R_t a, b \rangle$ is absolutely continuous in $t \geq 0$ for all $a, b \in H$. $\square$

Lemma 2.2 *Under the condition of Theorem 2.1, there is a family of nuclear operators $(U_s)_{s>0}$ on H such that (2.7) holds.*

Proof. Let $\{e_n : n = 1, 2, \ldots\}$ be an orthonormal basis of H. By Lemma 2.1, there are locally integrable functions $A_{m,n}$ on $[0, \infty)$ such that

$$\langle R_t e_m, e_n \rangle = \int_0^t A_{m,n}(s) ds, \qquad t \geq 0,\ m, n \geq 1. \tag{2.11}$$

From the symmetry of R_t we get

$$\int_0^t A_{m,n}(s) ds = \int_0^t A_{n,m}(s) ds, \tag{2.12}$$

while the positivity of R_t gives

$$\langle R_t a, a \rangle = \int_0^t \sum_{m,n=1}^\infty A_{m,n}(s) \langle a, e_m \rangle \langle a, e_n \rangle ds \geq 0 \tag{2.13}$$

for $a \in \mathrm{span}\{e_1, e_2, \ldots\}$. (The sum is actually finite!) In addition, since R_t is nuclear we have

$$\int_0^t \Big(\sum_{n=1}^\infty A_{n,n}(s) \Big) ds = \sum_{n=1}^\infty \langle R_t e_n, e_n \rangle = \mathrm{Tr}(R_t) < \infty. \tag{2.14}$$

Let F be the Borel subset of $[0, \infty)$ consisting of all $s \geq 0$ such that $A_{m,n}(s) = A_{n,m}(s)$ for $m, n \geq 1$ and

$$\sum_{m,n=1}^{\infty} A_{m,n}(s)\langle a, e_m\rangle\langle a, e_n\rangle \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} A_{n,n}(s) < \infty \tag{2.15}$$

for $a \in \operatorname{span}\{e_1, e_2, \ldots\}$ with rational coefficients. As observed in the proof of Lemma 2.1, $\langle R_t a, a\rangle$ is a non-decreasing function of $t \geq 0$. By (2.12), (2.13) and (2.14), F has full Lebesgue measure. For any $s \in F$,

$$U_s a = \sum_{m,n=1}^{\infty} A_{m,n}(s)\langle a, e_m\rangle e_n, \tag{2.16}$$

defines a positive-definite, symmetric linear operator on $\operatorname{span}\{e_1, e_2, \ldots\}$. Taking $b = xe_m + ye_n$, with x, y rational, we get

$$\langle U_s b, b\rangle = (x \;\; y)\begin{pmatrix} A_{m,m}(s) & A_{m,n}(s) \\ A_{n,m}(s) & A_{n,n}(s) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \geq 0,$$

so that the 2×2 matrix above is non-negative definite. Therefore, its determinant is non-negative, that is,

$$A_{m,n}(s)^2 \leq A_{m,m}(s)A_{n,n}(s). \tag{2.17}$$

Combined with the Cauchy-Schwarz inequality this gives,

$$\begin{aligned} \|U_s a\|^2 &= \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} A_{m,n}(s)\langle a, e_m\rangle\Big)^2 \\ &\leq \sum_{n=1}^{\infty} A_{n,n}(s)\Big(\sum_{m=1}^{\infty} A_{m,m}(s)^{1/2}|\langle a, e_m\rangle|\Big)^2 \\ &\leq \Big(\sum_{n=1}^{\infty} A_{n,n}(s)\Big)^2\|a\|^2 \end{aligned}$$

for $s \in F$ and $a \in \operatorname{span}\{e_1, e_2, \ldots\}$. This shows that U_s is a bounded operator and can be extended to the entire space H. In fact, U_s is a nuclear operator since

$$\operatorname{Tr}(U_s) = \sum_{n=1}^{\infty}\langle U_s e_n, e_n\rangle = \sum_{n=1}^{\infty} A_{n,n}(s) < \infty.$$

By (2.11) and (2.16), for $a \in \operatorname{span}\{e_1, e_2, \ldots\}$ we have

$$\langle R_t a, a\rangle = \sum_{m,n=1}^{\infty}\langle a, e_m\rangle\langle a, e_n\rangle\langle R_t e_m, e_n\rangle = \int_0^t \langle U_s a, a\rangle ds, \qquad t \geq 0. \tag{2.18}$$

Since $s \mapsto \operatorname{Tr}(U_s)$ is locally integrable, by dominated convergence we see that (2.18) holds for all $a \in H$. For $s \notin F$, we let U_s be the zero operator. $\square$

Proof of Theorem 2.1. Let $(U_s)_{s>0}$ be provided by Lemma 2.2. Note that (2.7) and the first equation of (2.5) imply

$$\int_0^r \langle U_{s+t}a, a\rangle ds = \int_0^r \langle U_s T_t^* a, T_t^* a\rangle ds, \qquad r, t \geq 0, a \in H.$$

Since H is separable, by Fubini's theorem, there are subsets G and G_s of $[0, \infty)$ with full Lebesgue measure such that

$$U_{s+t} = T_t U_s T_t^*, \qquad s \in G, t \in G_s.$$

Choose a decreasing sequence $s_n \in G$ with $s_n \to 0$, and define

$$\widetilde{U}_t := T_{t-s_n} U_{s_n} T_{t-s_n}^*, \qquad t > s_n.$$

Under this modification, $(\widetilde{U}_t)_{t>0}$ satisfies $\widetilde{U}_{r+t} = T_t \widetilde{U}_r T_t^*$ for all $r, t > 0$, while (2.7) remains unchanged. $\square$

Theorem 2.2 *If $(\mu_t)_{t\geq 0}$ is an SC-semigroup with decomposition (2.1), then we can write*

$$\int_{H^\circ} K(x, a) M_t(dx) = \int_0^t ds \int_{H^\circ} K(x, a) L_s(dx), \quad t \geq 0, a \in H, \tag{2.19}$$

where $L_s(dx)$ is a σ-finite kernel from $(0, \infty)$ to H° satisfying $L_{r+t} = (T_t L_r)|_{H^\circ}$ for all $r, t > 0$ and

$$\int_0^t ds \int_H (1 \wedge \|x\|^2) L_s(dx) < \infty, \qquad t \geq 0.$$

Proof. If $(\mu_t)_{t\geq 0}$ is an SC-semigroup given by (2.1), then $t \mapsto M_t$ is non-decreasing by the second equation in (2.5). Let $c \geq 1$ and $b \geq 0$ be as in the proof of Lemma 2.1 and let

$$h(t) := \int_{H^\circ} (1 \wedge \|x\|^2) M_t(dx), \qquad t \geq 0.$$

By (2.5) we have, for $r, t \geq 0$,

$$h(r+t) - h(r) = \int_{H^\circ} (1 \wedge \|T_r x\|^2) M_t(dx),$$

which is bounded above by $c^2 e^{2br} h(t)$. As in the proof of Lemma 2.1, one sees that $h(t)$ is absolutely continuous in $t \geq 0$. Since the family of finite measures $\nu_t(dx) := (1 \wedge \|x\|^2) M_t(dx)$ is non-decreasing and $t \mapsto h(t) = \nu_t(H^\circ)$ is absolutely continuous, $\nu([0, t], B) = \nu_t(B)$ defines a locally bounded Borel measure $\nu(\cdot, B)$ on $[0, \infty)$ for each $B \in \mathscr{B}(H^\circ)$. A monotone class argument shows that $\nu(A, \cdot)$ is a Borel measure on H° for each $A \in \mathscr{B}([0, \infty))$, so that $\nu(\cdot, \cdot)$ is a bimeasure. By [10, p.502], there is a probability kernel $J_s(dx)$ from $[0, \infty)$ to H° such that

$$\nu(A, B) = \int_A J_s(B) \nu(ds, H^\circ) = \int_A J_s(B) dh(s) = \int_A J_s(B) h'(s) ds,$$

where $h'(s)$ is the Radon-Nikodym derivative of $dh(s)$ relative to Lebesgue measure. Defining the σ-finite kernel $L_s(dx) := (1 \wedge \|x\|^2)^{-1}h'(s)J_s(dx)$ we obtain (2.19). By the second equation of (2.5) one can modify the definition of $(L_t)_{t>0}$ so that $L_{r+t} = (T_tL_r)|_{H^\circ}$ is satisfied for all $r, t > 0$. $\square$

We say the linear part $(b_t)_{t\geq 0}$ of (2.1) is *absolutely continuous* if there exists an H-valued path $(c_s)_{s>0}$ such that $\langle b_t, a\rangle = \int_0^t \langle c_s, a\rangle ds$ for all $t \geq 0$ and $a \in H$. The following theorem gives a Hilbert space version of Li [17, Theorem 2] or [21, Theorem 3.2] and extends the characterization of Bogachev *et al* [3, Lemma 2.6 and Proposition 4.3].

Theorem 2.3 *Suppose that $(\mu_t)_{t\geq 0}$ is a family of probability measures on H. If there is a family of infinitely divisible probabilities $(\nu_s)_{s>0}$ such that $\nu_{r+t} = T_t\nu_r$ for all $r, t > 0$ and*

$$\hat{\mu}_t(a) = \exp\left\{\int_0^t \log \hat{\nu}_s(a) ds\right\}, \qquad t \geq 0, a \in H, \tag{2.20}$$

then $(\mu_t)_{t\geq 0}$ is an SC-semigroup. Conversely, every SC-semigroup $(\mu_t)_{t\geq 0}$ with absolutely continuous linear part has representation (2.20).

Proof. If $(\mu_t)_{t\geq 0}$ is given by (2.20), it is clearly an SC-semigroup. Conversely, let $(\mu_t)_{t\geq 0}$ be an SC-semigroup and let $(U_s)_{s>0}$ and $(L_s)_{s>0}$ be provided by Theorems 2.1 and 2.2. Suppose that $\langle b_t, a\rangle = \int_0^t \langle c_s, a\rangle ds$. By (2.6), we can modify the definition of $(c_s)_{s>0}$ so that

$$c_{r+t} = T_tc_r + \int_{H^\circ} \left(\chi_{[0,1]}(\|T_tx\|) - \chi_{[0,1]}(\|x\|)\right) T_txL_r(dx), \quad r, t > 0.$$

Then we have the result by letting ν_s be the infinitely divisible probability defined by the triple (c_s, U_s, L_s). $\square$

We may call the family $(\nu_s)_{s>0}$ in Theorem 2.3 an *entrance law* for $(T_t)_{t\geq 0}$. (More precisely, it is an entrance law for the deterministic Markov process $\{T_tx : t \geq 0\}$, as, for example, in Sharpe [28].) If there is a probability measure ν_0 on H such that $\nu_s = T_s\nu_0$ for all $s > 0$, we say that $(\nu_s)_{s>0}$ is *closable*. In this case, the corresponding SC-semigroup $(\mu_t)_{t\geq 0}$ is given by

$$\hat{\mu}_t(a) = \exp\left\{\int_0^t \log \hat{\nu}_0(T_s^*a) ds\right\}, \quad t \geq 0, a \in H, \tag{2.21}$$

which belongs to the class (1.3). This explains the connection of our characterization with that of Bogachev *et al* [3].

Theorem 2.3 gives a characterization for all SC-semigroups under the assumption of absolute continuity on the linear part $(b_t)_{t\geq 0}$. This assumption cannot be removed since $(b_t)_{t\geq 0}$ can be discontinuous as pointed out in Schmuland and Sun [27]. The following example shows that it can even be continuous but nowhere differentiable.

Example 2.1 Consider $H = L^2([0, 2\pi))$ and let T_t be the shift operator by $t \geq 0$ (mod 2π). For $t \geq 0$ and $f \in L^2([0, 2\pi))$ set $b_t = (I - T_t)f$. Then $(\delta_{b_t})_{t\geq 0}$ is a constant SC-semigroup. Taking the inner product against f we obtain

$$\begin{aligned}\langle f, b_t\rangle &= \|f\|^2 - \frac{1}{2\pi}\int_0^{2\pi} f(x-t)f(x)dx \\ &= \|f\|^2 - |\hat{f}(0)|^2 - 2\sum_{n=1}^{\infty} |\hat{f}(n)|^2 \cos(nt),\end{aligned}$$

where $\hat{f}$ is the Fourier transform of f. Now let f be the function whose Fourier coefficients are given by

$$\hat{f}(n) = \begin{cases} 2^{-k/2} & \text{if } |n| = 2^k,\ k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\langle f, b_t \rangle = 2 - 2\sum_{k=1}^{\infty} 2^{-k} \cos(2^k t),$$

which is (up to a constant) Weierstrass's nowhere differentiable continuous function.

Let us consider another important special type of SC-semigroup given by (2.1) under the assumption:

$$\int_{H^\circ} (\|x\| \wedge \|x\|^2) M_t(dx) < \infty, \quad t \geq 0. \tag{2.22}$$

Since (2.2) holds automatically, (2.22) is only a first norm-moment condition on the restriction of M_t to $\{x \in H : \|x\| \geq 1\}$. We say the SC-semigroup $(\mu_t)_{t\geq 0}$ is *centered* if

$$\int_H \langle x, a\rangle \mu_t(dx) = 0, \quad t \geq 0, a \in H.$$

In this case, Theorem 2.3 implies that

$$\hat{\mu}_t(a) = \exp\bigg\{\int_0^t \bigg[-\frac{1}{2}\langle U_s a, a\rangle + \int_{H^\circ} K_1(x,a) L_s(dx)\bigg] ds\bigg\}, \qquad t \geq 0, a \in H, \tag{2.23}$$

where

$$K_1(x,a) := e^{i\langle x,a\rangle} - 1 - i\langle x, a\rangle, \qquad x, a \in H.$$

Construction and regularity of OU-processes defined by (2.23) are discussed systematically in Dawson and Li [8].

The characterizations (2.20) and (2.23) are of interest since they include some SC-semigroups arising naturally in applications which are not included in (1.3) and (2.21). We shall see in the next two sections that a rich class of such SC-semigroups arise in the study of fluctuation limits of catalytic branching superprocesses with immigration. Two particular examples are given below. We consider the Hilbert space $L^2(0,\infty)$. Let

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\{-x^2/2t\}, \qquad t > 0, x \in \mathbb{R}$$

and

$$p_t(x,y) = g_t(x-y) - g_t(x+y), \qquad t > 0,\ x, y \in (0,\infty). \tag{2.24}$$

Then the transition semigroup $(P_t)_{t\geq 0}$ of the ABM in $(0,\infty)$ is defined by $P_0 f = f$ and

$$P_t f(x) = \int_0^\infty p_t(x,y) f(y) dy, \qquad t > 0, x \in (0,\infty). \tag{2.25}$$

Let

$$k_t(y) = 2^{-1}(d/dx)p_t(x,y)|_{x=0^+} = yg_t(y)/t, \qquad t > 0, y \in (0,\infty). \tag{2.26}$$

It is not hard to check that

$$\int_0^\infty k_t(y)dt = 1 \quad \text{and} \quad k_{r+t}(y) = \int_0^\infty p_t(x,y)k_r(x)dx \tag{2.27}$$

for $r,t > 0$ and $y \in (0,\infty)$.

Example 2.2 Let $c > 0$ and $x_0 > 0$. By Theorem 4.1, there is a centered Gaussian SC-semigroup $(\mu_t)_{t\geq 0}$ on $L^2(0,\infty)$ given by

$$\hat{\mu}_t(f) = \exp\bigg\{ -c\int_0^t P_sf(x_0)^2 ds\bigg\}, \qquad t \geq 0, f \in L^2(0,\infty). \tag{2.28}$$

This is a special form of (2.20) and (2.23) with $(\nu_s)_{s>0}$ defined by

$$\hat{\nu}_s(f) = \exp\left\{-cP_sf(x_0)^2\right\}, \qquad s > 0, f \in L^2(0,\infty). \tag{2.29}$$

Observe that $f \mapsto P_sf(x_0)^2$ is a well-defined functional on $L^2(0,\infty)$ only for $s > 0$. Thus, the SC-semigroup (2.28) is not included in (1.3) and (2.21).

Example 2.3 Suppose that $(1 \vee |u|)m(du)$ is a finite measure on $\mathbb{R}^\circ := \mathbb{R} \setminus \{0\}$ and let

$$\varphi(z) = \int_{\mathbb{R}^\circ} \left(e^{iuz} - 1 - iuz\right) m(du), \quad z \in \mathbb{R}.$$

By Theorem 4.3,

$$\hat{\mu}_t'(f) = \exp\bigg\{ \int_0^t \varphi(\langle k_s, f\rangle)ds\bigg\}, \qquad t \geq 0, f \in L^2(0,\infty) \tag{2.30}$$

defines a centered SC-semigroup $(\mu_t')_{t\geq 0}$ on $L^2(0,\infty)$. By (2.27) one may check that $(\mu_t')_{t\geq 0}$ is included in (2.20) and (2.23). Unless $m(\mathbb{R}^\circ) = 0$, this SC-semigroup is not included in (1.3) and (2.21).

3 Fluctuation limits of superprocesses

In this section, we discuss small branching fluctuation limits of catalytic branching superprocesses with immigration, which lead to a class of OU-processes taking distribution values. Similar fluctuation limits for superprocesses with function-valued catalysts have been discussed in Gorostiza [13], Gorostiza and Li [14, 15], and Li [20]. We shall only give an outline of the arguments and refer the reader to the earlier papers for details. As pointed out in [20], the small branching fluctuation limit is typically equivalent to the high density and the large scale fluctuation limits. For simplicity, we restrict to the case where the underlying motion is an ABM in $D := (0,\infty)$. We write D instead of $(0,\infty)$ for the underlying space in the sequel since $(0,\infty)$ and $[0,\infty)$ will appear frequently with quite different meanings. This notation also suggests that some of the results can be modified to the case where D is a more general domain in $\mathbb{R}^d$.

Let $M(D)$ denote the space of finite Borel measures on D endowed with the topology of weak convergence. Let $\{B_t : t \geq 0\}$ be an ABM in D with transition semigroup $(P_t)_{t\geq 0}$ defined by (2.25). Let $\phi(\cdot, \cdot)$ be a function on $D \times [0, \infty)$ given by

$$\phi(x, z) = c(x)z^2 + \int_0^\infty (e^{-zu} - 1 + zu)m(x, du), \quad z \geq 0, x \in D, \tag{3.1}$$

where $c \in B(D)^+$ and $u^2 m(x, du)$ is a bounded kernel from D to $(0, \infty)$. For any $\eta \in B(D)^+$, there is a superprocess in $M(D)$ with transition semigroup $(Q_t)_{t\geq 0}$ determined by

$$\int_{M(D)} e^{-\nu(f)} Q_t(\mu, d\nu) = \exp\{-\mu(V_t f)\}, \qquad f \in B(D)^+, \tag{3.2}$$

where $(V_t)_{t\geq 0}$ is a semigroup of non-linear operators on $B(D)^+$ defined by

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_D \phi(y, V_s f(y))\eta(y) P_{t-s}(x, dy), \quad t \geq 0, x \in D, \tag{3.3}$$

see for example Dawson [4]. The superprocess describes the catalytic reaction of a large number of infinitesimal particles moving according to the transition law of the ABM and splitting according to the branching mechanism given by $\phi(\cdot, \cdot)$. The value $\eta(x)$ represents the density at $x \in D$ of a catalyst which causes the splitting. However, there are some catalytic reactions in which the catalyst is concentrated on a very small set and in that case the coefficient $\eta(\cdot)$ has to be replaced by an irregular one, as in Pagliaro and Taylor [26]. These lead to the study of a catalyst given not by a regular density function but rather by a measure $\eta \in M(D)$ with $\eta(dx) :=$ "catalytic mass in the volume element dx". Then we reformulate (3.3) as

$$V_t f(x) = P_t f(x) - \int_0^t ds \int_D \phi(y, V_s f(y)) p_{t-s}(x, y)\eta(dy), \qquad t \geq 0, x \in D, \tag{3.4}$$

where $p_t(x, y)$ is given by (2.24). A Markov process in $M(D)$ with transition semigroup $(Q_t)_{t\geq 0}$ defined by (3.2) and (3.4) is called a *catalytic branching super ABM* with parameters (η, ϕ).

Let $\{l_s(y) : s > 0, y > 0\}$ be a continuous version of the local time of $\{B_t : t \geq 0\}$. Then $K(r, t) = \eta(l_t) - \eta(l_r)$ defines an additive functional of $\{B_t : t \geq 0\}$. In view of (2.24) it is easy to check that

$$\boldsymbol{E}_x\{K(0, t)\} = \int_0^t ds \int_D p_s(x, y)\eta(dy) \leq \eta(1) \int_0^t \frac{1}{\sqrt{2\pi s}} ds \leq \eta(1)\sqrt{2t/\pi}.$$

Thus, $K(r, t)$ is admissible in the sense of [9, p.49] and the existence of the catalytic branching super ABM follows by [9, p.52]; see also [16]. The study of superprocesses with irregular catalysts was initiated by Dawson and Fleischmann [5, 6] and there has been a considerable development in the theory since then; see Dawson and Fleischmann [7] for a recent survey.

Set $\kappa_t(dx) = k_t(x)dx$. By (2.27), $(\kappa_t)_{t>0}$ forms an entrance law for the underlying semigroup $(P_t)_{t\geq 0}$, that is, $\kappa_r P_t = \kappa_{r+t}$ for all $r, t > 0$. Let

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t ds \int_D \phi(y, V_s f(y)) k_{t-s}(y)\eta(dy), \quad t > 0, f \in B(D)^+. \tag{3.5}$$

As in Li [18] one may see that

$$\int_{M(D)} e^{-\nu(f)} Q_t^\kappa(\mu, d\nu) = \exp\bigg\{ -\mu(V_t f) - \int_0^t S_r(\kappa, f) dr \bigg\}, \quad f \in B(D)^+ \tag{3.6}$$

defines the transition semigroup $(Q_t^\kappa)_{t\geq 0}$ of a Markov process $\{Y_t : t \geq 0\}$ in $M(D)$, which we shall call a *catalytic branching immigration super ABM with parameters* (η, ϕ, κ). By (3.4) and (3.5) it is not hard to check that

$$\frac{d}{d\theta} V_t(\theta f)(x) \bigg|_{\theta=0^+} = P_t f(x) \quad \text{and} \quad \frac{d}{d\theta} S_t(\kappa, \theta f) \bigg|_{\theta=0^+} = \kappa_t(f),$$

which, together with (3.6), imply that

$$\int_{M(D)} \nu(f) Q_t^\kappa(\mu, d\nu) = \mu(P_t f) + \int_0^t \kappa_r(f) dr. \tag{3.7}$$

By (2.27) and (3.7) it follows that if $Y_0 = \lambda$, then $\boldsymbol{E}\{Y_t(f)\} = \lambda(f)$ for all $t \geq 0$ and $f \in B(D)^+$, where λ denotes Lebesgue measure.

Now we consider a small branching fluctuation limit of the catalytic branching immigration ABM. For any $\theta > 0$, let $\phi_\theta(x, z) = \phi(x, \theta z)$ and $\mathscr{S}_\theta(D) = \{\mu - \theta^{-1}\lambda : \mu \in M(D)\}$. Suppose that $\{Y_t^\theta : t \geq 0\}$ is a catalytic branching immigration ABM with parameters $(\eta, \phi_\theta, \kappa)$ and $Y_0^\theta = \lambda$. As observed above, we have $\boldsymbol{E}\{Y_t^\theta(f)\} = \lambda(f)$ for all $t \geq 0$ and $f \in B(D)^+$. On the other hand, $\phi_\theta(x, z) \to 0$ as $\theta \to 0$. By (2.4), (2.5) and (2.6) we have $Y_t^\theta(f) \to \lambda(f)$ in distribution as $\theta \to 0$. We define the fluctuation process $\{Z_t^\theta : t \geq 0\}$ by

$$Z_t^\theta = \theta^{-1}[Y_t^\theta - \lambda], \qquad t \geq 0. \tag{3.8}$$

As in Gorostiza and Li [14] we see that $\{Z_t^\theta : t \geq 0\}$ is a centered signed-measure-valued Markov process with transition semigroup $(R_t^\theta)_{t\geq 0}$ determined by

$$\int_{\mathscr{S}_\theta(D)} e^{-\nu(f)} R_t^\theta(\mu, d\nu) = \exp\bigg\{ -\mu(\theta V_t^\theta(f/\theta)) + \int_0^t \eta(\phi(\theta V_s^\theta(f/\theta))) ds \bigg\},$$

where $(V_t^\theta)_{t\geq 0}$ is defined by

$$V_t^\theta f(x) + \int_0^t ds \int_D \phi_\theta(y, V_s^\theta f(y)) p_{t-s}(x, y) \eta(dy) = P_t f(x).$$

Let $\mathscr{S}(D)$ be the space of infinitely differentiable functions f on D such that

$$|||f|||_n := \max_{0\leq k\leq n} \sup_{u\in D} \bigg| (1 + u^2)^n \frac{d^k}{du^k} f(u) \bigg| < \infty, \quad n = 0, 1, 2, \ldots. \tag{3.9}$$

Then $\mathscr{S}(D)$ topologized by the norms $\{|||\cdot|||_n : n = 0, 1, 2, \ldots\}$ is a nuclear space. Let $\mathscr{S}'(D)$ denote the dual space of $\mathscr{S}(D)$. As in [14], the finite-dimensional distributions of $\{Z_t^\theta : t \geq 0\}$ converge as $\theta \to 0$ to those of an $\mathscr{S}'(D)$-valued Markov process $\{Z_t^0 : t \geq 0\}$ with transition semigroup $(R_t^0)_{t\geq 0}$ determined by

$$\int_{\mathscr{S}'(D)} e^{-\nu(f)} R_t^0(\mu, d\nu) = \exp\bigg\{ -\mu(P_t f) + \int_0^t \eta(\phi(P_s f)) ds \bigg\}, \quad f \in \mathscr{S}(D)^+. \tag{3.10}$$

Therefore, an OU-process with transition semigroup $(R_t^0)_{t\geq 0}$ is an approximation of the fluctuations of an immigration process around the average.

Theorem 3.1 *Let $\varphi(\cdot,\cdot)$ be a function on $D\times\mathbb{R}$ with the representation*

$$\varphi(x,z) = -c(x)z^2 + \int_{\mathbb{R}^\circ}(e^{izu}-1-izu)m(x,du), \quad x\in D, z\in\mathbb{R}, \tag{3.11}$$

where $c\in B(D)^+$ and $(|u|\wedge|u|^2)m(x,du)$ is a bounded kernel from D to $\mathbb{R}^\circ := \mathbb{R}\setminus\{0\}$. Then there is a transition semigroup $(R_t)_{t\geq 0}$ on $\mathscr{S}'(D)$ given by

$$\int_{\mathscr{S}'(D)} e^{i\nu(f)}R_t(\mu,d\nu) = \exp\left\{i\mu(P_tf) + \int_0^t \eta(\varphi(P_sf))ds\right\}, \quad t\geq 0, f\in\mathscr{S}(D). \tag{3.12}$$

Proof. The semigroup $(R_t)_{t\geq 0}$ can be obtained as in [14] by considering the difference of two Markov processes with transition semigroups of the form (3.10). $\square$

Heuristically, an OU-process with transition semigroup $(R_t)_{t\geq 0}$ is the mixture of the fluctuations of two immigration processes around their means. The branching mechanism of the processes is determined by the function $\varphi(\cdot,\cdot)$ given by (3.11) and the distribution of catalysts in D that cause the branching is given by η. A more singular transition semigroup is given by the following

Theorem 3.2 *Let φ be a function on $\mathbb{R}$ given by*

$$\varphi(z) = -cz^2 + \int_{\mathbb{R}^\circ}\left(e^{iuz}-1-iuz\right)m(du), \qquad z\in\mathbb{R}, \tag{3.13}$$

where $c\geq 0$ and $(|u|\wedge|u|^2)m(du)$ is a finite measure on $\mathbb{R}^\circ$. Then there is a transition semigroup $(R'_t)_{t\geq 0}$ on $\mathscr{S}'(D)$ given by

$$\int_{\mathscr{S}'(D)} e^{i\nu(f)}R'_t(\mu,d\nu) = \exp\left\{i\mu(P_tf) + \int_0^t \varphi(\kappa_s(f))ds\right\}, \quad t\geq 0, f\in\mathscr{S}(D). \tag{3.14}$$

Proof. This transition semigroup is obtained from the one in the last theorem by replacing $\varphi(x,z)$ and $\eta(dx)$ in (3.12) respectively by $\varphi(nz)$ and $\delta_{1/2n}(dx)$ and letting $n\to\infty$. $\square$

Roughly speaking, an OU-process with transition semigroup $(R'_t)_{t\geq 0}$ represents the fluctuations of a process over D that branches very actively only near the absorbing boundary.

4 OU-processes with function values

In this section, we show that, under suitable conditions, the OU-processes constructed in the last section take function values from $L^2(D,\lambda)$.

Suppose that η is a finite measure on D and $\varphi(\cdot,\cdot)$ is given by (3.11) with $u^2m(x,du)$ being a bounded kernel from D to $\mathbb{R}^\circ = \mathbb{R}\setminus\{0\}$. Let $W(ds,dx)$ be a white noise on $[0,\infty)\times D$ with covariance measure $2c(x)ds\eta(dx)$ and $N(ds,du,dx)$ be a Poisson random measure on $[0,\infty)\times\mathbb{R}^\circ\times D$ with intensity $dsm(x,du)\eta(dx)$. Suppose that $W(ds,dx)$ and $N(ds,du,dx)$ are defined on some complete probability space $(\Omega,\mathscr{F},\boldsymbol{P})$ and are independent of each other. Set $\tilde{N}(ds,du,dx) = N(ds,du,dx) - dsm(x,du)\eta(dx)$. Then we have

Theorem 4.1 *For each $t \geq 0$, the function*

$$Z_t^0(\omega, y) := \int_0^t \int_D p_{t-s}(x,y) W(\omega, ds, dx) + \int_0^t \int_{\mathbb{R}^\circ} \int_D u p_{t-s}(x,y) \tilde{N}(\omega, ds, du, dx) \qquad (4.1)$$

is well-defined in the $L^2(\Omega \times D, \boldsymbol{P} \times \lambda)$ sense and $\{Z_t^0 : t \geq 0\}$ is a Markov process with state space $L^2(D, \lambda)$, initial value zero and transition semigroup $(R_t)_{t\geq 0}$ given by

$$\int_{L^2(D,\lambda)} e^{i\langle h,f\rangle} R_t(g, dh) = \exp\left\{ i\langle g, P_t f\rangle + \int_0^t \eta(\varphi(P_s f)) ds \right\}, \quad f \in L^2(D, \lambda). \qquad (4.2)$$

Moreover, $Z_t^0(\omega, y)$ can be chosen as a function of (t, ω, y) belonging to $L^2([0,T]\times\Omega\times D, \lambda\times\boldsymbol{P}\times\lambda)$ for each $T > 0$.

Proof. By the inequality

$$\int_D p_{t-s}(x,y)^2 dy < \frac{1}{2\pi(t-s)} \int_{\mathbb{R}} \exp\left\{ - \frac{y^2}{(t-s)} \right\} dy = \frac{1}{2\sqrt{\pi(t-s)}},$$

we have

$$\begin{aligned} & \int_D \boldsymbol{E}\Big\{ \Big(\int_0^t \int_D p_{t-s}(x,y) W(ds, dx) \Big)^2 \Big\} dy \\ = \;& 2 \int_D dy \int_0^t ds \int_D p_{t-s}(x,y)^2 c(x) \eta(dx) \\ \leq \;& \int_0^t \frac{1}{\sqrt{\pi(t-s)}} ds \int_D c(x) \eta(dx) \\ < \;& \infty \end{aligned}$$

and

$$\begin{aligned} & \int_D \boldsymbol{E}\Big\{ \Big(\int_0^t \int_{\mathbb{R}^\circ} \int_D u p_{t-s}(x,y) \tilde{N}(ds, du, dx) \Big)^2 \Big\} dy \\ = \;& \int_D dy \int_0^t ds \int_D \eta(dx) \int_{\mathbb{R}^\circ} u^2 p_{t-s}(x,y)^2 m(x, du) \\ \leq \;& \int_0^t \frac{1}{2\sqrt{\pi(t-s)}} ds \int_D \eta(dx) \int_{\mathbb{R}^\circ} u^2 m(x, du) \\ < \;& \infty. \end{aligned}$$

Then the right hand side of (4.1) is well-defined in the $L^2(\Omega \times D, \boldsymbol{P} \times \lambda)$ sense. By the same reasoning, we see that it is also well-defined in the $L^2([0,T] \times \Omega \times D, \lambda \times \boldsymbol{P} \times \lambda)$ sense. For any $f \in L^2(D, \lambda)$, we have

$$\boldsymbol{E} \exp\left\{ i \int_0^t \int_D P_{t-s} f(x) W(ds, dx) \right\} = \exp\left\{ - \int_0^t ds \int_D c(x) [P_{t-s} f(x)]^2 \eta(dx) \right\}$$

and

$$\begin{aligned} & \boldsymbol{E} \exp\left\{ i \int_0^t \int_{\mathbb{R}^\circ} \int_D u P_{t-s} f(x) \tilde{N}(ds, du, dx) \right\} \\ = \;& \exp\left\{ \int_0^t ds \int_D c(x) \eta(dx) \int_{\mathbb{R}^\circ} \left(\exp\{ iu P_{t-s} f(x)\} - 1 - iu P_{t-s} f(x) \right) m(x, du) \right\}. \end{aligned}$$

Thus $\{Z_t^0 : t \geq 0\}$ has the asserted one-dimensional distributions. If $g \in L^2(D, \lambda)$, then $P_t g \in L^2(D, \lambda)$ for all $t \geq 0$. Clearly, the distribution $R_t(g, \cdot)$ of $P_t g + Z_t^0$ has characteristic functional given by (4.2) and $(R_t)_{t\geq 0}$ is a transition semigroup on $L^2(D, \lambda)$. The Markov property of $\{Z_t^0 : t \geq 0\}$ follows by a similar calculation of the characteristic functionals of the finite-dimensional distributions. $\square$

Suppose that $\varphi(\cdot)$ is given by (3.13) with $u^2 m(du)$ being a finite measure on $\mathbb{R}^\circ$. Set $\gamma(dx) = (1 - e^{-x^2})dx$ for $x \in D$. Let $\{B(t) : t \geq 0\}$ be a one-dimensional Brownian motion with increasing process $2ct$ and $N(ds, du)$ be a Poisson random measure on $[0, \infty) \times \mathbb{R}^\circ$ with intensity $ds m(du)$. Suppose that $\{B(t) : t \geq 0\}$ and $N(ds, du)$ are defined on some complete probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$ and are independent of each other. Set $\tilde{N}(ds, du) = N(ds, du) - ds m(du)$. Then we have

Theorem 4.2 *For each $t \geq 0$, the function*

$$Z_t^0(\omega, y) := \int_0^t k_{t-s}(y) B(\omega, ds) + \int_0^t \int_{\mathbb{R}^\circ} u k_{t-s}(y) \tilde{N}(\omega, ds, du) \tag{4.3}$$

is well-defined in the $L^2(\Omega \times D, \boldsymbol{P} \times \gamma)$ sense and $\{Z_t^0 : t \geq 0\}$ can be regarded as a Markov process with state space $\mathscr{S}'(\mathbb{R})$, initial value zero and transition semigroup $(R_t')_{t\geq 0}$ given by (3.14). Moreover, $Z_t^0(\omega, y)$ can be chosen as a function of (t, ω, y) belonging to $L^2([0, T] \times \Omega \times D, \lambda \times \boldsymbol{P} \times \gamma)$ for each $T > 0$.

Proof. For any $t > 0$,

$$\int_0^t k_s(y)^2 ds \leq \int_0^\infty \frac{y^2}{2\pi s^3} e^{-y^2/s} ds = \frac{1}{2\pi y^2}.$$

Then we have

$$\begin{aligned} & \int_D \boldsymbol{E}\Big\{\Big(\int_0^t k_{t-s}(y) B(ds)\Big)^2\Big\}\gamma(dy) \\ = \;& 2c \int_D \gamma(dy) \int_0^t k_{t-s}(y)^2 ds \\ < \;& \infty \end{aligned}$$

and

$$\begin{aligned} & \int_D \boldsymbol{E}\Big\{\Big(\int_0^t \int_{\mathbb{R}^\circ} u k_{t-s}(y) \tilde{N}(ds, du)\Big)^2\Big\}\gamma(dy) \\ = \;& \int_D \gamma(dy) \int_0^t k_{t-s}(y)^2 ds \int_{\mathbb{R}^\circ} u^2 m(du) \\ < \;& \infty. \end{aligned}$$

Thus, the right hand side of (4.3) is well-defined in the $L^2(\Omega \times D, \boldsymbol{P} \times \gamma)$ sense. Clearly, it is also well-defined in the $L^2([0, T] \times \Omega \times D, \lambda \times \boldsymbol{P} \times \gamma)$ sense. For any $f \in L^2(D, \lambda)$, we have

$$\boldsymbol{E} \exp\Big\{ i \int_0^t \langle k_{t-s}, f\rangle B(ds)\Big\} = \exp\Big\{ - \int_0^t c\langle k_{t-s}, f\rangle^2 ds\Big\}$$

and

$$
\begin{aligned}
& \boldsymbol{E} \exp \left\{ i \int_0^t \int_{\mathbb{R}^\circ} u \langle k_{t-s}, f \rangle \tilde{N}(ds, du) \right\} \\
= \quad & \exp \left\{ \int_0^t ds \int_{\mathbb{R}^\circ} (\exp\{iu\langle k_{t-s}, f\rangle\} - 1 - iu\langle k_{t-s}, f\rangle)\, m(du) \right\}.
\end{aligned}
$$

Therefore $\{Z_t^0 : t \geq 0\}$ has the correct one-dimensional distributions. The asserted Markov property follows by a calculation of the characteristic functionals of the finite-dimensional distributions. □

Theorem 4.3 *If $(1 \vee |u|)m(du)$ is a finite measure on $\mathbb{R}^\circ$, then for each $t \geq 0$ the function*

$$
Z_t^0(y) := \int_0^t \int_{\mathbb{R}^\circ} u k_{t-s}(y) \tilde{N}(ds, du) \tag{4.4}
$$

belongs to $L^2(D, \lambda)$ a.s. and $\{Z_t^0 : t \geq 0\}$ is a Markov process with state space $L^2(D, \lambda)$, initial value zero and transition semigroup $(R_t')_{t\geq 0}$ given by

$$
\int_{L^2(D,\lambda)} e^{i\langle h, f\rangle} R_t'(g, dh) = \exp\left\{ i\langle g, P_t f\rangle + \int_0^t \varphi(\langle k_s, f\rangle) ds \right\}, \quad f \in L^2(D, \lambda), \tag{4.5}
$$

where φ is given by (3.13) with $c = 0$.

Proof. For any $t > 0$, we have

$$
\int_0^t k_s(y) ds = \int_{y^2/2t}^\infty \frac{1}{\sqrt{\pi u}} e^{-u} du,
$$

which is bounded in $y \geq 0$ and dominated by

$$
\int_{y^2/2t}^\infty \frac{1}{\sqrt{\pi}} e^{-u} du = \frac{1}{\sqrt{\pi}} e^{-y^2/2t}
$$

for $y \geq \sqrt{2t}$. Therefore,

$$
\int_0^t ds \int_{\mathbb{R}^\circ} u k_{t-s} m(du)
$$

belongs to $L^2(D, \lambda)$ under our assumption. Since $k_t \in L^2(D, \lambda)$ for every $t > 0$ and a.s.

$$
\int_0^t \int_{\mathbb{R}^\circ} u k_{t-s} N(ds, du)
$$

is a finite sum, we have $Z_t^0 \in L^2(D, \lambda)$ a.s. If $g \in L^2(D, \lambda)$, then $P_t g \in L^2(D, \lambda)$ for all $t \geq 0$ and the distribution $R_t(g, \cdot)$ of $P_t g + Z_t^0$ has characteristic functional given by (4.5). Clearly, $(R_t)_{t\geq 0}$ is a transition semigroup on $L^2(D, \lambda)$. The Markov property of $\{Z_t^0 : t \geq 0\}$ follows by a calculation of the characteristic functionals of the finite-dimensional distributions. □

As in Li [18] one may see that the generalized OU-processes given by (4.2) and (4.5) usually do not have right continuous sample paths, neither do they have the strong Markov property. We shall prove in the next section that they do have those properties if we regard them as processes in another suitably chosen state space.

5 OU-processes with signed-measure values

In this section, we show that some of the generalized OU-processes given by (4.2) and (4.5) behave very regularly in the space of signed-measures. Indeed, from the proof of Theorem 5.1 we know that they are essentially special forms of the immigration processes studied in Li [17, 18].

Given a locally compact metric space E, we denote by $M(E)$ the space of finite Borel measures on E. Let $\{f_n\}_{n=1}^{\infty}$ be a dense subset of the space of all bounded uniformly continuous functions on E. We define the metric $r(\cdot,\cdot)$ on $M(E)$ by

$$r(\mu,\nu) = \sum_{n=1}^{\infty} 2^{-n}(1 \wedge |\mu(f_n) - \nu(f_n)|), \qquad \mu,\nu \in M(E). \tag{5.1}$$

Clearly, this metric is compatible with the topology of weak convergence in $M(E)$. Let $S(E) = \{\mu^+ - \mu^- : \mu^+, \mu^- \in M(E)\}$ be the space of finite signed-measures on E. Define a metric $\rho(\cdot,\cdot)$ on $S(E)$ by

$$\begin{aligned}\rho(\mu,\nu) \;=\; & \inf\{r(\mu^+,\nu^+) + r(\mu^-,\nu^-) : \mu^+,\mu^-,\nu^+,\nu^- \in M(E) \\ & \text{with } \mu^+ - \mu^- = \mu \text{ and } \nu^+ - \nu^- = \nu\}.\end{aligned} \tag{5.2}$$

Then $\mu_n \to \mu_0$ in $S(E)$ if and only if there are decompositions $\mu_n = \mu_n^+ - \mu_n^-$ and $\mu_0 = \mu_0^+ - \mu_0^-$ such that $\mu_n^+ \to \mu_0^+$ and $\mu_n^- \to \mu_0^-$ in $M(E)$. Below, we shall consider the metric space $(S(E),\rho)$ for $E = (0,\infty)$ or $[0,\infty)$.

Suppose that η is a finite measure, $\varphi(\cdot,\cdot)$ is given by (3.11) with $c(x) \equiv 0$, $|u|m(x,du)$ is a bounded kernel from D to $\mathbb{R}^\circ$, and that $N(ds,du,dx)$ is a Poisson random measure on $[0,\infty) \times \mathbb{R}^\circ \times D$ with intensity $dsm(x,du)\eta(dx)$. Let $\tilde{N}(ds,du,dx) = N(ds,du,dx) - dsm(x,du)\eta(dx)$ and

$$Y_t(f) := \int_0^t \int_{\mathbb{R}^\circ} \int_D uP_{t-s}f(x)\tilde{N}(ds,du,dx), \qquad t \geq 0, f \in B(D). \tag{5.3}$$

Theorem 5.1 *The process $\{Y_t : t \geq 0\}$ defined by (5.3) is an a.s. right continuous $S(D)$-valued strong Markov process with transition semigroup $(R_t)_{t\geq 0}$ defined by*

$$\int_{S(D)} e^{i\nu(f)} R_t(\mu,d\nu) = \exp\left\{ i\mu(P_t f) + \int_0^t \eta(\varphi(P_s f))ds \right\}, \quad f \in B(D). \tag{5.4}$$

Proof. We define the positive part $\{Y_t^+ : t \geq 0\}$ of $\{Y_t : t \geq 0\}$ by

$$Y_t^+(f) := \int_0^t \int_0^\infty \int_D uP_{t-s}f(x)N(ds,du,dx), \qquad t \geq 0, f \in B(D).$$

By the assumptions,

$$\begin{aligned}\boldsymbol{E}\{Y_t^+(f)\} &= \int_0^t ds \int_D \eta(dx) \int_0^\infty uP_{t-s}f(x)m(x,du) \\ &\leq t\|f\|\eta(D) \sup_{x\in D} \int_0^\infty um(x,du) \\ &< \infty.\end{aligned}$$

Then $\{Y_t^+ : t \geq 0\}$ is a well-defined $M(D)$-valued process, which is clearly a special form of the immigration process considered in [18] without branching. By [18, Theorem 4.1], $\{Y_t^+ : t \geq 0\}$ is a.s. right continuous. Similarly, the negative part $\{Y_t^- : t \geq 0\}$ of $\{Y_t : t \geq 0\}$ defined by

$$Y_t^-(f) := -\int_0^t \int_{-\infty}^0 \int_D uP_{t-s}f(x)N(ds,du,dx), \qquad t \geq 0, f \in B(D)$$

is also an a.s. right continuous immigration process. Then one can easily see that $\{Y_t : t \geq 0\}$ defined by (5.3) is an a.s. right continuous $S(D)$-valued Markov process with transition semigroup $(R_t)_{t\geq 0}$. The strong Markov property holds since $(R_t)_{t\geq 0}$ is clearly Feller. $\square$

Suppose that φ is given by (3.13) with $c = 0$, $|u|m(du)$ is a finite measure on $\mathbb{R}^\circ$, and $N(ds,du)$ is a Poisson random measure on $[0,\infty)\times\mathbb{R}^\circ$ with intensity $dsm(du)$. Let $\tilde{N}(ds,du) = N(ds,du) - dsm(du)$, and

$$Y_t(f) := \int_0^t \int_{\mathbb{R}^\circ} u\kappa_{t-s}(f)\tilde{N}(ds,du), \qquad t \geq 0, f \in B(D). \tag{5.5}$$

By an argument similar to that in the proof of Theorem 5.1 we get

Theorem 5.2 *The process $\{Y_t : t \geq 0\}$ defined by (5.5) is an $S(D)$-valued Markov process with transition semigroup $(R'_t)_{t\geq 0}$ defined by*

$$\int_{S(D)} e^{i\nu(f)}R'_t(\mu,d\nu) = \exp\left\{i\mu(P_tf) + \int_0^t \varphi(\kappa_s(f))ds\right\}, \quad f \in B(D). \tag{5.6}$$

As in [18] one can see that the process (5.5) does not have any right continuous modification. Observe that $h(x) := (1-e^{-x})$ is an excessive function of $(P_t)_{t\geq 0}$ and

$$T_tf(x) = \begin{cases} h(x)^{-1}P_t(hf)(x) & \text{for } t>0 \text{ and } x>0, \\ 2\kappa_t(hf) = (d/dx)P_t(hf)(0^+) & \text{for } t>0 \text{ and } x=0 \end{cases} \tag{5.7}$$

defines the transition semigroup $(T_t)_{t\geq 0}$ of a Markov process on $[0,\infty)$.

Theorem 5.3 *Let $\{Y_t : t \geq 0\}$ be defined by (5.5), $Z_t(\{0\}) = 0$, and $Z_t(dx) = (1-e^{-x})Y_t(dx)$ for $x > 0$. Then $\{Z_t : t \geq 0\}$ is an $S([0,\infty))$-valued Markov process with transition semigroup $(S_t)_{t\geq 0}$ defined by*

$$\int_{S([0,\infty))} e^{i\nu(f)}S_t(\mu,d\nu) = \exp\left\{i\mu(T_tf) + \int_0^t \varphi(\kappa_s(hf))ds\right\}, \quad f \in B([0,\infty)). \tag{5.8}$$

Moreover, $\{Z_t : t \geq 0\}$ has a right continuous strong Markov realization.

Proof. The first assertion holds by Theorem 5.2. Observe that

$$\int_{S([0,\infty))} e^{i\nu(f)}S_t(\mu,d\nu) = \exp\left\{i\mu(T_tf) + \int_0^t \varphi(2^{-1}T_sf(0))ds\right\}, \quad f \in B([0,\infty)),$$

by (5.7) and (5.8). Then the second assertion follows from [18, Theorem 4.1] as in the proof of Theorem 5.1. $\square$

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