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Construction of Immigration Superprocesses with Dependent Spatial Motion from One-Dimensional Excursions

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Abstract. A superprocess with dependent spatial motion and interactive immigration is constructed as the pathwise unique solution of a stochastic integral equation carried by a stochastic flow and driven by Poisson processes of one-dimensional excursions.

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1 Introduction

Let $M(\mathbb{R})$ denote the space of finite Borel measures on $\mathbb{R}$ endowed with a metric compatible with its topology of weak convergence. Let $C(\mathbb{R})$ be the set of bounded continuous functions on $\mathbb{R}$. For $f \in C(\mathbb{R})$ and $\mu \in M(\mathbb{R})$ set $\langle f, \mu\rangle = \int f d\mu$. Let $\{w_t : t \geq 0\}$ denote the coordinate process of $C([0,\infty), M(\mathbb{R}))$, which is furnished with the locally uniform convergence. Suppose that h is a continuously differentiable function on $\mathbb{R}$ such that both h and h' are square-integrable. Then the function

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R}, \tag{1.1}$$

is twice continuously differentiable with bounded derivatives ρ' and ρ''. Let $\sigma > 0$ be a constant. Based on the results of Dawson *et al* [2] and Wang [16], we shall prove that for each $\mu \in M(\mathbb{R})$

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there is a unique Borel probability measure $\boldsymbol{Q}_\mu$ on $C([0,\infty), M(\mathbb{R}))$ such that, for each $\phi \in C^2(\mathbb{R})$,

$$M_t(\phi) = \langle \phi, w_t\rangle - \langle \phi, \mu\rangle - \frac{1}{2}\rho(0)\int_0^t \langle \phi'', w_s\rangle ds, \quad t \geq 0, \tag{1.2}$$

under $\boldsymbol{Q}_\mu$ is a continuous martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, w_s\rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi', w_s\rangle^2 dz. \tag{1.3}$$

The system $\{\boldsymbol{Q}_\mu : \mu \in M(\mathbb{R})\}$ defines a diffusion process, which we shall call a *superprocess with dependent spatial motion* (SDSM). Here $\rho(0)$ is the migration rate and σ is the branching rate. The only difference between the SDSM and the super Brownian motion in $M(\mathbb{R})$ is the second term on the right hand side of (1.3), which comes from the dependence of the spatial motion. Because of the dependent spatial motion, the SDSM has properties rather different from those of the superprocess with independent spatial motion. It is well-known that the super Brownian motion started with an arbitrary initial state enters immediately the space of absolutely continuous measures and its density process satisfies a stochastic differential equation; see Konno and Shiga [7] and Reimers [11]. On the contrary, the SDSM lives in the space of purely atomic measures; see Wang [15] and Theorem 3.4 of this paper.

The main purpose of this paper is to construct a class of immigration diffusion processes associated with the SDSM. Let m be a non-trivial σ-finite Borel measure on $\mathbb{R}$ and let q be a Borel function on $M(\mathbb{R})\times\mathbb{R}$ satisfying certain regularity conditions to be specified. A modification of the SDSM is to replace (1.2) by

$$M_t(\phi) = \langle \phi, w_t\rangle - \langle \phi, \mu\rangle - \frac{1}{2}\rho(0)\int_0^t \langle \phi'', w_s\rangle ds - \int_0^t \langle \phi q(w_s,\cdot), m\rangle ds, \quad t \geq 0. \tag{1.4}$$

A solution of the martingale problem given by (1.3) and (1.4) can be interpreted as an SDSM with interactive immigration determined by $q(w_s,\cdot)$ and the reference measure m. Because of the dependence of $q(w_s,\cdot)$ on w_s, the duality method of Dawson *et al* [2] and Wang [16] fails and the uniqueness of the solution becomes a difficult problem. The next paragraph describes our approach to the construction of the immigration SDSM as a *diffusion process*.

Let $W = C([0,\infty), \mathbb{R}^+)$ and let $\tau_0(w) = \inf\{s > 0 : w(s) = 0\}$ for $w \in W$. Let W_0 be the set of paths $w \in W$ such that $w(0) = w(t) = 0$ for $t \geq \tau_0(w)$. We endow W and W_0 with the topology of locally uniform convergence. Let $\boldsymbol{Q}_\kappa$ be the excursion law of the Feller branching diffusion defined by (2.7). Let $W(dt, dy)$ be a time-space white noise on $[0,\infty)\times\mathbb{R}$ based on the Lebesgue measure; see e.g. Walsh [14]. Let $N_0(da, dw)$ be a Poisson random measure on $\mathbb{R}\times W_0$ with intensity $\mu(da)\boldsymbol{Q}_\kappa(dw)$ and $N(ds, da, du, dw)$ a Poisson random measure on $[0,\infty)\times\mathbb{R}\times[0,\infty)\times W_0$ with intensity $dsm(da)du\boldsymbol{Q}_\kappa(dw)$. We assume that $\{W(dt, dy)\}$, $\{N_0(da, dw)\}$ and $\{N(ds, da, du, dw)\}$ are defined on a complete standard probability space and are independent of each other. By Dawson *et al* [2, Lemma 3.1] or Wang [15, Lemma 1.3], for any $r \geq 0$ and $a \in \mathbb{R}$ the stochastic equation

$$x(t) = a + \int_r^t \int_{\mathbb{R}} h(y - x(s)) W(ds, dy), \quad t \geq r, \tag{1.5}$$

has a unique continuous solution $\{x(r,a,t) : t \geq r\}$, which is a Brownian motion with quadratic variation $\rho(0)dt$. Clearly, the system $\{x(r,a,t) : t \geq r; a \in \mathbb{R}\}$ determines an isotropic stochastic flow. Let us consider the following equation:

$$\begin{aligned} Y_t &= \int_{\mathbb{R}} \int_{W_0} w(t) \delta_{x(0,a,t)} N_0(da, dw) \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_0^{q(Y_s,a)} \int_{W_0} w(t-s) \delta_{x(s,a,t)} N(ds, da, du, dw), \quad t > 0. \end{aligned} \tag{1.6}$$

Our main result is that the above stochastic equation has a pathwise unique continuous solution $\{Y_t : t > 0\}$ and, if we set $Y_0 = \mu$, then $\{Y_t : t \geq 0\}$ is a diffusion process. We prove that the distribution of $\{Y_t : t \geq 0\}$ on $C([0,\infty), M(\mathbb{R}))$ solves the martingale problem (1.3) and (1.4). The application of equation (1.6) is essential in the construction of the immigration diffusion process since the uniqueness of solution of the martingale problem given by (1.3) and (1.4) still remains open. Our equation (1.6) also provides useful information on the structures of the sample paths of the immigration diffusion. For instance, from this formulation we can immediately read of the following properties:

(i) for any initial state $\mu \in M(\mathbb{R})$, the process $\{Y_t : t > 0\}$ consists of at most countably many atoms;

(ii) the spatial motion of the atoms of $\{Y_t : t \geq 0\}$ are determined by the flow $\{x(r,a,t) : t \geq r; a \in \mathbb{R}\}$;

(iii) the mass changes of the atoms are described by excursions of the Feller branching diffusion;

(iv) the immigration times and locations as well as the mass excursions are selected by the Poisson random measures $N_0(da, dw)$ and $N(ds, da, du, dw)$;

(v) there are infinitely but countably many immigration times in any non-trivial time interval if $\langle q(\nu,\cdot), m\rangle > 0$ for all $\nu \in M(\mathbb{R})$;

(vi) for any constants $t > r > 0$ there are at most only a finite number of atoms which have lived longer than r before time t.

A class of one-dimensional immigration diffusions was constructed in Pitman and Yor [10] as sums of excursions selected by Poisson point processes. Similar constructions in infinite-dimensional setting were discussed in Fu and Li [5], Li [8], Li and Shiga [9] and Shiga [12]. In particular, under stronger conditions on $q(\cdot,\cdot)$, Shiga [12, Corollary 5.3] constructed purely atomic measure-valued immigration branching diffusions *without spatial motion* as the unique solution of equation (1.6) with $\delta_{x(0,a,t)}$ and $\delta_{x(s,a,t)}$ replaced by δ_a. An extension of his result to non-trivial *independent spatial motion* was given recently in Fu and Li [5] by considering measure-valued excursions. In the present situation, we have to put up with the hardship brought about by the *dependent spatial motion*. Our approach consists of several parts. In Section 2, we prove some useful characterizations of the excursions of Feller branching diffusion processes. In Section 3, we treat the case where $\langle 1,\mu\rangle > 0$ and $\langle 1,m\rangle = 0$, i.e., we construct the SDSM without immigration by excursions. The results provide useful insights into the sample path structures of the SDSM and serve as preliminaries of the construction of immigration processes. In particular, from our construction it follows immediately that the SDSM is purely atomic and, under suitable conditional probabilities, the initial positions of its atoms are i.i.d. random variables with distribution $\langle 1, \mu \rangle^{-1} \mu(dx)$. In Section 4, we consider the case where $0 < \langle 1, m \rangle < \infty$ and $q(\cdot, \cdot) \equiv 1$, i.e., the case of deterministic immigration rate. In this case, the right hand side of (1.6) is actually independent of Y_s. The problem is to show the process $\{Y_t : t \geq 0\}$ defined by this formula is a diffusion process and solves the martingale problem (1.4) and (1.3) with $q(\cdot, \cdot) \equiv 1$. This is not so easy since the immigrants have dependent spatial motion and come infinitely many times at any non-trivial time interval. Because of the dependent spatial motion we cannot compute the Laplace functional of $\{Y_t : t \geq 0\}$ as in [5] and [12]. To resolve the difficulty, we chop off from every excursion a part with length $1/n$ and construct a right continuous strong Markov process $\{X_t^{(n)} : t \geq 0\}$, which is characterized as a SDSM with positive jumps. Then we obtain the desired martingale characterization of $\{Y_t : t \geq 0\}$ from that of $\{X_t^{(n)} : t \geq 0\}$ by letting $n \to \infty$. Based on those results, we construct a pathwise unique solution of the general equation (1.6) in Section 5 using the techniques developed in [5] and [12].

2 Excursions of Feller branching diffusions

Let $\beta > 0$ be a constant and $\{B(t) : t \geq 0\}$ a standard Brownian motion. For any initial condition $\xi(0) = x$ the stochastic differential equation

$$d\xi(t) = \sqrt{\beta \xi(t)} dB(t), \quad t \geq 0, \tag{2.1}$$

has a unique solution $\{\xi(t) : t \geq 0\}$, which is a diffusion process on $[0, \infty)$. The transition semigroup $(Q_t)_{t \geq 0}$ of the process is determined by

$$\int_0^\infty e^{-zy} Q_t(x, dy) = \exp\{-xz(1 + \beta tz/2)^{-1}\}, \quad t, x, z \geq 0; \tag{2.2}$$

see e.g. Ikeda and Watanabe [6, p.236]. In this paper, we call any diffusion process $\{\xi(t) : t \geq 0\}$ a *Feller branching diffusion with constant branching rate* β, or simply a β*-branching diffusion* if it has transition semigroup $(Q_t)_{t \geq 0}$. Letting $z \to \infty$ in (2.2) we get

$$Q_t(x, \{0\}) = \exp\{-2x/\beta t\}, \quad t > 0, x \geq 0. \tag{2.3}$$

In view of (2.1), $\{\xi(t) : t \geq 0\}$ is a continuous martingale with quadratic variation $\beta \xi(t) dt$. In general, if $\{\eta(t) : t \geq 0\}$ is a continuous martingale with quadratic variation $\sigma(t)\eta(t)dt$ for a predictable process $\{\sigma(t) : t \geq 0\}$, we call it a *Feller branching diffusion with branching rate* $\{\sigma(t) : t \geq 0\}$.

Let $Q_t^\circ(x, \cdot)$ denote the restriction of the measure $Q_t(x, \cdot)$ to $(0, \infty)$. Since the origin 0 is a trap for the β-branching diffusion process, the family of kernels $(Q_t^\circ)_{t \geq 0}$ also constitute a semigroup. In view of the infinite divisibility implied by (2.2), there is a family of *canonical measures* $(\kappa_t)_{t > 0}$ on $(0, \infty)$ such that

$$\int_0^\infty (1 - e^{-zy}) \kappa_t(dy) = z(1 + \beta tz/2)^{-1}, \quad t > 0, z \geq 0. \tag{2.4}$$

Indeed, we have

$$\kappa_t(dy) = 4(\beta t)^{-2} e^{-2y/\beta t} dy, \quad t > 0, x > 0. \tag{2.5}$$

Based on (2.2) and (2.4) one may check that

$$\int_0^\infty (1-e^{-zy})\kappa_{r+t}(dy) = \int_0^\infty \kappa_r(dy)\int_0^\infty (1-e^{-zy})Q_t^\circ(x,dy), \quad r,t>0, z\geq 0. \tag{2.6}$$

Then $\kappa_r Q_t^\circ = \kappa_{r+t}$ and hence $(\kappa_t)_{t>0}$ is an *entrance law* for $(Q_t^\circ)_{t\geq 0}$. It is known that there is a unique σ-finite measure $\boldsymbol{Q}_\kappa$ on $(W_0, \mathcal{B}(W_0))$ such that

$$\boldsymbol{Q}_\kappa\{w(t_1)\in dy_1,\cdots,w(t_n)\in dy_n\} = \kappa_{t_1}(dy_1)Q^\circ_{t_2-t_1}(y_1,dy_2)\cdots Q^\circ_{t_n-t_{n-1}}(y_{n-1},dy_n) \tag{2.7}$$

for $0<t_1<t_2<\cdots<t_n$ and $y_1,y_2,\cdots,y_n\in(0,\infty)$; see e.g. Pitman and Yor [10] for details. The measure $\boldsymbol{Q}_\kappa$ is known as the *excursion law* of the β-branching diffusion. Let $\mathcal{B}_t = \mathcal{B}_t(W_0)$ denote the σ-algebra on W_0 generated by $\{w(s): 0\leq s\leq t\}$. Roughly speaking, (2.7) asserts that $\{w(t): t>0\}$ is a β-branching diffusion relative to $(\boldsymbol{Q}_\kappa, \mathcal{B}_t)$ with one-dimensional distributions $\{\kappa_t: t>0\}$.

For $r>0$, let $\tau_r(w) = r\vee\tau_0(w)$, and let $\boldsymbol{Q}_{\kappa,r}$ denote the restriction of $\boldsymbol{Q}_\kappa$ to $W_r := \{w\in W_0: \tau_0(w)>r\}$. Observe that

$$\boldsymbol{Q}_\kappa(W_r) = \boldsymbol{Q}_{\kappa,r}(W_r) = \kappa_r(0,\infty) = 2/\beta r, \quad r>0. \tag{2.8}$$

The following theorem gives a stochastic equation for the excursions.

Theorem 2.1 *For any $r>0$, the coordinate process $\{w(t): t\geq r\}$ under $\boldsymbol{Q}_{\kappa,r}\{\,\cdot\,|\mathcal{B}_r\}$ is a β-branching diffusion. Moreover, there is a measurable mapping $B: W_0\to W$ such that $\{B(w,t)\equiv B(w)(t): t\geq r\}$ under $\boldsymbol{Q}_{\kappa,r}\{\,\cdot\,|\mathcal{B}_r\}$ is a Brownian motion stopped at time $\tau_r(w)$ and*

$$dw(t) = \sqrt{\beta w(t)}dB(w,t), \quad t\geq r. \tag{2.9}$$

Proof. By (2.7), $\{w(t): t\geq r\}$ under $\boldsymbol{Q}_{\kappa,r}\{\,\cdot\,|\mathcal{B}_r\}$ is a β-branching diffusion or, equivalently, a continuous martingale with quadratic variation $\beta w(t)dt$. Thus

$$\tilde{B}_r(w,t) := \int_r^{t\wedge\tau_r}\frac{1}{\sqrt{\beta w(s)}}dw(s), \quad t\geq r, \tag{2.10}$$

under $\boldsymbol{Q}_{\kappa,r}\{\,\cdot\,|\mathcal{B}_r\}$ is a Brownian motion stopped at time $\tau_r(w)$. We may define another Markov transition semigroup $(Q_t^h)_{t\geq 0}$ on $[0,\infty)$ by

$$Q_t^h(x,dy) = \begin{cases} x^{-1}yQ_t(x,dy) & \text{for } t>0, x>0, y\geq 0,\\ y\kappa_t(dy) & \text{for } t>0, x=0, y\geq 0,\end{cases}$$

which is a Doob's h-transform of $(Q_t)_{t\geq 0}$. It is not hard to check that $\bar{\boldsymbol{Q}}_{\kappa,r}(dw) := w(r)\boldsymbol{Q}_{\kappa,r}(dw)$ defines a probability measure. Moreover, (2.7) implies that $\{w(t): 0\leq t\leq r\}$ under $\bar{\boldsymbol{Q}}_{\kappa,r}$ is a diffusion process with transition semigroup $(Q_t^h)_{t\geq 0}$ and generator $2^{-1}\beta x d^2/dx^2 + \beta d/dx$. Thus $\{2\sqrt{w(t)/\beta}: 0\leq t\leq r\}$ under $\bar{\boldsymbol{Q}}_{\kappa,r}$ is a 4-dimensional Bessel diffusion. It follows that

$$m(t) := w(t) - \beta t, \quad 0\leq t\leq r, \tag{2.11}$$

is a continuous martingale with quadratic variation $\beta w(t)dt$ and the limits

$$\bar{M}_r(w,t) := \lim_{u\to 0^+}\int_u^t \frac{1}{\sqrt{\beta w(s)}}dm(s), \quad 0<t\leq r, \tag{2.12}$$

exist in $L^2(\bar{\boldsymbol{Q}}_{\kappa,r})$-sense. Let $\bar{M}_r(w,0) = 0$. Then $\{\bar{M}_r(w,t) : 0 \leq t \leq r\}$ under $\bar{\boldsymbol{Q}}_{\kappa,r}$ is a Brownian motion. By Shiga and Watanabe [13, Theorem 3.3.ii], for any $0 < \epsilon < 1$ we have $\bar{\boldsymbol{Q}}_{\kappa,r}\{\sqrt{w(t)} < t^{(1+\epsilon)/2}$ infinitely often as $t \to 0^+\} = 0$. By (2.11) and (2.12), the limits

$$\bar{B}_r(w,t) := \lim_{u \to 0^+} \int_u^t \frac{1}{\sqrt{\beta w(s)}} dw(s), \quad 0 < t \leq r,$$

also exist in $L^2(\bar{\boldsymbol{Q}}_{\kappa,r})$-sense. Indeed, setting $\bar{B}_r(w,0) = 0$ we have

$$\bar{B}_r(w,t) = \bar{M}_r(w,t) + \int_{0^+}^t \frac{\sqrt{\beta}}{\sqrt{w(s)}} ds, \quad 0 \leq t \leq r.$$

Now let $B(w,t) = \bar{B}_r(w,t)$ for $0 \leq t \leq r$ and $B(w,t) = \bar{B}_r(w,r) + \tilde{B}_r(w,t)$ for $t \geq r$. Since $\boldsymbol{Q}_{\kappa,r}$ is absolutely continuous relative to $\bar{\boldsymbol{Q}}_{\kappa,r}$, we see that $B(w,t)$ is uniquely defined on W_r out of a $\boldsymbol{Q}_\kappa$-null set and it does not depend on the particular choice of $r > 0$. Using this property, we can extend the definition of $B(w,t)$ to the whole space W_0 so that $\{B(w,t) : t \geq r\}$ under $\boldsymbol{Q}_{\kappa,r}\{\,\cdot\,|\mathcal{B}_r\}$ is a Brownian motion stopped at time $\tau_r(w)$ and satisfies (2.9). $\square$

3 SDSM without immigration

In this section, we give a rigorous construction of a purely atomic version of the SDSM with an arbitrary initial state. The results are useful in our study of the associated immigration processes. We consider a general branching density $\sigma(\cdot) \in C(\mathbb{R})^+$ and assume there is a constant $\epsilon > 0$ such that $\sigma(x) \geq \epsilon$ for all $x \in \mathbb{R}$. From the results in Dawson *et al* [2] and Wang [16] we know that the generator $\mathcal{L}$ of the SDSM with branching density $\sigma(\cdot)$ is expressed as

$$\begin{aligned}\mathcal{L}F(\nu) &= \frac{1}{2}\rho(0)\int_{\mathbb{R}} \frac{d^2}{dx^2}\frac{\delta F(\nu)}{\delta\nu(x)}\nu(dx) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} \rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\nu)}{\delta\nu(x)\delta\nu(y)}\nu(dx)\nu(dy) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} \sigma(x)\frac{\delta^2 F(\nu)}{\delta\nu(x)^2}\nu(dx). \end{aligned} \tag{3.1}$$

The domain of $\mathcal{L}$ include functions on $M(\mathbb{R})$ of the form $F_{n,f}(\nu) := \int f d\nu^n$ with $f \in C^2(\mathbb{R}^n)$ and functions of the form

$$F_{f,\{\phi_i\}}(\nu) := f(\langle\phi_1,\nu\rangle, \cdots, \langle\phi_n,\nu\rangle) \tag{3.2}$$

with $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$. Let $\mathcal{D}(\mathcal{L})$ denote the collection of all those functions.

Theorem 3.1 (Dawson *et al*, 2001; Wang, 1998) *For each $\mu \in M(\mathbb{R})$ there is a unique probability measure $\boldsymbol{Q}_\mu$ on the space $C([0,\infty), M(\mathbb{R}))$ such that $\boldsymbol{Q}_\mu\{w_0 = \mu\} = 1$ and under $\boldsymbol{Q}_\mu$ the coordinate process $\{w_t : t \geq 0\}$ solves the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem. Therefore, the system $\{\boldsymbol{Q}_\mu : \mu \in M(\mathbb{R})\}$ defines a diffusion process generated by the closure of $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$.*

Theorem 3.2 *A probability measure $\boldsymbol{Q}_\mu$ on $C([0,\infty), M(\mathbb{R}))$ is a solution of the $(\mathcal{L},\ \mathcal{D}(\mathcal{L}))$-martingale problem with $\boldsymbol{Q}_\mu\{w_0 = \mu\} = 1$ if and only if for each $\phi \in C^2(\mathbb{R})$,*

$$M_t(\phi) = \langle \phi, w_t\rangle - \langle \phi, \mu\rangle - \frac{1}{2}\rho(0)\int_0^t \langle \phi'', w_s\rangle ds, \quad t \geq 0, \tag{3.3}$$

under $\boldsymbol{Q}_\mu$ is a continuous martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, w_s\rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi', w_s\rangle^2 dz. \tag{3.4}$$

Proof. Suppose that $\boldsymbol{Q}_\mu$ is a probability measure on $C([0,\infty), M(\mathbb{R}))$ such that $\boldsymbol{Q}_\mu\{w_0 = \mu\} = 1$ and

$$F(w_t) - F(w_0) - \int_0^t \mathcal{L}F(w_s)ds, \quad t \geq 0, \tag{3.5}$$

is a continuous martingale for every $F \in \mathcal{D}(\mathcal{L})$. Comparing the martingales related to the functions $\mu \mapsto \langle \phi, \mu\rangle$ and $\mu \mapsto \langle \phi, \mu\rangle^2$ and using Itô's formula we see that (3.3) is a continuous martingale with quadratic variation process (3.4). Conversely, suppose that $\boldsymbol{Q}_\mu$ is a probability measure on $C([0,\infty), M(\mathbb{R}))$ under which (3.3) is a continuous martingale with quadratic variation process (3.4) for each $\phi \in C^2(\mathbb{R})$. Observe that for the function $F_{f,\{\phi_i\}}$ defined by (3.2) we have

$$\begin{aligned}\mathcal{L}F_{f,\{\phi_i\}}(\nu) \quad = \quad & \frac{1}{2}\rho(0)\sum_{i=1}^n f_i'(\langle \phi_1, \nu\rangle, \cdots, \langle \phi_n, \nu\rangle)\langle \phi_i'', \nu\rangle \\ & + \frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle \phi_1, \nu\rangle, \cdots, \langle \phi_n, \nu\rangle)\int_{\mathbb{R}^2} \rho(x-y)\phi_i'(x)\phi_j'(y)\nu(dx)\nu(dy) \\ & + \frac{1}{2}\sum_{i,j=1}^n f_{ij}''(\langle \phi_1, \nu\rangle, \cdots, \langle \phi_n, \nu\rangle)\langle \sigma\phi_i\phi_j, \nu\rangle.\end{aligned}$$

By Itô's formula we see that (3.5) is a continuous martingale if $F = F_{f,\{\phi_i\}}$. Then the theorem follows by an approximation of an arbitrary $F \in \mathcal{D}(\mathcal{L})$. □

We now consider the construction of the trajectories of the SDSM. Suppose that $(\Omega, \mathcal{F}, \boldsymbol{P})$ is a complete standard probability space on which we have a white noise $\{W(ds, dy)\}$ on $[0,\infty) \times \mathbb{R}$ based on the Lebesgue measure; see e.g. Walsh [14]. By Dawson *et al* [2, Lemma 3.1] or Wang [15, Lemma 1.3], for any $a \in \mathbb{R}$ the equation

$$x(t) = a + \int_0^t \int_{\mathbb{R}} h(y - x(s))W(ds, dy), \quad t \geq 0, \tag{3.6}$$

has a unique solution $\{x(a,t) : t \geq 0\}$. For a fixed constant $\beta > 0$ let

$$\psi(a,t) = \beta^{-1}\int_0^t \sigma(x(a,s))ds, \quad t \geq 0, a \in \mathbb{R}. \tag{3.7}$$

Suppose we also have on $(\Omega, \mathcal{F}, \boldsymbol{P})$ a sequence of independent β-branching diffusions $\{\xi_i(t) : t \geq 0; i = 1, 2, \cdots\}$ independent of $\{W(ds, dy)\}$. We assume each $\xi_i(0) \geq 0$ is deterministic and

$\sum_{i=1}^{\infty} \xi_i(0) < \infty$. Let $\{a_i : i = 1, 2, \cdots\}$ be a sequence of real numbers. Let $\xi_i(a, t) = \xi_i(\psi(a, t))$ and let $\mathcal{G}_t$ be the σ-algebra generated by all $\boldsymbol{P}$-null sets and the family of random variables

$$\{W([0, s] \times B) : 0 \leq s \leq t; B \in \mathcal{B}(\mathbb{R})\} \quad \text{and} \quad \{\xi_i(a_i, s) : 0 \leq s \leq t; i = 1, 2, \cdots\}. \tag{3.8}$$

For $r > 0$ let $n(r) = \#\{i : \xi_i(r) > 0\}$ and $n^\psi(r) = \#\{i : \xi_i(a_i, r) > 0\}$, where $\#\{\cdots\}$ denotes the number of elements of the set $\{\cdots\}$. Since zero is a trap for the β-branching diffusion, both $n(r)$ and $n^\psi(r)$ are a.s. non-increasing in $r > 0$.

Lemma 3.1 *We have $n(r) < \infty$ and $n^\psi(r) < \infty$ a.s. for each $r > 0$.*

Proof. In view of (2.3), we have

$$\sum_{i=1}^{\infty} \boldsymbol{P}\{\xi_i(r) > 0\} = \sum_{i=1}^{\infty} [1 - \exp\{-2\xi_i(0)/\beta r\}] \leq \frac{2}{\beta r} \sum_{i=1}^{\infty} \xi_i(0) < \infty.$$

Then an application of the Borel-Cantelli lemma yields that a.s. $n(r) < \infty$. By (3.7) and the assumption $\sigma(x) \geq \epsilon$, we have $\psi(a_i, t) \geq \epsilon t/\beta$. But 0 is a trap for $\{\xi_i(t) : t \geq 0\}$, so $n^\psi(r) < n(\epsilon r/\beta) < \infty$ a.s. for each $r > 0$. $\square$

Theorem 3.3 *The process $\{X_t : t \geq 0\}$ defined by*

$$X_t = \sum_{i=1}^{\infty} \xi_i(a_i, t) \delta_{x(a_i, t)}, \quad t \geq 0, \tag{3.9}$$

relative to $(\mathcal{G}_t)_{t \geq 0}$ is an SDSM.

Proof. By the assumption of independence, $\{\xi_i(t) : t \geq 0; i = 1, 2, \cdots\}$ and $\{W([0, t] \times B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\}$ are martingales relative to the filtration $(\mathcal{G}_t)_{t \geq 0}$. Given $\{W(ds, dy)\}$, the processes $\{\psi(a_i, \cdot) : i = 1, 2, \cdots\}$ are deterministic. Then the time-changed processes $\{\xi_i(a_i, \cdot) : i = 1, 2, \cdots\}$ are independent martingales under $\boldsymbol{P}\{\cdot | W\}$. Moreover, we have a.s.

$$\langle \xi_i(a_i) \rangle(t) = \int_0^{\psi(a_i, t)} \beta \xi_i(s) ds = \int_0^t \beta \xi_i(a_i, u) d\psi(a_i, u) = \int_0^t \sigma(x_i(u)) \xi_i(a_i, u) du$$

first under $\boldsymbol{P}\{\cdot | W\}$ and then under the non-conditional probability $\boldsymbol{P}$. By the same reasoning we get $\langle \xi_i(a_i), \xi_j(a_j) \rangle(t) \equiv 0$ a.s. under $\boldsymbol{P}$ for $i \neq j$. By Itô's formula,

$$\begin{aligned} \xi_i(a_i, t) \phi(x(a_i, t)) &= \xi_i(0) \phi(a_i) + \int_0^t \int_{\mathbb{R}} \xi_i(a_i, s) \phi'(x(a_i, s)) h(y - x(a_i, s)) W(ds, dy) \\ &\quad + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}} \xi_i(a_i, s) \phi''(x(a_i, s)) h(y - x(a_i, s))^2 dy \\ &\quad + \int_0^t \phi(x(a_i, s)) d\xi_i(a_i, s) \end{aligned}$$

for $\phi \in C^2(\mathbb{R})$. Taking the summation $\sum_{i=1}^{\infty}$ we get

$$\langle \phi, X_t \rangle = \langle \phi, X_0 \rangle + M_t(\phi) + \frac{1}{2} \rho(0) \int_0^t \langle \phi'', X_s \rangle ds, \quad t \geq 0,$$

where

$$M_t(\phi) := \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s\rangle W(ds,dy) + \sum_{i=1}^{\infty} \int_0^t \phi(x(a_i,s))d\xi_i(a_i,s),$$

is a continuous martingale relative to $(\mathcal{G}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi', X_s\rangle^2 dz + \int_0^t \langle \sigma\phi^2, X_s\rangle ds.$$

Then we have the desired result by Theorem 3.2. □

By Theorem 3.3, if the SDSM is started from an initial state in $M_a(\mathbb{R})$, purely atomic measures on $\mathbb{R}$, it lives in this space forever. More precisely, the position of its ith atom is described by $\{x(a_i,t) : t \geq 0\}$ and its mass by $\{\xi_i(a_i,t) : t \geq 0\}$. As observed in Wang [16, p.756], each $\{x(a_i,t) : t \geq 0\}$ is a Brownian motion with quadratic variation $\rho(0)dt$. If $a_i = a_j$, we have $x(a_i,t) = x(a_j,t)$ for all $t \geq 0$ by the uniqueness of solution of (3.6). On the contrary, if $a_i \neq a_j$, then $\{x(a_i,t) : t \geq 0\}$ and $\{x(a_j,t) : t \geq 0\}$ never hit each other.

To consider a more general initial state, we need a lemma for Poisson random measures. Suppose that E and F are metrizable topological spaces. Let $\mu \in M(E)$ and let $q(x,dy)$ be a Borel probability kernel from E to F. Then

$$\int_E \int_F h(x,y)\nu(dx,dy) = \int_E \mu(dx) \int_F h(x,y)q(x,dy), \quad h \in C(E\times F), \tag{3.10}$$

defines a measure $\nu \in M(E\times F)$. Let Y be a Poisson random measure on $E\times F$ with intensity ν and let $\eta = Y(E\times F)$. The following lemma shows that we can recover the kernel $q(x,dy)$ from the atoms of Y by a *suitable enumeration*.

Lemma 3.2 *In the situation described above, $X(\cdot) := Y(\cdot\times F)$ defines a Poisson random measure on E with intensity μ. Suppose that $\{(x_i,y_i) : i = 1,\cdots,\eta\}$ is an enumeration of the atoms of $Y(dx,dy)$ which only uses information from X. Then, given $\eta = k$ and $x_i = c_i \in E$ $(i = 1,\cdots,k)$, the sequence $\{y_i : i = 1,\cdots,k\}$ is formed of independent random variables with distributions $\{q(c_i,\cdot) : i = 1,\cdots,k\}$.*

Proof. We first consider a special version of the Poisson random measure Y constructed as follows. Let η be a Poisson random variable with parameter $\mu(E)$ and $\{(x_i,y_i) : i = 1,2,\cdots\}$ a sequence of i.i.d. random variables in $E\times F$ which are independent of η and have common distribution $\mu(E)^{-1}\nu$. Then

$$Y(dx,dy) := \sum_{i=1}^{\eta} \delta_{(x_i,y_i)}(dx,dy), \quad x\in E, y\in F,$$

is a Poisson random measure Y with intensity $\nu(dx,dy)$ and

$$X(dx) := \sum_{i=1}^{\eta} \delta_{x_i}(dx), \quad x\in E,$$

is a Poisson random measure on E with intensity $\mu(dx)$. Observe that, for any integer $k \geq 1$, any permutation $\{i_1,\cdots,i_k\}$ of $\{1,\cdots,k\}$ and any sequence $\{c_j : j = 1,\cdots,k\} \subset E$, under

$\boldsymbol{P}\{\,\cdot\,|x_{i_j} = c_j : j = 1, \cdots, k\}$ the sequence $\{y_{i_j} : j = 1, \cdots, k\}$ consists of independent random variables with distributions $\{q(c_j, \cdot) : j = 1, \cdots, k\}$. This proves the lemma for the special version of Y. The result for an arbitrary realization of the Poisson random measure holds by the uniqueness of distribution. $\square$

Now we consider the construction of the SDSM with a general initial state $\mu \in M(\mathbb{R})$ with $\langle 1, \mu\rangle > 0$. Suppose we have on some complete standard probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ a time-space white noise $W(ds, dy)$ on $[0, \infty) \times \mathbb{R}$ based on the Lebesgue measure and a Poisson random measure $N(da, dw)$ on $\mathbb{R} \times W_0$ with intensity $\mu(da)\boldsymbol{Q}_\kappa(dw)$, where $\boldsymbol{Q}_\kappa$ denotes the excursion law of the β-branching diffusion defined by (2.7). Assume that $\{W(ds, dy)\}$ and $\{N(da, dw)\}$ are independent. Let $m(r) = N(\mathbb{R} \times W_r)$ for $r > 0$. In view of (2.8), we have

$$\boldsymbol{E}\{m(r)\} = \langle 1, \mu\rangle \boldsymbol{Q}_\kappa(W_r) = 2\langle 1, \mu\rangle / r\beta,$$

and hence a.s. $m(r) < \infty$. Thus we can enumerate the atoms of $N(da, dw)$ into a sequence $\mathrm{supp}(N) = \{(a_i, w_i) : i = 1, 2, \cdots\}$ such that a.s. $\tau_0(w_{i+1}) < \tau_0(w_i)$ for all $i \geq 1$ and $\tau_0(w_i) \to 0$ as $i \to \infty$. Clearly, $m(r) = i$ for $\tau_0(w_{i+1}) \leq r < \tau_0(w_i)$, and $m(r) = 0$ for $r \geq \tau_0(w_1)$. Let $\psi(a, t)$ be defined by (3.7) and let $w(a, t) = w(\psi(a, t))$ for $w \in W_0$. For $r > 0$ let $\mathrm{supp}_r(N) = \{(a_i, w_i) : i = 1, \cdots, m(r)\}$, let $\mathrm{supp}_r^\psi(N) = \{(a_i, w_i) : w_i(a_i, r) > 0; i = 1, 2, \cdots\}$ and let $m^\psi(r) = \#\{\mathrm{supp}_r^\psi(N)\}$. As in the proof of Lemma 3.2, one can show that a.s. $m^\psi(r) < \infty$. Then we have the following

Lemma 3.3 *For each $r > 0$, we have a.s. $m(r) < \infty$ and $m^\psi(r) < \infty$.*

Let us see how to recover branching diffusions under some conditional probabilities by re-ordering suitably the atoms of $N(da, dw)$. For $t \geq 0$ let $\mathcal{G}_t$ be the σ-algebra generated by all $\boldsymbol{P}$-null sets and the families random variables

$$\{W([0, s] \times B) : 0 \leq s \leq t; B \in \mathcal{B}(\mathbb{R})\} \quad \text{and} \quad \{w_i(a_i, s) : 0 \leq s \leq t; i = 1, 2, \cdots\}. \tag{3.11}$$

Lemma 3.4 *For each $r > 0$ there is an enumeration $\{(a_{i_j}, w_{i_j}) : j = 1, \cdots, m^\psi(r)\}$ of $\mathrm{supp}_r^\psi(N)$ which only uses information from $\mathcal{G}_r$ and satisfies that $\{w_{i_j}(\psi(a_{i_j}, r)+t) : t \geq 0; j = 1, \cdots, m^\psi(r)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_r\}$ are independent β-branching diffusions which are independent of $\{W(dt, dy) : t \geq r; y \in \mathbb{R}\}$.*

Proof. As observed in the proof of Lemma 3.1 we have $\psi(a_i, r) \geq \epsilon r/\beta$. Since the finite measure $\kappa_{\epsilon r/\beta}(dy)$ on $(0, \infty)$ is absolutely continuous, the elements of the random set $\{w_i(\epsilon r/\beta) : i = 1, \cdots, m(\epsilon r/\beta)\}$ are a.s. mutually distinct. Let $\mathcal{F}_t := \sigma(\{w_i(s) : 0 \leq s \leq t; i = 1, 2, \cdots\})$ and let $\{(a_{k_j}, w_{k_j}) : j = 1, \cdots, m(\epsilon r/\beta)\}$ be the enumeration of $\mathrm{supp}_{\epsilon r/\beta}(N)$ so that $w_{k_1}(\epsilon r/\beta) < \cdots < w_{k_{m(\epsilon r/\beta)}}(\epsilon r/\beta)$. Note that this enumeration only uses information from $\mathcal{F}_{\epsilon r/\beta}$. An application of Theorem 2.1 and Lemma 3.2 shows that $\{w_{k_j}(\epsilon r/\beta + t) : t \geq 0; j = 1, \cdots, m(\epsilon r/\beta)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{F}_{\epsilon r/\beta}\}$ are independent β-branching diffusions. By the independence of $\{W(ds, dy)\}$ and $\{N(da, dw)\}$, we know that $\{w_{k_j}(\epsilon r/\beta + t) : t \geq 0; j = 1, \cdots, m(\epsilon r/\beta)\}$ are also independent β-branching diffusions under $\boldsymbol{P}\{\,\cdot\,|\mathcal{F}^e_{\epsilon r/\beta}\}$, where $\mathcal{F}^e_{\epsilon r/\beta}$ is the σ-algebra generated by $\mathcal{F}_{\epsilon r/\beta}$ and the family of random variables $\{W([0, s] \times B) : 0 \leq s \leq r; B \in \mathcal{B}(\mathbb{R})\}$. Moreover, $\{w_{k_j}(\epsilon r/\beta + t) : t \geq 0; j = 1, \cdots, m(\epsilon r/\beta)\}$ and $\{W(dt, dy) : t \geq r; y \in \mathbb{R}\}$ are independent under $\boldsymbol{P}\{\,\cdot\,|\mathcal{F}^e_{\epsilon r/\beta}\}$. Observe that $\mathcal{G}_r$ is generated by $\mathcal{F}^e_{\epsilon r/\beta}$, all $\boldsymbol{P}$-null sets and the family of random variables $\{w_i(s) : \epsilon r/\beta \leq s \leq \psi(a_i, r); i = 1, \cdots, m(\epsilon r/\beta)\}$. It follows that $\{w_{k_j}(\psi(a_{k_j}, r) + t) : t \geq$

$0; j = 1, \cdots, m(\epsilon r/\beta)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_r\}$ are also independent β-branching diffusions which are independent of $\{W(dt, dy) : t \geq r; y \in \mathbb{R}\}$. Finally, we remove the elements of $\{(a_{k_j}, w_{k_j}) : j = 1, \cdots, m(\epsilon r/\beta)\}$ with $w(a_{k_j}, r) = w_{k_j}(\psi(a_{k_j}, r)) = 0$ and relabel the remaining elements to get an enumeration $\{(a_{i_j}, w_{i_j}) : j = 1, \cdots, m^\psi(r)\}$ of $\mathrm{supp}_r^\psi(N)$ so that $w_{i_1}(\epsilon r/\beta) < \cdots < w_{i_{m^\psi(r)}}(\epsilon r/\beta)$. Clearly, this enumeration only uses information from $\mathcal{G}_r$ and has the desired property. $\square$

Theorem 3.4 *Let $\{X_t : t \geq 0\}$ be defined by $X_0 = \mu$ and*

$$X_t = \sum_{i=1}^{\infty} w_i(a_i, t)\delta_{x(a_i,t)} = \int_{\mathbb{R}} \int_{W_0} w(a, t)\delta_{x(a,t)} N(da, dw), \quad t > 0. \tag{3.12}$$

Then $\{X_t : t \geq 0\}$ relative to $(\mathcal{G}_t)_{t\geq 0}$ is an SDSM.

Proof. Let $(Q_t)_{t\geq 0}$ denote the transition semigroup of the SDSM. For $r > 0$ we see by Lemma 3.4 and Theorem 3.3 that $\{X_t : t \geq r\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_r\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. Thus $\{X_t : t > 0\}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$. We shall prove that the random measure X_t has distribution $Q_t(\mu, \cdot)$ for $t > 0$ so that the desired result follows from the uniqueness of distribution of the SDSM. Using the notation of the proof of Lemma 3.4, we have that $\{w_{k_j}(\epsilon r/\beta + t) : t \geq 0; j = 1, \cdots, m(\epsilon r/\beta)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{F}_{\epsilon r/\beta}\}$ are independent β-branching diffusions which are independent of $\{W(dt, dy) : t \geq 0; y \in \mathbb{R}\}$. By Theorem 3.3,

$$X_t^{(r)} := \sum_{j=1}^{m(\epsilon r/\beta)} w_{k_j}(\epsilon r/\beta + \psi(a_{k_j}, t))\delta_{x(a_{k_j},t)}, \quad t \geq 0, \tag{3.13}$$

under $\boldsymbol{P}\{\cdot\,|\mathcal{F}_{\epsilon r/\beta}\}$ is an SDSM with initial state

$$X_0^{(r)} = \sum_{j=1}^{m(\epsilon r/\beta)} w_{k_j}(\epsilon r/\beta)\delta_{a_{k_j}}.$$

This implies that $\{X_t^{(r)} : t \geq 0\}$ under the non-conditional probability $\boldsymbol{P}$ is an SDSM relative to the filtration $(\mathcal{H}_t^{(r)})_{t\geq 0}$, where $\mathcal{H}_t^{(r)}$ is generated by $\mathcal{F}_{\epsilon r/\beta}$ and $\{X_s^{(r)} : 0 \leq s \leq t\}$. For any $f \in C(\mathbb{R})^+$, we have by (2.4) that

$$\begin{aligned}
\boldsymbol{E}\exp\{-\langle f, X_0^{(r)}\rangle\} &= \boldsymbol{E}\exp\bigg\{-\int_{\mathbb{R}}\int_{W_0} w(\epsilon r/\beta)f(a)N(da, dw)\bigg\} \\
&= \exp\bigg\{-\int_{\mathbb{R}}\mu(da)\int_{W_0}(1 - e^{-w(\epsilon r/\beta)f(a)})\boldsymbol{Q}_\kappa(dw)\bigg\} \\
&= \exp\bigg\{-\int_{\mathbb{R}} f(a)(1 + \epsilon r f(a)/2)^{-1}\mu(da)\bigg\},
\end{aligned}$$

which converges to $\exp\{-\langle f, \mu\rangle\}$ as $r \to 0$. Thus $X_0^{(r)} \to \mu$ in distribution as $r \to 0$. Indeed, if we set $X_0^{(0)} = \mu$, then $\{X_0^{(r)} : r \geq 0\}$ is a measure-valued branching diffusion without migration;

see [12, Theorem 3.6]. By the Feller property of the SDSM, the distribution of $X_t^{(r)}$ converges to $Q_t(\mu,\cdot)$ as $r \to 0$. Since $\psi(a_{k_j},t) \geq \epsilon t/\beta$, we can rewrite (3.13) as

$$X_t^{(r)} := \sum_{i=1}^{m(\epsilon t/\beta)} w_i(\epsilon r/\beta + \psi(a_i,t))\delta_{x(a_i,t)}.$$

Then for fixed $t > 0$ we have $X_t^{(r)} \to X_t$ a.s. as $r \to 0$ and hence X_t has distribution $Q_t(\mu,\cdot)$. $\square$

By Theorem 3.4, the SDSM started with an arbitrary initial measure enters the space $M_a(\mathbb{R})$ of purely atomic measures immediately and lives in this space forever; see also Wang [15]. From Lemma 3.2 we know that for any $r > 0$ the family $\{a_{i_j} : j = 1,\cdots,m^\psi(r)\}$ under the regular conditional probability $\boldsymbol{P}\{\cdot|m^\psi(r)\}$ are i.i.d. random variables with distribution $\langle 1,\mu\rangle^{-1}\mu(dx)$. This gives an intuitive description of the locations $\{a_{i_j} : j = 1,\cdots,m^\psi(r)\}$ of the "ancestors" at the initial time of X_r. By Lemma 3.4, each $\{w_{i_j}(a_{i_j},t) : t \geq r\}$ under $\boldsymbol{P}\{\cdot|\mathcal{G}_r\}$ is a Feller branching diffusion with branching rate $\{\sigma(x(a_{i_j},t)) : t \geq r\}$. Then we have

$$dw_{i_j}(a_{i_j},t) = \sqrt{\sigma(x(a_{i_j},t))w_{i_j}(a_{i_j},t)}dB_{i_j}(r,t), \quad t \geq r, \tag{3.14}$$

for a Brownian motion $\{B_{i_j}(r,t) : t \geq r\}$ stopped at $\tau_0(w_{i_j}(a_{i_j}))$. However, under any enumeration the whole excursion, $\{w_{i_j}(a_{i_j},t) : t \geq 0\}$ is not a Feller branching diffusion, otherwise the initial condition $w_{i_j}(a_{i_j},0) = 0$ would imply $w_{i_j}(a_{i_j},t) = 0$ for all $t \geq 0$. Therefore, the constructions (3.9) and (3.12) of the SDSM are essentially different. Indeed, the purely atomic version of the SDSM with a general initial state can only be constructed by excursions, not usual Feller branching diffusions.

4 SDSM with deterministic immigration

In this section, we construct some immigration processes by one-dimensional excursions carried by stochastic flows. To simplify the discussion, we assume the branching density is a constant $\sigma > 0$. Suppose that $m \in M(\mathbb{R})$ satisfies $\langle 1,m\rangle > 0$. Let $\mathcal{L}$ be given by (3.1) and define

$$\mathcal{J}F(\nu) = \mathcal{L}F(\nu) + \int_{\mathbb{R}} \frac{\delta F(\nu)}{\delta\nu(x)}m(dx), \quad \nu \in M(\mathbb{R}). \tag{4.1}$$

Setting $\mathcal{D}(\mathcal{J}) = \mathcal{D}(\mathcal{L})$, we shall see that the $(\mathcal{J},\mathcal{D}(\mathcal{J}))$-martingale problem is equivalent with the one given by (1.3) and (1.4) with deterministic immigration rate $q(\cdot,\cdot) \equiv 1$.

Suppose that $(\Omega,\mathcal{F},\boldsymbol{P})$ is a complete standard probability space on which we have: (i) a white noise $W(ds,dy)$ on $[0,\infty)\times\mathbb{R}$ based on the Lebesgue measure; (ii) a sequence of independent σ-branching diffusions $\{\xi_i(t) : t \geq 0\}$ with $\xi_i(0) \geq 0$ $(i = 1,2,\cdots)$; (iii) a Poisson random measure $N(ds,da,dw)$ with intensity $ds m(da)\boldsymbol{Q}_\kappa(dw)$ on $[0,\infty)\times\mathbb{R}\times W_0$, where $\boldsymbol{Q}_\kappa$ denotes the excursion law of the σ-branching diffusion. We assume that $\sum_{i=1}^\infty \xi_i(0) < \infty$ and that $\{W(ds,dy)\}$, $\{\xi_i(t)\}$ and $\{N(ds,da,dw)\}$ are independent of each other. Given $(r,a) \in [0,\infty)\times\mathbb{R}$, let $\{x(r,a,t) : t \geq r\}$ denote the unique solution of

$$x(t) = a + \int_r^t \int_{\mathbb{R}} h(y - x(s))W(ds,dy), \quad t \geq r. \tag{4.2}$$

For $t \geq 0$ let $\mathcal{G}_t$ be the σ-algebra generated by all $\boldsymbol{P}$-null sets and the families of random variables

$$\{W([0,s] \times B), \xi_i(s) : 0 \leq s \leq t; B \in \mathcal{B}(\mathbb{R}), i = 1, 2, \cdots\} \tag{4.3}$$

and

$$\{N(J \times A) : J \in \mathcal{B}([0,s] \times \mathbb{R}); A \in \mathcal{B}_{t-s}(W_0); 0 \leq s \leq t\}. \tag{4.4}$$

For a sequence $\{a_i\} \subset \mathbb{R}$ let

$$Y_t = \sum_{i=1}^{\infty} \xi_i(t)\delta_{x(0,a_i,t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{x(s,a,t)} N(ds, da, dw), \quad t \geq 0. \tag{4.5}$$

(Here and in the sequel we make the convention that $\int_0^t = \int_{(0,t]} \cdot$.)

We shall prove that $\{Y_t : t \geq 0\}$ is a.s. continuous and solves the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$. Let $W_{1/n} = \{w \in W_0 : \tau_0(w) > 1/n\}$ and recall from (2.8) that $\boldsymbol{Q}_\kappa(W_{1/n}) = 2n/\sigma$. To prove the continuity of $\{Y_t : t \geq 0\}$ we consider the following approximating sequence:

$$Y_t^{(n)} = \sum_{i=1}^{\infty} \xi_i(t)\delta_{x(0,a_i,t)} + \int_0^t \int_{\mathbb{R}} \int_{W_{1/n}} w(t-s)\delta_{x(s,a,t)} N(ds, da, dw), \quad t \geq 0. \tag{4.6}$$

Lemma 4.1 *Both $\{Y_t : t \geq 0\}$ and $\{Y_t^{(n)} : t \geq 0\}$ are a.s. continuous, and for any $T > 0$ and $\phi \in C(\mathbb{R})^+$ we have a.s. $\{\langle \phi, Y_t^{(n)} \rangle : 0 \leq t \leq T\}$ converges to $\{\langle \phi, Y_t \rangle : 0 \leq t \leq T\}$ increasingly and uniformly as $n \to \infty$.*

Proof. Let $N_1(ds, dw)$ denote the image of $N(ds, da, dw)$ under the mapping $(s, a, w) \mapsto (s, w)$. Then $N_1(ds, dw)$ is a Poisson random measure on $[0, \infty) \times W_0$ with intensity $\langle m, 1 \rangle ds \boldsymbol{Q}_\kappa(dw)$ and is independent of $\{\xi_i(t) : t \geq 0; i = 1, 2, \cdots\}$. Note that

$$\langle 1, Y_t \rangle = \sum_{i=1}^{\infty} \xi_i(t) + \int_0^t \int_{W_0} w(t-s) N_1(ds, dw), \quad t \geq 0.$$

By Pitman and Yor [10, Theorem 4.1], $\{\langle 1, Y_t \rangle : t \geq 0\}$ is a diffusion process with generator $2^{-1}\sigma x d^2/dx^2 + \langle 1, m \rangle d/dx$. Let $\Omega_1 \in \mathcal{F}$ be a set with full $\boldsymbol{P}$-measure such that $\{\langle 1, Y_t(\omega) \rangle : t \geq 0\}$ is continuous and $N(\omega, [0,n] \times \mathbb{R} \times W_{1/n}) < \infty$ for all $n \geq 1$ and $\omega \in \Omega_1$. For $\omega \in \Omega_1$ and $\phi \in C(\mathbb{R})^+$, we have that $\{\langle \phi, Y_t^{(n)}(\omega) \rangle : t \geq 0\}$ is continuous and converges to $\{\langle \phi, Y_t(\omega) \rangle : t \geq 0\}$ increasingly as $n \to \infty$. Then $\{\langle \phi, Y_t(\omega) \rangle : t \geq 0\}$ is lower semi-continuous. The same reasoning shows that

$$\langle \|\phi\| - \phi, Y_t(\omega) \rangle = \|\phi\| \langle 1, Y_t(\omega) \rangle - \langle \phi, Y_t(\omega) \rangle, \quad t \geq 0,$$

is also lower semi-continuous. Since $\{\langle 1, Y_t(\omega) \rangle : t \geq 0\}$ is continuous, we conclude that $\{\langle \phi, Y_t(\omega) \rangle : t \geq 0\}$ is continuous, giving the desired results. $\square$

To show $\{Y_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$, we consider another approximating sequence $\{X_t^{(n)} : t \geq 0\}$ defined by

$$X_t^{(n)} = \sum_{i=1}^{\infty} \xi_i(t)\delta_{x(0,a_i,t)} + \int_0^t \int_{\mathbb{R}} \int_{W_{1/n}} w(t-s+1/n)\delta_{x(s,a,t)} N(ds, da, dw). \tag{4.7}$$

Note that the first parts of the excursions with length $1/n$ were chopped off in taking the second summation in (4.7).

Lemma 4.2 *The process $\{X_t^{(n)} : t \geq 0\}$ is a.s. càdlàg and for any $T > 0$ and $\phi \in C(\mathbb{R})$ we have a.s. $\{\langle \phi, X_t^{(n)} \rangle : 0 \leq t \leq T\}$ converges to $\{\langle \phi, Y_t \rangle : 0 \leq t \leq T\}$ uniformly as $n \to \infty$.*

Proof. As in the proof of Lemma 4.1, we may assume $\xi_i(\cdot) \equiv 0$ for all $i \geq 1$. The first assertion holds since $N([0,n] \times \mathbb{R} \times W_{1/n}) < \infty$ a.s. for all $n \geq 1$. Clearly, we have a.s. $\langle 1, X_t^{(n)} \rangle \leq \langle 1, Y_{t+1/n} \rangle$ simultaneously for all $t \geq 0$. By Lemma 4.1, there is a set $\Omega_2 \in \mathcal{F}$ with full $\boldsymbol{P}$-measure such that $\{\langle 1, Y_t^{(n)} \rangle : 0 \leq t \leq T\}$ converges to $\{\langle 1, Y_t \rangle : 0 \leq t \leq T\}$ uniformly as $n \to \infty$ for all $T > 0$ and $\omega \in \Omega_2$. Fix $T > 0$ and $\omega \in \Omega_2$. For $\varepsilon > 0$ let $m(\omega) \geq 1$ be an integer such that

$$\langle 1, Y_t(\omega) \rangle - \langle 1, Y_t^{(n)}(\omega) \rangle < \epsilon, \quad 0 \leq t \leq T+1,$$

for $n \geq m(\omega)$ or, equivalently,

$$\int_0^t \int_{\mathbb{R}} \int_{W_0 \setminus W_{1/m(\omega)}} w(t-s) N(\omega, ds, da, dw) < \epsilon, \quad 0 \leq t \leq T+1.$$

Since $N(\omega, [0,T] \times \mathbb{R} \times W_{1/m(\omega)}) < \infty$, there is an integer $M(\omega) \geq m(\omega)$ such that, for $n \geq M(\omega)$,

$$\int_0^t \int_{\mathbb{R}} \int_{W_{1/m(\omega)}} |w(t-s+1/n) - w(t-s)| N(\omega, ds, da, dw) < \epsilon, \quad 0 \leq t \leq T.$$

Then for $n \geq M(\omega)$ and $\phi \in C(\mathbb{R})$ we have

$$\begin{aligned}
& |\langle \phi, X_t^{(n)}(\omega) \rangle - \langle \phi, Y_t(\omega) \rangle| \\
\leq\ & \int_0^t \int_{\mathbb{R}} \int_{W_{1/m(\omega)}} \|\phi\| |w(t-s+1/n) - w(t-s)| N(\omega, ds, da, dw) \\
& + \int_0^t \int_{\mathbb{R}} \int_{W_0 \setminus W_{1/m(\omega)}} \|\phi\| |w(t-s+1/n) + w(t-s)| N(\omega, ds, da, dw) \\
<\ & 3\|\phi\|\epsilon
\end{aligned}$$

for $0 \leq t \leq T$. That is, $\{\langle \phi, X_t^{(n)}(\omega) \rangle : 0 \leq t \leq T\}$ converges to $\{\langle \phi, Y_t(\omega) \rangle : 0 \leq t \leq T\}$ uniformly as $n \to \infty$. $\square$

We can easily pick out β-branching diffusions in the process $\{X_t^{(n)} : t \geq 0\}$. Let $N_n(ds, da, dw)$ denote the restriction of $N(ds, da, dw)$ to $[0, \infty) \times \mathbb{R} \times W_{1/n}$. For $t \geq 0$ let $\eta_n(t) = N([0,t] \times \mathbb{R} \times W_{1/n})$ and let $\mathcal{G}_{n,t}$ be the σ-algebra generated by the families (4.3) and

$$\{N(J \times A) : J \in \mathcal{B}([0,s] \times \mathbb{R}); A \in \mathcal{B}_{t-s+1/n}(W_0); 0 \leq s \leq t\}. \tag{4.8}$$

Clearly, we can a.s. arrange the atoms of $N_n(ds, da, dw)$ into a sequence $\{(r_j, b_j, w_j) : j = 1, 2, \cdots,\}$ so that $0 < r_1 < r_2 < \cdots$. With this ordering we have

Lemma 4.3 *For any $r \geq 0$, the family $\{\xi_i(t+r), w_j(t+r-r_j+1/n) : t \geq 0; i = 1, 2, \cdots; j = 1, \cdots, \eta_n(r)\}$ under $\boldsymbol{P}\{\cdot | \mathcal{G}_{n,r}\}$ are independent σ-branching diffusions relative to $(\mathcal{G}_{n,r+t})_{t \geq 0}$.*

Proof. For $r \geq 0$ let $\mathcal{F}_{n,r}$ be the σ-algebra generated by

$$\{N_n(J \times A) : J \in \mathcal{B}([0,r] \times \mathbb{R}); A \in \mathcal{B}_{1/n}(W_0) \cap W_{1/n}\}.$$

By Theorem 2.1 and Lemma 3.2, $\{w_j(t+1/n) : t \geq 0; j = 1, \cdots, \eta_n(r)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{F}_{n,r}\}$ are independent σ-branching diffusions. Clearly, the same assertion is true for $\{w_j(t+r-r_j+1/n) : t \geq 0; j = 1, \cdots, \eta_n(r)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{F}'_{n,r}\}$, where $\mathcal{F}'_{n,r}$ is the σ-algebra generated by

$$\{N_n(J \times A) : J \in \mathcal{B}([0,s] \times \mathbb{R}); A \in \mathcal{B}_{r-s+1/n}(W_0) \cap W_{1/n}; 0 \leq s \leq r\}.$$

Then the result follows from the independence of $\{\xi_n(t)\}$, $\{W(ds,dy)\}$ and $\{N(ds,da,dw)\}$. □

Lemma 4.4 *The process $\{X_t^{(n)} : t \geq 0\}$ relative to $(\mathcal{G}_{n,t})_{t\geq 0}$ is a strong Markov process generated by the closure of $(\mathcal{J}_n, \mathcal{D}(\mathcal{J}_n))$, where*

$$\mathcal{J}_n F(\nu) = \mathcal{L}F(\nu) + \int_{\mathbb{R}} m(dx) \int_0^\infty [F(\nu + y\delta_x) - F(\nu)]\kappa_{1/n}(dy), \quad \nu \in M(\mathbb{R}), \tag{4.9}$$

for $F \in \mathcal{D}(\mathcal{J}_n) = \mathcal{D}(\mathcal{L})$.

Proof. Clearly, each r_k is a stopping times and for $0 \leq t < r_{k+1} - r_k$ we have

$$X_{t+r_k}^{(n)} = \sum_{i=1}^\infty \xi_i(t+r_k)\delta_{x_i(0,a_i,r_k+t)} + \sum_{j=1}^k w_j(t+r_k-r_j+1/n)\delta_{x(r_j,b_j,r_k+t)}.$$

Since $r \geq 0$ in Lemma 4.3 was arbitrary, $\{\xi_i(t+r_k), w_j(t+r_k-r_j+1/n) : t \geq 0; i = 1,2,\cdots; j = 1,\cdots,k\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_{n,r_k}\}$ are independent σ-branching diffusions relative to $(\mathcal{G}_{n,r_k+t})_{t\geq 0}$. By the independence of $\{\xi_n(t)\}$, $\{W(ds,dy)\}$ and $\{N(ds,da,dw)\}$ we may apply Theorem 3.3 to get that $\{X_{t+r_k}^{(n)} : 0 \leq t < r_{k+1} - r_k\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_{n,r_k}\}$ is a (killed) diffusion process relative to $(\mathcal{G}_{n,r_k+t})_{t\geq 0}$ with generator $\mathcal{L} - 2n\langle 1,m\rangle/\sigma$. Observe also that

$$\boldsymbol{P}\{F(X_{r_k}^{(n)})|\mathcal{G}_{n,r_k-}\} = \frac{\sigma}{2n\langle 1,m\rangle} \int_{\mathbb{R}} m(dx) \int_0^\infty F(X_{r_k-}^{(n)} + y\delta_x)\kappa_{1/n}(dy)$$

for any $F \in C(M(\mathbb{R}))$. Then $\{X_t^{(n)} : t \geq 0\}$ relative to $(\mathcal{G}_{n,t})_{t\geq 0}$ is a strong Markov process generated by the closure of $(\mathcal{J}_n, \mathcal{D}(\mathcal{J}_n))$. □

Theorem 4.1 *The process $\{Y_t : t \geq 0\}$ constructed by (4.5) is a.s. continuous and solves the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$.*

Proof. As observed in the proof of Lemma 4.1, $\{\langle 1, Y_t(\omega)\rangle : t \geq 0\}$ is a diffusion process with generator $2^{-1}\sigma x d^2/dx^2 + \langle 1,m\rangle d/dx$ and initial value $\langle 1,\mu\rangle$. Then $\boldsymbol{P}\{\langle 1,Y_t\rangle^n\}$ is a locally bounded function of $t \geq 0$ for every $n \geq 1$. For $F = F_{f,\{\phi_i\}}$ given by (3.2), Lemma 4.4 asserts that

$$F(X_t^{(n)}) - F(X_0^{(n)}) - \int_0^t \mathcal{J}_n F(X_s^{(n)})ds, \quad t \geq 0,$$

is a martingale relative to $(\mathcal{G}_{n,t})_{t\geq 0}$. Since $(\mathcal{G}_t)_{t\geq 0}$ is smaller than $(\mathcal{G}_{n,t})_{t\geq 0}$, for any $t \geq r \geq 0$ and any $G \in \mathrm{bp}\mathcal{G}_r$ we have

$$\boldsymbol{E}\bigg\{G\bigg[F(X_t^{(n)}) - F(X_r^{(n)}) - \int_r^t \mathcal{J}_n F(X_s^{(n)})ds\bigg]\bigg\} = 0. \tag{4.10}$$

By (2.5) and (4.9) it is not hard to check that $\mathcal{J}_n F(\nu) \to \mathcal{J}F(\nu)$ uniformly on the set $\{\nu \in M(\mathbb{R}) : \langle 1, \nu\rangle \leq a\}$ as $n \to \infty$ for each $a \geq 0$. By Lemma 4.2, letting $n \to \infty$ in (4.10) we get

$$\boldsymbol{E}\bigg\{G\bigg[F(Y_t) - F(Y_r) - \int_r^t \mathcal{J}F(Y_s)ds\bigg]\bigg\} = 0. \tag{4.11}$$

That is,

$$F(Y_t) - F(Y_0) - \int_0^t \mathcal{J}F(Y_s)ds, \quad t \geq 0,$$

is a martingale relative to $(\mathcal{G}_t)_{t\geq 0}$. Then the desired result follows by an approximation of an arbitrary $F \in \mathcal{D}(\mathcal{J})$. □

We can also pick out σ-branching diffusions in the process $\{Y_t : t \geq 0\}$ defined by (4.5). For $r > 0$ we can a.s. enumerate the atoms (s, a, w) of $N(ds, da, dw)$ satisfying $0 < s < r$ and $w(r - s) > 0$ into a sequence $\{(r_j, b_j, w_j) : j = 1, 2, \cdots\}$ so that $r_j < r_{j+1}$ for all $j \geq 1$. This enumeration gives the following

Theorem 4.2 *For any $r > 0$, the sequence $\{\xi_i(t + r), w_j(t + r - r_j) : t \geq 0; i = 1, 2, \cdots; j = 1, 2, \cdots\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_r\}$ are independent σ-branching diffusions relative to $(\mathcal{G}_{r+t})_{t\geq 0}$.*

Proof. Clearly, for any integer $n > 1/r$, we have $r_j < r-1/n$ if and only if $j \leq \eta_n(r-1/n)$. As in the proof of Lemma 4.3 we see that, the sequence $\{w_j(t+r-r_j) : t \geq 0; j = 1, \cdots, \eta_n(r-1/n)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{F}''_{n,r}\}$ are independent σ-branching diffusions, where $\mathcal{F}''_{n,r}$ is the σ-algebra generated by

$$\{N_n(J \times A) : J \in \mathcal{B}([0, s] \times \mathbb{R}); A \in \mathcal{B}_{r-s}(W_0) \cap W_{1/n}; 0 \leq s \leq r\}.$$

By the independence of $\{\xi_i(t)\}$, $\{W(ds, dy)\}$ and $\{N(ds, da, dw)\}$ we have that $\{\xi_i(t+r), w_j(t+r-r_j) : t \geq 0; i = 1, 2, \cdots; j = 1, \cdots, \eta_n(r-1/n)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_r\}$ are independent σ-branching diffusions. Since $\eta_n(r-1/n) \to \infty$ as $n \to \infty$, we have the desired result. □

Now let us consider an arbitrary initial state $\mu \in M(\mathbb{R})$. Suppose on the complete standard probability space $(\varOmega, \mathcal{F}, \boldsymbol{P})$ have: (i) a white noise $W(ds, dy)$ on $[0,\infty)\times\mathbb{R}$ based on the Lebesgue measure; (ii) a Poisson random measure $N_0(da, dw)$ on $\mathbb{R}\times W_0$ with intensity $\mu(da)\boldsymbol{Q}_\kappa(dw)$; and (iii) a Poisson random measure $N(ds, da, dw)$ on $[0,\infty)\times\mathbb{R}\times W_0$ with intensity $ds m(da)\boldsymbol{Q}_\kappa(dw)$. We assume that $\{W(ds, dy)\}$, $\{N_0(da, dw)\}$ and $\{N(ds, da, dw)\}$ are independent of each other. For $t \geq 0$ let $\mathcal{G}_t$ be the σ-algebra generated by all $\boldsymbol{P}$-null sets and the families of random variables

$$\{W([0, s] \times B), N_0(F \times A) : F \in \mathcal{B}(\mathbb{R}); A \in \mathcal{B}_t(W_0); B \in \mathcal{B}(\mathbb{R}); 0 \leq s \leq t\} \tag{4.12}$$

and

$$\{N(I \times B \times A) : I \in \mathcal{B}([0, s]); B \in \mathcal{B}(\mathbb{R}); A \in \mathcal{B}_{t-s}(W_0); 0 \leq s \leq t\}. \tag{4.13}$$

Given $(r,a) \in [0,\infty) \times \mathbb{R}$, let $\{x(r,a,t) : t \geq r\}$ denote the unique solution of (4.2). Let $Y_0 = \mu$ and for $t > 0$ let

$$Y_t = \int_{\mathbb{R}} \int_{W_0} w(t) \delta_{x(0,a,t)} N_0(da, dw) + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s) \delta_{x(s,a,t)} N(ds, da, dw). \tag{4.14}$$

Theorem 4.3 *The process $\{Y_t : t \geq 0\}$ defined above is a.s. continuous and solves the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$.*

Proof. By Theorem 3.4, the first term on the right hand side of (4.14) is a.s. continuous and converges to μ as $t \to 0$. By Lemma 4.1 the second term is also a.s. continuous. Thus $\{Y_t : t \geq 0\}$ is a.s. continuous. For $r > 0$ let $\{(a_i, u_i) : i = 1, \cdots, m(r)\}$ be set of atoms of $N_0(da, dw)$ satisfying $u_i(r) > 0$ and be arranged so that $u_1(r) < \cdots < u_{m(r)}(r)$. Let $\{(r_j, b_j, w_j) : j = 1, 2, \cdots\}$ be the set of atoms of $N(ds, da, dw)$ satisfying $0 < r_j < r$ and $w_j(r - r_j) > 0$ and be arranged so that $r_j < r_{j+1}$ for all $j \geq 1$. By Lemma 3.4, Theorem 4.2 and the independence assumption, $\{u_i(t+r), w_j(t+r-r_i) : t \geq 0; i = 1, \cdots, m(r); j = 1, 2, \cdots\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_r\}$ are independent σ-branching diffusions relative to $(\mathcal{G}_{r+t})_{t\geq 0}$. By Theorem 4.1 and the property of independent increments of $W(ds, dy)$ and $N(ds, da, dw)$, the continuous process $\{Y_{t+r} : t \geq 0\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_r\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem relative to the filtration $(\mathcal{G}_{t+r})_{t\geq 0}$. Since $r > 0$ was arbitrary in the above reasoning, we have the desired result. $\square$

Theorem 4.4 *The $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem has a unique solution.*

Proof. By Theorem 4.3, the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem has a solution. The approach to the uniqueness is similar to that in Dawson *et al* [2], so we only provide an outline. For $f \in C^2(\mathbb{R}^n)$ let

$$G^n f(x) = \frac{1}{2}\rho(0) \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f(x) + \frac{1}{2} \sum_{i,j=1, i\neq j}^n \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x), \quad x \in \mathbb{R}^n.$$

Define $\Phi_{ij} f \in C(\mathbb{R}^{n-1})$ by

$$\Phi_{ij} f(x_1, \cdots, x_{n-1}) = \sigma(x_{n-1}) f(x_1, \cdots, x_{n-1}, \cdots, x_{n-1}, \cdots, x_{n-2}),$$

where $x_{n-1} \in \mathbb{R}$ is in the places of the ith and the jth variables of f on the right hand side, and define $\Psi_i f \in C^2(\mathbb{R}^{n-1})$ by

$$\Psi_i f(x_1, \cdots, x_{n-1}) = \int_{\mathbb{R}} f(x_1, \cdots, x_{i-1}, x, x_i, \cdots, x_{n-1}) m(dx),$$

where $x \in \mathbb{R}$ is the ith variable of f on the right hand side. It is not hard to show that

$$\mathcal{J} F_{n,f}(\nu) = F_{n, G^n f}(\nu) + \frac{1}{2} \sum_{i,j=1, i\neq j}^n F_{n-1, \Phi_{ij} f}(\nu) + \sum_{i=1}^n F_{n-1, \Psi_i f}(\nu), \quad \nu \in M(\mathbb{R}).$$

Write $F_\nu(n, f) = F_{n,f}(\nu)$ and let

$$\begin{aligned} \mathcal{J}^* F_\nu(n, f) \;\; &= \;\; F_\nu(n, G^n f) + \frac{1}{2} \sum_{i,j=1, i\neq j}^n [F_\nu(n-1, \Phi_{ij} f) - F_\nu(n, f)] \\ &\quad + \sum_{i=1}^n [F_\nu(n-1, \Psi_i f) - F_\nu(n, f)]. \end{aligned} \tag{4.15}$$

Then we have

$$\mathcal{J}F_{n,f}(\nu) = \mathcal{J}^*F_\nu(n,f) + \frac{1}{2}n(n+1)F_\nu(n,f). \tag{4.16}$$

Guided by (4.15) we can construct a Markov process $\{(M_t, F_t) : t \geq 0\}$ with initial value $(M_0, F_0) = (n, f)$ and generator $\mathcal{J}^*$. Based on (4.16) one can prove that if $\{Y_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem with $Y_0 = \mu$, then

$$\boldsymbol{E}_\mu\{\langle f, Y_t^n\rangle\} = \boldsymbol{E}_{(n,f)}\bigg[\langle F_t, \mu^{M_t}\rangle \exp\bigg\{\frac{1}{2}\int_0^t M_s(M_s+1)ds\bigg\}\bigg], \quad t \geq 0;$$

see [4, p.195]. This duality determines the one-dimensional distributions of $\{Y_t : t \geq 0\}$ uniquely, and hence the conclusion follows by [4, p.184]. □

By the uniqueness of solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem, the immigration process constructed by (4.14) is a diffusion. From this construction we know that the immigration SDSM started with any initial state actually lives in the space purely atomic measures. The next theorem, which can be proved similarly as Theorem 3.2, gives a useful alternate characterization of the immigration SDSM.

Theorem 4.5 *A continuous $M(\mathbb{R})$-valued process $\{Y_t : t \geq 0\}$ is a solution of the $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem if and only if for each $\phi \in C^2(\mathbb{R})$,*

$$M_t(\phi) := \langle \phi, Y_t\rangle - \langle \phi, Y_0\rangle - \langle \phi, m\rangle t - \frac{1}{2}\rho(0)\int_0^t \langle \phi'', Y_s\rangle ds, \quad t \geq 0, \tag{4.17}$$

is a martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, Y_s\rangle ds + \int_0^t ds\int_{\mathbb{R}} \langle h(z-\cdot)\phi', Y_s\rangle^2 dz. \tag{4.18}$$

Under the condition of Theorem 4.5, the martingales $\{M_t(\phi) : t \geq 0\}$ defined by (4.17) and (4.18) form a system which is linear in $\phi \in C^2(\mathbb{R})$. Following the method of Walsh [14], we can define the stochastic integral

$$\int_0^t \int_{\mathbb{R}} \phi(s,x)M(ds,dx), \qquad t \geq 0,$$

if both $\phi(s,x)$ and $\phi'_x(s,x)$ are continuous on $[0,\infty) \times \mathbb{R}$. By a standard argument we get the following

Theorem 4.6 *In the situation described above, for any $t \geq 0$ and $\phi \in C^1(\mathbb{R})$ we have a.s.*

$$\langle \phi, Y_t\rangle = \langle P_t\phi, Y_0\rangle + \int_0^t \langle P_{t-s}\phi, m\rangle ds + \int_0^t \int_{\mathbb{R}} P_{t-s}\phi(x)M(ds,dx),$$

where $(P_t)_{t\geq 0}$ is the semigroup of the Brownian motion generated by $2^{-1}\rho(0)d^2/dx^2$.

5 SDSM with interactive immigration

In this section, we construct a diffusion solution of the martingale problem given by (1.3) and (1.4) with a general interactive immigration rate. This is done by solving a stochastic equation carried by a stochastic flow and driven by Poisson processes of excursions.

Let $\sigma > 0$ be a constant and let m be a non-trivial σ-finite Borel measure on $\mathbb{R}$. Suppose we have on a complete standard probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ the following: (i) a white noise $W(ds, dy)$ on $[0, \infty) \times \mathbb{R}$ based on the Lebesgue measure; (ii) a sequence of independent σ-branching diffusions $\{\xi_i(t) : t \geq 0\}$ with $\xi_i(0) \geq 0$ $(i = 1, 2, \cdots)$; and (iii) a Poisson random measure $N(ds, da, du, dw)$ on $[0, \infty) \times \mathbb{R} \times [0, \infty) \times W_0$ with intensity $dsm(da)du\boldsymbol{Q}_\kappa(dw)$, where $\boldsymbol{Q}_\kappa$ denotes the excursion law of the σ-branching diffusion. We assume that $\sum_{i=1}^\infty \xi_i(0) < \infty$ and that $\{W(ds, dy)\}$, $\{\xi_i(t)\}$ and $\{N(ds, da, du, dw)\}$ are independent of each other. For $t \geq 0$ let $\mathcal{G}_t$ be the σ-algebra generated by all $\boldsymbol{P}$-null sets and the families of random variables (4.3) and

$$\{N(J \times A) : J \in \mathcal{B}([0, s] \times \mathbb{R} \times [0, \infty)); A \in \mathcal{B}_{t-s}(W_0); 0 \leq s \leq t\}. \tag{5.1}$$

Let $\mathcal{P}$ be the σ-algebra on $[0, \infty) \times \mathbb{R} \times \Omega$ generated by functions of the form

$$g(s, x, \omega) = \eta_0(x, \omega)1_{\{0\}}(s) + \sum_{i=0}^\infty \eta_i(x, \omega)1_{(r_i, r_{i+1}]}(s), \tag{5.2}$$

where $0 = r_0 < r_1 < r_2 < \ldots$ and $\eta_i(\cdot, \cdot)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{G}_{r_i}$-measurable. We say a function on $[0, \infty) \times \mathbb{R} \times \Omega$ is *predictable* if it is $\mathcal{P}$-measurable.

We first construct an immigration process with purely atomic initial state and predictable immigration rate. Suppose that $q(\cdot, \cdot, \cdot)$ is a non-negative predictable function on $[0, \infty) \times \mathbb{R} \times \Omega$ such that $\boldsymbol{E}\{\langle q(t, \cdot), m\rangle^2\}$ is locally bounded in $t \geq 0$. For a sequence $\{a_i\} \subset \mathbb{R}$ let

$$Y_t = \sum_{i=1}^\infty \xi_i(t)\delta_{x(0,a_i,t)} + \int_0^t \int_{\mathbb{R}} \int_0^{q(s,a)} \int_{W_0} w(t-s)\delta_{x(s,a,t)} N(ds, da, du, dw), \quad t \geq 0. \tag{5.3}$$

Theorem 5.1 *The process $\{Y_t : t \geq 0\}$ defined by (5.3) has a continuous modification. For this modification and each $\phi \in C^2(\mathbb{R})$,*

$$M_t(\phi) := \langle \phi, Y_t\rangle - \langle \phi, Y_0\rangle - \frac{1}{2}\rho(0)\int_0^t \langle \phi'', Y_s\rangle ds - \int_0^t \langle q(s, \cdot)\phi, m\rangle ds, \quad t \geq 0, \tag{5.4}$$

is a continuous martingale relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, Y_s\rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)\phi', Y_s\rangle^2 dz. \tag{5.5}$$

Proof. Step 1) Suppose that $q(s, x, \omega) \equiv q(x)$ for a function $q \in L^1(\mathbb{R}, m)$. Let $N_q(ds, da, du, dw)$ denote the restriction of $N(ds, da, du, dw)$ to the set $\{(s, a, u, w) : s \geq 0; a \in \mathbb{R}; 0 \leq u \leq q(a); w \in W_0\}$ and let $N_q(ds, da, dw)$ be the image of $N_q(ds, da, du, dw)$ under the mapping $(s, a, u, w) \mapsto (s, a, w)$. Clearly, $N_q(ds, da, dw)$ is a Poisson measure on $[0, \infty) \times \mathbb{R} \times W_0$ with intensity $dsq(a)m(da)\boldsymbol{Q}_\kappa(dw)$ and (5.3) can be rewritten as

$$Y_t = \sum_{i=1}^\infty \xi_i(t)\delta_{x(0,a_i,t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{x(s,a,t)} N_q(ds, da, dw), \quad t > 0.$$

Then the results are reduced to those of Theorems 4.1 and 4.5. *Step 2)* Suppose that $q(\omega, s, x)$ is of the form (5.2). Note that $\eta_i(x)$ is actually deterministic under the conditional probability $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_{r_i}\}$. By the last step and Theorems 4.1 and 4.5, the results hold on each interval $[r_i, r_{i+1}]$ and hence on $[0, \infty)$. *Step 3)* The case of a general non-negative predictable function $q(\cdot, \cdot, \cdot)$ can be proved by approximating arguments similar to those in Fu and Li [5] and Shiga [12]. □

Let us consider a stochastic equation with purely atomic initial state. Suppose that $q(\cdot, \cdot)$ is a Borel function on $M(\mathbb{R}) \times \mathbb{R}$ such that there is a constant K such that

$$\langle q(\nu, \cdot), m\rangle \leq K(1 + \|\nu\|), \quad \nu \in M(\mathbb{R}), \tag{5.6}$$

and for each $R > 0$ there is a constant $K_R > 0$ such that

$$\langle |q(\nu, \cdot) - q(\gamma, \cdot)|, m\rangle \leq K_R\|\nu - \gamma\| \tag{5.7}$$

for ν and $\gamma \in M(\mathbb{R})$ satisfying $\langle 1, \nu\rangle \leq R$ and $\langle 1, \gamma\rangle \leq R$, where $\|\cdot\|$ denotes the total variation. For any sequence $\{a_i\} \subset \mathbb{R}$, consider the stochastic equation:

$$Y_t = \sum_{i=1}^{\infty} \xi_i(t)\delta_{x(0,a_i,t)} + \int_0^t \int_{\mathbb{R}} \int_0^{q(Y_s,a)} \int_{W_0} w(t-s)\delta_{x(s,a,t)} N(ds, da, du, dw), \quad t \geq 0. \tag{5.8}$$

Theorem 5.2 *Under the above conditions, there is a unique continuous solution $\{Y_t : t \geq 0\}$ of (5.8), which is a diffusion process. Moreover, for each $\phi \in C^2(\mathbb{R})$,*

$$M_t(\phi) = \langle \phi, Y_t\rangle - \langle \phi, Y_0\rangle - \frac{1}{2}\int_0^t \langle \phi'', Y_s\rangle ds - \int_0^t \langle q(Y_s, \cdot)\phi, m\rangle ds, \quad t \geq 0, \tag{5.9}$$

is a continuous martingale relative to the filtration $(\mathcal{G}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, Y_s\rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot)\phi', Y_s\rangle^2 dz, \quad t \geq 0. \tag{5.10}$$

Proof. Based on Theorem 5.1 and the results in the last section, it can be proved by iteration arguments similar to those in [5] and [12] that (5.8) has a unique solution and (5.9) is a continuous martingale with quadratic variation process (5.10). Let $\mu = \sum_{i=1}^\infty \xi_i(0)\delta_{a_i}$ and let $Q_t^q(\mu, \cdot)$ denote the distribution of Y_t defined by (5.8). For any bounded $(\mathcal{G}_t)$-stopping time $\tau \geq 0$, we can use the information from $\mathcal{G}_\tau$ to enumerate the atoms (s, a, u, w) of $N(ds, da, du, dw)$ satisfying $0 < s \leq \tau$ and $w(\tau - s) > 0$ into a sequence $\{(r_j, b_j, u_j, w_j) : j = 1, 2, \cdots\}$ so that $r_j \leq r_{j+1}$ for all $j \geq 1$. By the strong Markov property of σ-branching diffusions and a slight modification of the proof of Theorem 4.2, we can show that $\{\xi_i(t + \tau), w_j(t + \tau - r_j) : t \geq 0; i = 1, 2, \cdots; j = 1, 2, \cdots\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_\tau\}$ are independent σ-branching diffusions relative to $(\mathcal{G}_{\tau+t})_{t\geq 0}$. By the property of independent increments, $W_\tau(ds, dy) := W(ds + \tau, dy)$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_\tau\}$ is a white noise on $[0, \infty) \times \mathbb{R}$ based on the Lebesgue measure. Similarly, $N_\tau(ds, da, du, dw) := N(ds+\tau, da, du, dw)$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_\tau\}$ is a Poisson random measure with intensity $dsm(da)du\boldsymbol{Q}_\kappa(dw)$. Moreover, the families $\{\xi_i(t + \tau), w_j(t + \tau - r_j)\}$, $\{W_\tau(ds, dy)\}$ and $\{N_\tau(ds, da, du, dw)\}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_\tau\}$ are independent of each other. Observe that

$$Y_\tau = \sum_{i=1}^{\infty} \xi_i(\tau)\delta_{x(0,a_i,\tau)} + \sum_{j=1}^{\infty} w_j(\tau - r_j)\delta_{x(r_j,b_j,\tau)}$$

and

$$\begin{aligned} Y_{t+\tau} &= \sum_{i=1}^{\infty}\xi_i(t+\tau)\delta_{x(0,a_i,t+\tau)} + \sum_{j=1}^{\infty} w_j(t+\tau-r_j)\delta_{x(r_j,b_j,t+\tau)} \\ &\quad + \int_0^t\int_{\mathbb{R}}\int_0^{q(Y_{s+\tau},a)}\int_{W_0} w(t-s)\delta_{x(s+\tau,a,t+\tau)}N_\tau(ds,da,du,dw). \end{aligned}$$

By the uniqueness of solution of (5.8), $Y_{t+\tau}$ under $\boldsymbol{P}\{\,\cdot\,|\mathcal{G}_\tau\}$ has distribution $Q_t^q(Y_\tau,\cdot)$, giving the strong Markov property of $\{Y_t : t \geq 0\}$. □

We now consider a stochastic equation with a general initial state $\mu \in M(\mathbb{R})$. Suppose on the complete standard probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ we have the following: (i) a white noise $W(ds, dy)$ on $[0,\infty)\times\mathbb{R}$ based on the Lebesgue measure; (ii) a Poisson random measure $N_0(da,dw)$ on $\mathbb{R}\times W_0$ with intensity $\mu(dx)\boldsymbol{Q}_\kappa(dw)$; and (iii) a Poisson random measure $N(ds,da,du,dw)$ on $[0,\infty)\times\mathbb{R}\times[0,\infty)\times W_0$ with intensity $dsm(da)du\boldsymbol{Q}_\kappa(dw)$. We assume that $\{W(ds,dy)\}$, $\{N_0(da,dw)\}$ and $\{N(ds,da,dw)\}$ are independent of each other. For $t \geq 0$ let $\mathcal{G}_t$ be the σ-algebra generated by all $\boldsymbol{P}$-null sets and the families of random variables (4.12) and (5.1).

Theorem 5.3 *Suppose that $q(\cdot,\cdot)$ is a Borel function on $M(\mathbb{R})\times\mathbb{R}$ satisfying (5.6) and (5.7). Then the stochastic equation:*

$$\begin{aligned} Y_t &= \int_{\mathbb{R}}\int_{W_0} w(t)\delta_{x(0,a,t)}N_0(da,dw) \\ &\quad + \int_0^t\int_{\mathbb{R}}\int_0^{q(Y_s,a)}\int_{W_0} w(t-s)\delta_{x(s,a,t)}N(ds,da,du,dw), \quad t>0, \end{aligned} \tag{5.11}$$

has a unique continuous solution $\{Y_t : t > 0\}$. If we set $Y_0 = \mu$, then $\{Y_t : t \geq 0\}$ is a diffusion process and the martingale characterization of Theorem 5.2 holds.

Proof. If $q(\cdot,\cdot,\cdot)$ is a non-negative predictable function on $[0,\infty)\times\mathbb{R}\times\Omega$ such that $\boldsymbol{E}\{m(q(t,\cdot))^2\}$ is locally bounded in $t \geq 0$, it can be proved in three steps as in the proof of Theorem 5.1 that the process $\{Y_t : t \geq 0\}$ defined by $Y_0 = \mu$ and

$$\begin{aligned} Y_t &= \int_{\mathbb{R}}\int_{W_0} w(t)\delta_{x(0,a,t)}N_0(da,dw) \\ &\quad + \int_0^t\int_{\mathbb{R}}\int_0^{q(s,a)}\int_{W_0} w(t-s)\delta_{x(s,a,t)}N(ds,da,du,dw), \quad t>0, \end{aligned}$$

has a continuous modification and the results of Theorem 5.1 also hold for this process. Then one can show by iteration arguments that (5.11) has a unique solution and the martingale characterization of Theorem 5.2 holds. The strong Markov property of $\{Y_t : t \geq 0\}$ can be derived as in the proof of Theorem 5.2. □

The solution of (5.11) can be regarded as immigration processes associated with the SDSM with interactive immigration. It is not hard to show that the generator of the diffusion process $\{Y_t : t \geq 0\}$ is given by

$$\mathcal{J}F(\nu) = \mathcal{L}F(\nu) + \int_{\mathbb{R}} q(\nu,x)\frac{\delta F(\nu)}{\delta\nu(x)}m(dx), \quad \nu \in M(\mathbb{R}), \tag{5.12}$$

where $\mathcal{L}$ is defined by (3.1) and $q(\cdot,\cdot)$ is the interactive immigration rate. Note that the Markov property of $\{Y_t : t \geq 0\}$ was obtained from the uniqueness of solution of (5.8). This application of the stochastic equation is essential since the uniqueness of solution of the martingale problem given by (5.9) and (5.10) still remains open; see also Fu and Li [5] and Shiga [12].

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