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A degenerate stochastic partial differential equation for the purely atomic superprocess with dependent spatial motion

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Abstract

A purely atomic superprocess with dependent spatial motion is characterized as the pathwise unique solution of a stochastic partial differential equation, which is driven by a time-space white noise defining the spatial motion and a sequence of independent Brownian motions defining the branching mechanism.

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1 Introduction

A class of superprocesses with dependent spatial motion (SDSM) over the real line $\mathbb{R}$ were introduced and constructed in Wang (1997, 1998). The construction was then generalized in Dawson *et al* (2001). In this model, the spatial motion is defined by a system of differential equations driven by a sequence of independent Brownian motions, the individual noises, and a common time-space white noise, the common noise. In particular, if the coefficient of the individual noises is uniformly bounded away from zero, the SDSM is absolutely continuous and its density satisfies a stochastic differential equation (SPDE) that generalizes the Konno-Shiga SPDE satisfied by super Brownian motion over $\mathbb{R}$; see Dawson (1993), Dawson *et al* (2000), Dawson *et al* (2001) and Konno and Shiga (1988). On the contrary, if the individual noises vanish, the SDSM is purely atomic; see Wang (1997, 2002). A construction of the purely atomic SDSM in terms of one-dimensional excursions was given in Dawson and Li (2002), where some immigration diffusion processes associated with the SDSM were also constructed as pathwise unique solutions of stochastic equations with Poisson processes of one-dimensional excursions carried by a stochastic flow.

In this note, we establish an SPDE for the purely atomic SDSM. The SPDE is driven by a time-space white noise defining the spatial motion and a sequence of independent Brownian motions defining the branching mechanism. We show that the SDSM is a pathwise unique solution of the equation. The result is of interest since it contrasts with the well-known open problem of strong uniqueness for the Konno-Shiga equation and its generalization to the absolutely continuous SDSM.

2 Existence of solution of the atomic SPDE

Let $M(\mathbb{R})$ be the space of finite Borel measures on $\mathbb{R}$ endowed with the weak convergence topology. Let $C(\mathbb{R})$ be the set of bounded continuous functions on $\mathbb{R}$. Given a square-integrable function $h \in C(\mathbb{R})$, let

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R}. \tag{2.1}$$

We assume in addition that h is continuously differentiable with square-integrable derivative h'. Then ρ is twice continuously differentiable with bounded derivatives ρ' and ρ''. Let $\sigma(\cdot)$ be a bounded non-negative continuous function on $\mathbb{R}$ such that there is a constant $\epsilon > 0$ such that $\sigma(x) \geq \epsilon$ for all $x \in \mathbb{R}$. Then a continuous $M(\mathbb{R})$-valued process $\{X_t : t \geq 0\}$ is a realization of the purely atomic SDSM if and only if, for each $\phi \in C^2(\mathbb{R})$,

$$M_t(\phi) = \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - \frac{1}{2}\rho(0)\int_0^t \langle \phi'', X_s \rangle ds, \quad t \geq 0, \tag{2.2}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, X_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi', X_s \rangle^2 dz; \tag{2.3}$$

see e.g. Dawson and Li (2002, Theorem 3.2) or Wang (1998, Corollary 7.3 and 1997). SDSM is closely related to super-Brownian motion (SBM). The difference between SBM and SDSM is that SBM arises as the limit of a system of branching independent Brownian motions whereas the SDSM arises as the limit of a system of branching particles whose motions are driven by a common random medium (defined in terms of a Brownian sheet and the function $h(\cdot)$) and therefore the motions of the particles are dependent. In particular, the effect of the random medium on the flow of the resulting measure-valued process X gives rise to the second term on the right hand side of (2.3) which specifies the quadratic variation of the martingale defined in (2.2).

Let $\{W(t,x) : t \geq 0, x \in \mathbb{R}\}$ be a Brownian sheet and $\{B_1(t), B_2(t), \cdots : t \geq 0\}$ a sequence of independent one-dimensional Brownian motions which are independent of $\{W(t,x) : t \geq 0, x \in \mathbb{R}\}$. By Dawson *et al* (2001, Lemma 3.1) or Wang (1997, Lemma 1.3), given any $x_i(0) \in \mathbb{R}$ the stochastic equation

$$x_i(t) - x_i(0) = \int_0^t \int_{\mathbb{R}} h(y - x_i(s)) W(ds, dy), \quad t \geq 0, \tag{2.4}$$

have unique strong solutions $\{x_i(t) : t \geq 0\}$. Given $\{x_i(t) : t \geq 0\}$ and $\xi_i(0) \geq 0$, we consider the equation

$$\xi_i(t) - \xi_i(0) = \int_0^t \sqrt{\sigma(x_i(t))\xi_i(s)} dB_i(s), \qquad t \geq 0. \tag{2.5}$$

It is not hard to prove that equation (2.5) has a unique strong solution $\{\xi_i(t) : t \geq 0\}$; see e.g. Ikeda and Watanabe (1989).

Given a finite or countable set of positive integers I and a purely atomic finite measure $\nu = \sum_{i \in I} \xi_i(0)\delta_{x_i(0)}$ on $\mathbb{R}$, we define a purely atomic measure-valued process $\{X_t^\nu : t \geq 0\}$ by

$$X_t^\nu = \sum_{i \in I} \xi_i(t) \delta_{x_i(t)}, \quad t \geq 0. \tag{2.6}$$

By Itô's formula, for any $\phi \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned} \xi_i(t)\phi(x_i(t)) - \xi_i(0)\phi(x_i(0)) &= \int_0^t \int_{\mathbb{R}} \xi_i(s)\phi'(x_i(s)) h(y - x_i(s)) W(ds, dy) \\ &+ \frac{1}{2}\rho(0) \int_0^t \xi_i(s)\phi''(x_i(s)) ds + \int_0^t \phi(x_i(s)) \sqrt{\sigma(x_i(s))\xi_i(s)} dB_i(s), \end{aligned} \tag{2.7}$$

where we have used the fact

$$\int_{\mathbb{R}} h(y - x_i(s))^2 dy = \int_{\mathbb{R}} h(y)^2 dy = \rho(0).$$

Then taking the summation in (2.7) we get

$$\begin{aligned} \sum_{i \in I} \xi_i(t)\phi(x_i(t)) - \sum_{i \in I} \xi_i(0)\phi(x_i(0)) &= \int_0^t \int_{\mathbb{R}} \Big[\sum_{i \in I} \xi_i(s)\phi'(x_i(s)) h(y - x_i(s))\Big] W(ds, dy) \\ &+ \frac{1}{2}\rho(0) \int_0^t \Big[\sum_{i \in I} \xi_i(s)\phi''(x_i(s))\Big] ds + \sum_{i \in I} \int_0^t \phi(x_i(s)) \sqrt{\sigma(x_i(s))\xi_i(s)} dB_i(s). \end{aligned} \tag{2.8}$$

The above equation can be rewritten as

$$\langle\phi, X_t^\nu\rangle - \langle\phi, X_0^\nu\rangle = \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s^\nu\rangle W(ds,dy) + \frac{1}{2}\rho(0)\int_0^t \langle\phi'', X_s^\nu\rangle ds$$
$$+\sum_{i\in I}\int_0^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s). \tag{2.9}$$

Observe that

$$M_t^\nu(\phi) := \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s^\nu\rangle W(ds,dy) + \sum_{i\in I}\int_0^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s).$$

is a continuous martingale with quadratic variation process

$$\langle M^\nu(\phi)\rangle_t = \int_0^t ds\int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s^\nu\rangle^2 dy + \int_0^t \langle\sigma\phi^2, X_s^\nu\rangle ds.$$

Then we have proved the following

Theorem 2.1 *Given any purely atomic finite measure $\nu = \sum_{i\in I}\xi_i(0)\delta_{x_i(0)}$, equation (2.9) has a continuous and purely atomic measure-valued solution $\{X_t^\nu : t \geq 0\}$ in the form (2.6), which is a realization of the SDSM.*

Note that (2.9) gives a degenerate SPDE for the purely atomic measure-valued process $\{X_t^\nu : t \geq 0\}$, which parallels the SPDE of Dawson *et al* (2000).

3 Uniqueness of solution of the single-atomic SPDE

As a special case of the discussions in the last section, given the purely atomic finite measure $\xi_i(0)\delta_{x_i(0)}$, there is a continuous process $\{\xi_i(t)\delta_{x_i(t)} : t \geq 0\}$ satisfying the equation

$$\langle\phi, \xi_i(t)\delta_{x_i(t)}\rangle - \langle\phi, \xi_i(0)\delta_{x_i(0)}\rangle = \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', \xi_i(s)\delta_{x_i(s)}\rangle W(ds,dy)$$
$$+\frac{1}{2}\rho(0)\int_0^t \langle\phi'', \xi_i(s)\delta_{x_i(s)}\rangle ds + \int_0^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s). \tag{3.1}$$

The following theorem gives the uniqueness of solution of the above equation:

Theorem 3.1 *If $\{\xi_i(t)\delta_{x_i(t)} : t \geq 0\}$ is a solution of (3.1), then we have*

$$x_i(t) - x_i(0) = \int_0^t \int_{\mathbb{R}} h(y - x_i(s))W(ds,dy), \quad 0 \leq t < \tau_i, \tag{3.2}$$

and

$$\xi_i(t) - \xi_i(0) = \int_0^t \sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s), \quad t \geq 0, \tag{3.3}$$

where $\tau_i = \inf\{s \geq 0 : \xi_i(s) = 0\}$. Consequently, the solution $\{\xi_i(t)\delta_{x_i(t)} : t \geq 0\}$ of (3.1) is pathwise unique.

Proof. For each integer $n \geq 1$ let $\zeta_n = \inf\{s \geq 0 : \xi_i(s)\delta_{x_i(s)}([-n,n]^c) > 0\}$. (We here suppress the dependence of ζ_n on $i \in I$.) Since $\{\xi_i(t)\delta_{x_i(t)} : t \geq 0\}$ is continuous, we have $\lim_{n\to\infty} \zeta_n = \infty$. Choose any $\phi \in \mathcal{S}(\mathbb{R})$ such that $\phi(x) = 1$ when $|x| \leq n$. From (3.1) we get

$$\xi_i(t \wedge \zeta_n) - \xi_i(0) = \int_0^{t\wedge\zeta_n} \sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s), \quad t \geq 0. \tag{3.4}$$

Letting $n \to \infty$ we get equation (3.3). Let $b(t) = \xi_i(t)\phi(x_i(t))$. Then (3.1) implies

$$\begin{aligned} b(t) - b(0) = &\int_0^t \int_{\mathbb{R}} \xi_i(s)h(y - x_i(s))\phi'(x_i(s))W(ds, dy) + \frac{1}{2}\rho(0)\int_0^t \xi_i(s)\phi''(x_i(s))ds \\ &+ \int_0^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s). \end{aligned} \tag{3.5}$$

Let $\sigma_n = \inf\{s \geq 0 : \xi_i(s) \leq 1/n\}$ and $\gamma_n = \zeta_n \wedge \sigma_n$. Then clearly $\lim_{n\to\infty} \gamma_n = \tau_i$. By (3.3), (3.5) and Itô's formula, it is easy to find that

$$\begin{aligned} \phi(x_i(t \wedge \gamma_n)) - \phi(x_i(0)) &= b(t \wedge \gamma_n)/\xi_i(t \wedge \gamma_n) - b(0)/\xi_i(0) \\ &= \int_0^{t\wedge\gamma_n} \int_{\mathbb{R}} h(y - x_i(s))\phi'(x_i(s))W(ds, dy) + \frac{1}{2}\rho(0)\int_0^{t\wedge\gamma_n} \phi''(x_i(s))ds \end{aligned} \tag{3.6}$$

Choose any $\phi \in \mathcal{S}(\mathbb{R})$ such that $\phi(x) = x$ when $|x| \leq n$. From (3.6) we have

$$x_i(t \wedge \gamma_n) - x_i(0) = \int_0^{t\wedge\gamma_n} \int_{\mathbb{R}} h(y - x_i(s))W(ds, dy). \tag{3.7}$$

Letting $n \to \infty$ gives (3.2). Suppose that $\{\xi_i'(t)\delta_{x_i'(t)} : t \geq 0\}$ is another solution of (3.1). Let $\tau_i' = \inf\{s \geq 0 : \xi_i'(s) = 0\}$. Since the solution of (2.4) is pathwise unique, the above arguments show a.s. $x_i(t) = x_i'(t)$ for all $0 \leq t \leq \tau_i \wedge \tau_i'$. By the uniqueness of solution of (2.5) we have a.s. $\xi_i(t) = \xi_i'(t)$ for all $0 \leq t \leq \tau_i \wedge \tau_i'$, yielding a.s. $\tau_i = \tau_i'$. Therefore, the two solutions $\{\xi_i(t)\delta_{x_i(t)} : t \geq 0\}$ and $\{\xi_i'(t)\delta_{x_i'(t)} : t \geq 0\}$ a.s. coincide each other. $\square$

Note that the above theorem only gives the uniqueness of the position process $\{x_i(t) : t \geq 0\}$ up to the extinction time τ_i of the atom, which is sufficient to get the pathwise uniqueness of the solution $\{\xi_i(t)\delta_{x_i(t)} : t \geq 0\}$.

4 Uniqueness of solution of the multi-atomic SPDE

Suppose that $\{x_i(0) \in \mathbb{R} : i \in I\}$ is a collection of points which are all distinct. By the discussions in section 1, given the purely atomic finite initial measure $\nu = \sum_{i\in I} \xi_i(0)\delta_{x_i(0)}$, there is a continuous and purely atomic process $\{X_t : t \geq 0\}$ in the form

$$X_t = \sum_{i\in I} \xi_i(t)\delta_{x_i(t)}, \quad t \geq 0, \tag{4.1}$$

satisfying the equation

$$\begin{aligned} \langle\phi, X_t\rangle - \langle\phi, \nu\rangle = &\int_0^t \int_{\mathbb{R}} \langle h(y - \cdot)\phi', X_s\rangle W(ds, dy) + \frac{1}{2}\rho(0)\int_0^t \langle\phi'', X_s\rangle ds \\ &+ \sum_{i\in I} \int_0^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s). \end{aligned} \tag{4.2}$$

Let $\tau_i = \inf\{s \geq 0 : \xi_i(s) = 0\}$. We call each $\{x_i(t) : 0 \leq t \leq \tau_i\}$ a *position process* of the solution $\{X_t : t \geq 0\}$.

Theorem 4.1 *Suppose that the points $\{x_i(0) \in \mathbb{R} : i \in I\}$ are all distinct. Then, given the initial state $X_0 = \sum_{i\in I} \xi_i(0)\delta_{x_i(0)}$, the above equation has a pathwise unique continuous and purely atomic measure-valued solution $\{X_t : t \geq 0\}$ in the form (4.1). Moreover, $\{\xi_i(t) : t \geq 0\}$ and $\{x_i(t) : t \geq 0\}$ are given respectively by (3.2) and (3.3).*

We shall need the following result of Wang (1997, Lemma 1.2).

Lemma 4.1 *Let $\{x_i(t) : t \geq 0\}$ be the solution of (2.4). If $x_i(0) \neq x_j(0)$, then $x_i(t) \neq x_j(t)$ for all $t \geq 0$.*

Proof of Theorem 4.1. The existence of solution follows from Theorem 2.1. We first assume that I is a finite set and prove the pathwise uniqueness of the solution. For any solution $\{X_t : t \geq 0\}$ of (4.2),

$$M_t(\phi) = \langle \phi, X_t\rangle - \langle \phi, X_0\rangle - \frac{1}{2}\rho(0)\int_0^t \langle \phi'', X_s\rangle ds, \quad t \geq 0, \tag{4.3}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \langle \sigma\phi^2, X_s\rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi', X_s\rangle^2 dz. \tag{4.4}$$

By Dawson and Li ('02, Theorem 3.2) or Wang (1998, Corollary 7.3 and Theorem 4.1), the solution of the above martingale problem is unique, then the distribution of $\{X_t : t \geq 0\}$ must coincides with the process $\{X_t^\nu : t \geq 0\}$ constructed by (2.6). In particular, each $\{x_i(t) : 0 \leq t < \tau_i\}$ is a stopped 1-dimensional Brownian motion and any two of those Brownian motions never hit each other before their terminal time. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the one-point compactification of the real line and set $x_i(t) = \infty$ if $t \geq \tau_i$ for notational convenience. Let

$$\epsilon_0 = \inf\{|x_i(0) - x_j(0)| : \{x_i(0), x_j(0)\} \subset \mathbb{R} \text{ and } i \neq j \in I\}$$

and

$$\eta_1 = \inf\{t \geq 0 : x_i(t) \in \mathbb{R} \text{ and } |x_i(t) - x_i(0)| \geq \epsilon_0/3 \text{ for some } i \in I\}.$$

Then η_1 is a stopping time. Take $\phi \in \mathcal{S}(\mathbb{R})$ such that $\phi(x) = 0$ when $|x - x_i(0)| \geq 2\epsilon_0/3$. From (4.2) we get

$$\begin{aligned}\langle \phi, \xi_i(t)\delta_{x_i(t)}\rangle - \langle \phi, \xi_i(0)\delta_{x_i(0)}\rangle &= \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', \xi_i(s)\delta_{x_i(s)}\rangle W(ds, dy) \\ &+\frac{1}{2}\rho(0)\int_0^t \langle \phi'', \xi_i(s)\delta_{x_i(s)}\rangle ds + \int_0^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s)\end{aligned} \tag{4.5}$$

for $0 \leq t < \eta_1$. Since $|x_i(t) - x_i(0)| \leq \epsilon_0/3$ or $\xi_i(t) = 0$ for $0 \leq t \leq \eta_1$, the above equation actually holds for an arbitrary testing function $\phi \in \mathcal{S}(\mathbb{R})$. As in the proof of Theorem 3.1 we have

$$x_i(t) - x_i(0) = \int_0^t \int_{\mathbb{R}} h(y - x_i(s))W(ds, dy), \quad 0 \leq t < \eta_1 \wedge \tau_i, \tag{4.6}$$

and

$$\xi_i(t) - \xi_i(0) = \int_0^t \sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s), \quad 0 \le t < \eta_1, \tag{4.7}$$

By Lemma 4.1, $\{x_i(\eta_1) \in \mathbb{R} : i \in I\}$ are all distinct. Generally, once the stopping time η_n is defined with $\{x_i(\eta_n) \in \mathbb{R} : i \in I\}$ all distinct, we define

$$\epsilon_n = \inf\{|x_i(\eta_n) - x_j(\eta_n)| : \{x_i(\eta_n), x_j(\eta_n)\} \subset \mathbb{R} \text{ and } i \neq j \in I\}$$

and

$$\eta_{n+1} = \inf\{t \ge \eta_n : x_i(t) \in \mathbb{R} \text{ and } |x_i(t) - x_i(\eta_n)| \ge \epsilon_n/3 \text{ for some } i \in I\}.$$

If $\xi_i(\eta_n) > 0$, then $x_i(\eta_n) \in \mathbb{R}$ and $\eta_n < \tau_i$. By a time shift we get similarly as the above that

$$\begin{aligned}\langle \phi, \xi_i(t)\delta_{x_i(t)}\rangle - \langle \phi, \xi_i(\eta_n)\delta_{x_i(\eta_n)}\rangle &= \int_{\eta_n}^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', \xi_i(s)\delta_{x_i(s)}\rangle W(ds,dy) \\ &+ \frac{1}{2}\rho(0)\int_{\eta_n}^t \langle \phi'', \xi_i(s)\delta_{x_i(s)}\rangle ds + \int_{\eta_n}^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s)\end{aligned} \tag{4.8}$$

holds for $\eta_n \le t < \eta_{n+1}$ and an arbitrary testing function $\phi \in \mathcal{S}(\mathbb{R})$. Again as in the proof of Theorem 3.1 we have

$$x_i(t) - x_i(\eta_n) = \int_{\eta_n}^t \int_{\mathbb{R}} h(y - x_i(s))W(ds,dy), \quad \eta_n \le t < \eta_{n+1} \wedge \tau_i. \tag{4.9}$$

and

$$\xi_i(t) - \xi_i(\eta_n) = \int_{\eta_n}^t \sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s), \quad \eta_n \le t < \eta_{n+1}, \tag{4.10}$$

By Lemma 4.1 we conclude that $\{x_i(\eta_{n+1}) \in \mathbb{R} : i \in I\}$ are all distinct. For the same reason, we have $\lim_{n\to\infty} \eta_n = \infty$. Then get (3.2) and (3.3) and the pathwise uniqueness of $\{X_t : t \ge 0\}$ follows. Finally, for an infinite set I, we take a constant $t_0 > 0$. From (4.2) we know that $\{X_t : t \ge t_0\}$ is a continuous and purely atomic measure-valued solution of

$$\begin{aligned}\langle \phi, X_t\rangle - \langle \phi, X_{t_0}\rangle &= \int_{t_0}^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s\rangle W(ds,dy) + \frac{1}{2}\rho(0)\int_{t_0}^t \langle \phi'', X_s\rangle ds \\ &+ \sum_{i\in I}\int_{t_0}^t \phi(x_i(s))\sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s).\end{aligned} \tag{4.11}$$

As we assumed, there is a constant $\epsilon > 0$ such that $\sigma(x) \ge \epsilon$ for all $x \in \mathbb{R}$. By Dawson and Li (2002, Lemma 3.3 and Theorem 3.4) or Wang (2002, Lemma 2.2), $I(t_0)$ is a.s. a finite set. By a time shift of the result proved above, for each $i \in I(t_0)$ we have

$$x_i(t) - x_i(t_0) = \int_{t_0}^t \int_{\mathbb{R}} h(y - x_i(s))W(ds,dy), \quad t_0 \le t < \tau_i. \tag{4.12}$$

and

$$\xi_i(t) - \xi_i(t_0) = \int_{t_0}^t \sqrt{\sigma(x_i(s))\xi_i(s)}dB_i(s), \qquad t \geq t_0, \tag{4.13}$$

Obviously, $I = \cup_{t_0>0} I(t_0)$. Then for each $i \in I$ we have (4.12) and (4.13) when $t_0 > 0$ is sufficiently small. Since $t_0 > 0$ is arbitrary and since $\{\xi_i(t) : t \geq 0\}$ and $\{x_i(t) : t \geq 0\}$ are continuous, we may let $t_0 \to 0$ in the above and get respectively (3.2) and (3.3), which give the uniqueness of $\{X_t : t \geq 0\}$. □

5 An enriched version of the SPDE

In this section, we discuss an enriched version of the SPDE for the SDSM. Let $\{W(dt, dx)\}$ be a time-space white noise based on the Lebesgue measure and $\{B(a, \cdot) : a \in \mathbb{R}\}$ a family of independent one-dimensional Brownian motions which are independent of $\{W(dt, dx)\}$. Given $a \in \mathbb{R}$ and $\xi(a, 0) \geq 0$, let $\{x(a, t) : t \geq 0\}$ denote the unique strong solution of

$$x(t) - a = \int_0^t \int_{\mathbb{R}} h(y - x(s))W(ds, dy), \quad t \geq 0, \tag{5.1}$$

and let $\{\xi(a, t) : t \geq 0\}$ denote the unique strong solution of

$$\xi(t) - \xi(0) = \int_0^t \sqrt{\sigma(x(a, s))\xi(s)}dB(a, s), \quad t \geq 0. \tag{5.2}$$

Given a purely atomic finite initial measure $\nu = \sum_{i \in I} \xi_i \delta_{a_i}$ on $\mathbb{R}$, let

$$X_t^\nu = \sum_{i \in I} \xi(a_i, t)\delta_{x(a_i, t)}, \tag{5.3}$$

where $\{x(a_i, t) : t \geq 0\}$ is given by (5.1) and $\{\xi(a_i, t) : t \geq 0\}$ is the solution of (5.2) with $\xi(a_i, 0) = \xi_i$ and with a replaced by a_i. Let

$$X_t^e(da, db) = \sum_{i \in I} \xi(a_i, t)\delta_{x(a_i, 0)}(da)\delta_{x(a_i, t)}(db). \tag{5.4}$$

As in section 1, for any $\phi \in \mathcal{S}(\mathbb{R}^2)$ we have by Itô's formula

$$\begin{aligned}
&\xi(a_i, t)\phi(a_i, x(a_i, t)) - \xi(a_i, 0)\phi(a_i, a_i) \\
&\quad = \int_0^t \int_{\mathbb{R}} \xi(a_i, s)\phi'(a_i, x(a_i, s))h(y - x(a_i, s))W(ds, dy) \\
&\qquad + \frac{1}{2}\rho(0) \int_0^t \xi(a_i, s)\phi''(a_i, x(a_i, s))ds \\
&\qquad + \int_0^t \phi(a_i, x(a_i, s))\sqrt{\sigma(x(a_i, s))\xi(a_i, s)}dB(a_i, s).
\end{aligned}$$

Then taking summation we get

$$\begin{aligned}
\sum_{i\in I}\xi(a_i,t)\phi(a_i,x(a_i,t)) &- \sum_{i\in I}\xi(a_i,0)\phi(a_i,a_i) \\
&= \int_0^t\int_{\mathbb{R}}\Big[\sum_{i\in I}\xi(a_i,s)\phi'(a_i,x(a_i,s))h(y-x(a_i,s))\Big]W(ds,dy) \\
&\quad +\frac{1}{2}\rho(0)\int_0^t\Big[\sum_{i\in I}\xi(a_i,s)\phi''(a_i,x(a_i,s))\Big]ds \\
&\quad +\int_0^t\Big[\sum_{i\in I}\phi(a_i,x(a_i,s))\sqrt{\sigma(x(a_i,s))\xi(a_i,s)}\Big]dB(a_i,s).
\end{aligned}$$

The above equation can be written as

$$\begin{aligned}
\langle\phi,X_t^e\rangle-\langle\phi,X_0^e\rangle &= \int_0^t\int_{\mathbb{R}}\Big[\int_{\mathbb{R}^2}h(y-b)\phi'(a,b)X_s^e(da,db)\Big]W(ds,dy) \\
&+\frac{1}{2}\rho(0)\int_0^t\langle\phi'',X_s^e\rangle ds+\int_0^t\int_{\mathbb{R}}\phi(a,x(a,s))\sqrt{\sigma(x(a,s))\xi(a,s)}W^\nu(ds,da), \quad (5.5)
\end{aligned}$$

where $\{W^\nu(ds,dy)\}$ denotes the time-space white noise defined by

$$W^\nu((r,t]\times A)=\sum_{i\in I}(B(a_i,t)-B(a_i,r))1_A(a_i). \quad (5.6)$$

This gives an enriched SPDE of the SDSM, which contains more information of the process than equation (4.2).

Theorem 5.1 *Given any purely atomic finite measure $\nu=\sum_{i\in I}\xi(a_i,0)\delta_{x(a_i,0)}$, equation (5.1) has a unique continuous and purely atomic measure-valued solution $\{X_t^e: t\geq 0\}$ in the form (5.4).*

Proof. We have seen the existence of the solution. Let $\{X_t: t\geq 0\}$ be an arbitrary continuous and purely atomic measure-valued solution of the equation in form (5.1). If I is a finite set, for any $i\in I$ and $\phi_1\in\mathcal{S}(\mathbb{R})$ we may choose $\phi\in\mathcal{S}(\mathbb{R}^2)$ in a way so that $\phi(a_i,\cdot)\equiv\phi_1(\cdot)$ and $\phi(a_j,\cdot)\equiv 0$ for any $j\neq i$ from I. Then (5.1) becomes

$$\begin{aligned}
\langle\phi_1,\xi(a_i,t)\delta_{x(a_i,t)}\rangle-\langle\phi_1,\xi(a_i,0)\delta_{x(a_i,0)}\rangle &= \int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi_1',\xi(a_i,s)\delta_{x(a_i,s)}\rangle W(ds,dy) \\
+\frac{1}{2}\rho(0)\int_0^t\langle\phi_1'',\xi(a_i,s)\delta_{x(a_i,s)}\rangle ds &+ \int_0^t\phi(a_i,x(a_i,s))\sqrt{\sigma(x(a_i,s))\xi(a_i,s)}B(a_i,ds),
\end{aligned}$$

that is, $\{\xi(a_i,t)\delta_{x(a_i,t)}: t\geq 0\}$ satisfies the single-atomic equation. Thus the uniqueness follows from Theorem 3.1. For an infinite index set I, the conclusion can be obtained as in the proof of Theorem 4.1. □

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