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SKEW CONVOLUTION SEMIGROUPS AND RELATED IMMIGRATION PROCESSES [1]

Zeng-Hu LI

Department of Mathematics, Beijing Normal University,
Beijing 100875, People's Republic of China
E-mail: lizh@email.bnu.edu.cn

Abstract. A special type of immigration associated with measure-valued branching processes is formulated by using skew convolution semigroups. We give characterization for a general inhomogeneous skew convolution semigroup in terms of probability entrance laws. The related immigration process is constructed by summing up measure-valued paths in the Kuznetsov process determined by an entrance rule. The behavior of the Kuznetsov process is then studied, which provides insights into trajectory structures of the immigration process. Some well-known results on excessive measures are formulated in terms of stationary immigration processes.

Key words: measure-valued branching process; superprocess; immigration process; skew convolution semigroup; entrance law; entrance rule; excessive measure; Kuznetsov measure

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1 Introduction

Let E be a Lusin topological space, i.e., a homeomorph of a Borel subset of a compact metric space, with the Borel σ-algebra $\mathcal{B}(E)$. Let $B(E)$ denote the set of bounded $\mathcal{B}(E)$-measurable functions on E, and $B(E)^+$ the subspace of $B(E)$ of non-negative functions. We denote by $M(E)$ the space of finite measures on $(E, \mathcal{B}(E))$ endowed with the topology of weak convergence. For $f \in B(E)$ and $\mu \in M(E)$, write $\mu(f)$ for $\int_E f d\mu$. Suppose that

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$(P_t)_{t\geq 0}$ is the transition semigroup of a Borel right process ξ with state space E and $\phi(\cdot,\cdot)$ is a branching mechanism given by

$$\phi(x,z) = b(x)z + c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(x,\mathrm{d}u), \quad x \in E,\ z \geq 0, \tag{1.1}$$

where $b \in B(E)$, $c \in B(E)^+$ and $(u \wedge u^2)m(x,\mathrm{d}u)$ is a bounded kernel from E to $(0,\infty)$. Then for each $f \in B(E)^+$ the evolution equation

$$V_t f(x) + \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y)) P_{t-s}(x,\mathrm{d}y) = P_t f(x), \quad t \geq 0,\ x \in E, \tag{1.2}$$

has a unique solution $V_t f \in B(E)^+$, and there is a Markov semigroup $(Q_t)_{t\geq 0}$ on $M(E)$ such that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu,\mathrm{d}\nu) = \exp\{-\mu(V_t f)\} \tag{1.3}$$

for all $t \geq 0$, $\mu \in M(E)$ and $f \in B(E)^+$. A Markov process X having transition semigroup $(Q_t)_{t\geq 0}$ is called a *Dawson-Watanabe superprocess* with parameters (ξ,ϕ). Under our hypotheses, X has a Borel right realization; see Fitzsimmons [11] and [12]. The (ξ,ϕ)-superprocess is a mathematical model for the evolution of a population in some region; see e.g. Dawson [3] and [4]. If we consider a situation where there are additional sources of population from which immigration into the region occurs during the evolution, we need to introduce branching processes with immigration. This type of modification is familiar from the branching process literature; see e.g. [1], [5], [8], [21], [24] and [32].

A class of measure-valued immigration processes were formulated in Li [25] as follows. Let $(N_t)_{t\geq 0}$ be a family of probability measures on $M(E)$. We call $(N_t)_{t\geq 0}$ a *skew convolution semigroup* associated with X or $(Q_t)_{t\geq 0}$ provided

$$N_{r+t} = (N_r Q_t) * N_t, \quad r,t \geq 0, \tag{1.4}$$

where "$*$" denotes the convolution operation. The relation (1.4) is satisfied if and only if

$$Q_t^N(\mu,\cdot) := Q_t(\mu,\cdot) * N_t, \quad t \geq 0,\ \mu \in M(E), \tag{1.5}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. A Markov process is called an *immigration process* associated with X if it has transition semigroup $(Q_t^N)_{t\geq 0}$. The intuitive meaning of the immigration process is clear from (1.5), that is, $Q_t(\mu,\cdot)$ is the distribution of descendants of the people distributed as $\mu \in M(E)$ at time zero and N_t is the distribution of descendants of the people immigrating to E during the time interval $(0,t]$. Clearly, (1.5) gives the general formulation for the immigration independent of the inner population.

Needless to say, most of the theory of Dawson-Watanabe superprocesses carries over to their associated immigration processes and could be developed by techniques very close to those in [3] and [4]. It is interesting, however, that the immigration processes have many additional structures, as might be expected from (1.4) and (1.5). Note that

(1.5) is quite similar to the construction of Lévy's transition semigroup from a *usual* convolution semigroup. It is well-known that a convolution semigroup on the Euclidean space is uniquely determined by an infinitely divisible probability measure. As shown in Li [25], the skew convolution semigroup may be characterized in terms of an infinitely divisible probability entrance law. Therefore, the immigration process may be regarded as a generalized form of the celebrated Lévy process. Other examples of immigration processes are squares of Bessel diffusions and radial parts of Ornstein-Uhlenbeck diffusions; see [21] and [33]. The above formulation also includes new kinds of processes; see [26], [27] and [28].

In the immigration models studied before, authors usually assumed that the immigrants came to E according to a random measure on $I\!R \times E$. The scenes are not quite clear under the skew convolution semigroup formulation. This weak point has in fact motivated the present work. The main purpose of this paper is to give interpretations of the skew convolution semigroups by constructing and analyzing the trajectories of the related measure-valued immigration processes. We first prove that a general inhomogeneous skew convolution semigroup may be decomposed into three components which involve respectively a countable family of entrance laws, a countable family of closed entrance laws, and a continuum family of infinitely divisible probability entrance laws together with a diffuse measure on the index space. Then we give a construction for the immigration process defined above by picking up measure-valued paths with random times of birth and death. Our construction is based on the observation that any skew convolution semigroup is determined by a continuous increasing measure-valued path $(\gamma_t)_{t\geq 0}$ and an entrance rule $(G_t)_{t\geq 0}$. This fact yields a natural decomposition of the immigration into two parts; the deterministic part represented by $(\gamma_t)_{t\geq 0}$ and the random part determined by $(G_t)_{t\geq 0}$. The latter is usually an inhomogeneous immigration process and can be constructed by summing up paths $\{w_t : \alpha < t < \beta\}$ in the associated Kuznetsov process; see Kuznetsov [22]. By analyzing the asymptotic behavior of the paths $\{w_t : \alpha < t < \beta\}$ near the birth time $\alpha = \alpha(w)$, we show that almost all these paths start propagation in an extension E_D^T of the underlying space, including those growing up at points in this space from the null measure. Those combined with our construction of the immigration process give a full description of the phenomenon. In some special cases, the infinitely divisible entrance law for the (ξ, ϕ)-superprocess corresponds to a σ-finite entrance law and the associated immigration process can be constructed using a homogeneous path-valued Poisson random process whose characteristic measure is the Markov measure determined by the entrance law. The construction has been proved useful in studying the immigration processes; see e.g. [26], [29] and [32]. As additional applications of the construction, we give formulations of some well-known results on excessive measures in terms of stationary immigration processes.

The paper is organized as follows. The next section contains some preliminaries. In section 3, we prove the decomposition theorem for inhomogeneous skew convolution semigroups. The construction of immigration processes using Kuznetsov processes is given in section 4. Almost sure behavior of the Kuznetsov processes is studied in section 5. In section 6, we discuss stationary immigration processes determined by excessive measures.

2 Preliminaries

Recall that $M(E)$ is the space of finite Borel measures on the Lusin topological space E. It is well-known that $M(E)$ endowed with the weak convergence topology is also a Lusin space. Let $M(E)^\circ = M(E) \setminus \{0\}$, where 0 denotes the null measure. For a probability measure F on $M(E)$ we define its Laplace functional by

$$L_F(f) := \int_{M(E)} \mathrm{e}^{-\nu(f)} F(\mathrm{d}\nu), \quad f \in B(E)^+, \tag{2.1}$$

which determines F uniquely. It is well-known that F is infinitely divisible if and only if

$$L_F(f) = \exp\Big\{ -\eta(f) - \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big)\, H(\mathrm{d}\nu)\Big\}, \quad f \in B(E)^+, \tag{2.2}$$

where $\eta \in M(E)$ and $[1 \wedge \nu(E)]H(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$. See e.g. Kallenberg [20]. We write $F = I(\eta, H)$ if F is determined by (2.2).

Suppose that $X = (W, \mathcal{G}, \mathcal{G}_{r,t}, X_t, \boldsymbol{Q}_{r,\mu})$ is a Markov process in $M(E)$ with transition semigroup $(Q_{r,t})_{r\leq t}$. Let

$$V_{r,t}f(x) := -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_{r,t}(\delta_x, \mathrm{d}\nu), \quad r \leq t,\ x \in E,\ f \in B(E)^+, \tag{2.3}$$

where δ_x denote the unit mass concentrated at $x \in E$. In this paper, we always assume that, for each $r \leq t$ and $f \in B(E)^+$, the function $V_{s,t}f(x)$ of (s, x) restricted to $[r, t] \times E$ is bounded and measurable. We also assume that $V_{r,t}f(x)$ is right continuous in $t \geq r$ for $f \in C(E)^+$, continuous functions in $B(E)^+$. The process X is called a *measure-valued branching process* (MB-process) if its transition semigroup satisfies

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_{r,t}(\mu, \mathrm{d}\nu) = \exp\left\{-\mu(V_{r,t}f)\right\}, \quad r \leq t,\ f \in B(E)^+. \tag{2.4}$$

Under this hypothesis, $Q_{r,t}(\mu, \cdot)$ is infinitely divisible and $(V_{r,t})_{r\leq t}$ form a family of operators on $B(E)^+$ satisfying $V_{r,s}V_{s,t} = V_{r,t}$ for all $r \leq s \leq t$, which is called the *cumulant semigroup* of X. See e.g. Silverstein [34] and Watanabe [35]. The (ξ, ϕ)-superprocess defined in the introduction is a special form of the MB-process.

Suppose that $(Q_{r,t})_{r\leq t}$ is the transition semigroup of an MB-process and $(N_{r,t})_{r\leq t}$ is a family of probability measures on $M(E)$. We call $(N_{r,t})_{r\leq t}$ a *skew convolution semigroup* (SC-semigroup) associated with $(Q_{r,t})_{r\leq t}$ if

$$N_{r,t} = (N_{r,s}Q_{s,t}) * N_{s,t}, \quad r \leq s \leq t. \tag{2.5}$$

Theorem 2.1. *The equation (2.5) is fulfilled if and only if*

$$Q^N_{r,t}(\mu, \cdot) := Q_{r,t}(\mu, \cdot) * N_{r,t}, \quad r \leq t,\ \mu \in M(E), \tag{2.6}$$

defines a Markov semigroup $(Q^N_{r,t})_{r\leq t}$ *on* $M(E)$.

Proof. Let $(Q_{r,t})_{r\leq t}$ and $(Q^N_{r,t})_{r\leq t}$ be given by (2.4) and (2.6), respectively. It is not difficult to check that (2.5) is equivalent to the Chapman-Kolmogorov equation

$$Q^N_{r,t}(\mu,\cdot) = \int_{M(E)} Q^N_{r,s}(\mu,\mathrm{d}\nu)Q^N_{s,t}(\nu,\cdot), \quad r\leq s\leq t,\ \mu\in M(E),$$

from which the assertion follows. □

Let T be an interval and $(N_{r,t})_{r\leq t}$ be an SC-semigroup associated with $(Q_{r,t})_{r\leq t}$. If $\{Y : t\in T\}$ is a Markov process having transition semigroup $(Q^N_{r,t})_{r\leq t}$ defined by (2.6), we call it an *immigration process* associated with X. (Of course, SC-semigroups and immigration processes can also be formulated for some more general classes of Markov processes with state spaces possessing semigroup structures.)

It is known that a metric ϱ can be introduced into E so that the Borel σ-algebra on E induced by ϱ coincides with its original Borel σ-algebra; see e.g. Cohn [2, p275]. We write $M(E_\varrho)$ for the set $M(E)$ furnished with the topology of weak convergence on (E,ϱ). Then $M(E_\varrho)$ is locally compact and metrizable. Let $D(E_\varrho)^+$ be a countable dense subset of the space of *strictly positive* continuous functions on (E,ϱ).

Lemma 2.2. *Let* $\{F_n : n=1,2,\cdots\}$ *be a sequence of probabilities on* $M(E)$. *If the limit*

$$L(f) := \lim_{n\to\infty} L_{F_n}(f), \quad f\in D(E_\varrho)^+, \tag{2.7}$$

exists and $L(f)\to 1$ *as* $f\to 0$, *then there is a probability measure* F *on* $M(E)$ *such that* $L_F(f) = L(f)$ *for all* $f\in D(E_\varrho)^+$. *Moreover, if each* F_n *is infinitely divisible, so is* F.

Proof. Let $\bar{M}(E_\varrho) := M(E_\varrho)\cup\{\infty\}$ denote the one point compactification of $M(E_\varrho)$. Then the sequence $\{F_n\}$ viewed as probabilities on $\bar{M}(E_\varrho)$ is relatively compact. Let $\{F_{n_k}\}$ be a subsequence of $\{F_n\}$ which converges to some probability measure F on $\bar{M}(E_\varrho)$. By (2.7) and bounded convergence, we have

$$L(f) = \int_{\bar{M}(E_\varrho)} \mathrm{e}^{-\nu(f)} F(\mathrm{d}\nu), \quad f\in D(E_\varrho)^+,$$

where the integrand is defined as zero at ∞ by continuity. Since $L(f)\to 1$ as $f\to 0$, we have $F(M(E_\varrho)) = 1$ and hence the first assertion follows. The second assertion is immediate. □

For any $\alpha\in[-\infty,\infty)$, a family of σ-finite measures $(K_t)_{t>\alpha}$ on $M(E)$ is called an *entrance law* (at α) for the semigroup $(Q_t)_{r\leq t}$ if $K_rQ_{r,t} = K_t$ for all $t>r>\alpha$. It is called a *probability entrance law* if each K_t is a probability measure. An entrance law $(K_t)_{t>\alpha}$ is said to be *closable* if there is a σ-finite measure K_α on $M(E)$ such that $K_t = K_\alpha Q_{\alpha,t}$ for all $t>\alpha$. In this case, $(K_t)_{t\geq\alpha}$ is called a *closed entrance law* for $(Q_{r,t})_{r\leq t}$. An entrance law $(K_t)_{t>\alpha}$ is said to be *minimal* if every entrance law dominated by $(K_t)_{t>\alpha}$ is proportional to it. Those definitions are applicable to general transition semigroups under obvious modifications.

Example 2.1. Let $T_1 \subset I\!R$ be a countable set and $\{(K_{s,t})_{t>s} : s \in T_1\}$ be a family of probability entrance laws for $(Q_{r,t})_{r\leq t}$. Suppose that

$$-\sum_{s\in[r,t)\cap T_1} \log L_{K_{s,t}}(1) < \infty$$

for all $r \leq t \in I\!R$. By Lemma 2.2 we may see that

$$\log L_{N_{r,t}}(f) = \sum_{s\in[r,t)\cap T_1} \log L_{K_{s,t}}(f), \quad r \leq t,\ f \in B(E)^+, \tag{2.8}$$

defines a probability measure $N_{r,t}$ on $M(E)$. It is simple to check that $(N_{r,t})_{r\leq t}$ form an SC-semigroup. Suppose that for each $s \in T_1$ we have a Markov process $(X_{s,t})_{t>s}$ with transition semigroup $(Q_{r,t})_{r\leq t}$ and one-dimensional distributions $(K_{s,t})_{t>s}$ and that the family $\{(X_{s,t})_{t>s} : s \in T_1\}$ are independent. Then for any $t \geq \alpha$ the random measure

$$Y_t := \sum_{s\in[a,t)\cap T_1} X_{s,t}$$

is a.s. well-defined and $\{Y_t : t \geq \alpha\}$ is an immigration process corresponding to the SC-semigroup given by (2.8).

Example 2.2. Let $T_2 \subset I\!R$ be a countable set and $\{(K_{s,t})_{t\geq s} : s \in T_2\}$ be a family of closed probability entrance laws for $(Q_{r,t})_{r\leq t}$ such that

$$-\sum_{s\in(r,t]\cap T_2} \log L_{K_{s,t}}(1) < \infty$$

for all $r \leq t \in I\!R$. Then we may define an SC-semigroup $(N_{r,t})_{r\leq t}$ by

$$\log L_{N_{r,t}}(f) = \sum_{s\in(r,t]\cap T_2} \log L_{K_{s,t}}(f), \quad r \leq t,\ f \in B(E)^+. \tag{2.9}$$

The corresponding immigration process can be constructed similarly as in the last example.

Example 2.3. Suppose that $(N_{r,t})_{r\leq t}$ is a family of probability measures on $M(E)$ given by

$$\log L_{N_{r,t}}(f) = \int_r^t \log L_{K_{s,t}}(f)\zeta(\mathrm{d}s), \quad r \leq t,\ f \in B(E)^+, \tag{2.10}$$

where $\zeta(\mathrm{d}s)$ is a Radon measure on $I\!R$ and $\{(K_{s,t})_{t>s} : s \in I\!R\}$ is a family of infinitely divisible probability entrance laws for $(Q_{r,t})_{r\leq t}$. Then $(N_{r,t})_{r\leq t}$ is an SC-semigroup.

Example 2.4. Suppose we have a probability space on which the two processes $\{X_t^1 : t \geq 0\}$ and $\{X_t^2 : t \geq \alpha\}$ are defined, where $\{X_t^1 : t \geq 0\}$ is a superprocess with parameters (ξ_1, ϕ_1), and $\{X_t^2 : t \geq 0\}$ conditioned on $\{X_t^1 : t \geq 0\}$ is an immigration process corresponding to the SC-semigroup determined by

$$\log L_{N_{r,t}}(f) = -\int_r^t X_s(V_{t-s}^2 f)\mathrm{d}s, \quad r \leq t,\ f \in B(E)^+,$$

where $(V_t^2)_{t\geq 0}$ is defined by (1.2) with parameters (ξ_2, ϕ_2). Then $\{(X_t^1, X_t^2) : t \geq 0\}$ is also a Markov process. Intuitively, it describes the evolution of a population with two types of "particles" on E, where the second type can be produced by both of them; see Hong and Li [18]. More general forms of multi-type superprocesses were studied in Gorostiza and Lopez-Mimbela [16], Gorostiza and Roelly [17], Li [23], etc.

3 Decomposition of skew convolution semigroups

In this section, we prove a decomposition theorem for the inhomogeneous SC-semigroup, which appears even in the construction of homogeneous immigration processes. Let us consider the transition semigroup $(Q_{r,t})_{r\leq t}$ of an MB-process defined in the last section.

Theorem 3.1. *A family of probability measures $(N_{r,t})_{r\leq t}$ on $M(E)$ form an SC-semigroup associated with $(Q_{r,t})_{r\leq t}$ if and only if*

$$\begin{aligned}\log L_{N_{r,t}}(f) \quad &= \sum_{s\in[r,t)\cap T_1} \log L_{K_{s,t}^1}(f) + \sum_{s\in(r,t]\cap T_2} \log L_{K_{s,t}^2}(f) \\ &\qquad + \int_r^t \log L_{K_{s,t}^3}(f)\zeta(\mathrm{d}s), \qquad r \leq t,\ f \in B(E)^+, \qquad (3.1)\end{aligned}$$

where $T_1, T_2 \subset I\!R$ are countable sets, $\zeta(\mathrm{d}s)$ is a diffuse Radon measure on $I\!R$, $\{(K_{s,t}^1)_{t>s} : s \in T_1\}$ is a family of probability entrance laws, $\{(K_{s,t}^2)_{t\geq s} : s \in T_2\}$ is a family of closed probability entrance laws, and $\{(K_{s,t}^3)_{t>s} : s \in I\!R\}$ is a family of infinitely divisible probability entrance laws for $(Q_{r,t})_{r\leq t}$.

In principle, the entrance laws are obtained via applications of Lemma 2.2. The proof is a little tedious because we are not assuming the Feller property and the class $D(E_\varrho)^+$ is not preserved by the cumulant semigroup $(V_{r,t})_{r\leq t}$. We shall break the proof into several lemmas. Suppose that $(N_{r,t})_{r\leq t}$ is an SC-semigroup associated with $(Q_{r,t})_{r\leq t}$ and let

$$J_{r,t}(f) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_{r,t}(\mathrm{d}\nu), \quad r \leq t,\ f \in B(E)^+. \qquad (3.2)$$

Then the relation (2.5) is equivalent to

$$J_{r,t}(f) = J_{r,s}(V_{s,t}f) + J_{s,t}(f), \quad r \leq s \leq t,\ f \in B(E)^+. \qquad (3.3)$$

By (3.3) one sees that $J_{r,t}(f)$ is a non-increasing function of $r \leq t$ and $J_{t,t}(f)$ =0. Then there is a Radon measure $G_t(f, \cdot)$ on $(-\infty, t]$ such that $G_t(f, \{t\}) = \lim_{r\uparrow t} J_{r,t}(f)$ and

$$G_t(f, (r, s]) = \lim_{v\downarrow s} \lim_{u\downarrow r} [J_{u,t}(f) - J_{v,t}(f)] = \lim_{v\downarrow s} \lim_{u\downarrow r} J_{u,v}(V_{v,t}f), \qquad r \leq s < t. \qquad (3.4)$$

Lemma 3.2. *(i) If $f, g \in B(E)^+$ and $0 \leq g \leq cf$ for a constant $c \geq 1$, then $G_t(g, \cdot) \leq cG_t(f, \cdot)$. (ii) For $t \leq u$ and $f \in B(E)^+$, we have $G_u(f, \mathrm{d}s) = G_t(V_{t,u}f, \mathrm{d}s)$ on $(-\infty, r)$. (iii) For $t \leq u$ and $f \in B(E)^+$, we have $G_u(f, \mathrm{d}s) \ll G_t(1, \mathrm{d}s)$ on $(-\infty, t)$.*

Proof. For $r \leq s < t \leq u$, we use (2.4), (3.2) and Jensen's inequality to see that

$$\begin{aligned} G_t(g, (r, s]) &= \lim_{v \downarrow s} \lim_{w \downarrow r} J_{w,v}(V_{v,t}g) \leq \lim_{v \downarrow s} \lim_{w \downarrow r} J_{w,v}(V_{v,t}(cf)) \\ &\leq c \lim_{v \downarrow s} \lim_{w \downarrow r} J_{w,v}(V_{v,t}f) = cG_t(f, (r, s]). \end{aligned}$$

Similarly, we have $G_t(g, \{t\}) \leq cG_t(f, \{t\})$. Then (i) follows. Let $r \leq s < t \leq u$. By (3.4) and the semigroup property of $(V_{r,t})_{r \leq t}$,

$$G_u(f, (r, s]) = \lim_{v \downarrow s} \lim_{w \downarrow r} J_{w,v}(V_{v,t}V_{t,u}f) = G_t(V_{t,u}f, (r, s]),$$

yielding (ii). Combining (i) and (ii) we have

$$G_u(f, (r, s]) = G_t(V_{t,u}f, (r, s]) \leq (\|V_{t,u}f\| + 1)G_t(1, (r, s]),$$

from which (iii) follows. □

Clearly, we have the unique decomposition:

$$J_{r,t}(f) = J^1_{r,t}(f) + J^2_{r,t}(f) + J^3_{r,t}(f), \quad s \leq t, \ f \in B(E)^+, \tag{3.5}$$

where $J^1_{r,t}(f)$ is a left continuous non-increasing step function, $J^2_{r,t}(f)$ is a right continuous non-increasing step function and $J^3_{r,t}(f)$ is a continuous non-increasing function of $r \leq t$, and $J^i_{t,t}(f) = 0$ for $i = 1, 2, 3$. By the uniqueness, $(J^i_{r,t})_{r \leq t}$ also satisfies equation (3.3). Applying Lemma 2.2 we may get

$$N_{r,t} = N^1_{r,t} * N^2_{r,t} * N^3_{r,t}, \quad r \leq t, \tag{3.6}$$

where $(N^i_{r,t})_{r \leq t}$ is the SC-semigroup corresponding to the functional $(J^i_{r,t})_{r \leq t}$. Observe also that $J^1_{s,t}(f)$ and $J^2_{s,t}(f)$ yield atomic measures $G^1_t(f, \cdot)$ and $G^2_t(f, \cdot)$ on $(-\infty, t]$, respectively, and $J^3_{s,t}(f)$ yields a diffuse measure $G^3_t(f, \cdot)$ on $(-\infty, t]$.

Lemma 3.3. *There are countable sets $T_1, T_2 \subset I\!\!R$, probability entrance laws $\{(K^1_{r,t})_{t>s} : r \in T_1\}$ and closed probability entrance laws $\{(K^2_{r,t})_{t \geq r} : r \in T_2\}$ such that*

$$J^1_{r,t}(f) = -\sum_{s \in [r,t) \cap T_1} \log L_{K^1_{s,t}}(f), \quad r \leq t, \ f \in B(E)^+, \tag{3.7}$$

and

$$J^2_{r,t}(f) = -\sum_{s \in (r,t] \cap T_2} \log L_{K^2_{s,t}}(f), \quad r \leq t, \ f \in B(E)^+. \tag{3.8}$$

Proof. Since the arguments are similar, we only give the proof of (3.8). Recall that $J^2_{s,t}(f)$ is right continuous in $s \le t$. Consequently,

$$G^2_t(f, \{s\}) =\downarrow \lim_{r\uparrow s}[J^2_{r,t}(f) - J^2_{s,t}(f)] =\downarrow \lim_{r\uparrow s} J^2_{r,s}(V_{s,t}f), \quad f \in B(E)^+. \tag{3.9}$$

That is,

$$\exp\{-G^2_t(f, \{s\})\} =\uparrow \lim_{r\uparrow s} \int_{M(E)} \mathrm{e}^{-\nu(f)} N^2_{r,s} Q_{s,t}(\mathrm{d}\nu), \quad f \in B(E)^+.$$

By Lemma 2.2 we see that

$$\exp\{-G^2_t(f, \{s\})\} = \int_{M(E)} \mathrm{e}^{-\nu(f)} K^2_{s,t}(\mathrm{d}\nu), \quad f \in D(E_\varrho)^+, \tag{3.10}$$

for a probability measure $K^2_{s,t}$ on $M(E)$. Let $Q = \{u_i : i = 1, 2, \cdots\}$ be a countable dense subset of $I\!R$ and let T_2 be the collection of atoms of the measures $\{G^2_{u_i}(1, \cdot) : i = 1, 2, \cdots\}$. By (3.10) and Lemma 3.2,

$$\log L_{N^2_{r,t}}(f) = -G^2_t(f, (r, t]) = \log L_{K^2_{t,t}}(f) + \sum_{s\in(r,t)\cap T_2} \log L_{K^2_{s,t}}(f), \quad r \le t,$$

first for $f \in D(E_\varrho)^+$ and then for all $f \in B(E)^+$. In particular, (3.10) also holds for $f \in B(E)^+$. But, (3.9) implies that $G^2_t(f, \{s\}) = G^2_s(V_{s,t}f, \{s\})$. Then we must have $K_{s,s}Q_{s,t} = K_{s,t}$ for $s \le t$, that is $(K_{s,t})_{t\ge s}$ form a closed entrance law. If $-\log L_{K^2_{t,t}}(f) > 0$, then $-\log L_{K^2_{t,t}}(1) > 0$. By the continuity of $V_{t,u}1$ in $u \ge t$,

$$G^2_u(1, \{t\}) = -\log L_{K^2_{t,u}}(1) = -\log L_{K^2_{t,t}}(V_{t,u}1) > 0$$

for some $u \in Q$. Then we have $t \in Q$ and (3.8) follows. □

Lemma 3.4. *There is a diffuse Radon measure $\zeta(\mathrm{d}s)$ on $I\!R$ such that $G^3_t(f, \mathrm{d}s) \ll \zeta(\mathrm{d}s)$ for all $t \in I\!R$ and $f \in B(E)^+$.*

Proof. Let $Q = \{u_i : i = 1, 2, \cdots\}$ be a countable dense subset of $I\!R$ and choose $a_i > 0$ such that

$$\sum_{i=1}^{\infty} a_i G^3_{u_i}(1, (u_i - 1, u_i]) < \infty.$$

We may define a diffuse Radon (in fact finite) measure ζ on $I\!R$ by

$$\zeta(\mathrm{d}s) = \sum_{i=1}^{\infty} 1_{(u_i-1,u_i]}(s) G^3_{u_i}(1, \mathrm{d}s), \quad s \in I\!R. \tag{3.11}$$

If $G^3_t(f, B) > 0$ for a Borel set $B \subset (-\infty, t]$, then $G^3_t(f, (u_i - 1, u_i) \cap B) > 0$ for some $u_i \in (-\infty, t) \cap Q$. Therefore, Lemma 3.2 implies that $G^3_{u_i}(1, (u_i - 1, u_i) \cap B) > 0$, and hence $\zeta(B) > 0$. □

Lemma 3.5. *The Radon-Nikodym derivative $G_t^3(f, \mathrm{d}s)/\zeta(\mathrm{d}s)$ has a version $D_{s,t}(f)$ with the representation*

$$D_{s,t}(f) = -\log L_{F_{s,t}}(f), \quad s < t,\ f \in B(E)^+, \tag{3.12}$$

where $F_{s,t}$ is an infinitely divisible probability measure on $M(E)$.

Proof. Let $r < t$ and assume $\zeta(r, t] > 0$ to avoid triviality. We take an increasing sequence of ordered sets $\pi_n = \{s_0^n, s_1^n, \cdots, s_{m(n)}^n\}$ with $r = s_0^n < s_0^n < \cdots < s_{m(n)}^n = t$ and $\lim_{n\to\infty} \delta_n = 0$, where $\delta_n = \max\{s_i^n - s_{i-1}^n : 1 \le i \le m(n)\}$. Let $\mathcal{F}_n$ be the σ-algebra on $(r, t]$ generated by $(s_0^n, s_1^n], \cdots, (s_{m(n)-1}^n, s_{m(n)}^n]$ and let

$$M_n(f)(s) = \begin{cases} G_t^3(f, (s_{i-1}^n, s_i^n])/\zeta(s_{i-1}^n, s_i^n] & \text{if } s_{i-1}^n < s \le s_i^n \text{ and } \zeta(s_{i-1}^n, s_i^n] > 0, \\ 0 & \text{if } s_{i-1}^n < s \le s_i^n \text{ and } \zeta(s_{i-1}^n, s_i^n] = 0. \end{cases}$$

Then $\{M_n(f), \mathcal{F}_n : n \ge 1\}$ under the probability measure $\zeta(r, t]^{-1}\zeta$ is a martingale which is closed on the right by the Radon-Nikodym derivative $G_t^3(f, \mathrm{d}s)/\zeta(\mathrm{d}s)$. But, since $\{\mathcal{F}_n : n \ge 1\}$ generates the Borel σ-algebra on $(r, t]$, $M_n(f)(s)$ converges as $n \to \infty$ to $G_t^3(f, \mathrm{d}s)/\zeta(\mathrm{d}s)$ for ζ-a.e. $s \in (r, t]$ by the martingale convergence theorem. Then we may find a set $A_t \subset (-\infty, t]$ with full ζ-measure such that for any $s \in A_t$ there are sequences $r_k \uparrow s$ and $s_k \downarrow s$ satisfying

$$G_t^3(f, \mathrm{d}s)/\zeta(\mathrm{d}s) = \lim_{k\to\infty} G_t^3(f, (r_k, s_k])/\zeta(r_k, s_k], \quad f \in D(E_\varrho)^+.$$

Clearly, $G_t^3(f, (r_k, s_k]) = G_{s_k}(V_{s_k,t}f, (r_k, s_k])$ goes to zero as $k \to \infty$. It follows that

$$\begin{aligned} \frac{G_t^3(f, \mathrm{d}s)}{\zeta(\mathrm{d}s)} &= \lim_{k\to\infty} \frac{1 - \exp\{-G_{s_k}(V_{s_k,t}f, (r_k, s_k])\}}{\zeta(r_k, s_k]} \\ &= \lim_{k\to\infty} \frac{1}{\zeta(r_k, s_k]} \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) N_{r_k,s_k} Q_{s_k,t}(\mathrm{d}\nu) \end{aligned} \tag{3.13}$$

for $f \in D(E_\varrho)^+$. On the other hand, since

$$\int_r^t [G_t^3(f, \mathrm{d}s)/\zeta(\mathrm{d}s)]\zeta(\mathrm{d}s) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_{r,t}^3(\mathrm{d}\nu),$$

choosing a smaller full ζ-measure set $A_t \subset (-\infty, t]$ we may get $G_t^3(f, \mathrm{d}s)/\zeta(\mathrm{d}s) \downarrow 0$ as $f \downarrow 0$ for all $s \in A_t$. By (3.13) and Lemma 2.2 there are infinitely divisible probability measures $\{F_{s,t} : s \in A_t\}$ such that

$$G_t^3(f, \mathrm{d}s)/\zeta(\mathrm{d}s) = -\log L_{F_{s,t}}(f), \quad s \in A_t,\ f \in D(E_\varrho)^+.$$

Setting $F_{s,t} = \delta_0$ for $s \in (-\infty, t] \setminus A_t$ we have

$$G_t^3(f, (r, t]) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} N_{r,t}^3(\mathrm{d}\nu) = -\int_r^t \log L_{F_{s,t}}(f)\zeta(\mathrm{d}s),$$

first for $f \in D(E_\varrho)^+$ and then for all $f \in B(E)^+$. $\square$

Lemma 3.6. *There are infinitely divisible probability entrance laws* $\{(K_{s,t})_{t>s} : s \in I\!R\}$ *for the semigroup* $(Q_{r,t})_{r\leq t}$ *such that*

$$J^3_{r,t}(f) = -\int_r^t \log L_{K_{s,t}}(f)\zeta(\mathrm{d}s), \quad t \geq r, \; f \in B(E)^+. \tag{3.14}$$

Proof. By Lemma 3.2 we have $G^3_t(f, \mathrm{d}s) = G^3_r(V_{r,t}f, \mathrm{d}s)$ for $s < r \leq t$. It follows that

$$D_{s,t}(f) = D_{s,r}(V_{r,t}f), \quad \zeta\text{-a.e. } s \in (-\infty, r].$$

Let λ denote the Lebesgue measure on $I\!R$. By Fubini's lemma, there is a set $B(f) \subseteq I\!R$ with full ζ-measure and sets $C_s(f) \subseteq [s, \infty)$ and $C_{s,r}(f) \subseteq [r, \infty)$ with full λ-measure such that

$$D_{s,t}(f) = D_{s,r}(V_{r,t}f), \quad s \in B(f), \; r \in C_s(f), \; t \in C_{s,r}(f).$$

By Lemma 3.5, $D_{s,t}$ and $D_{s,r} \circ V_{r,t}$ are determined by their restrictions to the countable class $D(E_\varrho)^+$. Then for a set $B \subseteq I\!R$ with full ζ-measure and sets $C_s \subseteq (s, \infty)$ and $C_{s,r} \subseteq (r, \infty)$ with full λ-measures we have

$$D_{s,t} = D_{s,r} \circ V_{r,t}, \quad s \in B, \; r \in C_s, \; t \in C_{s,r}, \tag{3.15}$$

as operators on $B(E)^+$. For any $s \in B$, choose a sequence $\{s_k\} \subset C_s$ with $s_k \downarrow s$. By (3.15) we get

$$D_{s,s_k} \circ V_{s_k,t} = D_{s,s_{k+1}} \circ V_{s_{k+1},t} = D_{s,t}, \quad t \in C_{s,s_k} \cap C_{s,s_{k+1}}.$$

By Lemma 3.5 and our assumptions, $D_{s,s_k}(V_{s_k,t}f)$ and $D_{s,s_{k+1}}(V_{s_{k+1},t}f)$ are right continuous in $t \geq s_{k+1}$ for $f \in C(E)^+$. Then we have

$$D_{s,s_k} \circ V_{s_k,t} = D_{s,s_{k+1}} \circ V_{s_{k+1},t}, \quad t \geq s_k, \tag{3.16}$$

as operators on $B(E)^+$. For $t > s$ we take some $s_k \in (s, t)$ and let $I_{s,t} = D_{s,s_k} \circ V_{s_k,t}$, which is independent of the choice of s_k by (3.16). Correspondingly, let F_{s,s_k} be the infinitely divisible probability measure on $M(E)$ given by Lemma 3.5 and let $K_{s,t} = F_{s,s_k}Q_{s_k,t}$, which is independent of s_k ether. Clearly, $(K_{s,t})_{t>s}$ form an entrance law for $(Q_{r,t})_{r\leq t}$ and

$$I_{s,t} = -\log L_{F_{s,s_k}} \circ V_{s_k,t} = -\log L_{K_{s,t}}(f), \quad t > s. \tag{3.17}$$

From (3.15) we have $I_{s,t} = D_{s,t}$ for λ-a.e. $t \in (s, \infty)$. If $s \in I\!R \setminus B$, we define $I_{s,t} = 0$ for all $t > s$. Then Fubini's lemma yields the existence of a set $U \subseteq I\!R$ with full λ-measure such that for each $t \in U$ we have $I_{s,t} = D_{s,t}$ for ζ-a.e. $s \in (-\infty, t)$. Consequently, if $t \in U$,

$$J^3_{r,t}(f) = G^3_t(f, (r, t]) = \int_r^t I_{s,t}(f)\zeta(\mathrm{d}s), \quad r \leq t, \; f \in B(E)^+. \tag{3.18}$$

Recall that both ζ and $G^3_t(f,\cdot)$ are diffuse measures. For $t \in I\!R \setminus U$ we choose a sequence $\{t_k\} \subset U$ with $t_k \uparrow t$. By (3.18) and Lemma 3.2,

$$\begin{aligned} J^3_{r,t}(f) &= \lim_{k\to\infty} G^3_t(f,(r,t_k]) = \lim_{k\to\infty} G^3_{t_k}(V_{t_k,t}f,(r,t_k]) \\ &= \lim_{k\to\infty} \int_r^{t_k} I_{s,t_k}(V_{t_k,t}f)\zeta(\mathrm{d}s) = \lim_{k\to\infty} \int_r^{t_k} I_{s,t}(f)\zeta(\mathrm{d}s) \\ &= \int_r^t I_{s,t}(f)\zeta(\mathrm{d}s), \end{aligned}$$

yielding the desired result. □

Proof of Theorem 3.1. It is easy to see that, if $(N_{r,t})_{r\le t}$ is given by (3.1), it satisfies (2.5). Combining (3.5) and Lemmas 3.3 and 3.6 we see that any SC-semigroup has the decomposition (3.1). □

Let $(Q_t)_{t\ge 0}$ be the transition semigroup of a homogeneous MB-process. The following special case of Theorem 3.1 was proved in Li [25].

Theorem 3.7 (Li [25]). *A family of probability measures $(N_t)_{t\ge 0}$ on $M(E)$ is an SC-semigroup associated with $(Q_t)_{t\ge 0}$ if and only if there is an infinitely divisible probability entrance law $(K_t)_{t>0}$ for $(Q_t)_{t\ge 0}$ such that*

$$\log L_{N_t}(f) = \int_0^t [\log L_{K_s}(f)]\mathrm{d}s, \quad t\ge 0,\ f\in B(E)^+. \tag{3.19}$$

□

4 Construction of immigration processes

To make best use of the existing literature, we restrict in this and the subsequent sections to the transition semigroup $(Q_t)_{t\ge 0}$ of a homogeneous Borel right MB-process X. Let $(Q^\circ_t)_{t\ge 0}$ denote the restriction of $(Q_t)_{t\ge 0}$ to $M(E)^\circ$.

We first review some facts in potential theory; see e.g. Dellacherie et al [6] and Getoor [14]. A family of σ-finite measures $(J_t)_{t\in I\!R}$ is called an *entrance rule* for $(Q^\circ_t)_{t\ge 0}$ if $J_sQ^\circ_{t-s} \le J_t$ for $t > s \in I\!R$ and $J_sQ^\circ_{t-s} \uparrow J_t$ as $s \uparrow t$. Note that an entrance law $(H_t)_{t>r}$ at $r \in [-\infty,\infty)$ can be extended to an entrance rule by setting $H_t = 0$ for $t \le r$. Let $W(M(E))$ denote the space of paths $\{w_t : t \in I\!R\}$ that are $M(E)^\circ$-valued and right continuous on an open interval $(\alpha(w),\beta(w))$ and take the value of the null measure elsewhere. The path [0] constantly equal to 0 corresponds to (α,β) being empty. Set $\alpha([0]) = +\infty$ and $\beta([0]) = -\infty$. Let $(\mathcal{G}^\circ,\mathcal{G}^\circ_t)_{t\in I\!R}$ be the natural σ-algebras on $W(M(E))$ generated by the coordinate process. The shift operators $\{\sigma_t : t \in I\!R\}$ on $W(M(E))$ are defined by $\sigma_t w_s = w_{t+s}$. To any entrance rule $(J_t)_{t\in I\!R}$ there corresponds a unique σ-finite measure $\boldsymbol{Q}^J$ on $(W(M(E)),\mathcal{G}^\circ)$ under which the coordinate process $\{w_t : t \in I\!R\}$ is a

Markov process with one-dimensional distributions $(J_t)_{t\in I\!R}$ and semigroup $(Q_t^\circ)_{t\geq 0}$. That is,

$$\boldsymbol{Q}^J\{\alpha < t_1, w_{t_1} \in \mathrm{d}\nu_1, w_{t_2} \in \mathrm{d}\nu_2, \cdots, w_{t_n} \in \mathrm{d}\nu_n, t_n < \beta\}$$
$$= J_{t_1}(\mathrm{d}\nu_1) Q^\circ_{t_2-t_1}(\nu_1, \mathrm{d}\nu_2) \cdots Q^\circ_{t_n - t_{n-1}}(\nu_{n-1}, \mathrm{d}\nu_n) \tag{4.1}$$

for all $t_1 < \cdots < t_n \in I\!R$ and $\nu_1, \cdots, \nu_n \in M(E)^\circ$. The existence of this measure was proved by Kuznetsov [22]; see also Getoor and Glover [15]. The system $(W(M(E)), \mathcal{G}^\circ, \mathcal{G}^\circ_t, w_t, \boldsymbol{Q}^J)$ is now commonly called the *Kuznetsov process* determined by $(J_t)_{t\in I\!R}$, and $\boldsymbol{Q}^J$ is called the *Kuznetsov measure*. We have the representation

$$J_t = H_{-\infty,t} + \int_{I\!R} H_{s,t}\rho(\mathrm{d}s), \quad t \in I\!R, \tag{4.2}$$

where $\rho(\mathrm{d}s)$ is a Radon measure on $I\!R$ and $(H_{s,t})_{t\in I\!R}$ is an entrance law at $s \in [-\infty, \infty)$. This representation yields

$$\boldsymbol{Q}^J(\mathrm{d}w) = \boldsymbol{Q}_{-\infty}(\mathrm{d}w) + \int_{I\!R} \boldsymbol{Q}_s(\mathrm{d}w)\rho(\mathrm{d}s), \tag{4.3}$$

where $\boldsymbol{Q}_s(\mathrm{d}w)$ is the Kuznetsov measure determined by $(H_{s,t})_{t\in I\!R}$; see [15]. If $(J_t)_{t\in I\!R}$ is an entrance law at $r \in [-\infty, \infty)$, then $\boldsymbol{Q}^J$ is supported by $W_r(M(E))$, the subset of $W(M(E))$ comprising paths $\{w_t : t \in I\!R\}$ such that $\alpha(w) = r$. In particular, if F is an excessive measure for $(Q_t^\circ)_{t\geq 0}$ and $J_t \equiv F$, then $\boldsymbol{Q}^J$ is stationary, that is, $\boldsymbol{Q}^J \circ \sigma_t^{-1} = \boldsymbol{Q}^J$ for all $t \in I\!R$.

Now suppose that $(J_t)_{t\in I\!R}$ is an entrance rule for $(Q_t^\circ)_{t\geq 0}$ with the representation (4.2) and $N^J(\mathrm{d}w)$ is a Poisson random measure on $W(M(E))$ with intensity $\boldsymbol{Q}^J(\mathrm{d}w)$. It is easy to see that

$$Y_t^J := \int_{W(M(E))} w_t \, N^J(\mathrm{d}w) \tag{4.4}$$

is a.s. well-defined for each $t \in I\!R$.

Lemma 4.1. *In the situation described above, $\{Y_t^J : t \in I\!R\}$ is an immigration process corresponding to the (inhomogeneous) SC-semigroup $(N_{r,t})_{r\leq t}$ defined by*

$$\log L_{N_{r,t}}(f) = -\int_{[r,t)} \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_{s,t}(\mathrm{d}\nu)\rho(\mathrm{d}s), \quad r \leq t, f \in B(E)^+. \tag{4.5}$$

Proof. By (4.3), for any bounded Borel function F on $M(E)$ with $F(0) = 0$, we have

$$\boldsymbol{Q}^J\{F(w_t); r \leq \alpha < t\} = \int_{[r,t)} H_{s,t}(F)\, \rho(\mathrm{d}s). \tag{4.6}$$

Then the results follow from (4.6) and the Markov property of $\boldsymbol{Q}^J$. □

Let $(N_t)_{t\geq 0}$ an SC-semigroup associated with $(Q_t)_{t\geq 0}$ which is given by (3.19). Suppose that $K_t = I(\eta_t, H_t)$ and $N_t = I(\gamma_t, G_t)$ for $t > 0$ and $t \geq 0$, respectively.

Lemma 4.2. *Let $G_t = 0$ for $t < 0$. Then $(G_t)_{t\in\mathbb{R}}$ is an entrance rule for $(Q_t^\circ)_{t\geq 0}$.*

Proof. Recall that $Q_t(\mu, \cdot)$ is an infinitely divisible probability measure on $M(E)$ for all $t \geq 0$ and $\mu \in M(E)$. Suppose $Q_t(\delta_x, \cdot) = I(\lambda_t(x, \cdot), L_t(x, \cdot))$. By (1.4),

$$G_t = G_{t-r} + G_r Q_{t-r}^\circ + \int_E \gamma_r(\mathrm{d}x) L_{t-r}(x, \cdot), \quad t > r > 0,$$

and hence $G_r Q_{t-r} \leq G_t$. From (3.19) we have

$$G_t Q_{t-r}^\circ = \int_0^t H_s Q_{t-r}^\circ \mathrm{d}s = G_r Q_{t-r}^\circ + \int_r^t H_s Q_{t-r}^\circ \mathrm{d}s,$$

so $G_r Q_{t-r}^\circ \uparrow G_t$ as $r \uparrow t$. Therefore, $(G_t)_{t\in\mathbb{R}}$ is an entrance rule. $\square$

Now we give the construction of the immigration process corresponding to $(N_t)_{t\geq 0}$. The next theorem shows that, except the deterministic part $\{\gamma_t : t \geq 0\}$, both the entering times and the evolutions of the immigrants are decided by a Poisson random measure based on the Kuznetsov measure $\boldsymbol{Q}^G$.

Theorem 4.3. *Let Y_t^G be defined by (4.4) with $J = G$ and let $Y_t = \gamma_t + Y_t^G$. Then $\{Y_t : t \geq 0\}$ is an immigration process with one-dimensional distributions $(N_t)_{t\geq 0}$ and transition semigroup $(Q_t^N)_{t\geq 0}$.*

Proof. Suppose $(G_t)_{t\in\mathbb{R}}$ is represented by (4.2) with $G_{s,t}$ in place of $J_{s,t}$. Then, for $t \geq r \geq 0$,

$$\begin{aligned} \int_{[r,t)} G_{s,t}\, \rho(\mathrm{d}s) &= \int_{[0,t)} G_{s,t}\, \rho(\mathrm{d}s) - \int_{[0,r)} G_{r,s} Q_{t-r}^\circ \rho(\mathrm{d}s) \\ &= G_t - G_r Q_{t-r}^\circ \\ &= \int_0^t H_s \mathrm{d}s - \int_0^r H_s Q_{t-r}^\circ \mathrm{d}s. \end{aligned} \tag{4.7}$$

The relation $K_{s+t} = K_s Q_t$ yields

$$\eta_{s+t} = \int_E \eta_s(\mathrm{d}x) \lambda_t(x, \cdot), \quad H_{s+t} = \int_E \eta_s(\mathrm{d}x) L_t(x, \cdot) + H_s Q_t^\circ. \tag{4.8}$$

From the second equation in (4.8) we have

$$\int_0^r H_{s+t-r} \mathrm{d}s - \int_0^r H_s Q_{t-r}^\circ \mathrm{d}s = \int_0^r \mathrm{d}s \int_E \eta_s(\mathrm{d}x) L_{t-r}(x, \cdot).$$

Substituting this into (4.7) gives

$$\begin{aligned} \int_{[r,t)} G_t^s\, \rho(\mathrm{d}s) &= \int_0^{t-r} H_s \mathrm{d}s + \int_0^r \mathrm{d}s \int_E \eta_s(\mathrm{d}x) L_{t-r}(x, \cdot) \\ &= G_{t-r} + \int_E \gamma_r(\mathrm{d}x) L_{t-r}(x, \cdot). \end{aligned} \tag{4.9}$$

Since $\{\gamma_t : t \geq 0\}$ is deterministic, it is simple to check that $\{Y_t : t \geq 0\}$ is a Markov process with one-dimensional distributions $(N_t)_{t\geq 0}$. By Lemma 4.1 we have

$$\begin{aligned}
&\boldsymbol{E}[\exp\{-Y_t(f)\}|Y_s : 0 \leq s \leq r] \\
&\quad = \exp\Big\{ - Y_r^G(V_{t-r}f) - \gamma_t(f) \\
&\qquad\qquad - \int_{[r,t)} \rho(\mathrm{d}s) \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big)\, G_t^s(\mathrm{d}\nu)\Big\}.
\end{aligned} \tag{4.10}$$

Then we appeal the first equation in (4.8) to see that

$$\begin{aligned}
\gamma_t &= \int_0^{t-r} \eta_s \mathrm{d}s + \int_0^r \mathrm{d}s \int_E \eta_s(\mathrm{d}x)\lambda_{t-r}(x,\cdot) \\
&= \gamma_{t-r} + \int_E \gamma_r(\mathrm{d}x)\lambda_{t-r}(x,\cdot).
\end{aligned} \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) we get

$$\begin{aligned}
&\boldsymbol{E}\,[\exp\{-Y_t(f)\}|Y_s : 0 \leq s \leq r] \\
&\quad = \exp\Big\{ - Y_r(V_{t-r}f) - \gamma_{t-r}(f) - \int_{M(E)^\circ} \big(1 - \mathrm{e}^{-\nu(f)}\big)\, G_{t-r}(\mathrm{d}\nu)\Big\},
\end{aligned}$$

that is, $\{Y_t : t \geq 0\}$ is a Markov process with transition semigroup $(Q_t^N)_{t\geq 0}$. The theorem is proved. $\square$

We next consider the semigroup $(Q_t)_{t\geq 0}$ of the (ξ,ϕ)-superprocess. Let $\mathcal{K}^1(Q)$ denote the set of probability entrance laws $K = (K_t)_{t>0}$ for the semigroup $(Q_t)_{t\geq 0}$ such that

$$\int_0^1 \mathrm{d}s \int_{M(E)^\circ} \nu(E) K_s(\mathrm{d}\nu) < \infty. \tag{4.12}$$

Let $\mathcal{K}(P)$ be the set of entrance laws $\kappa = (\kappa_t)_{t>0}$ for the underlying semigroup $(P_t)_{t\geq 0}$ that satisfy $\int_0^1 \kappa_s(E)\mathrm{d}s < \infty$. For $\kappa \in \mathcal{K}(P)$, set

$$S_t(\kappa, f) = \kappa_t(f) - \int_0^t \mathrm{d}s \int_E \phi(y, V_s f(y))\kappa_{t-s}(\mathrm{d}y), \quad t > 0,\ f \in B(E)^+. \tag{4.13}$$

Note that $S_t(\kappa, f) = \mu(V_t f)$ if $(\kappa_t)_{t>0}$ is given by $\kappa_t = \mu P_t$. The following theorem characterizes completely the set of infinitely divisible probability entrance laws for $(Q_t)_{t\geq 0}$.

Theorem 4.4 (Li [26]). *Any $K \in \mathcal{K}^1(Q)$ is infinitely divisible if and only if it is given by*

$$\log L_{K_t}(f) = -S_t(\kappa, f) - \int_{\mathcal{K}(P)} (1 - \exp\{-S_t(\eta, f)\})\, F(\mathrm{d}\eta), \tag{4.14}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying

$$\int_0^1 \mathrm{d}s \int_{\mathcal{K}(P)} \eta_s(1) F(\mathrm{d}\eta) < \infty. \tag{4.15}$$

$\square$

Let $\mathcal{K}(Q^\circ)$ be the set of entrance laws K for $(Q_t^\circ)_{t\geq 0}$ satisfying (4.12). We can also give a general characterization for $\mathcal{K}(Q^\circ)$ as follows. See also Dynkin [7].

Theorem 4.5. *Any $H \in \mathcal{K}(Q^\circ)$ can be represented as*

$$\begin{aligned}&\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) \\ =\;\; & S_t(\kappa, f) + \int_{\mathcal{K}(P)} \left(1 - \exp\left\{-S_t(\eta, f)\right\}\right) F(\mathrm{d}\eta), \quad t > 0,\ f \in B(E)^+, \end{aligned} \tag{4.16}$$

where $\kappa \in \mathcal{K}(P)$ and F is a σ-finite measure on $\mathcal{K}(P)$ satisfying (4.15). If, in addition,

$$\int_a^\infty [\sup_{x\in E} |\phi(x,z)^{-1}|]\mathrm{d}z < \infty \tag{4.17}$$

for some constant $a > 0$, then (4.16) defines an entrance law $H \in \mathcal{K}(Q^\circ)$ for any $\kappa \in \mathcal{K}(P)$ and any σ-finite measure F on $\mathcal{K}(P)$ satisfying (4.15).

Proof. If $H \in \mathcal{K}(Q^\circ)$, then $(K)_{t>0} = I(0, H_t)_{t>0}$ defines an infinitely divisible probability entrance law $K \in \mathcal{K}^1(Q)$. Thus the representation (4.16) follows by (4.14). If (4.17) holds, there is a family of σ-finite measures $\{L_t(x,\cdot) : t > 0, x \in E\}$ on $M(E)^\circ$ such that $Q_t(\delta_x, \cdot) = I(0, L_t(x,\cdot))$; see Dawson [4, pp195-196]. Using this one can show that an arbitrary infinitely divisible probability entrance law $K \in \mathcal{K}^1(Q)$ may be given as $(K)_{t>0} = I(0, H_t)_{t>0}$ for some $H \in \mathcal{K}(Q^\circ)$. From (4.14) we know that (4.16) defines the entrance law $H \in \mathcal{K}(Q^\circ)$. $\square$

Let $H \in \mathcal{K}(Q^\circ)$ and let $\boldsymbol{Q}^H$ be corresponding the Kuznetsov measure supported by $W_0(M(E))$. If $N(\mathrm{d}s, \mathrm{d}w)$ is a Poisson random measure on $[0,\infty) \times W_0(M(E))$ with intensity $\mathrm{d}s \times \boldsymbol{Q}^H(\mathrm{d}w)$, then

$$Y_t = \int_{[0,t)} \int_{W_0(M(E))} w_{t-s}\, N(\mathrm{d}s, \mathrm{d}w), \quad t \geq 0, \tag{4.18}$$

defines an immigration process corresponding to the SC-semigroup $(N_t)_{t\geq 0}$ given by

$$\log L_{N_t}(f) = -\int_0^t \mathrm{d}s \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_s(\mathrm{d}\nu), \quad t \geq 0,\ f \in B(E)^+. \tag{4.19}$$

Clearly, (4.18) is essentially a special form of (4.4). This construction has been used in [26], [29] and [32]. It is simple to see from Theorems 3.7, 4.4 and 4.5 that, under condition (4.17), any homogeneous immigration process associated with the (ξ, ϕ)-superprocess can be constructed in the form (4.18). Another related work is Evans [10], where a conditioned (ξ, ϕ)-superprocess was constructed by adding up masses thrown off by an "immortal particle" moving around as a copy of ξ.

The construction using Kuznetsov process makes it possible to generalize some existing results for (ξ, ϕ)-superprocess to the immigration process. As an example, let us give a characterization for the "weighted occupation time" of the immigration process by using the construction (4.18). For simplicity we only consider a special case. It is known that if X is a (ξ, ϕ)-superprocess, then

$$\boldsymbol{Q}_\mu \exp\Big\{ - X_t(f) - \int_0^t X_s(g)\mathrm{d}s\Big\} = \exp\left\{-\mu(V_t(f,g))\right\}, \quad f, g \in B(E)^+, \tag{4.20}$$

where $V_t(f,g)(x) \equiv u_t(x)$ is the solution to

$$u_t(x) + \int_0^t \mathrm{d}s \int_E \phi(x, u_s(y)) P_{t-s}(x, \mathrm{d}y) = P_t f(x) + \int_0^t P_s g(x)\mathrm{d}s, \quad t \geq 0; \tag{4.21}$$

see e.g. Fitzsimmons [11] and Iscoe [19]. The formulas (4.20) and (4.21) characterize the joint distribution of X_t and the weighted occupation time $\int_0^t X_s\mathrm{d}s$. By Theorems 3.7 and 4.4 we know that

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t^\kappa(\mu, \mathrm{d}\nu) = \exp\Big\{ - \mu(V_t f) - \int_0^t S_r(\kappa, f)\mathrm{d}r\Big\}, \quad t \geq 0, f \in B(E)^+, \tag{4.22}$$

defines the transition semigroup $(Q_t^\kappa)_{t\geq 0}$ of an immigration process associated with the (ξ, ϕ)-superprocess. Let $h = \int_0^1 P_s 1\mathrm{d}s \in B(E)^+$. From the discussions in [26] we know that $(Q_t^\kappa)_{t\geq 0}$ has a realization $(W, \mathcal{G}, \mathcal{G}_t, Y_t, \boldsymbol{Q}_\mu^\kappa)$ such that for any $g \in B(E)^+$ the path $\{Y_t(g \wedge h) : t \geq 0\}$ is a.s. measurable and locally bounded, hence $\int_0^t Y_s(g)\mathrm{d}s$ can be defined a.s. by increasing limits.

Theorem 4.6. *Suppose that condition (4.17) holds. Let $(W, \mathcal{G}, \mathcal{G}_t, Y_t, \boldsymbol{Q}_\mu^\kappa)$ be the realization of $(Q_t^\kappa)_{t\geq 0}$ described above. Then we have*

$$\begin{aligned} &\boldsymbol{Q}_\mu^\kappa \exp\Big\{ - Y_t(f) - \int_0^t Y_s(g)\mathrm{d}s\Big\} \\ &\qquad = \exp\Big\{ - \mu(u_t) - \int_0^t S_r(\kappa, f, g)\mathrm{d}r\Big\}, \quad f, g \in B(E)^+, \end{aligned}$$

where $u_t(x)$ is defined by (4.21) and

$$S_t(\kappa, f, g) = \kappa_t(f) + \int_0^t \kappa_s(g)\mathrm{d}s - \int_0^t \kappa_{t-s}(\phi(u_s))\mathrm{d}s, \quad t > 0. \tag{4.23}$$

Proof. By Theorem 4.5 we have an entrance law $H \in \mathcal{K}(Q^\circ)$ such that

$$\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = S_t(\kappa, f), \quad t > 0, \ f \in B(E)^+. \tag{4.24}$$

Let $\boldsymbol{Q}^H$ be corresponding the Kuznetsov measure on $W(M(E))$. By (4.24) we have

$$\begin{aligned} & \boldsymbol{Q}^H\Big(1-\exp\Big\{-w_t(f)-\int_0^t w_s(g)\mathrm{d}s\Big\}\Big) \\ = & \lim_{r\downarrow 0}\boldsymbol{Q}^H\Big(1-\boldsymbol{Q}_{w_r}\exp\Big\{-X_{t-r}(f)-\int_0^{t-r}X_s(g)\mathrm{d}s\Big\}\Big) \\ = & \lim_{r\downarrow 0}\boldsymbol{Q}^H\left(1-\exp\{-w_r(u_{t-r})\}\right) \\ = & \lim_{r\downarrow 0}S_r(\kappa,u_{t-r}) \\ = & S_t(\kappa,f,g), \end{aligned}$$

where we have also appealed (4.21) and (4.23) to get the last equality. Then using the construction (4.18) we get

$$\begin{aligned} & \boldsymbol{Q}_0^\kappa\exp\Big\{-Y_t(f)-\int_0^t Y_s(g)\mathrm{d}s\Big\} \\ = & \exp\Big\{-\int_0^t\boldsymbol{Q}^H\Big(1-\exp\Big\{-w_{t-r}(f)-\int_r^t w_{s-r}(g)\mathrm{d}s\Big\}\Big)\mathrm{d}r\Big\} \\ = & \exp\Big\{-\int_0^t S_{t-r}(\kappa,f,g)\mathrm{d}r\Big\}, \end{aligned}$$

and the desired result follows by the relation $\boldsymbol{Q}_\mu^\kappa=\boldsymbol{Q}_\mu*\boldsymbol{Q}_0^\kappa$. $\square$

5 Almost sure behavior of Kuznetsov processes

In this section we study the behavior of Kuznetsov processes near their birth times. The discussion is of interest in providing insights into the trajectory structures of the immigration process. Again, the lack of Feller property makes the proof a little bit longer than expected. Let $(Q_t)_{t\geq 0}$ be the transition semigroup of a (ξ,ϕ)-superprocess.

We shall need to consider two topologies on the space E: the original topology and the Ray topology of ξ. We write E_r for the set E furnished with the Ray topology of ξ. The notation $M(E_r)$ is self-explanatory. Let $(P_t^b)_{t\geq 0}$ be the semigroup of bounded kernels on E defined by

$$P_t^b f(x)=\boldsymbol{P}_x f(\xi_t)\exp\Big\{-\int_0^t b(\xi_s)\mathrm{d}s\Big\},\quad x\in E,\ f\in B(E)^+. \tag{5.1}$$

It is simple to check that, for any $H\in\mathcal{K}(Q^\circ)$,

$$\gamma_t(f)=\int_{M(E)^\circ}\nu(f)H_t(\mathrm{d}\nu),\quad t>0,\ f\in B(E)^+, \tag{5.2}$$

defines an entrance law $\gamma=(\gamma_t)_{t>0}$ for $(P_t^b)_{t\geq 0}$.

We first consider a special σ-finite entrance law. Recall the general formula (4.16). Let $x \in E$ and suppose

$$\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) L_t(x, \mathrm{d}\nu) = V_t f(x), \quad t > 0,\ f \in B(E)^+, \tag{5.3}$$

defines an entrance law $L(x) \in \mathcal{K}(Q^\circ)$. Clearly, $(P_t^b(x, \cdot))_{t>0}$ is a minimal entrance law for $(P_t^b)_{t\geq 0}$, which may be given by (5.2) with $H_t(\mathrm{d}\nu)$ replaced by $L_t(x, \mathrm{d}\nu)$. From those facts it can be deduced easily that $L(x) \in \mathcal{K}(Q^\circ)$ is minimal.

Theorem 5.1. *Let $\boldsymbol{Q}^{L(x)}$ denote the Kuznetsov measure on $W(M(E))$ determined by $L(x) \in \mathcal{K}(Q^\circ)$. Then we have $w_t(E) \to 0$ and $w_t(E)^{-1} w_t \to \delta_x$ in $M(E_r)$ as $t \downarrow 0$ for $\boldsymbol{Q}^{L(x)}$-a.a. paths $w \in W(M(E))$.*

Proof. The results were proved in Li and Shiga [29] for the case where $(P_t)_{t\geq 0}$ is Feller and $\phi(x, z) \equiv z^2/2$ by a theorem of Perkins [30] which asserts that a conditioned (ξ, ϕ)-superprocess is a generalized Fleming-Viot superprocess. The calculations in [29] are complicated and cannot be generalized to the present situation. We here give a proof of the theorem based on an h-transform of the (ξ, ϕ)-superprocess. The Ray cone for the underlying process ξ plays an important role in our proof. We shall assume that $(P_t)_{t\geq 0}$ is conservative. The proof for a non-conservative underlying semigroup can be reduced to this case as in [29].

Let $\mathcal{R}$ be a countable Ray cone for ξ as constructed in Sharpe [31] and let $\bar{E}$ be the corresponding Ray-Knight compactification of E with the Ray topology. Note that each $f \in \mathcal{R}$ is continuous on E_r and admits a unique continuous extension $\bar{f}$ to $\bar{E}$. We regard $M(E_r)$ as a topological subspace of $M(\bar{E})$ in the usual way. Since $\bar{E}$ is a compact metric space, $M(\bar{E})$ is locally compact and separable. For any fixed $u > 0$,

$$U_t^r(\mu, \mathrm{d}\nu) = \mu(P_{u-r}^b 1)^{-1} \nu(P_{u-t}^b 1) Q_{t-r}(\mu, \mathrm{d}\nu), \quad 0 \leq r \leq t \leq u, \tag{5.4}$$

defines an inhomogeneous transition semigroup $(U_t^r)_{r\leq t}$ on $M(E)^\circ$. We define the probability measure $\boldsymbol{U}_u^{L(x)}(\mathrm{d}w)$ on $W(M(E))$ by

$$\boldsymbol{U}_u^{L(x)}(\mathrm{d}w) = P_u^b 1(x)^{-1} w_u(1) \boldsymbol{Q}^{L(x)}(\mathrm{d}w).$$

Then $\{w_t : 0 < t \leq u\}$ under $\boldsymbol{U}_u^{L(x)}$ is a Markov process with semigroup $(U_t^r)_{r\leq t}$ and one-dimensional distributions

$$H_t(x, \mathrm{d}\nu) := P_u^b 1(x)^{-1} \nu(P_{u-t}^b 1) L_t(x, \mathrm{d}\nu), \quad 0 < t \leq u. \tag{5.5}$$

Since $L(x) \in \mathcal{K}(Q^\circ)$ is minimal, $(H_t(x, \cdot))_{0<t\leq u}$ is a minimal (probability) entrance law for $(U_t^r)_{r\leq t}$. Take $f \in \mathcal{R}$. By (5.3) – (5.5) and the martingale convergence theorem we have $\boldsymbol{U}_u^{L(x)}$-a.s.

$$\begin{aligned} V_t f(x) &= \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) \nu(P_{u-t}^b 1)^{-1} H_t(x, \mathrm{d}\nu) P_u^b 1(x) \\ &= \lim_{r\downarrow 0} \int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) \nu(P_{u-t}^b 1)^{-1} U_t^r(w_r, \mathrm{d}\nu) P_u^b 1(x) \\ &= \lim_{r\downarrow 0} w_r(P_{u-r}^b 1)^{-1} \left(1 - \exp\left\{-w_r(V_{t-r} f)\right\}\right) P_u^b 1(x). \end{aligned} \tag{5.6}$$

By (5.1) and (5.6) it follows that $\boldsymbol{U}_u^{L(x)}$-a.s.

$$V_t f(x) \le \liminf_{r\downarrow 0} w_r(P^b_{u-r}1)^{-1}P^b_u 1(x) \le \liminf_{r\downarrow 0} \mathrm{e}^{2\|b\|u} w_r(1)^{-1}.$$

Note that $V_t f(x)$ is right continuous in $t \ge 0$. Then letting $t\downarrow 0$ and $f \uparrow \infty$ in the above inequality yields that $\boldsymbol{U}_u^{L(x)}$-a.s. $w_t(1) \to 0$ as $t \downarrow 0$. Since for each $u > 0$ the measures $\boldsymbol{U}_u^{L(x)}$ and $\boldsymbol{Q}^{L(x)}$ are mutually absolutely continuous on $\{w \in W_0(M(E)) : w_u(1) > 0\}$, we obtain the first assertion. By the same reasoning as (5.6) we have $\boldsymbol{U}_t^{L(x)}$-a.s.

$$\begin{aligned} P^b_t f(x) &= \int_{M(E)^\circ} \nu(f)\nu(1)^{-1} H_t(x, \mathrm{d}\nu) P^b_t 1(x) \\ &= \lim_{r\downarrow 0} w_r(P^b_{t-r}1)^{-1} w_r(P^b_{t-r}f) P^b_t 1(x). \end{aligned} \tag{5.7}$$

Clearly, $\boldsymbol{U}_u^{L(x)}$ is absolutely continuous relative to $\boldsymbol{U}_t^{L(x)}$ for $u \ge t > 0$. Since $f \in \mathcal{R}$ is an α-excessive function for $(P_t)_{t\ge 0}$ for some $\alpha = \alpha(f) \ge 0$, from (5.1) and (5.7) it follows that $\boldsymbol{U}_u^{L(x)}$-a.s.

$$\mathrm{e}^{-\|b\|t} P_t f(x) \le \liminf_{r\downarrow 0} \mathrm{e}^{(3\|b\|+\alpha)t} w_r(1)^{-1} w_r(f).$$

Take $w \in W_0(M(E))$ along which the above inequality holds for all $f \in \mathcal{R}$ and all rational $t \in (0, u]$. Let $r_k = r_k(w)$ be a sequence such that $r_k \downarrow 0$ and $w_{r_k}(1)^{-1} w_{r_k} \to \hat{w}_0$ in $M(\bar{E})$ as $k \to \infty$, where $\hat{w}_0$ is a probability measure on $\bar{E}$. Then we have

$$\mathrm{e}^{-\|b\|t} P_t f(x) \le \mathrm{e}^{(3\|b\|+\alpha)t} \hat{w}_0(\bar{f}).$$

Letting $t \downarrow 0$ gives $f(x) \le \hat{w}_0(\bar{f})$, so we have $\hat{w}_0 = \delta_x$. Those clearly imply $w_r(1)^{-1} w_r \to \delta_x$ in $M(E_r)$ as $r \downarrow 0$, and the second assertion follows immediately. $\square$

Now we consider an h-transform of the underlying semigroup $(P_t)_{t\ge 0}$. Let $h(x) = \int_0^1 P_s 1(x)\mathrm{d}s$ for $x \in E$. Since $h \in B(E)^+$ is an excessive function for $(P_t)_{t\ge 0}$, the formula

$$T_t f(x) = h(x)^{-1} \int_E f(y) h(y) P_t(x, \mathrm{d}y), \quad x \in E,\ f \in B(E)^+, \tag{5.8}$$

defines a Borel right semigroup $(T_t)_{t\ge 0}$ on E; see e.g. Sharpe [31]. Let $(T^\partial_t)_{t\ge 0}$ be a conservative extension of $(T_t)_{t\ge 0}$ to $E^\partial := E \cup \{\partial\}$, where ∂ is the cemetery point. Let E^∂_D denote the entrance space of $(T^\partial_t)_{t\ge 0}$ with the Ray topology. Let $E^T_D = E^\partial_D \setminus \{\partial\}$ and let $(\bar{T}_t)_{t\ge 0}$ be the Ray extension of $(T^\partial_t)_{t\ge 0}$ to E^T_D. Then $(\bar{T}_t)_{t\ge 0}$ is also a Borel right semigroups. Let $\kappa \in \mathcal{K}(P)$ be non-trivial and assume

$$\int_{M(E)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) H_t(\mathrm{d}\nu) = S_t(\kappa, f), \quad t > 0,\ f \in B(E)^+, \tag{5.9}$$

defines an entrance law $H := L\kappa \in \mathcal{K}(Q^\circ)$. Let $\boldsymbol{Q}^{L\kappa}$ denote the corresponding Kuznetsov measure on $W(M(E))$. Then we have

Theorem 5.2. *For $w \in W(M(E))$ define the $M(E_D^T)$-valued path $\{h\bar{w}_t : t \in I\!R\}$ by*

$$h\bar{w}_t(E_D^T \setminus E) = 0 \text{ and } h\bar{w}_t(\mathrm{d}x) = h(x)w_t(\mathrm{d}x) \text{ for } x \in E. \tag{5.10}$$

Then for $\boldsymbol{Q}^{L\kappa}$-a.a. $w \in W(M(E))$, $\{h\bar{w}_t : t > 0\}$ is right continuous in the topology of $M(E_D^T)$ and $h\bar{w}_t \to 0$ as $t \downarrow 0$. Moreover, for $\boldsymbol{Q}^{L\kappa}$-a.a. $w \in W(M(E))$ we have $w_t(h)^{-1}h\bar{w}_t \to \delta_{x(w)}$ for some $x(w) \in E_D^T$ as $t \downarrow 0$.

Proof. By the results in Fitzsimmons [11], if $f \in B(E)$ is finely continuous relative to $(P_t)_{t\geq 0}$, then $\{w_t(f) : t > 0\}$ is right continuous for a.a. $w \in W(M(E))$. Since the excessive function $h \in B(E)^+$ is finely continuous, so is fh for any bounded continuous function f on E. It follows that $\{hw_t : t > 0\}$ is right continuous for a.a. $w \in W(M(E))$. We may define a cumulant semigroup $(U_t)_{t\geq 0}$ by $U_t f = h^{-1}V_t(hf)$. Then $\{hw_t : t > 0\}$ is a Markov process with Borel right transition semigroup given by (1.3) with $(V_t)_{t\geq 0}$ replaced by $(U_t)_{t\geq 0}$. Let E_r^T denote the set E furnished with the relative topology from E_D^T. Applying the results in [11] again we conclude that $\{hw_t : t > 0\}$ is right continuous in $M(E_r^T)$ for a.a. $w \in W(M(E))$. Therefore, $\{h\bar{w}_t : t > 0\}$ is right continuous in $M(E_D^T)$ for a.a. $w \in W(M(E))$. Note that

$$\bar{\psi}(x,z) = \begin{cases} h(x)^{-1}\phi(x, h(x)z) & \text{if } x \in E, \\ 0 & \text{if } x \in E_D^T \setminus E, \end{cases}$$

defines a branching mechanism $\bar{\psi}(\cdot,\cdot)$ on E_D^T. Let $(\bar{U}_t)_{t\geq 0}$ be the cumulant semigroup given by

$$\bar{U}_t\bar{f}(x) + \int_0^t \mathrm{d}s \int_{E_D^T} \bar{\psi}(y, \bar{U}_s\bar{f}(y))\bar{T}_{t-s}(x,\mathrm{d}y) = \bar{T}_t\bar{f}(x), \quad t \geq 0, x \in E_D^T. \tag{5.11}$$

Then $(\bar{U}_t)_{t\geq 0}$ corresponds to Borel right transition semigroup $(\bar{Q}_t)_{t\geq 0}$ on $M(E_D^T)$. For any $t > 0$ and $x \in E_D^T$, the measure $\bar{T}_t(x,\cdot)$ is supported by E, so $\bar{T}_t\bar{f}(x)$ and $\bar{U}_t\bar{f}(x)$ are independent of the values of $\bar{f}$ on $E_D^T \setminus E$. Indeed, if $f = \bar{f}|_E$ for $f \in B(E_D^T)^+$, then $\bar{U}_t\bar{f}(x) = U_tf(x)$ for all $x \in E$. We may write $\bar{T}_tf$ and $\bar{U}_tf$ instead of $\bar{U}_t\bar{f}$ and $\bar{U}_t\bar{f}$ respectively. Clearly, the definitions of $\bar{T}_tf$ and $\bar{U}_tf$ can be extended to all non-negative Borel functions f on E by increasing limits. As shown in [26], there exists a measure $\rho \in M(E_D^T)$ such that $\kappa_t(f) = \rho(\bar{T}_t(h^{-1}f))$ and $S_t(\kappa, f) = \rho(\bar{U}_t(h^{-1}f))$. Then $\{h\bar{w}_t : t > 0\}$ is a Markov process with transition semigroup $(\bar{Q}_t)_{t\geq 0}$ and

$$\boldsymbol{Q}^{L\kappa}(1 - \mathrm{e}^{-h\bar{w}_t(\bar{f})}) = \rho(\bar{U}_t\bar{f}), \quad t > 0,\ f \in B(E_D^T)^+.$$

Now the results follow by Theorem 5.1 applied to $(\bar{U}_t)_{t\geq 0}$ and $(\bar{T}_t)_{t\geq 0}$. □

By (4.14), we have an entrance law $K := l\kappa \in \mathcal{K}^1(Q)$ given by

$$\int_{M(E)} \mathrm{e}^{-\nu(f)}K_t(\mathrm{d}\nu) = \exp\left\{-S_t(\kappa, f)\right\}, \quad t > 0,\ f \in B(E)^+. \tag{5.12}$$

It is easy to see that the restriction of K to $M(E)^\circ$ belongs $\mathcal{K}(Q^\circ)$. Let $\boldsymbol{Q}^{l\kappa}$ denote the corresponding Kuznetsov measure on $W(M(E))$.

Theorem 5.3. *For $\boldsymbol{Q}^{l\kappa}$-a.a. $w \in W(M(E))$, $\{h\bar{w}_t : t > 0\}$ is right continuous and $h\bar{w}_t \to \rho$ for some $\rho \in M(E_D^T)$ as $t \downarrow 0$.*

Proof. We use the notation introduced in the proof of Theorem 5.2. Clearly, $\{h\bar{w}_t : t > 0\}$ under $\boldsymbol{Q}^{l\kappa}$ is a Markov process with transition semigroup $(\bar{Q}_t)_{t\geq 0}$ and

$$\boldsymbol{Q}^{l\kappa} \exp\left\{-h\bar{w}_t(\bar{f})\right\} = \exp\left\{-\rho(\bar{U}_t\bar{f})\right\}, \quad t > 0,\ f \in B(E_D^T)^+.$$

Thus the assertions hold by the uniqueness of transition probability. □

Finally, we consider the path behavior of the Kuznetsov process determined by a general entrance rule. Let $(J_t)_{t\in\mathbb{R}}$ be an entrance rule for $(Q_t^\circ)_{t\geq 0}$ satisfying

$$\int_r^t \mathrm{d}s \int_{M(E)^\circ} \nu(E) J_s(\mathrm{d}\nu) < \infty, \quad r \leq t \in \mathbb{R}. \tag{5.13}$$

Then we may assume that $(J_t)_{t\in\mathbb{R}}$ is given by (4.2) with the entrance laws $\{(H_{s,s+t})_{t>0} : s \in \mathbb{R}\}$ taken from $\mathcal{K}(Q^\circ)$.

Theorem 5.4. *In the situation described above, for $\boldsymbol{Q}^J$-a.a. paths $w \in W(M(E))$ the process $\{h\bar{w}_t : t \in \mathbb{R}\}$ defined by (5.10) is right continuous in $M(E_D^T)^\circ$ on the interval $(\alpha(w), \beta(w))$ and $h\bar{w}_t \to h\bar{w}_\alpha$ for some $h\bar{w}_\alpha \in M(E_D^T)$ as $t \downarrow \alpha(w)$. Moreover, for $\boldsymbol{Q}^J$-a.a. paths $w \in W(M(E))$ with $h\bar{w}_\alpha = 0$, we have $w_t(h)^{-1}h\bar{w}_t \to \delta_{x(w)}$ for some $x(w) \in E_D^T$ as $t \downarrow \alpha(w)$.*

Proof. Let $\boldsymbol{Q}^H$ be the Kuznetsov measure on $W(M(E))$ corresponding to an entrance law $H \in \mathcal{K}(Q^\circ)$ represented by (4.16). Then we have

$$\boldsymbol{Q}^H(\mathrm{d}w) = \boldsymbol{Q}^{L\kappa}(\mathrm{d}w) + \int_{\mathcal{K}(P)} \boldsymbol{Q}^{l\eta}(\mathrm{d}w) F(\mathrm{d}\eta), \quad w \in W(M(E)).$$

By Theorems 5.2 and 5.3, for $\boldsymbol{Q}^H$-a.a. $w \in W(M(E))$ the process $\{h\bar{w}_t : t > 0\}$ is right continuous in $M(E_D^T)$ and $h\bar{w}_t \to h\bar{w}_0$ for some $h\bar{w}_0 \in M(E_D^T)$ as $t \downarrow 0$. Furthermore, for $\boldsymbol{Q}^H$-a.a. $w \in W(M(E))$ with $h\bar{w}_0 = 0$, we have $w_t(h)^{-1}h\bar{w}_t \to \delta_{x(w)}$ for some $x(w) \in E_D^T$ as $t \downarrow 0$. Then the desired result holds by the representation (4.3) of the measure $\boldsymbol{Q}^J(\mathrm{d}w)$. □

Clearly, (5.13) is satisfied by the entrance rule $(G_t)_{t\in\mathbb{R}}$ in Theorem 4.3. The following example shows that the consideration of $\{h\bar{w}_t : t \geq 0\}$ is necessary if one hopes to get the right limit of the path at $\alpha = \alpha(w)$ in the usual sense.

Example 5.1. Suppose that ξ is the minimal Brownian motion in a bounded domain $D \subset \mathbb{R}^d$ with smooth boundary ∂D. We also use ∂ to denote the inward normal derivative operator at ∂D. For any $\gamma \in M(\partial D)$, define $(G_t)_{t>0}$ by

$$\int_{M(D)^\circ} \left(1 - \mathrm{e}^{-\nu(f)}\right) G_t(\mathrm{d}\nu) = \int_0^t \left(1 - \exp\left\{-\gamma(\partial V_{t-s}f)\right\}\right) \mathrm{d}s, \quad f \in B(D)^+,$$

and set $G_t = 0$ for $t \leq 0$. Then $(G_t)_{t\in I\!R}$ form an entrance rule for the (ξ, ϕ)-superprocess. By simple modifications of the proofs of Theorems 5.3 and 5.4 one may see that $w_{\alpha+}(D) = \infty$ and $w_{\alpha+}(K) = 0$ for all compact sets $K \subset D$ and $\boldsymbol{Q}^G$-a.a. paths $w \in W(M(D))$.

6 Stationary immigration processes

The immigration processes formulated by SC-semigroups are closely related with the theory of excessive measures; see e.g. Fitzsimmons and Maisonneuve [13], Getoor [14] and Dellacherie et al [6]. In this section, we give formulations of some results on excessive measures in terms of stationary immigration processes.

We first consider the semigroup $(Q_t)_{t\geq 0}$ of a general Borel right MB-process. Given two probability measures F_1 and F_2 on $M(E)$, we write $F_1 \preceq F_2$ if there is some probability G such that $F_1 * G = F_2$. Let $\mathcal{E}^*(Q)$ denote the set of all probability measures F on $M(E)$ such that

$$\int_{M(E)^\circ} \nu(1) F(\mathrm{d}\nu) < \infty \tag{6.1}$$

and $FQ_t \preceq F$ for all $t \geq 0$. We write $F \in \mathcal{E}_i^*(Q)$ if $F \in \mathcal{E}^*(Q)$ is a stationary distribution for $(Q_t)_{t\geq 0}$, and write $F \in \mathcal{E}_p^*(Q)$ if $F \in \mathcal{E}^*(Q)$ and $\lim_{t\to\infty} FQ_t = \delta_0$. Clearly, we have $\delta_0 \in \mathcal{E}_i^*(Q)$, but there can be other non-trivial stationary distributions although we are considering the state space $M(E)$.

Let $\mathcal{E}(Q^\circ)$ denote the class of all excessive measures F for $(Q_t^\circ)_{t\geq 0}$ satisfying (6.1). Let $\mathcal{E}_i(Q^\circ)$ be the subset of $\mathcal{E}(Q^\circ)$ comprising invariant measures, and $\mathcal{E}_p(Q^\circ)$ the subset of purely excessive measures. The classes $\mathcal{E}(Q^\circ)$ and $\mathcal{E}^*(Q)$ are closely related. Indeed, $F \in \mathcal{E}^*(Q)$ is infinitely divisible if and only if $F = I(\rho, J)$ for $\rho \in M(E)$ and $J \in \mathcal{E}(Q^\circ)$ satisfying

$$\int_E \rho(\mathrm{d}x)\lambda_t(x,\cdot) \leq \rho \quad \text{and} \quad \int_E \rho(\mathrm{d}x)L_t(x,\cdot) + JQ_t^\circ \leq J. \tag{6.2}$$

Under the condition (4.17), $F \in \mathcal{E}^*(Q)$ is infinitely divisible if and only if $F = I(0, J)$ for some $J \in \mathcal{E}(Q^\circ)$.

The following theorem shows that $\mathcal{E}^*(Q)$ is identical with the totality of stationary distributions of immigration processes associated with X.

Theorem 6.1. *Let $F \in \mathcal{E}^*(Q)$. Then it may be written uniquely as $F = F_i * F_p$, where $F_i = \lim_{t\to\infty} FQ_t \in \mathcal{E}_i^*(Q)$ and $F_p \in \mathcal{E}_p^*(Q)$. Moreover, there is a unique SC-semigroup $(N_t)_{t\geq 0}$ such that $\lim_{t\to\infty} N_t = F_p$.*

Proof. Let $(N_t)_{t\geq 0}$ be the distributions on $M(E)$ satisfying $F = (FQ_t) * N_t$. By the branching property of the semigroup $(Q_t)_{t\geq 0}$ one checks for any $r \geq 0$ and $t \geq 0$,

$$\begin{aligned}(FQ_{r+t}) * N_{r+t} &= F = (FQ_t) * N_t = \{[(FQ_r) * N_r]Q_t\} * N_t \\ &= (FQ_{r+t}) * (N_rQ_t) * N_t.\end{aligned} \tag{6.3}$$

It follows that $(N_t)_{t\geq 0}$ satisfies the relation (1.4), so it is an SC-semigroup associated with $(Q_t)_{t\geq 0}$. By the definition of $\mathcal{E}^*(Q)$, we have $FQ_{r+t} \preceq FQ_t$, so the following limits exist and give the Laplace functionals of two probability measures F_i and F_p:

$$L_{F_i}(f) = \uparrow \lim_{t\uparrow\infty} L_{FQ_t}(f), \quad L_{F_p}(f) = \downarrow \lim_{t\uparrow\infty} L_{N_t}(f), \quad f \in B(E)^+. \tag{6.4}$$

Clearly, $F_i \in \mathcal{E}_i^*(Q)$ and $F = F_i * F_p$. On the other hand,

$$F_i * F_p = F = (FQ_t) * N_t = F_i * (F_pQ_t) * N_t,$$

so $F_p = (F_pQ_t) * N_t$. Therefore $F_p \in \mathcal{E}^*(Q)$ and $\lim_{t\to\infty} F_pQ_t = \delta_0$. The uniqueness of the decomposition is immediate. □

It is well-known that any $J \in \mathcal{E}(Q^\circ)$ has the Riesz type decomposition $J = J_i + J_p$, where $J_i \in \mathcal{E}_i(Q^\circ)$ and $J_p \in \mathcal{E}_p(Q^\circ)$ may be represented as $J_p = \int_0^\infty H_t\,\mathrm{d}t$ for some $H \in \mathcal{K}(Q^\circ)$. Let $\boldsymbol{Q}^J$ be the Kuznetsov measure on $W(M(E))$ determined by J and let $N^J(\mathrm{d}w)$ be a Poisson random measure with intensity $\boldsymbol{Q}^J(\mathrm{d}w)$. By Lemma 4.1,

$$Y_t^J := \int_{W(M(E))} w_t\, N^J(\mathrm{d}w), \quad t \in I\!R, \tag{6.5}$$

defines a stationary immigration process with one-dimensional distribution $I(0, J)$ which corresponds to the SC-semigroup $(N_t)_{t\geq 0}$ given by (4.19). The Kuznetsov measures determined by J_i and J_p are restrictions of $\boldsymbol{Q}^J$ to $\{w \in W(M(E)) : \alpha(w) = -\infty\}$ and $\{w \in W(M(E)) : \alpha(w) > -\infty\}$, respectively; see Fitzsimmons and Maisonneuve [13]. It follows that

$$Y_t^{(p)} = \int_{W(M(E))} w_t 1_{\{\alpha > -\infty\}} N^J(\mathrm{d}w), \quad t \in I\!R,$$

defines a stationary immigration process having one-dimensional distribution $I(0, J_p)$ and SC-semigroup $(N_t)_{t\geq 0}$. Intuitively, $\{Y_t^{(p)} : t \in I\!R\}$ is the "purely immigrative" part of the population. On the contrary,

$$Y_t^{(i)} = \int_{W(M(E))} w_t 1_{\{\alpha = -\infty\}} N^J(\mathrm{d}w), \quad t \in I\!R,$$

is a stationary (ξ, ϕ)-superprocess with one-dimensional distribution $I(0, J_i)$, which represents the "native" part of the population.

The immigration process defined by (6.5) is usually not right continuous, but it may have right a continuous modification. For $w \in W(M(E))$, we set

$$w_{t+} = \begin{cases} \lim_{s\downarrow t} w_s & \text{if the limit exists in } M(E), \\ 0 & \text{if the above limit does not exist in } M(E), \end{cases} \tag{6.6}$$

and define the process $\{\bar{Y}_t^J : t \in I\!R\}$ by

$$\bar{Y}_t^J = \int_{W(M(E))} w_{t+} N^J(\mathrm{d}w), \quad t \in I\!R, \tag{6.7}$$

Then $\bar{Y}_t^J = Y_t^J$ a.s. since $\boldsymbol{Q}^J\{w_{t+} \neq w_t\} = \boldsymbol{Q}^J\{\alpha = t\} = 0$; see [13]. In other words, $\{\bar{Y}_t^J : t \in I\!R\}$ is a modification for $\{Y_t^J : t \in I\!R\}$.

Theorem 6.2. *Suppose $(Q_t)_{t\geq 0}$ is the transition semigroup of a (ξ,ϕ)-superprocess. (i) If $J \in \mathcal{E}_i(Q^\circ)$, then $\{\bar{Y}_t^J \equiv Y_t^J : t \in I\!R\}$ is a.s. right continuous. (ii) If $J \in \mathcal{E}_p(Q^\circ)$ is a measure potential, that is,*

$$\begin{aligned}\int_{M(E)^\circ} & (1 - \mathrm{e}^{-\nu(f)}) J(\mathrm{d}\nu) \\ &= \int_0^\infty \mathrm{d}s \int_{M(E)^\circ} (1 - \exp\{-\nu(V_s f)\}) G(\mathrm{d}\nu), \quad f \in B(E)^+,\end{aligned}$$

where $\nu(1)G(\mathrm{d}\nu)$ is a finite measure on $M(E)^\circ$, then $\{\bar{Y}_t^J : t \in I\!R\}$ is a.s. right continuous.

Proof. Since (i) is simple, we only give the proof of (ii). For $k = 1, 2, \cdots$ let

$$W_k(M(E)) = \{w \in W(M(E)) : w_{\alpha+}(E) \geq 1/k\}.$$

By the results in [13], the path $\{w_{t+} : t \in I\!R\}$ is right continuous for $\boldsymbol{Q}^J$-a.a. $w \in W(M(E))$ and $\boldsymbol{Q}^J([\cup_{k=1}^\infty W_k(M(E))]^c) = 0$. Let

$$\bar{Y}_t^{(k)} = \int_{W_k(M(E))} X_t(w, \cdot) 1_{\{\alpha(w) \geq -k\}} N^J(\mathrm{d}w), \quad t \in I\!R.$$

Clearly, $\{\bar{Y}_t^{(k)} : t \geq -k\}$ is an immigration process corresponding to the SC-semigroup $(N_t^{(k)})_{t\geq 0}$ given by

$$\begin{aligned}\int_{M(E)} & \mathrm{e}^{-\nu(f)} N_t^{(k)}(\mathrm{d}\nu) \\ &= \exp\Big\{ - \int_0^t \mathrm{d}s \int_{M(E)} \big(1 - \mathrm{e}^{-\nu(V_s f)}\big) 1_{\{\nu(E) \geq 1/k\}} G(\mathrm{d}\nu) \Big\}.\end{aligned}$$

Observe that for each $l > -k$ the process $\{\bar{Y}_t^{(k)} : -k \leq t \leq l\}$ is a.s. a finite sum of right continuous paths and $\bar{Y}_t^{(k)} \to \bar{Y}_t^J$ increasingly as $k \to \infty$, so the result follows as in [26]. $\square$

Theorem 6.3. *Suppose $(Q_t)_{t\geq 0}$ is the transition semigroup of a (ξ,ϕ)-superprocess. Let $J \in \mathcal{E}(Q^\circ)$ and let $\{Y_t^J : t \in I\!R\}$ be defined by (6.5). For each $r > 0$, let*

$$Y_t^{r,J} = \int_{W(M(E))} w_t 1_{\{t \geq \alpha + r\}} N(\mathrm{d}w), \quad t \in I\!R.$$

Then $\{Y_t^{r,J} : t \in I\!R\}$ is an a.s. right continuous stationary immigration process and $Y_t^{r,J} \to Y_t^J$ increasingly a.s. as $r \downarrow 0$ for every $t \in I\!R$.

Proof. Clearly, $JQ^\circ_r \in \mathcal{E}(Q^\circ)$ and

$$JQ^\circ_r = J_i + J_p Q^\circ_r = J_i + \int_0^\infty H_r Q^\circ_s \mathrm{d}s.$$

Using (4.1) one may check that the Kuznetsov measure on $W(M(E))$ determined by $JQ^\circ_r \in \mathcal{E}(Q^\circ)$ is the image of $\boldsymbol{Q}^J$ under the mapping $\{w_t : t \in \mathbb{R}\} \mapsto \{w_t 1_{\{t>\alpha+r\}} : t \in \mathbb{R}\}$. It follows that $\{Y^{r,J}_t : t \in \mathbb{R}\}$ is a stationary immigration process corresponding to JQ°_r. By Theorem 6.2, $\{Y^{r,J}_t : t \in \mathbb{R}\}$ is a.s. right continuous. The second assertion is immediate. $\square$

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