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Non-local Branching Superprocesses and Some Related Models

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Abstract

A new formulation of non-local branching superprocesses is given from which we derive as special cases the rebirth, the multitype, the mass-structured, the multilevel and the age-reproduction-structured superprocesses and the superprocess-controlled immigration process. This unified treatment simplifies considerably the proof of existence of the old classes of superprocesses and also gives rise to some new ones.

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1 Introduction

Measure-valued branching processes or superprocesses constitute a rich class of infinite dimensional processes currently under rapid development. Such processes arose in applications as high density limits of branching particle systems; see e.g. Dawson (1992, 1993), Dynkin (1993, 1994), Watanabe (1968). The development of this subject has been stimulated from different subjects including branching processes, interacting particle systems, stochastic partial differential equations and non-linear partial differential equations. The study of superprocesses has also led to a better understanding of some results in those subjects. In the literature, several different types of superprocess have been introduced and studied. In particular, Dawson and Hochberg (1991), Dawson et al (1990) and Wu (1994) studied multilevel branching superprocesses, Gorostiza and Lopez-Mimbela (1990), Gorostiza and Roelly (1991), Gorostiza et al (1992) and Li (1992b) studied multitype superprocesses, Dynkin (1993, 1994) and Li (1992a, 1993) studied non-local branching superprocesses, Gorostiza (1994) studied mass-structured superprocesses, Hong and Li (1999) and Li (2002) studied superprocess-controlled immigration processes, and Bose and Kaj (2000) studied age-reproduction-structured superprocesses. Those models arise in different circumstances of application and are of their own theoretical interests.

In this paper, we provide a unified treatment of the above models. We first give a new formulation of the non-local branching superprocess as the high density limit of some specific branching particle systems. Then we derive from this superprocess the multitype, the mass-structured, the multilevel and the age-reproduction-structured superprocesses and superprocess-controlled immigration processes. Another related model, the so-called rebirth superprocesses, is also introduced to explain the non-local branching mechanism. This unified treatment simplifies considerably the proof of existence of the old classes of superprocesses and also gives rise to some new ones. We think that this treatment may give some useful perspectives for those models. The unification is done by considering an enriched underlying state space $E \times I$ instead of E. In this way, the mutation in types of the offspring can be modeled by jumps in the I-coordinates so that the multitype superprocess can be derived. The superprocess-controlled immigration process is actually a special form of the multitype superprocess. To get the mass-structured superprocess we let $I = (0, \infty)$, which represents the mass or size of the infinitesimal particles. For the age-reproduction-structured superprocess, we take $I = [0, \infty) \times I\!\!N$, where $I\!\!N$ is the set of non-negative integers, to keep the information on ages and numbers of offspring of the particles. In this model we have of course that any particle has non-decreasing $[0, \infty) \times I\!\!N$-coordinates starting from $(0, 0)$ at its birth time. To get a two level superprocess, we simply assume that $I = M(S)^\circ$ is the space of nontrivial finite measures on another space S, and the $M(S)^\circ$-coordinate of the underlying process is a superprocess itself. For two-level branching systems, what has been done so far for the second level branching is local, that is, when a superparticle branches, superoffspring are produced as exact copies of their parent. Since the superparticles have an internal dynamics and evolve as branching systems themselves, it is desirable to have the possibility that the superoffspring have internal structures different from those of their parents, which requires a non-local branching mechanism. Models of this type have potential applications in genetics, population dynamics and other complex multilevel systems; see e.g. Dawson (2000) and Jagers (1995).

Notation and basic setting: Suppose that E is a Lusin topological space, i.e., a homeomorph of a Borel subset of some compact metric space, with Borel σ-algebra $\mathcal{B}(E)$. Let $M(E)$ denote the space of finite Borel measures on E topologized by the weak convergence topology, so it is also a Lusin topological space. Let $N(E)$ be the subspace of $M(E)$ consisting of integer-valued measures on E and let $M(E)^\circ = M(E)\setminus\{0\}$, where 0 denotes the null measure on E. The unit mass concentrated at a point $x \in E$ is denoted by δ_x. Let

$$\begin{aligned} B(E) &= \{ \text{ bounded } \mathcal{B}(E)\text{-measurable functions on } E \}, \\ C(E) &= \{ f\colon f \in B(E) \text{ is continuous } \}, \\ B_a(E) &= \{ f\colon f \in B(E) \text{ and } \|f\| \leq a \}, \end{aligned}$$

where $a \geq 0$ and "$\|\cdot\|$"denotes the supremum norm. The subsets of positive members of the function spaces are denoted by the superscript "+"; e.g., $B^+(E)$, $C^+(E)$. For $f \in B(E)$ and $\mu \in M(E)$, we write $\mu(f)$ for $\int_E f \mathrm{d}\mu$.

2 Non-local branching particle systems

Non-local branching particle systems have been considered by many authors. We here adapt the model of Dynkin (1993). Let $\xi = (\Omega, \xi_t, \mathcal{F}, \mathcal{F}_t, \boldsymbol{P}_x)$ be a right continuous strong Markov process with state space E and transition semigroup $(P_t)_{t\geq 0}$. Let $\gamma \in B^+(E)$ and let $F(x, \mathrm{d}\nu)$ be a Markov kernel from E to $N(E)$ such that

$$\sup_{x\in E} \int_{N(E)} \nu(1) F(x, \mathrm{d}\nu) < \infty. \tag{2.1}$$

A *branching particle system* with parameters (ξ, γ, F) is described by the following properties:

(2.A) The particles in E move randomly according to the law given by the transition probabilities of ξ.

(2.B) For a particle which is alive at time r and follows the path $(\xi_t)_{t\geq r}$, the conditional probability of survival during the time interval $[r, t]$ is $\rho(r, t) := \exp\{-\int_r^t \gamma(\xi_s)\mathrm{d}s\}$.

(2.C) When a particle dies at a point $x \in E$, it gives birth to a random number of offspring in E according to the probability kernel $F(x, \mathrm{d}\nu)$. The offspring then start to move from their locations. (Thus the name "non-local branching" is used.)

In the model, it is assumed that the migrations, the lifetimes and the branchings of the particles are independent of each other. Let $X_t(B)$ denote the number of particles in $B \in \mathcal{B}(E)$ that are alive at time $t \geq 0$ and assume $X_0(E) < \infty$. Then $\{X_t : t \geq 0\}$ is a Markov process with state space $N(E)$. For $\sigma \in N(E)$, let $\boldsymbol{Q}_\sigma$ denote the conditional law of $\{X_t : t \geq 0\}$ given $X_0 = \sigma$. For $f \in B^+(E)$, put

$$u_t(x) \equiv u_t(x, f) = -\log \boldsymbol{Q}_{\delta_x} \exp\{-X_t(f)\}. \tag{2.2}$$

The independence hypotheses imply that

$$\boldsymbol{Q}_\sigma \exp\{-X_t(f)\} = \exp\{-\sigma(u_t)\}. \tag{2.3}$$

Moreover, we have the following fundamental equation

$$\mathrm{e}^{-u_t(x)} = \boldsymbol{P}_x\{\rho(0,t)\mathrm{e}^{-f(\xi_t)}\} + \boldsymbol{P}_x\bigg\{\int_0^t \bigg[\rho(0,s)\gamma(\xi_s)\int_{N(E)} \mathrm{e}^{-\nu(u_{t-s})}F(\xi_s,\mathrm{d}\nu)\bigg]\mathrm{d}s\bigg\}. \tag{2.4}$$

This equation is obtained by thinking that if a particle starts moving from point x at time 0, it follows a path of ξ and does not branch before time t, or it first splits at time $s \in (0,t]$. By a standard argument one sees that equation (2.4) is equivalent to

$$\begin{aligned} \mathrm{e}^{-u_t(x)} &= \boldsymbol{P}_x\mathrm{e}^{-f(\xi_t)} - \boldsymbol{P}_x\bigg\{\int_0^t \gamma(\xi_s)\mathrm{e}^{-u_{t-s}(\xi_s)}\mathrm{d}s\bigg\} \\ &\quad + \boldsymbol{P}_x\bigg\{\int_0^t \bigg[\gamma(\xi_s)\int_{N(E)} \mathrm{e}^{-\nu(u_{t-s})}F(\xi_s,\mathrm{d}\nu)\bigg]\mathrm{d}s\bigg\}; \end{aligned} \tag{2.5}$$

see e.g. Dawson (1992, 1993) and Dynkin (1993, 1994). It is sometimes more convenient to denote

$$v_t(x) \equiv v_t(x,f) = 1 - \exp\{-u_t(x)\}, \tag{2.6}$$

and rewrite (2.5) into the form

$$\begin{aligned} v_t(x) &= \boldsymbol{P}_x\Big\{1 - \mathrm{e}^{-f(\xi_t)}\Big\} - \boldsymbol{P}_x\bigg\{\int_0^t \gamma(\xi_s)v_{t-s}(\xi_s)\mathrm{d}s\bigg\} \\ &\quad + \boldsymbol{P}_x\bigg\{\int_0^t \bigg[\gamma(\xi_s)\int_{N(E)} (1-\mathrm{e}^{-\nu(u_{t-s})})F(\xi_s,\mathrm{d}\nu)\bigg]\mathrm{d}s\bigg\}. \end{aligned} \tag{2.7}$$

3 Non-local branching superprocesses

In this section, we prove a limit theorem for a sequence of non-local branching particle systems. Although the particle systems considered here are very specific, they lead to the same class of non-local branching superprocesses constructed in Dynkin (1993, 1994) and Li (1992a) with a slightly different formulation. We shall give some details of the derivation to clarify the meaning of the parameters, which is needed in understanding the connections of non-local branching with other related models.

Let $\{X_t(k) : t \geq 0\}$, $k = 1, 2, \dots$ be a sequence of branching particle systems with parameters (ξ, γ_k, F_k). Then for each k,

$$\{X_t^{(k)} := k^{-1}X_t(k) : t \geq 0\} \tag{3.1}$$

defines a Markov process in $M_k(E) := \{k^{-1}\sigma : \sigma \in N(E)\}$. For $\sigma \in M_k(E)$, let $\boldsymbol{Q}_\sigma^{(k)}$ denote the conditional law of $\{X_t^{(k)} : t \geq 0\}$ given $X_0^{(k)} = \sigma$. By (2.3) we have

$$\boldsymbol{Q}_\sigma^{(k)} \exp\Big\{-X_t^{(k)}(f)\Big\} = \exp\Big\{-\sigma(ku_t^{(k)})\Big\}, \tag{3.2}$$

where $u_t^{(k)}(x)$ is determined by

$$v_t^{(k)}(x) = k[1 - \exp\{-u_t^{(k)}(x)\}]. \tag{3.3}$$

and

$$\begin{aligned} v_t^{(k)}(x) &= \boldsymbol{P}_x\left\{k(1-\mathrm{e}^{-f(\xi_t)/k})\right\} - \boldsymbol{P}_x\bigg\{\int_0^t \gamma_k(\xi_s) v_{t-s}^{(k)}(\xi_s)\mathrm{d}s\bigg\} \\ &\quad + \boldsymbol{P}_x\bigg\{\int_0^t \Big[k\gamma_k(\xi_s)\int_{N(E)}(1-\mathrm{e}^{-\nu(u_{t-s}^{(k)})})F_k(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\bigg\}. \end{aligned} \tag{3.4}$$

For $\mu \in M(E)$, let $\sigma_{k\mu}$ be a Poisson random measure on E with intensity $k\mu$, and let $\boldsymbol{Q}_{(\mu)}^{(k)}$ denote the conditional law of $\{X_t^{(k)} : t \geq 0\}$ given $X_0^{(k)} = k^{-1}\sigma_{k\mu}$. From (3.2) we get

$$\boldsymbol{Q}_{(\mu)}^{(k)} \exp\left\{-X_t^{(k)}(f)\right\} = \exp\left\{-\mu(v_t^{(k)})\right\}. \tag{3.5}$$

It is natural to treat separately the offspring that start their motion from the death sites of their parents. Suppose that $g_k \in B^+(E \times [0,1])$ and, for each $x \in E$,

$$g_k(x,z) = \sum_{i=0}^{\infty} p_i^{(k)}(x) z^i, \qquad z \in [0,1],$$

is a probability generating function with $\sup_{x\in E}(\mathrm{d}/\mathrm{d}z)g_k(x,1^-) < \infty$. Let α_k and $\beta_k \in B^+(E)$ and assume $\gamma_k(x) := \alpha_k(x) + \beta_k(x)$ is strictly positive. Let $F^{(k)}(x,\mathrm{d}\nu)$ be another probability kernel from E to $N(E)$ satisfying (2.1). We may replace $F_k(x,\mathrm{d}\nu)$ by

$$\gamma_k(x)^{-1}\Big[\alpha_k(x)\sum_{i=0}^{\infty} p_i^{(k)}(x)F_0^{(i)}(x,\mathrm{d}\nu) + \beta_k(x)F^{(k)}(x,\mathrm{d}\nu)\Big], \tag{3.6}$$

where $F_0^{(i)}(x,\mathrm{d}\nu)$ denotes the unit mass concentrated at $i\delta_x$. Intuitively, as a particle splits at $x \in E$, the branching is of local type with probability $\alpha_k(x)/\gamma_k(x)$ and is of non-local type with probability $\beta_k(x)/\gamma_k(x)$. If it chooses the local branching type, the distribution of the offspring number is $\{p_i^{(k)}(x)\}$. The non-local branching at $x \in E$ is described by the kernel $F^{(k)}(x,\mathrm{d}\nu)$. Now (3.4) turns into

$$\begin{aligned} v_t^{(k)}(x) &= \boldsymbol{P}_x\left\{k(1-\mathrm{e}^{-f(\xi_t)/k})\right\} - \boldsymbol{P}_x\bigg\{\int_0^t \gamma_k(\xi_s)v_{t-s}^{(k)}(\xi_s)\mathrm{d}s\bigg\} \\ &\quad + \boldsymbol{P}_x\bigg\{\int_0^t k\alpha_k(\xi_s)[1-g_k(\xi_s,\mathrm{e}^{-u_{t-s}^{(k)}(\xi_s)})]\mathrm{d}s\bigg\} \\ &\quad + \boldsymbol{P}_x\bigg\{\int_0^t \Big[k\beta_k(\xi_s)\int_{N(E)}[1-\mathrm{e}^{-\nu(u_{t-s}^{(k)})}]F^{(k)}(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\bigg\}, \end{aligned} \tag{3.7}$$

or equivalently

$$v_t^{(k)}(x) + \int_0^t \boldsymbol{P}_x[\phi_k(\xi_s, v_{t-s}^{(k)}(\xi_s)) + \psi_k(\xi_s, v_{t-s}^{(k)})]\mathrm{d}s = \boldsymbol{P}_x k[1-\mathrm{e}^{-f(\xi_t)/k}], \tag{3.8}$$

where

$$\phi_k(x,z) = k\alpha_k(x)[g_k(x, 1-z/k) - (1-z/k)] \tag{3.9}$$

and

$$\psi_k(x,f) = \beta_k(x)[f(x) - \zeta_k(x,f)], \tag{3.10}$$

where

$$\zeta_k(x,f) = \int_{N(E)} k(1-\exp\{\nu(\log(1-f/k))\})F^{(k)}(x,\mathrm{d}\nu). \tag{3.11}$$

Let $M_0(E)$ denote the set of all Borel probability measures on E. Suppose that $h_k \in B^+(E\times M_0(E)\times[0,1])$ and, for each $(x,\pi)\in E\times M_0(E)$,

$$h_k(x,\pi,z) = \sum_{i=0}^{\infty} q_i^{(k)}(x,\pi)z^i, \qquad z\in[0,1],$$

is a probability generating function with $\sup_{x,\pi}(\mathrm{d}/\mathrm{d}z)h_k(x,\pi,1^-) < \infty$. Suppose that $G(x,\mathrm{d}\pi)$ is a probability kernel from E to $M_0(E)$. We may consider a special form of the second term in (3.6) by letting

$$F^{(k)}(x,\mathrm{d}\nu) = \int_{M_0(E)} \Big[\sum_{i=0}^{\infty} q_i^{(k)}(x,\pi)(l\pi)^{*i}(\mathrm{d}\nu)\Big] G(x,\mathrm{d}\pi), \tag{3.12}$$

where $l\pi(\mathrm{d}\nu)$ denotes the image of π under the map $y\mapsto\delta_y$ from E to $M(E)$ and $(l\pi)^{*i}$ denotes the i-fold convolution of $l\pi$. Now we have

$$\zeta_k(x,f) = \int_{M_0(E)} k[1-h_k(x,\pi,1-\pi(f)/k)]G(x,\mathrm{d}\pi). \tag{3.13}$$

Intuitively, if a parent particle at $x\in E$ chooses non-local branching, it first selects an offspring-location-distribution $\pi(x,\cdot)\in M_0(E)$ according to the probability kernel $G(x,\mathrm{d}\pi)$, then gives birth to a random number of offspring according to the distribution $\{q_i^{(k)}(x,\pi(x,\cdot))\}$, and those offspring choose their locations in E independently of each other according to $\pi(x,\cdot)$. A similar non-local branching mechanism was considered in Li (1992a, 1993).

In view of (3.5) and (3.8), it is natural to assume the sequences $\{\phi_k\}$, $\{\beta_k\}$ and $\{\zeta_k\}$ to converge if one hopes to obtain convergence of $\{X_t^{(k)} : t\geq 0\}$ to some process $\{X_t : t\geq 0\}$ as $k\to\infty$.

Lemma 3.1 *(i) Suppose that*

$$\sum_{i=0}^{\infty} iq_i^{(k)}(x,\pi) \leq 1 \tag{3.14}$$

and that $\zeta_k(x,f) \to \zeta(x,f)$ uniformly on $E \times B_a^+(E)$ for each $a \geq 0$, then $\zeta(x,f)$ has representation

$$\zeta(x,f) = \lambda(x,f) + \int_{M(E)^\circ} (1 - \mathrm{e}^{-\nu(f)}) \varGamma(x, \mathrm{d}\nu), \tag{3.15}$$

where $\lambda(x, \mathrm{d}y)$ is a bounded kernel on E, and $\nu(1)\varGamma(x,\mathrm{d}\nu)$ is a bounded kernel from E to $M(E)^\circ$ with

$$\lambda(x,1) + \int_{M(E)^\circ} \nu(1)\varGamma(x,\mathrm{d}\nu) \leq 1. \tag{3.16}$$

(ii) A functional $\zeta(x,f)$ can be given by (3.15) and (3.16) if and only if it has representation

$$\zeta(x,f) = \int_{M_0(E)} \Big[d(x,\pi)\pi(f) + \int_0^\infty (1 - \mathrm{e}^{-u\pi(f)}) n(x,\pi,\mathrm{d}u) \Big] G(x,\mathrm{d}\pi), \tag{3.17}$$

where $d \in B^+(E \times M_0(E))$, $un(x,\pi,\mathrm{d}u)$ is a bounded kernel from $E \times M_0(E)$ to $(0,\infty)$ and $G(x,\mathrm{d}\pi)$ is a probability kernel from E to $M_0(E)$ with

$$d(x,\pi) + \int_0^\infty un(x,\pi,\mathrm{d}u) \leq 1. \tag{3.18}$$

(iii) To each function $\zeta(\cdot,\cdot)$ given by (3.17) and (3.18) there corresponds a sequence of the form (3.13) satisfying the requirement of (i).

Proof. (i) Note that $k(1 - \mathrm{e}^{-f/k})$ converges to f uniformly in $B_a^+(E)$. Then

$$\zeta_k(x, k(1 - \mathrm{e}^{-f/k})) = \int_{N(E)} k(1 - \exp\{\nu(f)/k\}) F^{(k)}(x,\mathrm{d}\nu)$$

converges to $\zeta(x,f)$ uniformly on $E \times B_a^+(E)$. It is known that a metric r can be introduced into E so that (E,r) becomes a compact metric space while the Borel σ-algebra induced by r coincides with $\mathcal{B}(E)$; see e.g. Parthasarathy (1967, p.14). Now $M(E)$ endowed with weak convergence topology is a locally compact metrizable space. Let $\bar{M}(E) = M(E) \cup \{\partial\}$ be the one-point-compactification of $M(E)$. By (3.14), $\{k\nu(1)F^{(k)}(x, \mathrm{d}(k^{-1}\nu)) : x \in E, k \geq 1\}$ viewed as a family of finite measures on $\bar{M}(E)$ is tight. Fix $x \in E$ and take $\{n_i\} \subset \{n\}$ such that $k_i\nu(1)F_{k_i}(x, \mathrm{d}(k_i^{-1}\nu))$ converges to some finite measure $G(x,\mathrm{d}\nu)$ on $\bar{M}(E)$ as $i \to \infty$. It follows that

$$\zeta(x,f) = \int_{M(E)^\circ} (1 - \mathrm{e}^{-\nu(f)}) \nu(1)^{-1} G(x,\mathrm{d}\nu),$$

first for $f \in C^+(E,r)$ and then for all $f \in B^+(E)$. Now (3.15) follows by a simple change of the measure and (3.16) follows from (3.14). (ii) is immediate. To get (iii) we may set

$$h_k(x,\pi,z) = 1 + d(x,\pi)(z-1) + k^{-1} \int_0^\infty (\mathrm{e}^{ku(z-1)} - 1) n(x,\pi,\mathrm{d}u).$$

Observe that

$$\frac{\mathrm{d}^i}{\mathrm{d}z^i}h_k(x,\pi,0)\geq 0,\quad i=1,2,\ldots,$$

and (3.14) assures that $h_k(x,\pi,0)\geq 0$. Thus for fixed $(x,a)\in E\times M_0(E)$, $h_k(x,\pi,\cdot)$ is a probability generating function. Then we define $\zeta_k(x,f)$ by (3.13) so that $\zeta_k(x,f)=\zeta(x,f)$ for $(x,f)\in E\times B^+_{1/k}(E)$. $\square$

Lemma 3.2 (Li, 1992c) *(i) Suppose that, for each $l\geq 0$, the sequence $\phi_k(x,z)$ is uniformly Lipschitz in z on the set $E\times[0,l]$ and that $\phi_k(x,z)$ converges to some $\phi(x,z)$ uniformly as $k\to\infty$, then $\phi(x,z)$ has the representation*

$$\phi(x,z)=b(x)z+c(x)z^2+\int_0^\infty(\mathrm{e}^{-zu}-1+zu)m(x,\mathrm{d}u),\quad x\in E,z\geq 0,\tag{3.19}$$

where $b\in B(E)$, $c\in B^+(E)$ and $(u\wedge u^2)m(x,\mathrm{d}u)$ is a bounded kernel from E to $(0,\infty)$.

(ii) To each function $\phi(\cdot,\cdot)$ given by (3.19) there corresponds a sequence of the form (3.9) satisfying the requirement of (i).

Based on Lemmas 3.1 and 3.2, the following result can be proved similarly as in Dawson (1992, 1993), Dynkin (1993, 1994) and Li (1992a, c).

Lemma 3.3 *If the conditions of Lemma 3.1 (i) and Lemma 3.2 (i) are fulfilled and if $\beta_k\to\beta\in B^+(E)$ uniformly as $k\to\infty$, then for each $a\geq 0$ both $v_t^{(k)}(x,f)$ and $ku_t^{(k)}(x,f)$ converge boundedly and uniformly on the set $[0,a]\times E\times B^+_a(E)$ of (t,x,f) to the unique bounded positive solution $V_tf(x)$ to the evolution equation*

$$V_tf(x)+\int_0^t\Big\{\int_E[\phi(y,V_{t-s}f(y))+\psi(y,V_{t-s}f)]P_s(x,\mathrm{d}y)\Big\}\mathrm{d}s=P_tf(x),\quad t\geq 0,\tag{3.20}$$

where

$$\psi(x,f)=\beta(x)[f(x)-\zeta(x,f)],\qquad x\in E,f\in B^+(E).\tag{3.21}$$

By Lemma 3.3 and Dawson (1993, p.42),

$$\int_{M(E)}\mathrm{e}^{-\nu(f)}Q_t(\mu,\mathrm{d}\nu)=\exp\{-\mu(V_tf)\},\qquad f\in B^+(E),\tag{3.22}$$

defines a transition semigroup $(Q_t)_{t\geq 0}$ on $M(E)$. A Markov process $\{X_t:t\geq 0\}$ with state space $M(E)$ is called a *non-local branching superprocess* with parameters (ξ,ϕ,ψ) if it has transition semigroup $(Q_t)_{t\geq 0}$. Condition (3.14) means that the corresponding branching particle system has subcritical non-local branching. In terms of the limiting superprocess, this condition is expressed as (3.16), which is of course a restriction of the class of $\zeta(\cdot,\cdot)$ given by (3.15). However, since $\phi(x,z)+\beta(x)z$ belongs to the class defined by (3.19), and since $\beta \in B^+(E)$ is arbitrary, (3.16) does not put any restriction on the generality of

$$\phi(x, f(x)) + \psi(x, f) = \phi(x, f(x)) + \beta(x)f(x) - \beta(x)\zeta(x, f).$$

Therefore, the class of non-local branching superprocesses given by (3.20) and (3.22) coincides with those constructed Dynkin (1993, 1994) and Li (1992a), where the first term of $\psi(x, \cdot)$ was written into $\phi(x, \cdot)$. In principle, (3.20) and (3.22) give the most general non-local branching superprocesses constructed in the literature up to now. A more general class of non-local branching superprocesses were discussed in Dynkin et al (1994), but their existence has not been established. The next theorem follows similarly as in Li (1992a, c).

Theorem 3.1 *Let $\{X_t^{(k)} : t \geq 0\}$ be the sequence of renormalized branching particle systems determined by (3.5) and (3.8), and let $\{X_t : t \geq 0\}$ be the non-local branching superprocess with transition semigroup $(Q_t)_{t\geq 0}$ given by (3.20) and (3.22). Assume that the conditions of Lemma 3.1 (i) and Lemma 3.2 are fulfilled. Then for every $\mu \in M(E)$, $0 \leq t_1 < \ldots < t_n$ and $a \geq 0$, as $k \to \infty$,*

$$\boldsymbol{Q}_{(\mu)}^{(k)} \exp\Big\{ - \sum_{i=1}^n X_{t_i}^{(k)}(f_i)\Big\} \;\to\; \boldsymbol{Q}_\mu \exp\Big\{ - \sum_{i=1}^n X_{t_i}(f_i)\Big\}$$

uniformly on $f_1, \ldots, f_n \in B_a^+(E)$.

Naturally, we may regard $\{X_t^{(k)} : t \geq 0\}$ as a process in the space $M(E)$. Then the above theorem shows that the finite dimensional distributions of $\{X_t^{(k)} : t \geq 0\}$ under $\boldsymbol{Q}_{(\mu)}^{(k)}$ converge as $k \to \infty$ to those of $\{X_t : t \geq 0\}$ under $\boldsymbol{Q}_\mu$. Therefore, the non-local branching superprocess is a small particle approximation for the non-local branching particle system. Heuristically, ξ gives the law of the migration of the "particles", $\phi(x, \cdot)$ describes the amount of offspring born at $x \in E$ by a parent that dies at this point, and $\psi(x, \cdot)$ describes the amount of the offspring born by this parent that are displaced randomly into the space according to distributions π randomly chosen by $G(x, d\pi)$. Thus the locations of non-locally displaced offspring involve two sources of randomness.

Replacing f in (3.21) and (3.22) by θf and differentiating at $\theta = 0$ we see that the first moments of the superprocess are given by

$$\int_{M(E)} \nu(f) Q_t(\mu, \mathrm{d}\nu) = \mu(T_t f), \qquad t \geq 0, f \in B^+(E), \tag{3.23}$$

where $(T_t)_{t\geq 0}$ is a locally bounded semigroup of kernels on E determined by

$$T_t f(x) + \int_0^t \Big\{ \int_E b(y) T_{t-s} f(y) + \beta(y)[T_{t-s} f(y) - m(y, T_{t-s} f)] P_s(x, \mathrm{d}y)\Big\} \mathrm{d}s = P_t f(x) \tag{3.24}$$

and $m(x, \mathrm{d}y)$ is the bounded kernel on E defined by

$$m(x, f) = \lambda(x, f) + \int_{M(E)^\circ} \nu(f) \varGamma(x, \mathrm{d}\nu). \tag{3.25}$$

In particular, we may define another locally bounded semigroup of kernels $(U_t)_{t\geq 0}$ on E by

$$U_t f(x) + \int_0^t \Big\{ \int_E \beta(y)[U_{t-s}f(y) - m(y, U_{t-s}f)] P_s(x, \mathrm{d}y) \Big\} \mathrm{d}s = P_t f(x), \tag{3.26}$$

which has weak generator G such that

$$Gf(x) = Af(x) + \beta(x)[m(x,f) - f(x)], \qquad f \in \mathcal{D}(A), \tag{3.27}$$

where A denotes the weak generator of $(P_t)_{t\geq 0}$. Now the generator B of $(T_t)_{t\geq 0}$ can be expressed as

$$Bf(x) = Gf(x) - b(x)f(x), \qquad f \in \mathcal{D}(A). \tag{3.28}$$

By a comparison theorem we have $T_t f \leq \mathrm{e}^{\|b\|t} U_t f$ for all $t \geq 0$ and $f \in B(E)^+$. From this and (3.23) we have

$$\int_{M(E)} \nu(f) Q_t(\mu, \mathrm{d}\nu) \leq \mathrm{e}^{\|b\|t} \mu(U_t f), \qquad t \geq 0, f \in B^+(E). \tag{3.29}$$

Note that (3.29) implies that $\nu \mapsto \nu(1)$ is a $\|b\|$-excessive function for $(Q_t)_{t\geq 0}$.

To conclude this section, let us consider briefly the special, and possibly more desirable, case where $G(x, \mathrm{d}\pi) \equiv$ unit mass at some $\pi(x, \cdot) \in M_0(E)$, that is, the non-locally displaced offspring born at $x \in E$ choose their locations independently according to the (non-random) distribution $\pi(x, \cdot)$. In this case, the non-local branching mechanism is given by

$$\psi(x, f) = \beta(x)[f(x) - \zeta(x, \pi(x, f))], \qquad x \in E, f \in B^+(E), \tag{3.30}$$

where

$$\zeta(x, z) = d(x)z + \int_0^\infty (1 - e^{-zu}) n(x, \mathrm{d}u), \qquad x \in E, z \geq 0, \tag{3.31}$$

where $d \in B^+(E)$ and $un(x, \mathrm{d}u)$ is a bounded kernel from E to $(0, \infty)$ with

$$m(x) := d(x) + \int_0^\infty un(x, \mathrm{d}u) \leq 1, \qquad x \in E. \tag{3.32}$$

In particular, if $\zeta(x, z) \equiv z$, we may rewrite (3.20) formally as

$$\frac{\mathrm{d}}{\mathrm{d}t} V_t f(x) = AV_t f(x) - \phi(x, V_t f(x)) + \beta(x)[\pi(x, V_t f) - V_t f(x)], \quad t \geq 0, x \in E,$$

with initial condition $V_0 f = f$. This equation corresponds to a superprocess with underlying generator A and non-trivial local and non-local branching mechanisms. Alternatively, we may also think that the superprocess has underlying generator $Af(x) + \beta(x)[\pi(x, f) - f(x)]$ and only non-trivial local branching mechanism. Since the generator B of a general

Markov process in E is the limit of a sequence of operators of the type $\beta(x)[\pi(x,f)-f(x)]$, in principle a superprocess with more general underlying generator $A+B$ and only local branching mechanism can be approximated by a sequence of superprocesses with underlying generator A and non-trivial local and non-local branching mechanisms. Under suitable conditions it is also possible to establish convergence of branching particle systems with underlying generator A and non-trivial local and non-local branching mechanisms to the superprocess with underlying generator $A+B$ and with only non-trivial local branching mechanism, which has been done in a particular setting in Gorostiza (1994); see also section 6.

4 Rebirth superprocesses

We may consider a modification of the branching particle system described in the last two sections. Let (ξ,γ,F) be given as in section 2. A *rebirth branching particle system* with parameters (ξ,γ,F) is described by (2.A), (2.B) and the following

(4.C) When a particle dies at a point $x\in E$, it gives birth to a random number of offspring in E according to the probability kernel $F(x,\mathrm{d}\nu)$. In addition, the parent particle itself is replaced by an extra offspring at site $x\in E$, that is, the parent particle is reborn. All the offspring then start to move from their locations.

Let $\{X_t : t\geq 0\}$ be the process defined in the same way as in section 2. Then $\{X_t : t\geq 0\}$ is still a Markov process with state space $N(E)$. We also have (2.2) and (2.3), but (2.4) is now replaced by

$$\begin{aligned}\mathrm{e}^{-u_t(x)} &= \boldsymbol{P}_x\{\rho(0,t)\mathrm{e}^{-f(\xi_t)}\} \\ &\quad +\boldsymbol{P}_x\Big\{\int_0^t\Big[\rho(0,s)\gamma(\xi_s)\int_{N(E)}\mathrm{e}^{-u_{t-s}(\xi_s)}\mathrm{e}^{-\nu(u_{t-s})}F(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\}.\end{aligned}\tag{4.1}$$

This is equivalent to

$$\begin{aligned}\mathrm{e}^{-u_t(x)} &= \boldsymbol{P}_x\mathrm{e}^{-f(\xi_t)}-\boldsymbol{P}_x\Big\{\int_0^t\gamma(\xi_s)\mathrm{e}^{-u_{t-s}(\xi_s)}\mathrm{d}s\Big\} \\ &\quad +\boldsymbol{P}_x\Big\{\int_0^t\Big[\gamma(\xi_s)\int_{N(E)}\mathrm{e}^{-u_{t-s}(\xi_s)}\mathrm{e}^{-\nu(u_{t-s})}F(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\},\end{aligned}$$

or

$$\begin{aligned}1-\mathrm{e}^{-u_t(x)} &= \boldsymbol{P}_x\Big\{1-\mathrm{e}^{-f(\xi_t)}\Big\}-\boldsymbol{P}_x\Big\{\int_0^t\gamma(\xi_s)(1-\mathrm{e}^{-u_{t-s}(\xi_s)})\mathrm{d}s\Big\} \\ &\quad +\boldsymbol{P}_x\Big\{\int_0^t\Big[\gamma(\xi_s)\int_{N(E)}(1-\mathrm{e}^{-u_{t-s}(\xi_s)})\mathrm{e}^{-\nu(u_{t-s})}F(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\} \\ &\quad +\boldsymbol{P}_x\Big\{\int_0^t\Big[\gamma(\xi_s)\int_{N(E)}(1-\mathrm{e}^{-\nu(u_{t-s})})F(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\}.\end{aligned}$$

Let $v_t(x)\equiv v_t(x,f)$ be defined by (2.6). Then we have

$$v_t(x) = \boldsymbol{P}_x\Big\{1-\mathrm{e}^{-f(\xi_t)}\Big\}-\boldsymbol{P}_x\Big\{\int_0^t\gamma(\xi_s)v_{t-s}(\xi_s)\mathrm{d}s\Big\}$$

$$+\boldsymbol{P}_x\Big\{\int_0^t\Big[\gamma(\xi_s)\int_{N(E)}v_{t-s}(\xi_s)\mathrm{e}^{-\nu(u_{t-s})}F(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\} \tag{4.2}$$
$$+\boldsymbol{P}_x\Big\{\int_0^t\Big[\gamma(\xi_s)\int_{N(E)}(1-\mathrm{e}^{-\nu(u_{t-s})})F(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\}.$$

We now consider a sequence of rebirth branching particle systems $\{X_t(k): t\geq 0\}$ with parameters (ξ,γ_k,F_k). Define $\{X_t^{(k)}: t\geq 0\}$ and choose F_k as in section 3 with $\alpha_k(x)\equiv 0$ and $\gamma_k(x)\equiv\beta_k(x)$. Then (3.5) remains valid if we replace (3.7) by

$$\begin{aligned}v_t^{(k)}(x) \;=\; &\boldsymbol{P}_x\left\{k(1-\mathrm{e}^{-f(\xi_t)/k})\right\}-\boldsymbol{P}_x\Big\{\int_0^t\beta_k(\xi_s)v_{t-s}^{(k)}(\xi_s)\mathrm{d}s\Big\}\\ &+\boldsymbol{P}_x\Big\{\int_0^t\Big[\beta_k(\xi_s)\int_{N(E)}v_{t-s}^{(k)}(\xi_s)\mathrm{e}^{-\nu(u_{t-s}^{(k)})}F_k(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\}\\ &+\boldsymbol{P}_x\Big\{\int_0^t\Big[k\beta_k(\xi_s)\int_{N(E)}[1-\mathrm{e}^{-\nu(u_{t-s}^{(k)})}]F_k(\xi_s,\mathrm{d}\nu)\Big]\mathrm{d}s\Big\},\end{aligned} \tag{4.3}$$

or equivalently

$$v_t^{(k)}(x)+\int_0^t\boldsymbol{P}_x[\beta_k(\xi_s)\phi_k(\xi_s,v_{t-s}^{(k)})+\psi_k(\xi_s,v_{t-s}^{(k)})]\mathrm{d}s=\boldsymbol{P}_xk[1-\mathrm{e}^{-f(\xi_t)/k}], \tag{4.4}$$

where ψ_k is given by (3.10) and (3.13), and

$$\phi_k(x,f)=-f(x)\int_{M_0(E)}h_k(x,\pi,1-\pi(f)/k)G(x,\mathrm{d}\pi). \tag{4.5}$$

Lemma 4.1 *If the conditions of Lemma 3.1 (i) are fulfilled and if $\beta_k\to\beta\in B^+(E)$ uniformly as $k\to\infty$, then, for each $a\geq 0$, we have $\phi_k(x,f)\to f(x)$ uniformly on $E\times B_a^+(E)$ and the solution $v_t^{(k)}(x,f)$ to (4.4) converges boundedly and uniformly on the set $[0,a]\times E\times B_a^+(E)$ of (t,x,f) to the unique bounded positive solution $V_tf(x)$ to the evolution equation*

$$V_tf(x)-\int_0^t\Big[\int_E\beta(y)\zeta(y,V_{t-s}f)P_s(x,\mathrm{d}y)\Big]\mathrm{d}s=P_tf(x), \tag{4.6}$$

where $\zeta(\cdot,\cdot)$ is defined by (3.15) and (3.21).

Based on this lemma, one can show as in section 3 that the finite dimensional distributions of $\{X_t^{(k)}: t\geq 0\}$ under $\boldsymbol{Q}_{(\mu)}^{(k)}$ converge as $k\to\infty$ to those of the process $\{X_t: t\geq 0\}$ with semigroup $(Q_t)_{t\geq 0}$ defined by (3.22) and (4.6). Since $\alpha_k(x)\equiv 0$ in the approximating sequence, we call $\{X_t: t\geq 0\}$ a *rebirth superprocess*. Note that (4.6) is the special form of (3.20) with local branching mechanism $\phi(x,z)\equiv-\beta(x)z$, which exactly compensates the death factor in the non-local branching mechanism. This observation might be helpful in understanding the non-local branching mechanism given by (3.21).

5 Multitype superprocesses

In this section, we deduce the existence of a class of multitype superprocesses from that of the non-local branching superprocess constructed in section 3 following the arguments in Li (1992b). Let E and I be two Lusin topological spaces and let $\xi = \{\Omega, (\eta_t, \alpha_t), \mathcal{F}, \mathcal{F}_t, \boldsymbol{P}_{(x,a)}\}$ be a right continuous strong Markov process with state space $E \times I$. Let $\phi(\cdot,\cdot,\cdot)$ and $\zeta(\cdot,\cdot,\cdot)$ be given by (3.19) and (3.31), respectively, with $x \in E$ replaced by $(x,a) \in E \times I$. Let $\beta(\cdot,\cdot) \in E \times I$ and let $\pi(x,a,\mathrm{d}b)$ be a probability kernel from $E \times I$ to I. As a special form of the model given in section 3, we have a non-local branching superprocess $\{X_t(dx,da) : t \geq 0\}$ in $M(E \times I)$ with transition probabilities determined by

$$\boldsymbol{Q}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \qquad t \geq 0, f \in B^+(E \times I), \tag{5.1}$$

where $V_t f$ is the unique bounded positive solution to

$$\begin{aligned} V_t f(x,a) \;&+\; \int_0^t \boldsymbol{P}_{(x,a)}[\phi(\eta_s, \alpha_s, V_{t-s}f(\eta_s,\alpha_s)) + \beta(\eta_s,\alpha_s)V_{t-s}f(\eta_s,\alpha_s)]\mathrm{d}s \\ &-\; \int_0^t \boldsymbol{P}_{(x,a)}[\beta(\eta_s,\alpha_s)\zeta(\eta_s,\alpha_s,\pi(\eta_s,\alpha_s,V_{t-s}f(\eta_s,\cdot)))]\mathrm{d}s \\ &=\; \boldsymbol{P}_{(x,a)}[f(\eta_t,\alpha_t)]. \end{aligned} \tag{5.2}$$

We may call $\{X_t : t \geq 0\}$ a *multitype superprocess* with type space I. Heuristically, $\{\eta_t : t \geq 0\}$ gives the law of migration of the "particles", $\{\alpha_t : t \geq 0\}$ represents the mutation of their types, $\phi(x,a,\cdot)$ describes the amount of the a-type offspring born when an a-type parent dies at $x \in E$, $\zeta(x,a,\cdot)$ describes the amount of the offspring born by this parent that change into new types randomly according to the kernel $\pi(x,a,\mathrm{d}b)$, and $\beta(x,a)$ represents the birth rate of the changing-type offspring at $x \in E$. It is assumed that all of the offspring start migrating from the death site of their parent. Note that the migration process $\{\eta_t : t \geq 0\}$ and the mutation process $\{\alpha_t : t \geq 0\}$ are not necessarily independent.

Now let us consider a special case which has been studied in the literature. Suppose that $I = \{1,\dots,k\}$ and for each $i \in I$, $\eta^{(i)}$ is a right continuous strong Markov process in E with semigroup $(P_t^{(i)})_{t\geq 0}$, $\phi^{(i)}$ belongs to the class given by (3.19) and $\zeta^{(i)}$ belongs to the class given by (3.31). Let ξ be a right continuous strong Markov process in the product space $E \times I$ with transition semigroup $(P_t)_{t\geq 0}$ defined by

$$P_t f(x,i) = \int_E f(y,i) P_t^{(i)}(x,\mathrm{d}y), \qquad f \in B^+(E \times I).$$

Let $\phi((x,i),z) = \phi^{(i)}(x,z)$. Suppose that $\beta^{(i)} \in B^+(E)$ and $\pi(x,i,\cdot)$ is a Markov kernel from $E \times I$ to I having the decomposition

$$\pi(x,i,\cdot) = \sum_{j=1}^k p_j^{(i)}(x)\delta_j(\cdot),$$

where $p_j^{(i)}(x) \geq 0$, $\sum_{j=0}^k p_j^{(i)}(x) \equiv 1$ and δ_j denotes the unit mass at $j \in I$. Then we have a multitype superprocess $\{X_t : t \geq 0\}$ in $M(E \times I)$ by (5.1) and (5.2). For $i \in I$ and $\mu \in M(E\times I)$ we define $\mu^{(i)} \in M(E)$ by $\mu^{(i)}(B) = \mu(B \times \{i\})$. The map $\mu \mapsto (\mu^{(1)}, \dots, \mu^{(k)})$ is clearly a homeomorphism between $M(E\times I)$ and the k-dimensional product space $M(E)^k$. Therefore, $\{(X_t^{(1)}, \dots, X_t^{(k)}) : t \geq 0\}$ is a Markov process in the space $M(E)^k$, which may be called a *k-type superprocess*. Clearly, this class of k-type superprocesses coincides with the one defined in Li (1992b). Heuristically, $\eta^{(i)}$ gives the law of the migration of the ith type "particles", $\phi^{(i)}(x, \cdot)$ describes the amount of the ith type offspring born when an ith type parent dies at point $x \in E$, $\zeta^{(i)}(x, \cdot)$ describes the amount of the offspring born by this parent that change into new types randomly according to the discrete distribution $\{p_1^{(i)}(x), \dots, p_k^{(i)}(x)\}$, and $\beta^{(i)}(x)$ represents the birth rate of the changing-type offspring at $x \in E$. The study of multitype superprocesses was initiated by Gorostiza and Lopez-Mimbela (1990); see also Gorostiza and Roelly (1991) and Gorostiza et al (1992).

6 Superprocess-controlled immigration

By the discussions in the last section, we have a special 2-type superprocess $\{(X_t^{(1)}, X_t^{(2)}) : t \geq 0\}$ in $M(E)^2$ with transition probabilities determined by

$$\boldsymbol{Q}_{(\mu^{(1)},\mu^{(2)})} \exp\left\{-X_t^{(1)}(f^{(1)}) - X_t^{(2)}(f^{(2)})\right\} = \exp\left\{-\mu^{(1)}(v_t^{(1)}) - \mu^{(2)}(v_t^{(2)})\right\}, \qquad (6.1)$$

where $v_t^{(1)}(\cdot)$ and $v_t^{(2)}(\cdot)$ are defined uniquely by

$$v_t^{(1)}(x) + \int_0^t \left[\int_E \left(\phi^{(1)}(y, v_{t-s}^{(1)}(y)) - v_{t-s}^{(2)}(y)\right) P_s^{(1)}(x, \mathrm{d}y)\right]\mathrm{d}s = P_t^{(1)} f^{(1)}(x), \qquad (6.2)$$

and

$$v_t^{(2)}(x) + \int_0^t \left[\int_E \phi^{(2)}(y, v_{t-s}^{(2)}(y)) P_s^{(1)}(x, \mathrm{d}y)\right]\mathrm{d}s = P_t^{(2)} f^{(2)}(x). \qquad (6.3)$$

In particular, if $f^{(2)} \equiv 0$, we have $v_t^{(2)} \equiv 0$ and

$$\boldsymbol{Q}_{(\mu^{(1)},\mu^{(2)})} \exp\left\{-X_t^{(1)}(f^{(1)})\right\} = \exp\left\{-\mu^{(1)}(v_t^{(1)})\right\}, \qquad (6.4)$$

where $v_t^{(1)}(\cdot)$ is given by

$$v_t^{(1)}(x) + \int_0^t \left[\int_E \phi^{(1)}(y, v_{t-s}^{(1)}(y)) P_s^{(1)}(x, \mathrm{d}y)\right]\mathrm{d}s = P_t^{(1)} f^{(1)}(x). \qquad (6.5)$$

Thus $\{X_t^{(1)} : t \geq 0\}$ is a superprocess in $M(E)$ with parameters $(\eta^{(1)}, \phi^{(1)})$. On the other hand, by an expression of weighted occupation times, the value in (6.1) is equal to

$$\boldsymbol{Q}_{(\mu^{(1)},\mu^{(2)})} \exp\left\{-X_t^{(1)}(f^{(1)})\right\} \exp\left\{-\mu^{(2)}(v_t^{(2)}) - \int_0^t X_s^{(1)}(v_{t-s}^{(2)})\mathrm{d}s\right\};$$

see e.g. Dawson (1993) and Dynkin (1993, 1994). Then we see that

$$\boldsymbol{Q}_{(\mu^{(1)},\mu^{(2)})}\left[\exp\left\{-X_t^{(2)}(f^{(2)})\right\}\Big|X_s^{(1)}:s\geq 0\right]=\exp\left\{-\mu^{(2)}(v_t^{(2)})-\int_0^t X_s^{(1)}(v_{t-s}^{(2)})\mathrm{d}s\right\}.$$

That is, given $\{X_t^{(1)}:t\geq 0\}$, the second coordinate $\{X_t^{(2)}:t\geq 0\}$ is a superprocess with parameters $(\eta^{(2)},\phi^{(2)})$ and *with immigration controlled by* $\{X_t^{(1)}:t\geq 0\}$. A special class of superprocess-controlled immigration processes have been studied in Hong and Li (1999) and their relation with multitype superprocesses has been pointed out in Li (2002).

7 Mass-structured superprocesses

A multitype superprocess $\{X_t(\mathrm{d}x,\mathrm{d}a):t\geq 0\}$ with type space $I=(0,\infty)$ can be called a *mass-structured superprocess* if we interpret $x\in E$ and $a>0$ as the coordinates of position and mass, respectively. For the mass-structured superprocess, we may consider its *aggregated process* $\{Y_t:t\geq 0\}$ defined by

$$Y_t(\mathrm{d}x):=\int_0^\infty aX_t(\mathrm{d}x,\mathrm{d}a),\qquad x\in E.\tag{7.1}$$

Since the integrand on the right hand side is unbounded, $\{Y_t:t\geq 0\}$ is only well-defined under some restrictions. In general, $\{Y_t:t\geq 0\}$ is not Markovian. Let A be the weak generator of $\{(\eta_t,\alpha_t):t\geq 0\}$ and $(T_t)_{t\geq 0}$ the locally bounded semigroup of finite kernels on $E\times I$ with generator

$$Bf(x,a)=Af(x,a)+\beta(x,a)[m(x,a)\pi(a,f(x,\cdot))-f(x,a)]-b(x,a)f(x,a),\tag{7.2}$$

where $b(x,a)$ is the coefficient of the linear term of $\phi(x,a,z)$ and $m(x,a)$ is defined by (3.32) with $x\in E$ replaced by $(x,a)\in E\times I$. Indeed, (7.2) is of the same form as (3.28) with (x,a) instead of x. By the discussions in section 3, the first moments of the superprocess are given by

$$\boldsymbol{Q}_\mu\{X_t(f)\}=\mu(T_tf),\qquad f\in B^+(E\times I).\tag{7.3}$$

In practice, we may have that a newborn offspring is no larger than its parent, which corresponds to the assumption that $\pi(a,\cdot)$ is supported by $(0,a)$. Let $H(x,a)=a$ and suppose that $H\in\mathcal{D}(A)$ is a c_1-excessive function of $\{(\eta_t,\alpha_t):t\geq 0\}$ for some constant $c_1>0$. In this case, we have $BH(x,a)\leq(c_1+\|b\|)H(x,a)$ and hence $H\in\mathcal{D}(B)$ is a $(c_1+\|b\|)$-excessive function of $(T_t)_{t\geq 0}$. It follows from (7.3) that

$$\boldsymbol{Q}_\mu\{X_t(H)\}\leq \mathrm{e}^{(c_1+\|b\|)t}\mu(H).\tag{7.4}$$

Then we may change the state space slightly and take any σ-finite measure μ on $E\times(0,\infty)$ satisfying $\mu(H)<\infty$ as the initial state of $\{X_t:t\geq 0\}$; see e.g. El Karoui and Roelly (1991) or Li (1992c). In this case, (7.4) implies that $X_t(H)<\infty$ a.s. for all $t\geq 0$ so that (7.1) defines an aggregated process $\{Y_t:t\geq 0\}$ with finite measure values.

A special type of mass-structured superprocess with Markovian aggregated process has been studied by Gorostiza (1994). Assume that $\beta(\cdot,\cdot) \equiv 0$ and $\alpha_t = g(t, \alpha_0)$ for a deterministic mapping $g(\cdot,\cdot)$ from $[0,\infty)\times(0,\infty)$ to $(0,\infty)$. Let $\boldsymbol{P}_x$ denote the conditional law of $\{\eta_t : t \geq 0\}$ given $\eta_0 = x$. For $f \in B^+(E)$, (5.2) becomes

$$V_t f(x,a) + \int_0^t \boldsymbol{P}_x[\phi(\eta_s, g(s,a), V_{t-s}f(\eta_s, g(s,a)))]\mathrm{d}s = \boldsymbol{P}_x f(\eta_t). \tag{7.5}$$

Since the motion of $\alpha_t = g(t, \alpha_0)$ is deterministic, if X_0 is supported by $E \times \{a\}$, then X_t is supported by $E \times \{g(t,a)\}$ and $Y_t = g(t,a)X_t$. In this case, $\{Y_t : t \geq 0\}$ is a Markov process since the transformation $X_t \mapsto Y_t$ loses no information. For $B \in \mathcal{B}(E)$, let $X_t^a(B) = X_t(B \times \{g(t,a)\})$. Then $\{X_t^a : t \geq 0\}$ is an inhomogeneous superprocess with cumulant semigroup $(V_{r,t}^a)_{t \geq r \geq 0}$ defined by $V_{r,t}^a f(x) := V_{t-r}f(x, g(r,a))$, which has underlying process $\{\eta_t : t \geq 0\}$ and time-dependent branching mechanism $\phi(x, g(t,a), \cdot)$. This gives a representation of the aggregated process in terms of an inhomogeneous superprocess. A representation of this type was first given by Gorostiza (1994) in the case where $\alpha_t = \alpha_0 \mathrm{e}^{ct}$ for a constant $c \in \mathbb{R}$. Gorostiza (1994) obtained the process as high density limit of a sequence of branching particle systems where the mass of each offspring is equal to that of its parent multiplied by a fixed positive constant factor, and the mass of any particle does not change during its lifetime, realizing in a particular case the program mentioned at the end of section 3.

8 Multilevel superprocesses

Multilevel superprocesses arise as limits of multilevel branching particle systems. In a two level system, objects at the higher level consist of non-trivial sub-populations of objects at the lower level and both lower level and higher level objects can branch. A lower level object consisting of a population can be described by a measure on some space S. We can then view a two level system as a multitype system with $I = M(S)^\circ$, the space of non-trivial finite Borel measures on S. Non-local branching is natural in this context. For example, at the particle level the offspring of a second order object consisting of a set of particles could consist of a subset of the particles or include more than one copy of the original particles.

To make this precise, we may let S be a topological Lusin space and $\{\alpha_t : t \geq 0\}$ be the Markov process with state space $M(S)^\circ$ obtained by killing a superprocess at its extinction time. Then $\{X_t : t \geq 0\}$ is a Markov process with state space $M(E \times M(S)^\circ)$, which can be called a *multilevel superprocess* generalizing the model of Dawson and Hochberg (1991), Dawson et al (1990) and Wu (1994). For the multilevel process, it is also natural to study the *aggregated process* $\{Y_t : t \geq 0\}$ defined by

$$Y_t(A \times B) := \int_A \int_{M(S)} a(B) X_t(\mathrm{d}x, \mathrm{d}a), \qquad A \in \mathcal{B}(E), B \in \mathcal{B}(S). \tag{8.1}$$

To illustrate the possibilities of non-local branching, consider the case in which E is a singleton. In this case, we may view $\{X_t : t \geq 0\}$ as a superprocess with state space

$M(M(S)^\circ)$. A possible non-local branching mechanisms is obtained by taking $\pi(\mu, \mathrm{d}\nu)$ to be the law of

$$\frac{\mu(S)}{N}\sum_{i=1}^{N}\delta_{Z_i}, \tag{8.2}$$

where $N \geq 1$ is an integer-valued random variable, and $\{Z_1, Z_2, \cdots\}$ are i.i.d. random variables in S with distribution $\mu(S)^{-1}\mu(\cdot)$. That is, the offspring of a level two object $\mu \in M(S)$ is a single point measure with the same total mass as μ and its location is selected randomly according to the empirical measure of a sample from the normalized parent distribution.

Another possibility is given by

$$\pi(\mu, \mathrm{d}\nu) = \delta_{\mu_B}(\mathrm{d}\nu), \tag{8.3}$$

where $B \in \mathcal{B}(S)$ and $\mu_B \in M(S)$ is defined by $\mu_B(A) = \mu(A \cap B)$. In this case the offspring is a level two object in which only level one individuals falling in the set $B \subset S$ are present.

In the case in which E is a countable set, we may interpret the $M(E \times M(S)^\circ)$-valued process $\{X_t : t \geq 0\}$ as a population in a sequence of islands. The E-coordinate tells in which island the $M(S)^\circ$-valued objects $\{\alpha_t : t \geq 0\}$ in the first level are located. The non-local branching is given by $\pi(x, \mu, d\nu) = \pi(\mu, d\nu)$, which only acts on the $M(S)^\circ$-coordinate at the higher level. Jumps in E described by $\{\eta_t : t \geq 0\}$ correspond to the independent migration of "clans" (families of the lower level) between the islands. Suggestively, we may call $\{X_t : t \geq 0\}$ a *stepping stone type superprocess.*

Properties and applications of multilevel superprocesses involving local branching have been studied extensively in the literature; see e.g. Dawson and Hochberg (1991), Dawson et al (1990, 1994, 1995), Etheridge (1993), Gorostiza (1996), Gorostiza et al (1995), Hochberg (1995), Wu(1994) and the references therein.

9 Age-reproduction-structured superprocesses

Let E be a Lusin topological space and let $\xi = \{\Omega, (\eta_t, \alpha_t, \theta_t), \mathcal{F}, \mathcal{F}_t, \boldsymbol{P}_{(y,a,z)}, \gamma\}$ be a Borel Markov process with state space $E \times \mathbb{R}^+ \times \mathbb{N}^+$, where γ is a terminal time; see Sharpe (1988, p.65). We assume that both α_t and θ_t are non-decreasing processes. Let $\beta(\cdot,\cdot,\cdot) \in B(E \times \mathbb{R}^+ \times \mathbb{N}^+)$ and let $\zeta(\cdot,\cdot,\cdot,\cdot)$ be given by (3.31) with $x \in E$ replaced by $(x, a, z) \in E \times \mathbb{R}^+ \times \mathbb{N}^+$. As a special form of the models given in sections 4 and 5, we have a rebirth multitype superprocess $\{X_t : t \geq 0\}$ in $M(E \times \mathbb{R}^+ \times \mathbb{N}^+)$ with transition probabilities determined by

$$\boldsymbol{Q}_\mu \exp\{-X_t(f)\} = \exp\{-\mu(V_t f)\}, \qquad t \geq 0, f \in B^+(E \times \mathbb{R}^+ \times \mathbb{N}^+), \tag{9.1}$$

where $V_t f$ is the unique bounded positive solution to

$$\begin{aligned} V_t f(y,a,z) \quad &- \quad \int_0^t \boldsymbol{P}_{(y,a,z)}[\beta(\eta_s, \alpha_s, \theta_s)\zeta(\eta_s, \alpha_s, \theta_s, V_{t-s}f(\eta_s, 0, 0))1_{\{\alpha_s<\gamma\}}]\mathrm{d}s \\ &= \quad \boldsymbol{P}_{(y,a,z)}[f(\eta_t, \alpha_t, \theta_t)1_{\{\alpha_s<\gamma\}}]. \end{aligned} \tag{9.2}$$

It is not hard to check that the first moments of the superprocess are given by

$$\boldsymbol{Q}_\mu\{X_t(f)\} = \mu(T_t f), \qquad f \in B^+(E \times I\!R^+ \times I\!N^+), \tag{9.3}$$

where $(T_t)_{t\geq 0}$ is a semigroup of bounded linear operators on $B^+(E \times I\!R^+ \times I\!N^+)$ defined by

$$\begin{aligned} T_t f(y,a,z) &- \int_0^t \boldsymbol{P}_{(y,a,z)}[\beta(\eta_s,\alpha_s,\theta_s)m(\eta_s,\alpha_s,\theta_s)T_{t-s}f(\eta_s,0,0)1_{\{\alpha_s<\gamma\}}]\mathrm{d}s \\ &= \boldsymbol{P}_{(y,a,z)}[f(\eta_t,\alpha_t,\theta_t)1_{\{\alpha_s<\gamma\}}], \end{aligned} \tag{9.4}$$

where $m(\cdot,\cdot,\cdot)$ is given by (3.32) with $x \in E$ replaced by $(x,a,z) \in E \times I\!R^+ \times I\!N^+$. Using $(T_t)_{t\geq 0}$ we may rewrite (9.2) into

$$V_t f(y,a,z) + \int_0^t T_s[\beta m V_{t-s}f - \zeta(\cdot,\cdot,\cdot,V_{t-s}f(\cdot,0,0))](y,a,z)\mathrm{d}s = T_t f(y,a,z). \tag{9.5}$$

In the case $\alpha_t \equiv \alpha_0 + t$, we may call $\{X_t : t \geq 0\}$ an *age-reproduction-structured superprocess*. Heuristically, η_t represents the location of a "particle", α_t its age and θ_t the number of its offspring born in the time interval $(t-\alpha_t, t]$. At each branching time, the particle gives birth to a random number of offspring whose motions start from the branching site and whose ages and reproduction numbers start from zero. The particle does not disappear at its branching times, it is removed from the population only when its age exceeds the lifetime γ. An interesting limit theorem for age-reproduction-structured branching particle systems was proved in Bose and Kaj (2000) which leads to the superprocess in the special case where E is a singleton and $\eta_t \equiv \eta_0$. (Compare (9.5) and their equation (2.8).)

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