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Fluctuations of a super-Brownian motion with randomly controlled immigration [1]

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Abstract. We study the fluctuations around mean of a super Brownian motion with immigration controlled by the trajectory of a stationary immigration process. The main result is a central limit theorem which holds for all dimension numbers and leads to some Gaussian random fields.

Key words: super-Brownian motion, immigration, stationary process, central limit theorem.

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A variety of limit theorems have been proved for Dawson-Watanabe superprocesses. Dawson [1] obtained a spatial central limit theorem for the stationary state of a (α, d, β)-superprocess with underlying dimension number $d > \alpha/\beta$. Iscoe [6] proved central limit theorems for the associated weighted occupation time process in the same situation. A central limit theorem of super-Brownian motion was given in Li [10], which leads to non-degenerate limit distributions for all dimension numbers. Immigration structures associated with Dawson-Watanabe superprocesses have been studied by several authors; see Gorostiza and Lopez-Mimbela [3], Li [7, 8, 9], Li and Wang [12] and the references therein. Limit theorems for immigration processes were studied in Li and Shiga [11], where the immigration is governed by a deterministic measure. Hong and Li [5] considered a super-Brownian motion with immigration governed by the trajectory of another super-Brownian motion and proved a central limit theorem for such process, which lead to Gaussian random fields for dimension numbers $d \geq 3$. For $d = 3$ the field is spatially uniform; for $d \geq 5$ its covariance is given by the potential operator of the underlying Brownian motion; and for $d = 4$ the limit field involves a mixture of the two kinds of

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fluctuations mentioned above, which exhibits a departure from the phenomenon in Li [10] and Li and Shiga [11]. Hong [4] investigated the asymptotic behavior of the model for $d = 2$.

To find new situations where non-degenerate limit theorems for a superprocess can be obtained, we consider in this paper a super-Brownian motion with immigration controlled by the trajectory of a stationary immigration process. The main result is a central limit theorem for the process. We shall see that the limit theorem gives the same limit laws as the ones in Li [10] and Li and Shiga [11], in the contrast to the result of Hong and Li [5]. The study has been stimulated by the work of Dawson and Fleischmann [2], who studied a super-Brownian motion with random branching mechanism governed by another super-Brownian motion. The process considered here can also be regarded as a special form of the multi-type branching-immigration model studied by Gorostiza and Lopez-Mimbela [3] and Li [7].

1. Super Brownian motion with immigration

Let $C(I\!\!R^d)$ denote the space of continuous bounded functions on $I\!\!R^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in I\!\!R^d$. Let $C_p(I\!\!R^d) := \{f \in C(I\!\!R^d) : |f(x)| \leq \mathrm{const}\cdot\phi_p(x)\}$. In duality, let $M_p(I\!\!R^d)$ be the space of Radon measures μ on $I\!\!R^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(I\!\!R^d)$. We endow $M_p(I\!\!R^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(I\!\!R^d)$. Then $M_p(I\!\!R^d)$ is metrizable. Throughout this note, λ denotes the Lebesgue measure on $I\!\!R^d$.

Suppose that $(P_t)_{t\geq 0}$ is the semigroup of a standard Brownian motion in $I\!\!R^d$. For any $b \geq 0$ we let $P^b_t = \mathrm{e}^{-bt}P_t$. Let $\gamma := \{\gamma_t : t \geq 0\}$ be a continuous path from $[0, \infty)$ to $M_p(I\!\!R^d)$. In this note, a Markov process $\{X^\gamma_t : t \geq 0\}$ with state space $M_p(I\!\!R^d)$ is called a *non-supercritical super-Brownian motion with immigration controlled by* γ if it has transition semigroup $(Q^\gamma_{r,t})_{t\geq r\geq 0}$ such that

$$\int_{M_p(I\!\!R^d)} \mathrm{e}^{-\langle \nu, f\rangle} Q^\gamma_{r,t}(\mu, d\nu) = \exp\Big\{ - \langle \mu, v(t-r, \cdot)\rangle - \int_r^t \langle \gamma_s, v(t-s, \cdot)\rangle ds\Big\} \tag{1}$$

for $f \in C^+_p(I\!\!R^d)$, where $v(\cdot, \cdot)$ is the unique solution of the evolution equation

$$v(t, x) = P^b_t f(x) - \int_0^t P^b_{t-s} v(s, \cdot)^2(x) ds, \quad t \geq 0; \tag{2}$$

see e.g. Dawson [2] and Li and Wang [12]

Let $\mathbf{Q}^\gamma_\mu$ denote the conditional law of $\{X^\gamma_t : t \geq 0\}$ given that $X^\gamma_0 = \mu$. Suppose that $\{\phi(t, \cdot) : t \geq 0\}$ is a continuous path from $[0, \infty)$ to $C^+_p(I\!\!R^d)$ bounded above by $\mathrm{const}\cdot\phi_p$.

By an approximating procedure as Iscoe [6], one may show

$$\mathbf{Q}_\mu^\gamma \exp\Big\{ -\int_0^t \langle X_s^\gamma, \phi(s)\rangle ds\Big\} = \exp\Big\{ -\langle \mu, w(t,\cdot)\rangle - \int_0^t \langle \gamma_s, w(s,\cdot)\rangle ds\Big\}, \tag{3}$$

for $f \in C_p^+(I\!\!R^d)$, where $w(\cdot,\cdot)$ satisfies

$$w(r,x) = \int_0^r P_{r-s}^b \phi(t-s)(x)ds - \int_0^r P_{r-s}^b w(s,\cdot)^2(x)ds, \quad 0 \le r \le t. \tag{4}$$

In particular, (1) defines a homogeneous semigroup $(Q_t^\lambda)_{t\ge 0}$ if $\gamma_t \equiv \lambda$. Observe that, when $b > 0$, we have

$$\int_0^\infty \langle \lambda, v(s,\cdot)\rangle ds \le \int_0^\infty \langle \lambda, P_t^b f\rangle ds = \langle \lambda, f\rangle / b < \infty$$

for all $f \in C_p^+(I\!\!R^d)$. It follows that $Q_t^\lambda(0,\cdot) \to \mathcal{Q}^\lambda$ as $t \to \infty$, where $\mathcal{Q}^\lambda$ is a stationary distribution for $(Q_t^\lambda)_{t\ge 0}$ given by

$$\int_{M_p(I\!\!R^d)} \mathrm{e}^{-\langle \nu, f\rangle} \mathcal{Q}^\lambda(d\nu) = \exp\Big\{ -\int_0^\infty \langle \lambda, v(s,\cdot)\rangle ds\Big\}. \tag{5}$$

Now it is not difficult to construct a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ on which the two processes $\{\varrho_t : t \ge 0\}$ and $\{Y_t : t \ge 0\}$ are defined, where $\{\varrho_t : t \ge 0\}$ is a stationary subcritical super-Brownian motion with immigration having one-dimensional distribution $\mathcal{Q}^\lambda$, and given $\{\varrho_t : t \ge 0\}$ the process $\{Y_t : t \ge 0\}$ is a critical super-Brownian motion with immigration controlled by $\{\varrho_t : t \ge 0\}$ and $Y_0 = 0$.

One particular choice for the space $(\Omega, \mathcal{F}, \mathbf{Q})$ is given as follows. Let $C_{[0,\infty)}$ denote the totality of continuous paths $\{w(\cdot) : t \ge 0\}$ from $[0,\infty)$ to $M_p(I\!\!R^d)$, with the Skorokhod topology and the Borel σ-algebra $\mathcal{G}$. Suppose that Q^λ is the distribution on $(C_{[0,\infty)}, \mathcal{G})$ of the stationary immigration process with one-dimensional distribution $\mathcal{Q}^\lambda$ and that Q_0^γ is the distribution of the critical super-Brownian motion with immigration controlled by $\{\gamma_t : t \ge 0\}$ and $Y_0 = 0$. Let $\Omega = C_{[0,\infty)} \times C_{[0,\infty)}$ and $\mathcal{F} = \mathcal{G} \times \mathcal{G}$ and define the probability measure $\mathbf{Q}$ on $\mathcal{F}$ by

$$\mathbf{Q}(dw_1, dw_2) = Q^\lambda(dw_1) Q_0^{w_1}(dw_2), \quad w_1, w_2 \in C_{[0,\infty)}.$$

Let $\varrho_t(w_1,w_2) = w_1(t)$ and $Y_t(w_1,w_2) = w_2(t)$. Then $\{(\varrho_t, Y_t) : t \ge 0\}$ has the pre-described distribution property.

By (1) we have

$$\mathbf{Q}[\exp\{-\langle Y_t, f\rangle\} | \{\varrho_t : t \ge 0\}] = \exp\Big\{ -\int_0^t \langle \varrho_s, u(t-s)\rangle ds\Big\}, \tag{6}$$

where $u(\cdot,\cdot)$ is the solution of

$$u(t,x) = P_t f(x) - \int_0^t P_{t-s} u(s,\cdot)^2(x) ds, \quad t \geq 0. \tag{7}$$

Taking the expectation of (6) and using (3) and (5) we get

$$\begin{aligned} \mathbf{Q}\exp\{-\langle Y_t, f\rangle\} &= \int_{M_p(I\!R^d)} \exp\Big\{ - \langle \mu, w(t,\cdot)\rangle - \int_0^t \langle \lambda, w(r,\cdot)\rangle dr \Big\} \mathcal{Q}^\lambda(d\mu) \\ &= \exp\Big\{ - \int_0^\infty \langle \lambda, v(r,\cdot)\rangle dr - \int_0^t \langle \lambda, w(r,\cdot)\rangle dr \Big\}, \end{aligned} \tag{8}$$

where $w(\cdot,\cdot)$ and $v(\cdot,\cdot)$ are defined respectively by

$$w(r,x) = \int_0^r P^b_{r-s} u(s,\cdot)(x) ds - \int_0^r P^b_{r-s} w^2(s,\cdot)(x) ds, \quad r \geq 0, \tag{9}$$

and

$$v(r,x) = P^b_r w(t,\cdot)(x) - \int_0^r P^b_{r-s} v^2(s,\cdot)(x) ds, \quad r \geq 0. \tag{10}$$

2. A central limit theorem

We present here a central limit theorem for the process $\{Y_t : t \geq 0\}$ defined in the last section. It is not difficult to check by using (7) – (10) that $\mathbf{Q}\{Y_t(f)\} = t\lambda(f)/b$ for $t \geq 0$ and $f \in C_p(I\!R^d)$. Let $\mathcal{S}(I\!R^d)$ be the space of rapidly decreasing, infinitely differentiable functions on $I\!R^d$ whose all partial derivatives are also rapidly decreasing, and let $\mathcal{S}'(I\!R^d)$ be the dual space of $\mathcal{S}(I\!R^d)$. We define the $\mathcal{S}'(I\!R^d)$-valued process $\{Z_t : t > 0\}$ by

$$\langle Z_t, f\rangle := a_d(t)^{-1}[\langle Y_t, f\rangle - t\langle \lambda, f\rangle/b], \quad f \in \mathcal{S}(I\!R^d), \tag{11}$$

where $a_1(t) = t^{3/4}$, $a_2(t) = (t\log t)^{1/2}$ and $a_d(t) = t^{1/2}$ for $d \geq 3$. Then we have

Theorem 1. *As $t \to \infty$, the distribution of Z_t converges to a centered Gaussian random variable Z_∞ in $\mathcal{S}'(I\!R^d)$ with covariance*

$$\mathbf{Cov}(Z_\infty, f\rangle, \langle Z_\infty, g\rangle) = \begin{cases} 2\langle \lambda, f\rangle\langle \lambda g\rangle/3b\pi^{1/2}, & d = 1, \\ \langle \lambda, f\rangle\langle \lambda, g\rangle/4\pi b, & d = 2, \\ \langle \lambda, fGg\rangle/2b, & d \geq 3, \end{cases}$$

where G denotes the potential operator of the Brownian motion.

Now we proceed to the proof of Theorem 1 by an argument adapted from [10]. Let $f_t := a_d(t)^{-1} f$. In the following lemmas and proofs, $u_t(s), w_t(s)$ and $v_t(s)$ are the solutions of equations (7), (9) and (10), respectively, with f being replaced by f_t, and C denotes a constant which may take different values in different lines.

Lemma 2. *For $f \in \mathcal{S}(I\!\!R^d)^+$ let*

$$A_d(t,f) := \int_0^t dr \int_0^r ds \int_0^s \mathrm{e}^{-b(r-s)} \langle \lambda, (P_{s-q} f_t)^2 \rangle dq.$$

Then we have

$$\lim_{t\to\infty} A_d(t,f) = \begin{cases} 2\langle \lambda, f\rangle^2/3b\sqrt{\pi}, & d=1, \\ \langle \lambda, f\rangle^2/4\pi b, & d=2, \\ \langle \lambda, fGf\rangle/2b, & d\geq 3. \end{cases}$$

Proof. We have clearly

$$A_d(t,f) = a_d(t)^{-2} \int_0^t \mathrm{e}^{-br} dr \int_0^r \mathrm{e}^{bs} ds \int_0^s dq \int_{I\!\!R^d} P_{s-q} f(x)^2 dx.$$

When $d \geq 3$, we use l'Hospital's rule to get

$$\begin{aligned} \lim_{t\to\infty} A_d(t,f) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{e}^{-br} dr \int_0^r \mathrm{e}^{bs} ds \int_0^s dq \int_{I\!\!R^d} P_{s-q} f(x)^2 dx \\ &= \lim_{t\to\infty} \frac{1}{e^{bt}} \int_0^t e^{bs} ds \int_0^s dq \int_{I\!\!R^d} P_{s-q} f(x)^2 dx \\ &= \lim_{t\to\infty} \frac{1}{b} \int_0^t dq \int_{I\!\!R^d} P_{s-q} f(x)^2 dx \\ &= \langle \lambda, fGf\rangle/2b, \end{aligned}$$

For $d=1$ we have

$$\begin{aligned} \lim_{t\to\infty} A_1(t,f) &= \lim_{t\to\infty} \frac{1}{t^{3/2}} \int_0^t \mathrm{e}^{-br} dr \int_0^r \mathrm{e}^{bs} ds \int_0^s dq \int_{I\!\!R^d} P_{s-q} f(x)^2 dx \\ &= \lim_{t\to\infty} \frac{2}{3\sqrt{t}\,\mathrm{e}^{bt}} \int_0^t \mathrm{e}^{bs} ds \int_0^s dq \int_{I\!\!R^d} P_{s-q} f(x)^2 dx \\ &= \lim_{t\to\infty} \frac{2}{3b\sqrt{t}} \int_0^t dq \int_{I\!\!R^d} P_q f(x)^2 dx \\ &= \lim_{t\to\infty} \frac{2}{3b\sqrt{t}} \int_0^t \frac{1}{\sqrt{4\pi q}} dq \int_{I\!\!R^2} \exp\{-\frac{(y-x)^2}{4q}\} f(x) f(y) dx dy \\ &= \lim_{t\to\infty} \frac{2}{3b} \int_0^1 \frac{1}{\sqrt{4\pi r}} dr \int_{I\!\!R^2} \exp\{-\frac{(y-x)^2}{4tr}\} f(x) f(y) dx dy \\ &= 2\langle \lambda, f\rangle^2/3b\sqrt{\pi}, \end{aligned}$$

where we used the change of variables $q = tr$ in the fifth step. Similarly, by setting $q = t^{1-r}$ for $d=2$, one may see that $\lim_{t\to\infty} A_2(t,f) = \langle \lambda, f\rangle^2/4\pi b$. $\square$

Lemma 3. *For $f \in \mathcal{S}(I\!R^d)^+$ let*

$$B_d(t,f) := \int_0^t dr \int_0^r ds \int_0^s \mathrm{e}^{-b(r-s)} \langle \lambda, (P_{s-q} f_t)^2 - u_t(s-q,\cdot)^2 \rangle dq.$$

Then we have $\lim_{t\to\infty} B_d(t,f) = 0$.

Proof. Note that for any $f \in \mathcal{S}(I\!R^d)^+$ we have

$$||P_s f|| \le C \cdot (1 \wedge s^{-d/2}),$$

where $C = C(f) \ge 0$. From equation (7) we can see that

$$\begin{aligned}
(P_r f_t)^2 - u_t(r)^2 &= 2u_t(r) \int_0^r P_{r-s} u_t(s)^2 ds + \Big(\int_0^r P_{r-s} u_t(s)^2 ds \Big)^2 \\
&\le 3 P_r f_t \cdot \int_0^r P_{r-s} (P_s f_t)^2 ds \\
&\le C \cdot a_d(t)^{-3} (P_r f)^2 \cdot \int_0^r (1 \wedge s^{-d/2}) ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
B_d(t,f) &\le C \cdot a_d(t)^{-3} \int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)} ds \int_0^s dq \int_{I\!R^d} P_{s-q} f(x)^2 dx \int_0^t (1 \wedge l^{-d/2}) dl \\
&\le C \cdot a_d(t)^{-3} \int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)} ds \int_0^t (1 \wedge q^{-d/2}) dq \int_0^t (1 \wedge l^{-d/2}) dl.
\end{aligned}$$

Then we have for dimension one

$$\begin{aligned}
\limsup_{t\to\infty} B_1(t,f) &\le C \cdot \limsup_{t\to\infty} \frac{1}{t^{9/4}} \int_0^t dr \int_0^r \mathrm{e}^{-bs} ds \int_0^t (1 \wedge q^{-d/2}) dq \int_0^t (1 \wedge l^{-d/2}) dl \\
&\le C \cdot \limsup_{t\to\infty} \frac{1}{t^{5/4}} \int_0^t (1 \wedge q^{-d/2}) dq \int_0^t (1 \wedge l^{-d/2}) dl \\
&= 0.
\end{aligned}$$

The proof for other dimension numbers are similar. □

Proof of Theorem 1. From (7) – (9) and (11) we get the Laplace functional

$$\begin{aligned}
\mathbf{Q} \exp\{-\langle Z_t, f\rangle\} &= \exp\Big\{ t\langle \lambda, f_t\rangle/b - \int_0^\infty \langle \lambda, v_t(r)\rangle dr - \int_0^t \langle \lambda, w_t(r)\rangle dr \Big\} \\
&= \exp\Big\{ t\langle \lambda, f_t\rangle/b - \int_0^\infty \langle \lambda, v_t(r)\rangle dr - \int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)} \langle \lambda, u_t(s)\rangle ds \\
&\qquad + \int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)} \langle \lambda, w_t(s)^2 \rangle ds \Big\} \\
&= \exp\Big\{ t\langle \lambda, f_t\rangle/b - \int_0^\infty \langle \lambda, v_t(r)\rangle dr - \int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)} \langle \lambda, f_t\rangle ds \\
&\qquad + \int_0^t dr \int_0^r ds \int_0^s \mathrm{e}^{-b(r-s)} \langle \lambda, u_t(s-q)^2 \rangle dq \\
&\qquad + \int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)} \langle \lambda, w_t(s)^2 \rangle ds \Big\}, \qquad (12)
\end{aligned}$$

where

$$t\langle\lambda, f_t\rangle/b - \int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, f_t\rangle ds = b^{-1}\int_0^t \mathrm{e}^{-br}\langle\lambda, f_t\rangle ds \to 0 \tag{13}$$

as $t \to \infty$. By equations (7), (9) and (10), we have

$$v_t(s) \le P_s^b w_t(t) \le \int_0^t P_{s+t-r}^b u_t(r)dr \le \mathrm{e}^{-bs}\int_0^t \mathrm{e}^{-b(t-r)}P_{s+t}f_t dr \le \mathrm{e}^{-bs}P_{s+t}f_t.$$

It follows that

$$\limsup_{t\to\infty}\int_0^\infty \langle\lambda, v_t(s)\rangle ds \le \lim_{t\to\infty} a_d(t)^{-1}\langle\lambda, f\rangle = 0. \tag{14}$$

Similarly, one may check that

$$\lim_{t\to\infty}\int_0^t dr \int_0^r \mathrm{e}^{-b(r-s)}\langle\lambda, w_t(s)^2\rangle ds = 0. \tag{15}$$

On the other hand, combining Lemmas 2 and 3, we have

$$\lim_{t\to\infty}\int_0^t dr \int_0^r ds \int_0^s \mathrm{e}^{-b(r-s)}\langle\lambda, u_t(s-q)^2\rangle dq = \begin{cases} 2\langle\lambda, f\rangle^2/3b\pi^{1/2}, & d=1, \\ \langle\lambda, f\rangle^2/4\pi b, & d=2, \\ \langle\lambda, fGf\rangle/2b, & d\ge 3. \end{cases} \tag{16}$$

Combining (12) – (16) we obtain the desired convergence. □

An immediate consequence of Theorem 1 is the following

Corollary 4. *For $d \ge 1$ we have $t^{-1}Z_t \to \lambda$ in probability.*

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