Fluctuations of a super-Brownian motion with randomly controlled immigration

Wenming HONG
Institute of Mathematics, Fudan University,
Shanghai 200433, P. R. CHINA

Zenghu LI
Department of Mathematics, Beijing Normal University
Beijing 100875, P. R. China

Abstract. We study the fluctuations around mean of a super Brownian motion with immigration controlled by the trajectory of a stationary immigration process. The main result is a central limit theorem which holds for all dimension numbers and leads to some Gaussian random fields.

Key words: super-Brownian motion, immigration, stationary process, central limit theorem.

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A variety of limit theorems have been proved for Dawson-Watanabe superprocesses. Dawson [1] obtained a spatial central limit theorem for the stationary state of a \((\alpha, d, \beta)\)-superprocess with underlying dimension number \(d > \alpha/\beta\). Iscoe [6] proved central limit theorems for the associated weighted occupation time process in the same situation. A central limit theorem of super-Brownian motion was given in Li [10], which leads to non-degenerate limit distributions for all dimension numbers. Immigration structures associated with Dawson-Watanabe superprocesses have been studied by several authors; see Gorostiza and Lopez-Mimbela [3], Li [7, 8, 9], Li and Wang [12] and the references therein. Limit theorems for immigration processes were studied in Li and Shiga [11], where the immigration is governed by a deterministic measure. Hong and Li [5] considered a super-Brownian motion with immigration governed by the trajectory of another super-Brownian motion and proved a central limit theorem for such process, which lead to Gaussian random fields for dimension numbers \(d \geq 3\). For \(d = 3\) the field is spatially uniform; for \(d \geq 5\) its covariance is given by the potential operator of the underlying Brownian motion; and for \(d = 4\) the limit field involves a mixture of the two kinds of

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fluctuations mentioned above, which exhibits a departure from the phenomenon in Li [10] and Li and Shiga [11]. Hong [4] investigated the asymptotic behavior of the model for $d = 2$.

To find new situations where non-degenerate limit theorems for a superprocess can be obtained, we consider in this paper a super-Brownian motion with immigration controlled by the trajectory of a stationary immigration process. The main result is a central limit theorem for the process. We shall see that the limit theorem gives the same limit laws as the ones in Li [10] and Li and Shiga [11], in the contrast to the result of Hong and Li [5]. The study has been stimulated by the work of Dawson and Fleischmann [2], who studied a super-Brownian motion with random branching mechanism governed by another super-Brownian motion. The process considered here can also be regarded as a special form of the multi-type branching-immigration model studied by Gorostiza and Lopez-Mimbela [3] and Li [7].

1. Super Brownian motion with immigration

Let $C(\mathbb{R}^d)$ denote the space of continuous bounded functions on $\mathbb{R}^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1 + |x|^2)^{-p/2}$ for $x \in \mathbb{R}^d$. Let $C_p(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) : |f(x)| \leq \text{const} \cdot \phi_p(x) \}$. In duality, let $M_p(\mathbb{R}^d)$ be the space of Radon measures $\mu$ on $\mathbb{R}^d$ such that $\langle \mu, f \rangle := \int f(x) \mu(dx) < \infty$ for all $f \in C_p(\mathbb{R}^d)$. We endow $M_p(\mathbb{R}^d)$ with the $p$-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f \rangle \to \langle \mu, f \rangle$ for all $f \in C_p(\mathbb{R}^d)$. Then $M_p(\mathbb{R}^d)$ is metrizable. Throughout this note, $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Suppose that $(P_t)_{t \geq 0}$ is the semigroup of a standard Brownian motion in $\mathbb{R}^d$. For any $b \geq 0$ we let $P_t^b = e^{-bt}P_t$. Let $\gamma := \{ \gamma_t : t \geq 0 \}$ be a continuous path from $[0, \infty)$ to $M_p(\mathbb{R}^d)$. In this note, a Markov process $\{X_t^\gamma : t \geq 0\}$ with state space $M_p(\mathbb{R}^d)$ is called a non-supercritical super-Brownian motion with immigration controlled by $\gamma$ if it has transition semigroup $(Q_{r,t}^\gamma)_{t \geq r \geq 0}$ such that

$$\int_{M_p(\mathbb{R}^d)} e^{-\langle \mu, f \rangle} Q_{r,t}^\gamma(\mu, d\nu) = \exp \left\{ -\langle \mu, v(t-r, \cdot) \rangle - \int_r^t \langle \gamma_s, v(t-s, \cdot) \rangle ds \right\}$$

for $f \in C_p^+(\mathbb{R}^d)$, where $v(\cdot, \cdot)$ is the unique solution of the evolution equation

$$v(t, x) = P_t^b f(x) - \int_0^t P_{t-s}^b v(s, \cdot)^2(x) ds, \quad t \geq 0;$$

see e.g. Dawson [2] and Li and Wang [12].

Let $Q_{t}^\mu$ denote the conditional law of $\{X_t^\gamma : t \geq 0\}$ given that $X_0^\gamma = \mu$. Suppose that $\{\phi(t, \cdot) : t \geq 0\}$ is a continuous path from $[0, \infty)$ to $C_p^+(\mathbb{R}^d)$ bounded above by $\text{const} \cdot \phi_p$. 

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By an approximating procedure as Iscoe [6], one may show
\[ Q_t^\gamma \exp \left\{ - \int_0^t \langle X_s^\gamma, \phi(s) \rangle ds \right\} = \exp \left\{ - \langle \mu, w(t, \cdot) \rangle - \int_0^t \langle \gamma_s, w(s, \cdot) \rangle ds \right\}, \tag{3} \]
for \( f \in C^+_p(\mathbb{R}^d) \), where \( w(\cdot, \cdot) \) satisfies
\[ w(r, x) = \int_0^r P^b_{r-s} \phi(t-s)(x)ds - \int_0^r P^b_{r-s} w(s, \cdot)^2(x)ds, \quad 0 \leq r \leq t. \tag{4} \]

In particular, (1) defines a homogeneous semigroup \((Q^\lambda_t)_{t \geq 0}\) if \( \gamma_t \equiv \lambda \). Observe that, when \( b > 0 \), we have
\[ \int_0^\infty \langle \lambda, v(s, \cdot) \rangle ds \leq \int_0^\infty \langle \lambda, P^b_t f \rangle ds = \langle \lambda, f \rangle / b < \infty \]
for all \( f \in C^+_p(\mathbb{R}^d) \). It follows that \( Q^\lambda_t(0, \cdot) \to Q^\lambda \) as \( t \to \infty \), where \( Q^\lambda \) is a stationary distribution for \((Q^\lambda_t)_{t \geq 0}\) given by
\[ \int_{M_p(\mathbb{R}^d)} e^{-(\nu, f)} Q^\lambda(d\nu) = \exp \left\{ - \int_0^\infty \langle \lambda, v(s, \cdot) \rangle ds \right\}. \tag{5} \]

Now it is not difficult to construct a probability space \((\Omega, \mathcal{F}, Q)\) on which the two processes \( \{ \theta_t : t \geq 0 \} \) and \( \{ Y_t : t \geq 0 \} \) are defined, where \( \{ \theta_t : t \geq 0 \} \) is a stationary subcritical super-Brownian motion with immigration having one-dimensional distribution \( Q^\lambda \), and given \( \{ \theta_t : t \geq 0 \} \) the process \( \{ Y_t : t \geq 0 \} \) is a critical super-Brownian motion with immigration controlled by \( \{ \theta_t : t \geq 0 \} \) and \( Y_0 = 0 \).

One particular choice for the space \((\Omega, \mathcal{F}, Q)\) is given as follows. Let \( C_{[0,\infty)} \) denote the totality of continuous paths \( \{ w(\cdot) : t \geq 0 \} \) from \([0,\infty)\) to \( M_p(\mathbb{R}^d) \), with the Skorokhod topology and the Borel \( \sigma \)-algebra \( \mathcal{G} \). Suppose that \( Q^\lambda \) is the distribution on \((C_{[0,\infty)}, \mathcal{G})\) of the stationary immigration process with one-dimensional distribution \( Q^\lambda \) and that \( Q^\gamma_0 \) is the distribution of the critical super-Brownian motion with immigration controlled by \( \{ \gamma_t : t \geq 0 \} \) and \( Y_0 = 0 \). Let \( \Omega = C_{[0,\infty)} \times C_{[0,\infty)} \) and \( \mathcal{F} = \mathcal{G} \times \mathcal{G} \) and define the probability measure \( Q \) on \( \mathcal{F} \) by
\[ Q(dw_1, dw_2) = Q^\lambda(dw_1)Q^\gamma_0(w_1)(dw_2), \quad w_1, w_2 \in C_{[0,\infty)}. \]

Let \( \theta_t(w_1, w_2) = w_1(t) \) and \( Y_t(w_1, w_2) = w_2(t) \). Then \( \{ (\theta_t, Y_t) : t \geq 0 \} \) has the pre-described distribution property.

By (1) we have
\[ Q[\exp\{-\langle Y_t, f \rangle\} \mid \{ \theta_t : t \geq 0 \}] = \exp \left\{ - \int_0^t \langle \gamma_s, u(t-s) \rangle ds \right\}. \tag{6} \]
where \(u(\cdot, \cdot)\) is the solution of
\[
u(t, x) = P_t f(x) - \int_0^t P_{t-s}^2 u(s, \cdot)(x) ds, \quad t \geq 0.
\]
Taking the expectation of (6) and using (3) and (5) we get
\[
Q\exp\{-\langle Y_t, f \rangle\} = \int_{M_p(\mathbb{R}^d)} \exp\left\{-\langle \mu, w(t, \cdot) \rangle - \int_0^t \langle \lambda, v(r, \cdot) \rangle dr\right\} Q^\lambda(d\mu)
\]
where \(w(\cdot, \cdot)\) and \(v(\cdot, \cdot)\) are defined respectively by
\[
w(r, x) = \int_0^r P^b_{r-s} u(s, \cdot)(x) ds - \int_0^r P^b_{r-s} w^2(s, \cdot)(x) ds, \quad r \geq 0,
\]
and
\[
v(r, x) = P^b_r w(t, \cdot)(x) - \int_0^r P^b_{r-s} v^2(s, \cdot)(x) ds, \quad r \geq 0.
\]

2. A central limit theorem

We present here a central limit theorem for the process \(\{Y_t : t \geq 0\}\) defined in the last section. It is not difficult to check by using (7) – (10) that
\[
Q\{Y_t(f)\} = \frac{t \lambda(f)}{b} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad f \in C_p(\mathbb{R}^d).
\]
Let \(\mathcal{S}(\mathbb{R}^d)\) be the space of rapidly decreasing, infinitely differentiable functions on \(\mathbb{R}^d\) whose all partial derivatives are also rapidly decreasing, and let \(\mathcal{S}'(\mathbb{R}^d)\) be the dual space of \(\mathcal{S}(\mathbb{R}^d)\). We define the \(\mathcal{S}'(\mathbb{R}^d)\)-valued process \(\{Z_t : t > 0\}\) by
\[
\langle Z_t, f \rangle := a_d(t)^{-1}\langle Y_t, f \rangle - t \langle \lambda, f \rangle / b, \quad f \in \mathcal{S}(\mathbb{R}^d),
\]
where \(a_1(t) = t^{3/4}, \quad a_2(t) = (t \log t)^{1/2}\) and \(a_d(t) = t^{1/2}\) for \(d \geq 3\). Then we have

**Theorem 1.** As \(t \to \infty\), the distribution of \(Z_t\) converges to a centered Gaussian random variable \(Z_\infty\) in \(\mathcal{S}'(\mathbb{R}^d)\) with covariance
\[
\text{Cov}(Z_\infty, f), (Z_\infty, g)) = \begin{cases}
2\langle \lambda, f \rangle \langle \lambda g \rangle / 3b \pi^{1/2}, & d = 1, \\
\langle \lambda, f \rangle \langle \lambda, g \rangle / 4\pi b, & d = 2, \\
\langle \lambda, fGg \rangle / 2b, & d \geq 3,
\end{cases}
\]
where \(G\) denotes the potential operator of the Brownian motion.

Now we proceed to the proof of Theorem 1 by an argument adapted from [10]. Let
\(f_t := a_d(t)^{-1} f\). In the following lemmas and proofs, \(u_t(s), w_t(s)\) and \(v_t(s)\) are the solutions
of equations (7), (9) and (10), respectively, with \( f \) being replaced by \( f_t \), and \( C \) denotes a constant which may take different values in different lines.

**Lemma 2.** For \( f \in S(\mathbb{R}^d)^+ \) let

\[
A_d(t, f) := \int_0^t dr \int_0^r ds \int_0^s e^{-b(r-s)} \langle \lambda, (P_s - q f_t)^2 \rangle dq.
\]

Then we have

\[
\lim_{t \to \infty} A_d(t, f) = \begin{cases} 
2\langle \lambda, f \rangle^2 / 3b\sqrt{\pi}, & d = 1, \\
\langle \lambda, f \rangle^2 / 4\pi b, & d = 2, \\
\langle \lambda, fGf \rangle / 2b, & d \geq 3.
\end{cases}
\]

**Proof.** We have clearly

\[
A_d(t, f) = a_d(t)^{-2} \int_0^t e^{-br} dr \int_0^r e^{bs} ds \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx.
\]

When \( d \geq 3 \), we use l'Hospital’s rule to get

\[
\lim_{t \to \infty} A_d(t, f) = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-br} dr \int_0^r e^{bs} ds \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{bs} ds \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx
\]

\[
= \langle \lambda, fGf \rangle / 2b,
\]

For \( d = 1 \) we have

\[
\lim_{t \to \infty} A_1(t, f) = \lim_{t \to \infty} \frac{1}{t^{3/2}} \int_0^t e^{-br} dr \int_0^r e^{bs} ds \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx
\]

\[
= \lim_{t \to \infty} \frac{2}{3\sqrt{t}} \int_0^t e^{bs} ds \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx
\]

\[
= \lim_{t \to \infty} \frac{2}{3b\sqrt{t}} \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx
\]

\[
= \lim_{t \to \infty} \frac{2}{3b\sqrt{t}} \int_0^s \frac{1}{\sqrt{4\pi q}} dq \int_{\mathbb{R}^2} \exp\left\{ -\frac{(y-x)^2}{4q} \right\} f(x)f(y) dydx
\]

\[
= \lim_{t \to \infty} \frac{2}{3b} \int_0^t \frac{1}{\sqrt{4\pi r}} dr \int_{\mathbb{R}^2} \exp\left\{ -\frac{(y-x)^2}{4tr} \right\} f(x)f(y) dydx
\]

\[
= 2\langle \lambda, f \rangle^2 / 3b\sqrt{\pi},
\]

where we used the change of variables \( q = tr \) in the fifth step. Similarly, by setting \( q = t^{1-r} \) for \( d = 2 \), one may see that \( \lim_{t \to \infty} A_2(t, f) = \langle \lambda, f \rangle^2 / 4\pi b \). □
Lemma 3. For \( f \in \mathcal{S}({\mathbb{R}}^d)^+ \) let
\[
B_d(t, f) := \int_0^t dr \int_0^r ds \int_0^s e^{-b(r-s)} \langle \lambda, (P_{s-q} f_t)^2 - u_t(s - q, \cdot)^2 \rangle dq.
\]
Then we have \( \lim_{t \to \infty} B_d(t, f) = 0 \).

**Proof.** Note that for any \( f \in \mathcal{S}({\mathbb{R}}^d)^+ \) we have
\[
\|P_t f\| \leq C \cdot (1 \wedge s^{-d/2}),
\]
where \( C = C(f) \geq 0 \). From equation (7) we can see that
\[
(P_r f_t)^2 - u_t(r)^2 = 2u_t(r) \int_0^r P_{r-s} u_t(s)^2 ds + \left( \int_0^r P_{r-s} u_t(s)^2 ds \right)^2 \\
\leq 3P_r f_t \cdot \int_0^r P_{r-s} (P_s f_t)^2 ds \\
\leq C \cdot a_d(t)^{-3} (P_r f)^2 \cdot \int_0^r (1 \wedge s^{-d/2}) ds.
\]
It follows that
\[
B_d(t, f) \leq C \cdot a_d(t)^{-3} \int_0^t dr \int_0^r e^{-b(r-s)} ds \int_0^s dq \int_{\mathbb{R}^d} P_{s-q} f(x)^2 dx \int_0^r (1 \wedge l^{-d/2}) dl \\
\leq C \cdot a_d(t)^{-3} \int_0^t dr \int_0^r e^{-b(r-s)} ds \int_0^s dq \int_0^t (1 \wedge q^{-d/2}) dq \int_0^t (1 \wedge l^{-d/2}) dl.
\]
Then we have for dimension one
\[
\lim_{t \to \infty} B_d(t, f) \leq C \cdot \lim_{t \to \infty} \frac{1}{t^{9/4}} \int_0^t dr \int_0^r e^{-b s} ds \int_0^s dq \int_0^t (1 \wedge q^{-d/2}) dq \int_0^t (1 \wedge l^{-d/2}) dl \\
\leq C \cdot \lim_{t \to \infty} \frac{1}{t^{5/4}} \int_0^t (1 \wedge q^{-d/2}) dq \int_0^t (1 \wedge l^{-d/2}) dl \\
= 0.
\]
The proof for other dimension numbers are similar. \( \square \)

**Proof of Theorem 1.** From (7) – (9) and (11) we get the Laplace functional
\[
\mathbb{Q} \exp\{-\langle Z_t, f \rangle\} = \exp \left\{ t \langle \lambda, f_t \rangle / b - \int_0^\infty \langle \lambda, v_t(r) \rangle dr - \int_0^t \langle \lambda, w_t(r) \rangle dr \right\} \\
= \exp \left\{ t \langle \lambda, f_t \rangle / b - \int_0^\infty \langle \lambda, v_t(r) \rangle dr - \int_0^t dr \int_0^r e^{-b(r-s)} \langle \lambda, u_t(s) \rangle ds \\
+ \int_0^t dr \int_0^r e^{-b(r-s)} \langle \lambda, w_t(s)^2 \rangle ds \right\} \\
= \exp \left\{ t \langle \lambda, f_t \rangle / b - \int_0^\infty \langle \lambda, v_t(r) \rangle dr - \int_0^t dr \int_0^r e^{-b(r-s)} \langle \lambda, f_t \rangle ds \\
+ \int_0^t dr \int_0^r ds \int_0^s e^{-b(r-s)} \langle \lambda, u_t(s - q)^2 \rangle dq \\
+ \int_0^t dr \int_0^r e^{-b(r-s)} \langle \lambda, w_t(s)^2 \rangle ds \right\},
\]
(12)
where
\[
t(\lambda, f_t)/b - \int_0^t dr \int_0^r e^{-b(r-s)}(\lambda, f_t)ds = b^{-1} \int_0^t e^{-br}(\lambda, f_t)ds \rightarrow 0
\] (13)
as \( t \rightarrow \infty \). By equations (7), (9) and (10), we have
\[
v_t(s) \leq P^b_s w_t(t) \leq \int_0^t P^b_{s+t-r} u_t(r)dr \leq e^{-bs} \int_0^t e^{-b(t-r)}P^b_{s+t} f_t dr \leq e^{-bs} P^b_{s+t} f_t.
\]
It follows that
\[
\limsup_{t \rightarrow \infty} \int_0^\infty (\lambda, v_t(s))ds \leq \lim_{t \rightarrow \infty} a_d(t)^{-1}(\lambda, f) = 0.
\] (14)
Similarly, one may check that
\[
\lim_{t \rightarrow \infty} \int_0^t dr \int_0^r e^{-b(r-s)}(\lambda, w_t(s)^2)ds = 0.
\] (15)
On the other hand, combining Lemmas 2 and 3, we have
\[
\lim_{t \rightarrow \infty} \int_0^t dr \int_0^s ds \int_0^r e^{-b(r-s)}(\lambda, u_t(s-q)^2)dq = \begin{cases} 2(\lambda, f)^2/3b\pi^{1/2}, & d = 1, \\ (\lambda, f)^2/4\pi b, & d = 2, \\ (\lambda, fGf)/2b, & d \geq 3. \end{cases}
\] (16)
Combining (12) – (16) we obtain the desired convergence. □

An immediate consequence of Theorem 1 is the following

**Corollary 4.** For \( d \geq 1 \) we have \( t^{-1}Z_t \rightarrow \lambda \) in probability.

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**References**


