Published in: *Journal of Applied Probability* **37** *(2000), 627–634.*

ORNSTEIN-UHLENBECK TYPE PROCESSES AND BRANCHING PROCESSES WITH IMMIGRATION

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Abstract. It is shown that an Ornstein-Uhlenbeck type process associated with a spectrally positive Lévy process can be obtained as the fluctuation limits of both discrete state and continuous state branching processes with immigration.

Key words: branching process; immigration; Lévy process; Ornstein-Uhlenbeck type process; fluctuation limit

AMS 1991 Subject Classifications: Primary 60J80; Secondary 60F05.

1. Introduction

Suppose that $b \geq 0$ and $c \geq 0$ are constants and that $m(\mathrm{d}u)$ is a σ-finite measure on $(0, \infty)$ such that

$$\int_0^\infty (u \wedge u^2) m(\mathrm{d}u) < \infty. \tag{1.1}$$

Let φ be a function on $[0, \infty)$ with the representation

$$\varphi(\lambda) = c\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda u} - 1 + \lambda u) m(\mathrm{d}u), \qquad \lambda \geq 0. \tag{1.2}$$

By an Ornstein-Uhlenbeck type process we mean a real-valued cádlág Markov process $\{z_t : t \geq 0\}$ with transition function $\pi_t(x, \mathrm{d}y)$ given by

$$\int_R \mathrm{e}^{-\lambda y} \pi_t(x, \mathrm{d}y) = \exp\bigg\{ - x\mathrm{e}^{-bt}\lambda + \int_0^t \varphi(\mathrm{e}^{-bs}\lambda)\mathrm{d}s \bigg\}, \qquad \lambda \geq 0. \tag{1.3}$$

It is well-known that $\{z_t : t \geq 0\}$ is the unique solution of the stochastic differential equation

$$\mathrm{d}z_t = \mathrm{d}l_t - bz_t\mathrm{d}t, \qquad t \geq 0,$$

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[1]Research supported by the National Natural Science Foundation of China (Grant No.19361060).

where $\{l_t : t \geq 0\}$ is a spectrally positive cádlág Lévy process with $l_0 = 0$ and with increment distributions determined by

$$\mathbf{E} \exp\{-\lambda(l_{r+t} - l_r)\} = \exp\{-t\varphi(\lambda)\}, \quad \lambda \geq 0.$$

Ornstein-Uhlenbeck type processes associated with general Lévy processes were introduced by Sato and Yamazato (1984) and Wolfe (1982) in studying the class of limit distributions of sums of certain random variables and have been studied by many authors since then; see e.g. Hadjiev (1985), Samorodnitsky and Taqqu (1994) and Shiga (1990).

In this note, we show that the Ornstein-Uhlenbeck type process may arise as the fluctuation limits of both discrete state and continuous state branching processes with immigration, giving interpretations for the process from the viewpoint of applications. The particular case $b = 0$ corresponds to the situation where the immigration rate is zero and $\{z_t : t \geq 0\}$ itself is a Lévy process. Therefore, the results reveal new connections between branching processes and Lévy processes; see Bingham (1976), Lamperti (1967) and Le Gall and Le Jan (1998a,b) for earlier results on the connections between those processes. We refer to Bertoin (1996) and Sato (1990) for the basic theory of Lévy processes.

2. Branching process with immigration

Let $\gamma > 0$ be a constant and $g(s) = \sum_{i=0}^{\infty} p_i s^i$ be a probability generating function. We shall always assume that g is non-supercritical, that is $g'(1^-) \leq 1$. Then there is a unique solution $F_t(s)$ to the equation

$$\frac{\partial}{\partial t} F_t(s) = \gamma[g(F_t(s)) - F_t(s)], \quad F_0(s) = s, \qquad t \geq 0, s \in [0,1]. \tag{2.1}$$

Moreover, there is a continuous time Markov transition function $p_t(i,j)$ on N defined by

$$\sum_{j=0}^{\infty} p_t(i,j) s^j = F_t(s)^i, \qquad t \geq 0, s \in [0,1]. \tag{2.2}$$

A Markov chain $\{x_t : t \geq 0\}$ with transition function $p_t(i,j)$ is called a discrete state branching process (DB-process) with parameters (γ, g); see e.g. Arthreya and Ney (1972; pp102-106). Instead of the generating function given by (2.1) and (2.2), it is sometimes more convenient to consider the Laplace transform of the transition function. So we rewrite (2.2) as

$$\sum_{j=0}^{\infty} p_t(i,j) \mathrm{e}^{-j\lambda} = \mathrm{e}^{-iu_t(\lambda)}, \qquad t \geq 0, \lambda \geq 0. \tag{2.3}$$

where

$$u_t(\lambda) = -\log F_t(\mathrm{e}^{-\lambda}), \qquad t \geq 0, \lambda \geq 0. \tag{2.4}$$

Let $a \geq 0$ be another constant. A Markov chain $\{y_t : t \geq 0\}$ is called a discrete state branching process with immigration (DBI-process) with parameters (γ, g, a) if it has transition semigroup $q_t(i,j)$ defined by

$$\sum_{j=0}^{\infty} q_t(i,j)\mathrm{e}^{-j\lambda} = \exp\bigg\{ - iu_t(\lambda) - a\int_0^t w_s(\lambda)\mathrm{d}s\bigg\}, \qquad t \geq 0, \lambda \geq 0, \tag{2.5}$$

where $w_t(\lambda) = 1 - \exp\{-u_t(\lambda)\}$. It is clear that the immigration of $\{y_t : t \geq 0\}$ here is governed by a Poisson process with parameter $a \geq 0$. By (2.1) and (2.4) we get the equation

$$\frac{\partial}{\partial t} w_t(\lambda) = -\psi(w_t(\lambda)), \quad w_0(\lambda) = 1 - \mathrm{e}^{-\lambda}, \qquad t \geq 0, \lambda \geq 0, \tag{2.6}$$

where

$$\psi(z) = \gamma[g(1-z) - (1-z)], \qquad 0 \leq z \leq 1.$$

Integrating both sides of (2.6) we get

$$w_t(\lambda) + \int_0^t \psi(w_s(\lambda))\mathrm{d}s = 1 - \mathrm{e}^{-\lambda}, \qquad t \geq 0, \lambda \geq 0. \tag{2.7}$$

Let $b = \gamma[1 - g'(1)]$ and let $\psi_0(z) = \psi(z) - bz$. We may rewrite (2.7) into the following equivalent form

$$w_t(\lambda) + \int_0^t \mathrm{e}^{-b(t-s)}\psi_0(w_s(\lambda))\mathrm{d}s = \mathrm{e}^{-bt}(1 - \mathrm{e}^{-\lambda}), \qquad t \geq 0, \lambda \geq 0. \tag{2.8}$$

By (2.5) and (2.8) it is easy to check that

$$\sum_{j=1}^{\infty} jq_t(i,j) = i\mathrm{e}^{-bt} + a\int_0^t \mathrm{e}^{-bs}\mathrm{d}s, \qquad t \geq 0. \tag{2.9}$$

Now let us describe a continuous state analogue of the DBI-process introduced by Kawazu and Watanabe (1971). Suppose that φ is given by (1.2) and let $a \geq 0$ and $b \geq 0$ be constants. A Markov process $\{y_t : t \geq 0\}$ on $[0, \infty)$ is called a continuous state branching process with immigration (CBI-process) with parameters (b, φ, a) if it has transition function $q_t(x, \mathrm{d}y)$ given by

$$\int_0^{\infty} \mathrm{e}^{-\lambda y} q_t(x, \mathrm{d}y) = \exp\bigg\{ - xv_t(\lambda) - a\int_0^t v_s(\lambda)\mathrm{d}s\bigg\}, \qquad t \geq 0, \lambda \geq 0, \tag{2.10}$$

where $v_t(\lambda)$ is the unique positive solution to the equation

$$\frac{\partial}{\partial t} v_t(\lambda) = -bv_t(\lambda) - \varphi(v_t(\lambda)), \quad v_0(\lambda) = \lambda, \qquad t \geq 0, \lambda \geq 0; \tag{2.11}$$

see Kawazu and Watanabe (1971). Note that (2.11) is equivalent to the integral equation

$$v_t(\lambda) + \int_0^t \mathrm{e}^{-b(t-s)} \varphi(v_s(\lambda))\mathrm{d}s = \mathrm{e}^{-bt}\lambda, \qquad t \geq 0, \lambda \geq 0. \tag{2.12}$$

Based on (2.10) and (2.12) it is easy to check that

$$\int_0^\infty y q_t(x, \mathrm{d}y) = x\mathrm{e}^{-bt} + a\int_0^t \mathrm{e}^{-bs}\mathrm{d}s, \qquad t \geq 0. \tag{2.13}$$

Suppose that $\{y_{k,t} : t \geq 0\}$ is a sequence of DBI-processes with parameters (γ_k, g_k, ka). We assume that $y_{k,0}$ is a Poisson random variable with parameter kx for some $x \geq 0$. Let

$$\phi_k(z) = k\gamma_k[g_k(1 - z/k) - (1 - z/k)], \qquad 0 \leq z \leq k. \tag{2.14}$$

Lemma 2.1. *If $\phi(z) = \lim_{k\to\infty} \phi_k(z)$ uniformly on $[0, l]$ for each finite $l \geq 0$, then the limit function must be of the form $\phi(z) = bz + \varphi(z)$, where $b \geq 0$ and φ is a function with the representation* (1.2)*.*

Proof. By a result of Li (1991) the limit function has the representation

$$\phi(z) = b_1 z + cz^2 + \int_0^\infty \left(\mathrm{e}^{-zu} - 1 + \frac{zu}{1+u^2}\right) m(\mathrm{d}u), \qquad z \geq 0,$$

for some constants $c \geq 0$ and b_1, and a σ-finite measure $m(\mathrm{d}u)$ on $(0, \infty)$ such that

$$\int_0^\infty (1 \wedge u^2) m(\mathrm{d}u) < \infty.$$

By hypothesis, each g_k is non-supercritical, so $\phi_k(z)$ and hence $\phi(z)$ is non-decreasing in $z \geq 0$. Then we have

$$\phi'(0^+) = b_1 - \int_0^\infty \frac{u^3}{1+u^2} m(\mathrm{d}u) \geq 0.$$

Therefore, $m(\mathrm{d}u)$ satisfies the integral condition (1.1), so the assertion follows with $b = \phi'(0^+)$. $\square$

Under the condition of the above lemma, if we assume further that the sequence $\{\phi_k\}$ is uniformly Lipschitz on each finite interval $[0, l]$, then $\{k^{-1}y_{k,t} : t \geq 0\}$ converges as $k \to \infty$ to the CBI-process $\{y_t : t \geq 0\}$ with $y_0 = x$; see e.g. Li (1998) for the proof of this convergence in the measure-valued setting. This describes a connection between the DBI- and the CBI-processes.

3. High density fluctuation limits

In this section, we show that the Ornstein-Uhlenbeck type process may arise as the high density fluctuation limit of a suitable sequence of DBI-processes. The arguments here have been inspired by those of Holley and Stroock (1978) and Li (1999). Let us consider a sequence of DBI-processes $\{y_{k,t} : t \geq 0\}$ with parameters $(\gamma_k, g_k, kb_k a)$, where $b_k = \gamma_k[1 - g_k'(1^-)] \geq 0$. Assume that $y_{k,0}$ is a Poisson random variable with parameter ka. By (2.9) one may check that $\mathbf{E}y_{k,t} = ka$ for all $t \geq 0$. Let $u_{k,t}(\lambda)$ be defined by (2.1) and (2.4) with (γ, g) replaced by (γ_k, g_k) and let $w_{k,t}(\lambda) = 1 - \exp\{-u_{k,t}(\lambda)\}$. Then $w_{k,t}(\lambda)$ is the solution to

$$w_{k,t}(\lambda) + \int_0^t \mathrm{e}^{-b_k(t-s)}\psi_k(w_{k,s}(\lambda))\mathrm{d}s = \mathrm{e}^{-b_k t}(1 - \mathrm{e}^{-\lambda}), \qquad t \geq 0, \lambda \geq 0, \tag{3.1}$$

where

$$\psi_k(z) = \gamma_k[g_k(1-z) - (1-z)] - b_k z, \qquad 0 \leq z \leq 1.$$

Lemma 3.1. *Let $N_k = \{-ka, -ka+1, \cdots\}$. Define the N_k-valued process $\{z_{k,t} : t \geq 0\}$ by $z_{k,t} = y_{k,t} - ka$. Then we have*

$$\begin{aligned} &\mathbf{E}\exp\{-\lambda z_{k,t}\} \\ =&\exp\left\{ka(\lambda - 1 + \mathrm{e}^{-\lambda}) + ka\int_0^t \psi_k(w_{k,s}(\lambda))\mathrm{d}s\right\}. \qquad t \geq 0, \lambda \geq 0. \end{aligned} \tag{3.2}$$

Furthermore, $\{z_{k,t} : t \geq 0\}$ is a Markov process with transition function $q_{k,t}(x, \mathrm{d}y)$ determined by

$$\begin{aligned} &\int_{N_k} \mathrm{e}^{-\lambda y} q_{k,t}(x, \mathrm{d}y) \\ =&\exp\left\{ - xu_{k,t}(\lambda) + a_{k,t}(\lambda) + ka\int_0^t \psi_k(w_{k,s}(\lambda))\mathrm{d}s\right\}, \qquad t \geq 0, \lambda \geq 0. \end{aligned} \tag{3.3}$$

where

$$a_{k,t}(\lambda) = ka(\lambda - 1 + \mathrm{e}^{-\lambda}) - ka(u_{k,t}(\lambda) - w_{k,t}(\lambda)).$$

Proof. We first observe that, by (3.1),

$$\begin{aligned} &b_k\int_0^t w_{k,s}(\lambda)\mathrm{d}s \\ =&b_k\int_0^t (1 - \mathrm{e}^{-\lambda})\mathrm{e}^{-b_k s}\mathrm{d}s - b_k\int_0^t \mathrm{d}s\int_0^s \mathrm{e}^{-b_k(s-u)}\psi_k(w_{k,u}(\lambda))\mathrm{d}u \\ =&(1 - \mathrm{e}^{-b_k t})(1 - \mathrm{e}^{-\lambda}) - \int_0^t (1 - \mathrm{e}^{-b_k(t-u)})\psi_k(w_{k,u}(\lambda))\mathrm{d}u \\ =&(1 - \mathrm{e}^{-\lambda}) - w_{k,t}(\lambda) - \int_0^t \psi_k(w_{k,u}(\lambda))\mathrm{d}u. \end{aligned} \tag{3.4}$$

By the definition of $z_{k,t}$ we have

$$\mathbf{E}\exp\{-\lambda z_{k,t}\} = \exp\bigg\{ka\lambda - kaw_{k,t}(\lambda) - kab_k\int_0^t w_{k,s}(\lambda)\mathrm{d}s\bigg\}.$$

Then (3.2) follows from (3.4). On the other hand, from (2.5) we see that $\{z_{k,t} : t \geq 0\}$ is a Markov process with transition function $q_{k,t}(x, \mathrm{d}y)$ such that

$$\int_{N_k} \mathrm{e}^{-\lambda y} q_{k,t}(x, \mathrm{d}y) = \exp\bigg\{ka\lambda - (x+a)u_{k,t}(\lambda) - kab_k\int_0^t w_{k,s}(\lambda)\mathrm{d}s\bigg\},$$

which yields (3.3) by (3.4). $\Box$

Let $N^{(k)} = \{(-ka)/\sqrt{k}, (-ka+1)/\sqrt{k}, \cdots\}$. What we are really interested is the $N^{(k)}$-valued Markov process $\{z_t^{(k)} : t \geq 0\}$ defined by

$$z_t^{(k)} = (y_{k,t} - ka)/\sqrt{k}, \qquad t \geq 0. \tag{3.5}$$

To give a characterization of this process let

$$w_t^{(k)}(\lambda) = \sqrt{k}(1 - \exp\{-u_{k,t}(\lambda/\sqrt{k})\}), \qquad t \geq 0, \lambda \geq 0. \tag{3.6}$$

Then we have

$$w_t^{(k)}(\lambda) + \frac{1}{\sqrt{k}}\int_0^t \mathrm{e}^{-b_k(t-s)}\varphi_k(w_s^{(k)}(\lambda))\mathrm{d}s = \sqrt{k}\mathrm{e}^{-b_k t}(1 - \mathrm{e}^{-\lambda/\sqrt{k}}), \qquad t \geq 0, \tag{3.7}$$

where

$$\varphi_k(z) = k\{\gamma_k[g_k(1 - z/\sqrt{k}) - (1 - z/\sqrt{k})] - b_k z/\sqrt{k}\}, \qquad 0 \leq z \leq \sqrt{k}. \tag{3.8}$$

By Lemma 3.1 we see that

$$\begin{aligned}&\mathbf{E}\exp\{-\lambda z_t^{(k)}\}\\ &= \exp\bigg\{ka(\lambda/\sqrt{k} - 1 + \mathrm{e}^{-\lambda/\sqrt{k}}) + a\int_0^t \varphi_k(w_s^{(k)}(\lambda))\mathrm{d}s\bigg\}.\end{aligned} \tag{3.9}$$

and $\{z_t^{(k)} : t \geq 0\}$ is a Markov process with transition function $q_t^{(k)}(x, \mathrm{d}y)$ determined by

$$\begin{aligned}&\int_{N^{(k)}} \mathrm{e}^{-\lambda y} q_t^{(k)}(x, \mathrm{d}y)\\ &= \exp\bigg\{ - x\sqrt{k}u_{k,t}(\lambda/\sqrt{k}) + a_t^{(k)}(\lambda) + a\int_0^t \varphi_k(w_s^{(k)}(\lambda))\mathrm{d}s\bigg\}.\end{aligned} \tag{3.10}$$

where

$$a_t^{(k)}(\lambda) = ka(\lambda/\sqrt{k} - 1 + \mathrm{e}^{-\lambda/\sqrt{k}}) - ka(u_{k,t}(\lambda/\sqrt{k}) - w_{k,t}(\lambda/\sqrt{k})).$$

Lemma 3.2. *Assume that $\varphi(z) = \lim_{k\to\infty} \varphi_k(z)$ uniformly on $[0, l]$ for each finite $l \geq 0$ and $\varphi'(0^+) = 0$, then the limit function has the representation* (1.2).

Proof. Let $\beta_k = 1 - g_k'(1^-)$ and $f_k(s) = g_k(s) + \beta_k s - \beta_k$. Then f_k is a critical generating function and

$$\varphi_k(z) = k\gamma_k[f_k(1 - z/\sqrt{k}) - (1 - z/\sqrt{k})], \qquad 0 \leq z \leq \sqrt{k}.$$

By applying Lemma 2.1 to the sequence $\{\varphi_{k^2}\}$ and using the assumption $\varphi'(0^+) = 0$ we see that φ has the representation (1.2). $\square$

Theorem 3.1. *Under the conditions of Lemma 3.2, assume further that $b = \lim_{k\to\infty} b_k$. Then $\{z_t^{(k)} : t \geq 0\}$ converges weakly in $D([0,\infty), R)$ to an Ornstein-Uhlenbeck type process$\{z_t : t \geq 0\}$ such that*

$$\mathbf{E}\exp\{-\lambda z_t\} = \exp\left\{a\lambda^2 + a\int_0^t \varphi(\mathrm{e}^{-bs})\mathrm{d}s\right\}, \qquad t \geq 0, \lambda \geq 0, \tag{3.11}$$

and the transition semigroup $\pi_t(x, \mathrm{d}y)$ of $\{z_t : t \geq 0\}$ is given by

$$\begin{aligned}&\int_{I\!R} \mathrm{e}^{-\lambda y}\pi_t(x, \mathrm{d}y)\\ &= \exp\left\{ - x\mathrm{e}^{-bt}\lambda + a\int_0^t [\varphi(\mathrm{e}^{-bs}\lambda) + b\mathrm{e}^{-2bt}\lambda^2]\mathrm{d}s\right\}, \qquad t \geq 0, \lambda \geq 0.\end{aligned} \tag{3.12}$$

Proof. By (3.6) and (3.7) it is easy to see that

$$\lim_{k\to\infty} w_t^{(k)}(\lambda) = \lim_{k\to\infty} \sqrt{k}u_{k,t}(\lambda/\sqrt{k}) = \mathrm{e}^{-bt}\lambda.$$

It follows that

$$\lim_{k\to\infty} a_t^{(k)}(\lambda) = (1 - \mathrm{e}^{-2bt})\lambda^2/2.$$

Then the desired convergence follows by an application of the result of Ethier and Kurtz (1986; p172). $\square$

We conclude this section by observing that, for any $b \geq 0$ and any φ given by (1.2), we can always choose $\{\gamma_k\}$ and $\{g_k\}$ as described above so that $b = \lim_{k\to\infty} \gamma_k[1 - g_k'(1^-)]$ and the sequence φ_k defined by (3.8) converges uniformly on each finite interval $[0, l]$ to φ. In order to do so, take any

$$\gamma_k > b + 2c + \frac{1}{\sqrt{k}}\int_0^\infty \left(\mathrm{e}^{-\sqrt{k}u} - 1 + \sqrt{k}u\right) m(\mathrm{d}u),$$

and define

$$\begin{aligned} g_k(s) = & s + b(1-s)/\gamma_k + c(1-s)^2/\gamma_k \\ & + \frac{1}{k\gamma_k}\int_0^\infty \Big(\exp\{-\sqrt{k}(1-s)u\} - 1 + \sqrt{k}(1-s)u\Big)\, m(\mathrm{d}u). \end{aligned}$$

Then one may check that $\{g_k\}$ is a sequence of probability generating functions, $b = \gamma_k[1 - g_k'(1^-)]$ and

$$k\{\gamma_k[g_k(1-z/\sqrt{k}) - (1-z/\sqrt{k})] - bz/\sqrt{k}\} = \varphi(z), \qquad 0 \le z \le \sqrt{k},$$

providing more than we wanted.

4. Low density fluctuation limits

In this section, we show that the Ornstein-Uhlenbeck type process may arise as the small branching low density fluctuation limit of the CBI-process. This kind of fluctuation limits of branching models have been considered by Gorostiza (1996) and Li (1999). Let $a \ge 0$ and $b \ge 0$ and φ be given by (1.2). For any integer $k \ge 1$ let $\varphi_k(z) = \varphi(z/k)$. Clearly, we have $\varphi_k(z) \to 0$ as $k \to \infty$. Let $\{y_{k,t} : t \ge 0\}$ be a continuous state branching process with immigration with parameters (b, φ_k, ba) and let $z_{k,t} = k(y_{k,t} - a)$. By similar arguments as in the last section one may check that $\{z_{k,t} : t \ge 0\}$ is a Markov process on $[-ka, \infty)$ with transition function $\tau_{k,t}(x, \mathrm{d}y)$ determined by

$$\int_{-ka}^\infty \mathrm{e}^{-\lambda y}\tau_{k,t}(x,\mathrm{d}y) = \exp\bigg\{ - xv_{k,t}(k\lambda)/k + a\int_0^t \varphi(v_{k,s}(k\lambda)/k)\mathrm{d}s\bigg\}, \tag{4.1}$$

where $v_{k,t}(k\lambda)/k$ satisfies

$$v_{k,t}(k\lambda)/k + k^{-1}\int_0^t \mathrm{e}^{-b(t-s)}\varphi(v_{k,s}(k\lambda)/k)\mathrm{d}s = \mathrm{e}^{-bt}\lambda, \qquad t \ge 0, \lambda \ge 0.$$

It is clear that $v_{k,t}(k\lambda)/k \le \mathrm{e}^{-bt}\lambda$ and $v_{k,t}(k\lambda)/k \to \mathrm{e}^{-bt}\lambda$ as $k \to \infty$. By the result of Ethier and Kurtz (1986; p172) we get

Theorem 4.1. *Suppose that $z_{k,0} \to z_0$ in distribution as $k \to \infty$. Then $\{z_{k,t} : t \ge 0\}$ converges weakly in $D([0,\infty), R)$ to an Ornstein-Uhlenbeck type process $\{z_t : t \ge 0\}$ with transition function $\pi_t(x, \mathrm{d}y)$ given by*

$$\int_{I\!R} \mathrm{e}^{-\lambda y}\pi_t(x,\mathrm{d}y) = \exp\bigg\{ - x\mathrm{e}^{-bt}\lambda + a\int_0^t \varphi(\mathrm{e}^{-bs}\lambda)\mathrm{d}s\bigg\}, \qquad \lambda \ge 0, x \in R.$$

Acknowledgment. I would like to thank the referee for bringing to my attention a number of references on OU-type processes.

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