Published in: Stochastic Models (Ottawa, Ontario, 1998), CMS Conference Proceedings, Series **26**, 159–171, Edited by Gorostiza, L.G. and Ivanoff, B.G., AMS, Providence, RI.

High-Density Fluctuations of Immigration Branching Particle Systems

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Dedicated to Professor Donald A. Dawson

Abstract. We obtain a class of generalized Ornstein-Uhlenbeck processes as high–density fluctuation limits of branching particle systems with immigration. We consider in particular the stationary case.

2000 Mathematics Subject Classification. Primary 60J80; Secondary 60G20, 60G10

Keywords and phrases. immigration particle system; entrance law; Ornstein-Uhlenbeck process; Langevin equation; fluctuation limit; stationary distribution

1 Introduction

Fluctuation limits of particle systems (with or without branching) have been studied extensively. Usually they lead to generalized Ornstein-Uhlenbeck processes; see e.g. Bojdecki and Gorostiza [1, 2], Dawson et al [5], Dawson and Gorostiza [6], Gorostiza [10], Holley and Stroock [12], and Itô [14]. Since the branching particle systems can be unstable, non-stationary scalings are employed in studying their fluctuation limits, which yield time-inhomogeneous Ornstein-Uhlenbeck processes. Fluctuation limits of measure-valued branching processes with immigration were studied in Gorostiza and Li [11], and Li [16], which gave time-homogeneous Ornstein-Uhlenbeck processes. The measure-valued processes considered in [11, 16] are superprocess-type limits of a class of branching particle systems with immigration. In this paper we consider high-density fluctuation limits for the branching particle systems with immigration. The Ornstein-Uhlenbeck processes obtained here are different from the ones in [11, 16], and we shall see that the superprocess-type limit and the high-density fluctuation limit are interchangeable. We will consider in particular the stationary case.

Let us recall some basic facts on the branching particle systems. We refer the reader to Dawson [4] for the necessary background. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. We denote by $C(\mathbb{R}^d)$ the set of bounded continuous functions on $\mathbb{R}^d$, and $C_0(\mathbb{R}^d)$ the set of functions in $C(\mathbb{R}^d)$ vanishing at infinity. Suppose that $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \xi_t, \mathbf{P}_x)$ is a diffusion process with semigroup $(P_t)_{t\geq 0}$ generated by a differential operator A. Throughout the paper we fix a strictly positive reference function $\rho \in \mathcal{D}(A)$ with $A\rho \in C_\rho(\mathbb{R}^d)$, where $C_\rho(\mathbb{R}^d)$ denotes the set of functions $f \in C(\mathbb{R}^d)$ satisfying $|f| \leq \text{const} \cdot \rho$. We assume further that $\rho^{-1}g$ is bounded for every rapidly decreasing function $g \in C(\mathbb{R}^d)$. The subsets of non-negative elements of the above function spaces are indicated by the superscript '+', e.g. $C_\rho(\mathbb{R}^d)^+$. Let $M_\rho(\mathbb{R}^d)$ be the space of

σ-finite Borel measures μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu(f) := \int f \mathrm{d}\mu < \infty$ for all $f \in C_\rho(\mathbb{R}^d)^+$. We equip $M_\rho(\mathbb{R}^d)$ with the topology defined by $\mu_k \to \mu$ if and only if $\mu_k(f) \to \mu(f)$ for all $f \in C_\rho(\mathbb{R}^d)$. Let $N_\rho(\mathbb{R}^d)$ be the subspace of $M_\rho(\mathbb{R}^d)$ consisting of integer-valued measures.

Let $\beta \in C(\mathbb{R}^d)^+$ and let $g(x,z)$ be a continuous function of $(x,z) \in \mathbb{R}^d \times [0,1]$. Suppose that for each fixed $x \in \mathbb{R}^d$, $g(x,\cdot)$ coincides on $[0,1]$ with the probability generating function of a critical branching law. Throughout this paper we assume that $c(x) := \beta(x) g''(x, 1^-)$ is a bounded continuous function on $\mathbb{R}^d$. For any $f \in C_\rho(\mathbb{R}^d)$ the evolution equation

$$\begin{aligned}\exp\{-u_t(x)\} &= \mathbf{P}_x \exp\{-f(\xi_t)\} - \int_0^t \mathbf{P}_x \left[\beta(\xi_{t-s}) \exp\{-u_s(\xi_{t-s})\}\right] \mathrm{d}s \\ &\quad + \int_0^t \mathbf{P}_x \left[\beta(\xi_{t-s}) g(\xi_{t-s}, \exp\{-u_s(\xi_{t-s})\})\right] \mathrm{d}s, \quad t \geq 0,\end{aligned}$$

has a unique positive solution $u_t = U_t f \in C_\rho(\mathbb{R}^d)$. In the sequel we shall simply write the above equation as

$$\mathrm{e}^{-u_t} = P_t \mathrm{e}^{-f} - \int_0^t P_{t-s} \left[\beta \left(\mathrm{e}^{-u_s} - g(\mathrm{e}^{-u_s})\right)\right] \mathrm{d}s, \quad t \geq 0. \tag{1.1}$$

It is well-known that the formula

$$\int_{N_\rho(\mathbb{R}^d)} \mathrm{e}^{-\nu(f)} Q_t(\sigma, \mathrm{d}\nu) = \exp\{-\sigma(U_t f)\}, \quad f \in C_\rho(\mathbb{R}^d), \tag{1.2}$$

defines a transition semigroup $(Q_t)_{t\geq 0}$ on $N_\rho(\mathbb{R}^d)$. A Markov process X is called a *branching particle system* with parameters (ξ, β, g) if its transition probabilities are determined by (1.1) and (1.2).

For $f \in C_\rho(\mathbb{R}^d)$ let

$$J_t f(x) = 1 - \exp\{-U_t f(x)\}, \quad t \geq 0, x \in \mathbb{R}^d. \tag{1.3}$$

Then from (1) we have

$$J_t f(x) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \varphi(x, J_s f(y)) P_{t-s}(x, \mathrm{d}y) = P_t \left(1 - \mathrm{e}^{-f}\right)(x), \tag{1.4}$$

where

$$\varphi(x,z) = \beta(x)[g(x, 1-z) - (1-z)], \quad x \in \mathbb{R}^d, 0 \leq z \leq 1. \tag{1.5}$$

Note that $\varphi''(x, 0^+) = c(x)$ by our assumption.

2 Immigration particle systems

Suppose that X is a branching particle system with transition semigroup $(Q_t)_{t\geq 0}$ as described in the last section. Let $(N_t)_{t\geq 0}$ be a family of probability measures on $N_\rho(\mathbb{R}^d)$. We call $(N_t)_{t\geq 0}$ a *skew convolution semigroup* associated with X or $(Q_t)_{t\geq 0}$ provided that

$$N_{r+t} = (N_r Q_t) * N_t, \quad r, t \geq 0, \tag{2.6}$$

where '$*$' denotes the operation of convolution. The relation (2.6) is necessary and sufficient to ensure that

$$Q_t^N(\sigma,\cdot) := Q_t(\sigma,\cdot) * N_t, \quad t \geq 0,\ \sigma \in N_\rho(\mathbb{R}^d), \tag{2.7}$$

defines a transition semigroup $(Q_t^N)_{t\geq 0}$ on $N_\rho(\mathbb{R}^d)$. A Markov process Y is naturally called an *immigration particle system* associated with X if it has $(Q_t^N)_{t\geq 0}$ as transition semigroup. The intuitive meaning of the immigration particle system is clear from (2.7), that is, $Q_t(\sigma,\cdot)$ is the distribution of descendants of the people distributed at time zero as $\sigma \in N_\rho(\mathbb{R}^d)$, and N_t is the distribution of descendants of the people immigrating to $\mathbb{R}^d$ during the time interval $(0,t]$. We refer to Li [15] for the basic facts and characterizations about skew convolution semigroups and immigration particle systems. (In [15] only particle systems taking finite measure values were considered, but all the results there remain valid in the present context under obvious modifications.)

Now we consider a particular form of the immigration particle system. Let $\gamma \in M_\rho(\mathbb{R}^d)$ be a purely excessive measure for ξ. Then there is an entrance law $(\kappa_t)_{t>0}$ for ξ such that $\gamma = \int_0^\infty \kappa_t \mathrm{d}t$; see Dynkin [8]. For $t > 0$ and $f \in C_\rho(\mathbb{R}^d)$ we let

$$R_t(\kappa, f) = \kappa_t\left(1 - \mathrm{e}^{-f}\right) - \int_0^t \kappa_{t-s}\left(\varphi(J_s f)\right)\mathrm{d}s. \tag{2.8}$$

Let $N_\rho(\mathbb{R}^d)^\circ = N_\rho(\mathbb{R}^d) \setminus \{0\}$, where 0 denotes the null measure, and let $(Q_t^\circ)_{t\geq 0}$ be the restriction of $(Q_t)_{t\geq 0}$ to $N_\rho(\mathbb{R}^d)^\circ$. By Theorem 3.3 of [15] we see that

$$\int_{N_\rho(\mathbb{R}^d)^\circ} \mathrm{e}^{-\nu(f)} K_t(\mathrm{d}\nu) = \exp\{-R_t(\kappa, f)\}, \quad f \in C_\rho(\mathbb{R}^d),$$

defines an infinitely divisible probability entrance law $(K_t)_{t>0}$ for $(Q_t^\circ)_{t\geq 0}$. It follows from Theorem 3.2 of [15] that there is a finite entrance law $(H_t)_{t>0}$ such that

$$\int_{N_\rho(\mathbb{R}^d)^\circ} (1 - e^{\nu(f)}) H_t(d\nu) = R_t(\kappa, f),\ f \in C_\rho(\mathbb{R}^d).$$

Then using Theorem 3.1 of [15] we see that the formula

$$\int_{N_\rho(\mathbb{R}^d)} \mathrm{e}^{-\nu(f)} Q_t^{(\kappa)}(\sigma, \mathrm{d}\nu) = \exp\left\{-\sigma(U_t f) - \int_0^t R_s(\kappa, f)\mathrm{d}s\right\},\ f \in C_\rho(\mathbb{R}^d), \tag{2.9}$$

defines a transition semigroup $(Q_t^{(\kappa)})_{t\geq 0}$ on $N_\rho(\mathbb{R}^d)$, which is a special case of the one defined by (2.7). In the sequel, a Markov process Y will be called an *immigration particle system* with parameters (ξ, β, g, κ) if it has transition semigroup determined by (2.9).

A special excessive measure for ξ is given by $\gamma = \int_0^\infty \nu P_t \mathrm{d}t$ for some $\nu \in M_\rho(\mathbb{R}^d)$. In this case we have $R_t(\kappa, f) = \nu(J_t f)$ by (1.4) and (2.8), and the immigration is governed by a time-space Poisson random measure with intensity $\mathrm{d}s\nu(\mathrm{d}x)$; see e.g. Dawson and Ivanoff [7].

We will need the following lemmas.

Lemma 2.1 *For any $t \geq 0$, $\sigma \in N_\rho(\mathbb{R}^d)$ and $f \in C_\rho(\mathbb{R}^d)$ we have*

$$\int_{N_\rho(\mathbb{R}^d)} \nu(f) Q_t^{(\kappa)}(\sigma, \mathrm{d}\nu) = \sigma(P_t f) + \int_0^t \kappa_s(f)\mathrm{d}s. \tag{2.10}$$

Proof. From (1.1) and (2.8) we get

$$\frac{\partial}{\partial\theta}U_t(\theta f)|_{\theta=0} = P_t f, \quad \frac{\partial}{\partial\theta}R_t(\kappa,\theta f)|_{\theta=0} = \kappa_t(f).$$

Using these results and (2.9) we obtain (2.10). □

Let $\mathbf{Q}^{(\kappa)}_{(\mu)}$ denote the law of the immigration particle system $\{Y_t : t \geq 0\}$ given that Y_0 is a Poisson random measure with intensity $\mu \in M_\rho(\mathbb{R}^d)$. Then we have from (2.4)

$$\mathbf{Q}^{(\kappa)}_{(\mu)}\exp\{-Y_t(f)\} = \exp\bigg\{-\mu(J_t f) - \int_0^t R_s(\kappa, f)\mathrm{d}s\bigg\}, \quad f \in C_\rho(\mathbb{R}^d). \tag{2.11}$$

Lemma 2.2 *For $t \geq 0$ and $f \in C_\rho(\mathbb{R}^d)$ we have $\mathbf{Q}^{(\kappa)}_{(\gamma)}Y_t(f) = \gamma(f)$ and*

$$\mathbf{Q}^{(\kappa)}_{(\gamma)}\{Y_t(f)^2\} = \gamma(f)^2 + \gamma(f^2) + \int_0^t \gamma(c(P_s f)^2)\mathrm{d}s. \tag{2.12}$$

Proof. This is similar to the proof of Lemma 2.1. From (1.4) and (2.8) it follows that

$$\begin{aligned}
\frac{\partial}{\partial\theta}J_t(\theta f)|_{\theta=0} = P_t f, &\quad \frac{\partial}{\partial\theta}R_t(\kappa,\theta f)|_{\theta=0} = \kappa_t(f), \\
-\frac{\partial^2}{\partial\theta^2}J_t(\theta f)|_{\theta=0} &= P_t(f^2) + \int_0^t P_{t-u}(c(P_u f)^2)\mathrm{d}u, \\
-\frac{\partial^2}{\partial\theta^2}R_t(\kappa,\theta f)|_{\theta=0} &= \kappa_t(f^2) + \int_0^t \kappa_{t-u}(c(P_u f)^2)\mathrm{d}u.
\end{aligned}$$

Replacing f by θf in (2.11), differentiating with respect to θ at zero and using (2.8) we get

$$\mathbf{Q}^{(\kappa)}_{(\mu)}Y_t(f) = \mu(P_t f) + \int_0^t \kappa_s(f)\mathrm{d}s,$$

and

$$\begin{aligned}
\mathbf{Q}^{(\kappa)}_{(\mu)}\{Y_t(f)^2\} &= \bigg[\mu(P_t f) + \int_0^t \kappa_s(f)\mathrm{d}s\bigg]^2 + \mu(P_t(f^2)) + \int_0^t \mu P_{t-s}(c(P_s f)^2)\mathrm{d}s \\
&\quad + \int_0^t \bigg[\kappa_s(f^2) + \int_0^s \kappa_{s-u}(c(P_u f)^2)\mathrm{d}u\bigg]\mathrm{d}s.
\end{aligned}$$

Setting $\mu = \gamma$, the results are clear by the entrance law property $\kappa_s P_t = \kappa_{s+t}$ and the integral representation for γ. □

Let $\mathcal{S}(\mathbb{R}^d)$ be the space of infinitely differentiable, rapidly decreasing functions all of whose derivatives are also rapidly decreasing.

Lemma 2.3 *Suppose that $\{Y_t, \mathcal{F}_t : t \geq 0\}$ is a realization of the immigration particle system with parameters (ξ, β, g, κ). If $f \in \mathcal{S}(\mathbb{R}^d)$, then the limit $\kappa_{0+}(f) := \lim_{r\downarrow 0}\kappa_r(f)$ exists and*

$$M_t(f) := Y_t(f) - \int_0^t Y_s(Af)\mathrm{d}s - t\kappa_{0+}(f), \quad t \geq 0, \tag{2.13}$$

is a martingale. In particular $\{Y_t(f), t \geq 0\}$ has a right–continuous modification for any $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. Under the hypothesis we have $Af \in \mathcal{S}(\mathbb{R}^d)$ and $f = P_t f - \int_0^t P_s A f \mathrm{d}s$ for any $t \geq 0$. It follows immediately from the entrance law property that

$$\kappa_{0^+}(f) = \kappa_t(f) - \int_0^t \kappa_s(Af)\mathrm{d}s, \quad t \geq 0. \tag{2.14}$$

If $t \geq r \geq 0$, using (2.10) we get

$$\begin{aligned}&\mathbf{E}\bigg\{Y_t(f) - \int_0^t Y_s(Af)\mathrm{d}s - t\kappa_{0^+}(f)\bigg|\mathcal{F}_r\bigg\} \\ &= Y_r(P_{t-r}f) + \int_0^{t-r}\kappa_s(f)\mathrm{d}s - \int_0^r Y_s(Af)\mathrm{d}s - \int_r^t \mathbf{E}\{Y_s(Af)|\mathcal{F}_r\}\mathrm{d}s - t\kappa_{0^+}(f).\end{aligned}$$

By (2.10) and (2.14) it follows that

$$\begin{aligned}\int_r^t \mathbf{E}\{Y_s(Af)|\mathcal{F}_r\}\mathrm{d}s &= \int_r^t \bigg[Y_r(P_{s-r}Af) + \int_0^{s-r}\kappa_u(Af)\mathrm{d}u\bigg]\mathrm{d}s \\ &= Y_r(P_{t-r}f - f) + \int_r^t [\kappa_{s-r}(f) - \kappa_{0^+}(f)]\mathrm{d}s \\ &= Y_r(P_{t-r}f - f) + \int_0^{t-r}\kappa_s(f)\mathrm{d}s - (t-r)\kappa_{0^+}(f).\end{aligned}$$

Now it is clear that

$$\mathbf{E}\bigg\{Y_t(f) - \int_0^t Y_s(Af)\mathrm{d}s - t\kappa_{0^+}(f)\bigg|\mathcal{F}_r\bigg\} = Y_r(f) - \int_0^r Y_s(Af)\mathrm{d}s - r\kappa_{0^+}(f).$$

That is, (2.13) is a martingale. □

We conclude this section by observing that the measure-valued immigration processes considered in [16] arise as superprocess-type limits of the immigration particle systems. Let $\{Y_t(k) : t \geq 0\}$, $k = 1, 2, \dots$ be a sequence of immigration particle systems with parameters $(\xi, k\beta, g, k\kappa)$. Suppose that $Y_0(k)$ is a Poisson random measure with intensity $k\gamma \in M_\rho(\mathbb{R}^d)$. Since for any $l \geq 0$ we have

$$k^2\varphi(x, z/k) = k^2\beta(x)[g(x, 1 - z/k) - (1 - z/k)] \to c(x)z^2/2$$

uniformly on the set $\mathbb{R}^d \times [0, l]$ as $k \to \infty$, by a theorem in [15] the sequence $\{k^{-1}Y_t(k) : t \geq 0\}$ converges as $k \to \infty$ to a Markov process $\{Y_t^{(0)} : t \geq 0\}$ with $Y_0^{(0)} = \gamma$ and with transition semigroup $(Q_t^\kappa)_{t\geq 0}$ determined by

$$\int_{M_\rho(\mathbb{R}^d)} \mathrm{e}^{-\nu(f)}Q_t^\kappa(\mu, \mathrm{d}\nu) = \exp\bigg\{-\mu(V_t f) - \int_0^t S_u(\kappa, f)\mathrm{d}u\bigg\}, f \in C_\rho(\mathbb{R}^d)^+, \tag{2.15}$$

where $V_t f$ is the solution to

$$V_t f(x) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} c(y)V_s f(y)^2 P_{t-s}(x, \mathrm{d}y) = P_t f(x), \quad t \geq 0, x \in \mathbb{R}^d,$$

and $S_u(\kappa, f)$ is defined by

$$S_u(\kappa, f) = \kappa_u(f) - \frac{1}{2}\int_0^u \kappa_{u-s}(c(V_s f)^2)\mathrm{d}s, \quad u > 0, f \in C_\rho(\mathbb{R}^d)^+.$$

The process $\{Y_t^{(0)} : t \geq 0\}$ has a diffusion realization, which is called a *measure-valued immigration diffusion process*; see [16].

3 Fluctuation limits

In this section we consider the high density fluctuation limits of the immigration particle systems. Let γ and κ be given as in the last section and let $\{Y_t^{(k)} : t \geq 0\}, k = 1, 2, \dots$ be a sequence of immigration particle systems with corresponding parameters $(\xi, \beta, g, k\kappa)$. Suppose that $Y_0^{(k)}$ is a Poisson random measure with intensity $k\gamma$. We define the fluctuation process $\{Z_t^{(k)} : t \geq 0\}$ by

$$Z_t^{(k)} = \frac{1}{\sqrt{k}}[Y_t^{(k)} - k\gamma], \quad t \geq 0. \tag{3.16}$$

Then $\{Z_t^{(k)} : t \geq 0\}$ is a Markov process taking signed-measure values from the space $N_k(\mathbb{R}^d) := \{\mu/\sqrt{k} - \sqrt{k}\gamma : \mu \in N_\rho(\mathbb{R}^d)\}$.

Lemma 3.1 *The Markov process $\{Z_t^{(k)} : t \geq 0\}$ has transition semigroup $(R_t^{(k)})_{t\geq 0}$ which is determined by*

$$\begin{aligned}\int_{N_k(\mathbb{R}^d)} & \mathrm{e}^{-\nu(f)} R_t^{(k)}(\mu, \mathrm{d}\nu) = \\ &= \exp\bigg\{ - \mu(U_t^{(k)} f) + A_t^{(k)}(f) + \int_0^t k\gamma(\varphi(J_s(f/\sqrt{k})))\mathrm{d}s\bigg\},\end{aligned} \tag{3.17}$$

where $U_t^{(k)} f = \sqrt{k} U_t(f/\sqrt{k})$ and

$$A_t^{(k)}(f) = k\gamma(f/\sqrt{k} - 1 + \mathrm{e}^{-f/\sqrt{k}}) - k\gamma(U_t(f/\sqrt{k}) - J_t(f/\sqrt{k})). \tag{3.18}$$

Proof. Let us compute the conditional Laplace functional of the process $\{Z_t^{(k)} : t \geq 0\}$. Take $t \geq 0$ and $r \geq 0$. Using the Markov property of $\{Y_t^{(k)} : t \geq 0\}$ and (9) we have

$$\begin{aligned} &\mathbf{E}\left[\exp\{-Z_{r+t}^{(k)}(f)\}|Z_s^{(k)} : s \leq r\right] \\ &= \exp\left\{\sqrt{k}\gamma(f)\right\} \mathbf{E}\left[\exp\{-Y_{r+t}^{(k)}(f/\sqrt{k})\}|Y_s^{(k)} : s \leq r\right] \\ &= \exp\left\{\sqrt{k}\gamma(f)\right\} \exp\bigg\{ - Y_r^{(k)}(U_t(f/\sqrt{k})) - \int_0^t R_s(k\kappa, f/\sqrt{k})\mathrm{d}s\bigg\} \\ &= \exp\bigg\{\sqrt{k}\gamma(f) - Z_r^{(k)}(\sqrt{k}U_t(f/\sqrt{k})) - \gamma(kU_t(f/\sqrt{k})) - \int_0^t R_s(k\kappa, f/\sqrt{k})\mathrm{d}s\bigg\}. \end{aligned}$$

That is, $\{Z_t^{(k)} : t \geq 0\}$ is a Markov process with transition semigroup $(R_t^{(k)})_{t\geq 0}$ given by

$$\begin{aligned}\int_{N_k(\mathbb{R}^d)} \mathrm{e}^{-\nu(f)} R_t^{(k)}(\mu, \mathrm{d}\nu) &= \exp\bigg\{ - \mu(U_t^{(k)} f) + \gamma(\sqrt{k}f) - \gamma(kU_t(f/\sqrt{k})) \\ &\qquad - \int_0^t R_s(k\kappa, f/\sqrt{k})\mathrm{d}s\bigg\}.\end{aligned} \tag{3.19}$$

In addition to $A_t^{(k)}(f)$ given by (3.18), let

$$B_t^{(k)}(f) = k\gamma(1 - \mathrm{e}^{-f/\sqrt{k}}) - k\gamma(J_t(f/\sqrt{k})) - \int_0^t R_s(k\kappa, f/\sqrt{k})\mathrm{d}s,$$

where J_t is defined by (1.3). We may rewrite (3.19) as

$$\int_{N_k(\mathbb{R}^d)} \mathrm{e}^{-\nu(f)} R_t^{(k)}(\mu, \mathrm{d}\nu) = \exp\Big\{ -\mu(U_t^{(k)} f) + A_t^{(k)}(f) + B_t^{(k)}(f)\Big\}. \tag{3.20}$$

Using the equation (1.4) we have

$$\begin{aligned}
&\int_t^\infty k\kappa_r(1-\mathrm{e}^{-f/\sqrt{k}})\mathrm{d}r - k\gamma(J_t(f/\sqrt{k})) \\
&\quad = \int_0^\infty k\kappa_r(P_t(1-\mathrm{e}^{-f/\sqrt{k}}))\mathrm{d}r - \int_0^\infty k\kappa_r(J_t(f/\sqrt{k}))\mathrm{d}r \\
&\quad = \int_0^\infty \mathrm{d}r \int_0^t \kappa_{r+t-s} k(\varphi(J_s(f/\sqrt{k})))\mathrm{d}s \\
&\quad = \int_0^t \mathrm{d}s \int_{t-s}^\infty k\kappa_u(\varphi(J_s(f/\sqrt{k})))\mathrm{d}u.
\end{aligned}$$

On the other hand, by (2.8) it follows that

$$\begin{aligned}
\int_0^t k\kappa_r(1-\mathrm{e}^{-f/\sqrt{k}})\mathrm{d}r - \int_0^t R_r(k\kappa, f/\sqrt{k})\mathrm{d}r &= \int_0^t \mathrm{d}r \int_0^r k\kappa_{r-s}(\varphi(J_s(f/\sqrt{k})))\mathrm{d}s \\
&= \int_0^t \mathrm{d}s \int_0^{t-s} k\kappa_u(\varphi(J_s(f/\sqrt{k})))\mathrm{d}u.
\end{aligned}$$

Summing the two last equations we get

$$B_t^{(k)}(f) = \int_0^t k\gamma(\varphi(J_s(f/\sqrt{k})))\mathrm{d}s. \tag{3.21}$$

Then (3.17) follows from (3.21) and (3.20). $\square$

Lemma 3.2 *The one-dimensional distributions of the process $\{Z_t^{(k)} : t \geq 0\}$ are determined by*

$$\begin{aligned}
&\mathbf{E}\exp\{-Z_t^{(k)}(f)\} = \\
&\quad = \exp\left\{k\gamma(f/\sqrt{k} - 1 + \mathrm{e}^{-f/\sqrt{k}}) + \int_0^t k\gamma(\varphi(J_s(f/\sqrt{k})))\mathrm{d}s\right\}.
\end{aligned} \tag{3.22}$$

Proof. By (2.11) and the present assumption we have

$$\mathbf{E}\exp\{-Y_t^{(k)}(f)\} = \exp\Big\{ -k\gamma(J_t f) + \int_0^t R_s(k\kappa, f)\mathrm{d}s\Big\}.$$

Then for (3.16) we get

$$\mathbf{E}\exp\{-Z_t^{(k)}(f)\} = \exp\Big\{ -k\gamma(J_t(f/\sqrt{k}) - f/\sqrt{k}) - \int_0^t R_s(k\kappa, f/\sqrt{k})\mathrm{d}s\Big\}.$$

Using the notation in the proof of the last lemma we have

$$\mathbf{E}\exp\{-Z_t^{(k)}(f)\} = \exp\Big\{k\gamma(f/\sqrt{k} - 1 + \mathrm{e}^{-f/\sqrt{k}}) + B_t^{(k)}(f)\Big\}.$$

Then (3.22) follows by (3.21). $\square$

Let $\mathcal{S}'(\mathbb{R}^d)$ be the dual space of $\mathcal{S}(\mathbb{R}^d)$ and write $\langle\ ,\ \rangle$ for the duality on $(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$. We may also regard $\{Z_t^{(k)} : t \geq 0\}$ as a process in $\mathcal{S}'(\mathbb{R}^d)$.

Theorem 3.3 *The finite dimensional distributions of* $\{Z_t^{(k)} : t \geq 0\}$ *converge as* $k \to \infty$ *to those of a Markov process* $\{Z_t : t \geq 0\}$ *with state space* $\mathcal{S}'(\mathbb{R}^d)$*. The transition semigroup* $(R_t^{(\kappa)})_{t\geq 0}$ *of* $\{Z_t : t \geq 0\}$ *is given by*

$$\begin{aligned}&\int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{-\langle \nu, f\rangle} R_t^{(\kappa)}(\mu, \mathrm{d}\nu) \\ &= \exp\bigg\{ -\langle \mu, P_t f\rangle + \frac{1}{2}\gamma(f^2 - (P_t f)^2) + \frac{1}{2}\int_0^t \gamma(c(P_s f)^2)\mathrm{d}s\bigg\},\end{aligned} \tag{3.23}$$

and its one dimensional distribution is determined by

$$\mathbf{E}\exp\{-\langle Z_t, f\rangle\} = \exp\bigg\{\frac{1}{2}\gamma(f^2) + \frac{1}{2}\int_0^t \gamma(c(P_s f)^2)\mathrm{d}s\bigg\}. \tag{3.24}$$

Proof. Take any bounded sequence $\{f_k\} \in \mathcal{S}(\mathbb{R}^d)$ such that $f_k \to f \in \mathcal{S}(\mathbb{R}^d)$. Using the equations (1.1) and (1.4), and criticality of g one can check that

$$\lim_{k\to\infty} \sqrt{k} U_t(f_k/\sqrt{k}) = \lim_{k\to\infty} \sqrt{k} J_t(f_k/\sqrt{k}) = P_t f. \tag{3.25}$$

By Taylor's expansion,

$$\begin{aligned}\lim_{k\to\infty} k[U_t(f_k/\sqrt{k}) - J_t(f_k/\sqrt{k})] &= \lim_{k\to\infty} k[U_t(f_k/\sqrt{k}) - 1 + \exp\{-U_t(f_k/\sqrt{k})\}] \\ &= \frac{1}{2}(P_t f)^2.\end{aligned}$$

Then by (3.18) it follows that

$$\lim_{k\to\infty} A_t^{(k)}(f_k) = A_t(f) := \frac{1}{2}\gamma(f^2) - \frac{1}{2}\gamma((P_t f)^2). \tag{3.26}$$

Since $\varphi''(x, 0^+) = c(x)$ by the assumption, using (3.26) and Taylor's expansion we get

$$\lim_{k\to\infty} B_t^{(k)}(f_k) = \lim_{k\to\infty} \int_0^t k\gamma(\varphi(J_s(f_k/\sqrt{k})))\mathrm{d}s = \frac{1}{2}\int_0^t \gamma(c(P_s f)^2)\mathrm{d}s. \tag{3.27}$$

By (3.22) the one-dimensional distributions of $\{Z_t^{(k)} : t \geq 0\}$ converge to those of $\{Z_t : t \geq 0\}$.

For $0 = t_0 \leq t_1 < \cdots < t_n$ and $f_1, \cdots, f_n \in \mathcal{S}(\mathbb{R}^d)$ let

$$h_j^{(k)} = f_j + U_{t_{j+1}-t_j}^{(k)}(f_{j+1} + \cdots + U_{t_n - t_{n-1}}^{(k)} f_n).$$

Using (3.17) inductively we get

$$\begin{aligned}\mathbf{E}\exp\bigg\{ -\sum_{j=1}^n \langle Z_{t_j}^{(k)}, f_j\rangle\bigg\} &= \exp\bigg\{k\gamma(h_1^{(k)}/\sqrt{k} - 1 + \mathrm{e}^{-h_1^{(k)}/\sqrt{k}}) \\ &+ \sum_{j=1}^n A_{t_j - t_{j-1}}^{(k)}(h_j^{(k)}) + \sum_{j=1}^n \int_0^{t_j - t_{j-1}} k\gamma(\varphi(J_s(h_j^{(k)}/\sqrt{k}))\mathrm{d}s\bigg\}.\end{aligned} \tag{3.28}$$

By (3.25) it is clear that

$$h_j^{(k)} \to h_j := f_j + P_{t_{j+1}-t_j}(f_{j+1} + \cdots + P_{t_n - t_{n-1}} f_n) \tag{3.29}$$

boundedly as $k \to \infty$. Applying (3.26), (3.27) and (3.29) to (3.28) we have

$$\begin{aligned} &\lim_{k\to\infty} \mathbf{E} \exp\left\{ -\sum_{j=1}^n Z_{t_j}^{(k)}(f_j)\right\} \\ &= \exp\left\{ \frac{1}{2}\gamma(f_1^2) + \sum_{j=1}^n A_{t_j - t_{j-1}}(h_j) + \frac{1}{2}\sum_{j=2}^n \int_0^{t_j - t_{j-1}} \gamma(c(P_s h_j)^2)\mathrm{d}s\right\}. \end{aligned}$$

As in Iscoe [13], we see that the finite-dimensional distributions of $\{Z_t^{(k)} : t \geq 0\}$ converge to those of the Markov process $\{Z_t : t \geq 0\}$. □

Observe that if $\{Z_t(k) : t \geq 0\}$ is given by the last theorem with the parameters (ξ, c, γ) replaced by $(\xi, kc, k\gamma)$, then $\{k^{-1}Z_t(k) : t \geq 0\}$ converges to a Markov process $\{Z_t^{(0)} : t \geq 0\}$ with $Z_0^{(0)} = 0$ and with semigroup $(R_t^\kappa)_{t\geq 0}$ given by

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{-\langle \nu, f\rangle} R_t^\kappa(\mu, \mathrm{d}\nu) = \exp\left\{ -\langle \mu, P_t f\rangle + \frac{1}{2}\int_0^t \gamma(c(P_s f)^2)\mathrm{d}s\right\}.$$

This together with the result in [16] shows that the superprocess–type limit and the fluctuation limit are interchangeable.

For any $f \in \mathcal{S}(\mathbb{R}^d)$ define $Qf \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle Qf, g\rangle = \gamma(cfg) - \gamma(fAg + gAf), \quad g \in \mathcal{S}(\mathbb{R}^d). \tag{3.30}$$

Then we have

Theorem 3.4 *The fluctuation limit process $\{Z_t : t \geq 0\}$ obtained in Theorem 3.3 has a continuous realization which solves the Langevin equation*

$$\begin{aligned} \mathrm{d}Z_t &= A^* Z_t \mathrm{d}t + \mathrm{d}W_t, \quad t \geq 0, \\ Z_0 &= \textit{white noise based on } \gamma, \end{aligned} \tag{3.31}$$

where A^* denotes the adjoint operator of A and $\{W_t : t \geq 0\}$ is an $\mathcal{S}'(\mathbb{R}^d)$-valued Wiener process with covariance functional

$$\mathbf{E}\{\langle W_r, f\rangle\langle W_t, g\rangle\} = (r \wedge t)\langle Qf, g\rangle, \quad r, t \geq 0,\ f, g \in \mathcal{S}(\mathbb{R}^d). \tag{3.32}$$

Proof. Observe that $\{Z_t : t \geq 0\}$ is an $\mathcal{S}'(\mathbb{R}^d)$-valued mean zero Gaussian process. Set $K(r, f; t, g) = \mathbf{E}\{\langle Z_r, f\rangle\langle Z_t, g\rangle\}$. By a standard argument one may check from (3.8) that

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \langle \nu, f\rangle R_t^{(\kappa)}(\mu, \mathrm{d}\nu) = \mu(P_t f),$$

and

$$K(t, f; t, g) = \gamma(fg) + \int_0^t \gamma(cP_s f P_s g)\mathrm{d}s.$$

It follows from these results and the Markov property that

$$K(r, f; t, g) = \gamma(fP_{t-r}g) + \int_0^r \gamma(cP_s f P_{t-r+s} g)\mathrm{d}s, \quad t \geq r \geq 0. \tag{3.33}$$

By (3.33) and the fact $\|f - P_{t-r}f\| \leq \|Af\|(t-r)$ (where $\| \quad \|$ denotes sup norm) one easily sees that

$$\begin{aligned}
\mathbf{E}\{|\langle Z_t, f\rangle - \langle Z_r, f\rangle|^2\} &= 2\gamma(f[f - P_{t-r}f]) + \int_r^t \gamma(c(P_s f)^2)\mathrm{d}s \\
&\quad + 2\int_0^r \gamma(cP_s f[P_s f - P_{t-r+s}f])\mathrm{d}s \\
&\leq 2\|Af\|\gamma(|f|)(t-r) + \mathrm{const}\cdot\|cf\|\gamma(\rho)(t-r) \\
&\quad + 2\|cAf\|(t-r)\int_0^r \gamma(P_s f)\mathrm{d}s
\end{aligned}$$

for $t \geq r \geq 0$. Then $\{Z_t : t \geq 0\}$ has a continuous realization; see e.g. Walsh [17, p. 274]. Observe that

$$\int_0^t \gamma(c[P_s f P_s A g + P_s g P_s A f])\mathrm{d}s = \gamma(cP_t f P_t g) - \gamma(cfg). \tag{3.34}$$

By (3.33) one checks that

$$\frac{\partial}{\partial t}K(t, f; t, g) = \gamma(cP_t f P_t g). \tag{3.35}$$

Using (3.33), (3.34) and (3.35) we get

$$\frac{\partial}{\partial t}K(t, f; t, g) - K(t, Af; t, g) - K(t, f; t, Ag) = \gamma(cfg) - \gamma(fAg + gAf).$$

By the results of [1, p. 234] (see also [2]) we conclude that $\{Z_t : t \geq 0\}$ satisfies the generalized Langevin equation (3.16) with $\{W_t : t \geq 0\}$ given by (3.17). □

Since it may happen that $c = 0$, (3.30) and (3.32) indicate that $\gamma(fAf) \leq 0$. To see that this is true observe that

$$\gamma(fAf) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\gamma((P_t f)^2)\Big|_{t=0}$$

and

$$\gamma((P_t f)^2) \leq \gamma(P_t(f^2)) \leq \gamma(f^2),$$

where the second inequality holds because γ is an excessive measure for $(P_t)_{t\geq 0}$.

Note that the Ornstein-Uhlenbeck process Z is different from the ones obtained in [11, 16] as small branching fluctuation limits, where the distribution of the driving process $\{W_t : t \geq 0\}$ does not involve the generator A.

4 Weak convergence

We already have the convergence of the finite-dimensional distributions. Since the limit process is continuous, the tightness and consequently the weak convergence of the sequence $\{Z_t^{(k)} : t \geq 0\}$ in the cadlag space $D([0,\infty), \mathcal{S}'(\mathbb{R}^d))$ can be obtained easily as follows.

Theorem 4.1 *The sequence $\{Z_t^{(k)} : t \geq 0\}$ converges weakly to the process $\{Z_t : t \geq 0\}$ in the space $D([0,\infty), \mathcal{S}'(\mathbb{R}^d))$.*

Proof. By a theorem in [9], due to the continuity of the limit and the martingale structure in Lemma 2.3 it suffices to show that

$$\sup_{k\geq 1} \mathbf{E} \sup_{0\leq s\leq t} \{\langle Z_s^{(k)}, f\rangle^2\} < \infty \tag{4.36}$$

for all $t > 0$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Let

$$N_t^{(k)}(f) := \langle Z_t^{(k)}, f\rangle - \int_0^t \langle Z_s^{(k)}, Af\rangle \mathrm{d}s, \quad t \geq 0. \tag{4.37}$$

Since $\gamma(f) = \int_0^\infty \kappa_s(f)\mathrm{d}s < \infty$, we have $\lim_{t\to\infty} \kappa_t(f) = 0$. Letting $t \to \infty$ in (2.14) gives that $\gamma(Af) = -\kappa_{0+}(f)$. Then (3.1) and (4.2) yield that

$$\begin{aligned}
\sqrt{k} N_t^{(k)}(f) &= Y_t^{(k)}(f) - k\gamma(f) - \int_0^t Y_s^{(k)}(Af)\mathrm{d}s + tk\gamma(Af), \\
&= Y_t^{(k)}(f) - k\gamma(f) - \int_0^t Y_s^{(k)}(Af)\mathrm{d}s - tk\kappa_{0+}(f).
\end{aligned}$$

By Lemma 2.3 we see that $\{N_t^{(k)}(f) : t \geq 0\}$ is a martingale. On the other hand, by Lemma 2.2 we get

$$\mathbf{E}\{\langle Z_t^{(k)}, f\rangle^2\} = \gamma(f^2) + \int_0^t \gamma(c(P_s f)^2)\mathrm{d}s, \quad t \geq 0, f \in C_\rho(\mathbb{R}^d). \tag{4.38}$$

By (4.2) and Doob's inequality we see that

$$\begin{aligned}
\mathbf{E} \sup_{0\leq s\leq t} \{\langle Z_s^{(k)}, f\rangle^2\} &\leq 2\mathbf{E} \sup_{0\leq s\leq t} \{N_s^{(k)}(f)^2\} + 2\mathbf{E}\left\{ \sup_{0\leq s\leq t} \left[\int_0^s \langle Z_u^{(k)}, Af\rangle \mathrm{d}u \right]^2 \right\} \\
&\leq 8\mathbf{E}\{N_t^{(k)}(f)^2\} + 2\mathbf{E}\left\{ \sup_{0\leq s\leq t} s \int_0^s \langle Z_u^{(k)}, Af\rangle^2 \mathrm{d}u \right\} \\
&\leq 16\mathbf{E}\{\langle Z_t^{(k)}, f\rangle^2\} + 16t\mathbf{E}\left\{ \int_0^t \langle Z_u^{(k)}, Af\rangle^2 \mathrm{d}u \right\} + 2t\mathbf{E}\left\{ \int_0^t \langle Z_u^{(k)}, Af\rangle^2 \mathrm{d}u \right\} \\
&\leq \mathrm{const} \cdot \mathbf{E}\{\langle Z_t^{(k)}, f\rangle^2\} + \mathrm{const} \cdot t\mathbf{E}\left\{ \int_0^t \langle Z_u^{(k)}, Af\rangle^2 \mathrm{d}u \right\}.
\end{aligned}$$

Then (4.1) follows from (4.3). □

Example. A typical example is where $A = \Delta - b$ is the generator of a killed Brownian motion with $b \in C(\mathbb{R}^d)^+$ bounded away from zero. In this case, we may let $\rho(x) = 1/(1 + |x|^p)$ for any $p > d$ and let $\gamma \in M_\rho(\mathbb{R}^d)$ be the Lebesgue measure.

As in [11] and [16] one may take a sequence $b_k \downarrow 0$ and replace A by $A - b_k$ in taking the fluctuation limit. By doing so one includes the situation where $\gamma \in M_\rho(\mathbb{R}^d)$ is a general excessive (not necessarily purely excessive) measure. In particular, one may include the case $A = \Delta$ and $\gamma =$ Lebesgue measure in the above example.

5 Stationary processes

We now give a brief discussion of the fluctuation limit for stationary particle systems. Let $(Q_t^{(\kappa)})_{t\geq 0}$ be the semigroup determined by (2.9). By the definition (2.8) it is easy to check that

$$\int_0^\infty R_t(\kappa, f)\mathrm{d}t = \gamma(1 - \mathrm{e}^{-f}) - \int_0^\infty \gamma(\varphi(J_s f))\mathrm{d}s.$$

It follows from (2.4) and the fact $U_t\rho \leq P_t\rho$ that if $\sigma(P_t\rho) \to 0$ as $t \to \infty$, then $Q_t^{(\kappa)}(\sigma, \cdot) \to Q_\infty^{(\kappa)}$ as $t \to \infty$, where $Q_\infty^{(\kappa)}$ is the stationary distribution of $(Q_t^{(\kappa)})_{t\geq 0}$ given by

$$\int_{M_\rho(\mathbb{R}^d)} \mathrm{e}^{-\nu(f)} Q_\infty^{(\kappa)}(\mathrm{d}\nu) = \exp\left\{ - \gamma(1 - \mathrm{e}^{-f}) + \int_0^\infty \gamma(\varphi(U_s f))\mathrm{d}s \right\}, \ f \in C_\rho(\mathbb{R}^d)^+.$$

On the other hand, if $\{Z_t : t \geq 0\}$ is the process obtained in Theorem 3.3, then from (3.8) the distribution of Z_t converges as $t \to \infty$ to $R_\infty^{(\kappa)}$ given by

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \mathrm{e}^{-\langle \nu, f\rangle} R_\infty^{(\kappa)}(\mathrm{d}\nu) = \exp\left\{ \frac{1}{2}\gamma(f^2) + \frac{1}{2}\int_0^\infty \gamma(c(P_s f)^2)\mathrm{d}s \right\}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

It follows that $R_\infty^{(\kappa)}$ is a stationary distribution of the semigroup $(R_t^{(\kappa)})_{t\geq 0}$ given by (3.23). Moreover, if $\langle \mu, P_t\rho\rangle \to 0$ as $t \to \infty$, then $R_t^{(\kappa)}(\mu, \cdot) \to R_\infty^{(\kappa)}$ as $t \to \infty$.

If we consider a sequence of stationary immigration processes $\{Y_t^{(k)} : t \geq 0\}$ with semigroup $(Q_t^{(k\kappa)})_{t\geq 0}$ and one-dimensional distribution $Q_\infty^{(k\kappa)}$, and if we take the fluctuation limit as in section 3, then we get a stationary $\mathcal{S}'(\mathbb{R}^d)$-valued Markov process with semigroup $(R_t^{(\kappa)})_{t\geq 0}$ and one-dimensional distribution $R_\infty^{(\kappa)}$. That is, the fluctuation limit and the long–time limit are interchangeable. We refer the reader to Bojdecki and Jakubowski [3] for discussions on invariant measures of generalized Ornstein-Uhlenbeck processes in conuclear spaces.

Acknowledgments. The first named author thanks CIMAT (Guanajuato, Mexico) for its hospitality, and acknowledges support of the Consejo Nacional de Ciencia y Tecnología (Mexico, Grant No. 27932-E). The second named author acknowledges supports of the National Natural Science Foundation (Grant No. 19361060) and the Mathematical Center of the State Education Commission of China.

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