```
> #A MAPLE program for computing $\widehat{\bigtriangleup}_1$
> G:={1,a,b,c,d,h}; # the symmetric group of degree 3, where a^3=1,
> b^2=1, c=ab, d=a^2, and h=a^2b
```

$$G := \{1, a, b, c, d, h\}$$

```
> v:=z->if z=1 then 1 elif z=a then a elif z=b then b elif z=c
then c
> elif z=d then d elif z=h then h elif z=-1 then 1 elif z=-a then
a elif
> z=-b then b elif z=-c then c elif z=-d then d elif z=-h then
h end if;
> # define the absolute values of the elements of G and -G
```

$v := z \rightarrow$
if $z = 1$ **then** 1
elif $z = a$ **then** a
elif $z = b$ **then** b
elif $z = c$ **then** c
elif $z = d$ **then** d
elif $z = h$ **then** h
elif $z = -1$ **then** 1
elif $z = -a$ **then** a
elif $z = -b$ **then** b
elif $z = -c$ **then** c
elif $z = -d$ **then** d
elif $z = -h$ **then** h
end if

```
> i:=z->if z=1 then 1 elif z=a then d elif z=b then b elif z=c
then c
> elif z=d then a elif z=h then h elif z=-1 then -1 elif z=-a then
-d
> elif z=-b then -b elif z=-c then -c elif z=-d then -a elif z=-h
then
> -h end if; # define the inverse on G and -G
```

$i := z \rightarrow$
if $z = 1$ **then** 1
elif $z = a$ **then** d
elif $z = b$ **then** b
elif $z = c$ **then** c
elif $z = d$ **then** a
elif $z = h$ **then** h
elif $z = -1$ **then** -1
elif $z = -a$ **then** $-d$
elif $z = -b$ **then** $-b$
elif $z = -c$ **then** $-c$
elif $z = -d$ **then** $-a$
elif $z = -h$ **then** $-h$
end if

```
>  m:=(x,y)->piecewise(x=1 and y=1,1,x=1 and y=a,a,x=1 and y=b,b,x=1
and
>  y=c,c,x=1 and y=d,d,x=1 and y=h,h,x=a and y=1,a,x=a and y=a,d,x=a
and
>  y=b,c,x=a and y=c,h,x=a and y=d,1,x=a and y=h,b,x=b and y=1,b,x=b
and
>  y=a,h,x=b and y=b,1,x=b and y=c,d,x=b and y=d,c,x=b and y=h,a,x=c
and
>  y=1,c,x=c and y=a,b,x=c and y=b,a,x=c and y=c,1,x=c and y=d,h,x=c
and
>  y=h,d,x=d and y=1,d,x=d and y=a,1,x=d and y=b,h,x=d and y=c,b,x=d
and
>  y=d,a,x=d and y=h,c,x=h and y=1,h,x=h and y=a,c,x=h and y=b,d,x=h
and
>  y=c,a,x=h and y=d,b,x=h and y=h,1); # define the multiplication
on
>  G
```

$$m := (x, y) \to \text{piecewise}(x = 1 \textbf{ and } y = 1, 1, x = 1 \textbf{ and } y = a, a, x = 1 \textbf{ and } y = b, b, x = 1 \textbf{ and } y = c, c, x = 1 \textbf{ and } y = d, d, x = 1 \textbf{ and } y = h, h, x = a \textbf{ and } y = 1, a, x = a \textbf{ and } y = a, d, x = a \textbf{ and } y = b, c, x = a \textbf{ and } y = c, h, x = a \textbf{ and } y = d, 1, x = a \textbf{ and } y = h, b, x = b \textbf{ and } y = 1, b, x = b \textbf{ and } y = a, h, x = b \textbf{ and } y = b, 1, x = b \textbf{ and } y = c, d, x = b \textbf{ and } y = d, c, x = b \textbf{ and } y = h, a, x = c \textbf{ and } y = 1, c, x = c \textbf{ and } y = a, b, x = c \textbf{ and } y = b, a, x = c \textbf{ and } y = c, 1, x = c \textbf{ and } y = d, h, x = c \textbf{ and } y = h, d, x = d \textbf{ and } y = 1, d, x = d \textbf{ and } y = a, 1, x = d \textbf{ and } y = b, h, x = d \textbf{ and } y = c, b, x = d \textbf{ and } y = d, a, x = d \textbf{ and } y = h, c, x = h \textbf{ and } y = 1, h, x = h \textbf{ and } y = a, c, x = h \textbf{ and } y = b, d, x = h \textbf{ and } y = c, a, x = h \textbf{ and } y = d, b, x = h \textbf{ and } y = h, 1)$$

```
>  mu:=(x,y)->if member(x,G) and member(y,G) then m(x,y) elif
>  member(-x,G) and member(-y,G) then m(v(x),v(y)) else -m(v(x),v(y))
end
>  if; # define the multiplication on G and -G
```

$$\begin{aligned}&\mu := (x, y) \to\\ &\textbf{if}\,\text{member}(x, G) \textbf{ and } \text{member}(y, G)\,\textbf{then}\,\text{m}(x, y)\\ &\textbf{elif}\,\text{member}(-x, G) \textbf{ and } \text{member}(-y, G)\,\textbf{then}\,\text{m}(\text{v}(x), \text{v}(y))\\ &\textbf{else} - \text{m}(\text{v}(x), \text{v}(y))\\ &\textbf{end if}\end{aligned}$$

```
>  theta1:=x_1->if x_1=1 then 0 elif x_1=a then -e_2-beta elif x_1=b
>  then 0 elif x_1=c then -e_2-beta elif x_1=d then e_2-beta elif
x_1=h
>  then e_2-beta end if; # define a map from G to
>  {0,-e_2-beta,e_2+beta,e_2-beta,-e_2+beta}
```

$\theta 1 := x\_1 \rightarrow$
if $x\_1 = 1$ **then** 0
elif $x\_1 = a$ **then** $-e\_2 - \beta$
elif $x\_1 = b$ **then** 0
elif $x\_1 = c$ **then** $-e\_2 - \beta$
elif $x\_1 = d$ **then** $e\_2 - \beta$
elif $x\_1 = h$ **then** $e\_2 - \beta$
end if

```
>  theta2:=(x_1,x_2)->if x_1=d and theta1(x_2)=e_2-beta then e_2
elif
>  x_1=c and theta1(x_2)=e_2-beta then -e_2 elif x_1=a and
>  theta1(x_2)=-e_2-beta then -e_2 elif x_1=h and theta1(x_2)=-e_2-beta
>  then e_2   else 0 end if; # define a map from G^2 to {0,e_2,-e_2}
```

$\theta 2 := (x\_1, x\_2) \rightarrow$
if $x\_1 = d$ **and** $\theta 1(x\_2) = e\_2 - \beta$ **then** e_2
elif $x\_1 = c$ **and** $\theta 1(x\_2) = e\_2 - \beta$ **then** $-e\_2$
elif $x\_1 = a$ **and** $\theta 1(x\_2) = -e\_2 - \beta$ **then** $-e\_2$
elif $x\_1 = h$ **and** $\theta 1(x\_2) = -e\_2 - \beta$ **then** e_2
else 0
end if

```
>  theta3:=(x_1,x_2,x_3)->if x_1=a and theta2(x_2,x_3)=e_2 then
>  -e_1-alpha elif x_1=c and theta2(x_2,x_3)=e_2 then e_1+alpha
elif
>  x_1=d and theta2(x_2,x_3)=e_2 then e_1-alpha elif x_1=h and
>  theta2(x_2,x_3)=e_2 then -e_1+alpha elif x_1=a and
>  theta2(x_2,x_3)=-e_2 then e_1+alpha elif x_1=c and
>  theta2(x_2,x_3)=-e_2 then -e_1-alpha elif x_1=d and
>  theta2(x_2,x_3)=-e_2 then -e_1+alpha elif x_1=h and
>  theta2(x_2,x_3)=-e_2 then e_1-alpha else 0 end if; # define a
map from
>  G^3 to {0,-e_1-alpha,e_1+alpha,e_1-alpha,-e_1+alpha}
```

$\theta 3 := (x\_1, x\_2, x\_3) \rightarrow$
if $x\_1 = a$ **and** $\theta 2(x\_2, x\_3) = e\_2$ **then** $-e\_1 - \alpha$
elif $x\_1 = c$ **and** $\theta 2(x\_2, x\_3) = e\_2$ **then** $e\_1 + \alpha$
elif $x\_1 = d$ **and** $\theta 2(x\_2, x\_3) = e\_2$ **then** $e\_1 - \alpha$
elif $x\_1 = h$ **and** $\theta 2(x\_2, x\_3) = e\_2$ **then** $-e\_1 + \alpha$
elif $x\_1 = a$ **and** $\theta 2(x\_2, x\_3) = -e\_2$ **then** $e\_1 + \alpha$
elif $x\_1 = c$ **and** $\theta 2(x\_2, x\_3) = -e\_2$ **then** $-e\_1 - \alpha$
elif $x\_1 = d$ **and** $\theta 2(x\_2, x\_3) = -e\_2$ **then** $-e\_1 + \alpha$
elif $x\_1 = h$ **and** $\theta 2(x\_2, x\_3) = -e\_2$ **then** $e\_1 - \alpha$
else 0
end if

```
>  theta4:=(x_1,x_2,x_3,x_4)->if x_1=a and theta3(x_2,x_3,x_4)=e_1+alpha
>  then e_1 elif x_1=h and theta3(x_2,x_3,x_4)=e_1+alpha then e_1
elif
>  x_1=c and theta3(x_2,x_3,x_4)=e_1-alpha then e_1 elif x_1=d and
>  theta3(x_2,x_3,x_4)=e_1-alpha then e_1  elif x_1=a and
>  theta3(x_2,x_3,x_4)=-e_1-alpha then -e_1 elif x_1=h and
>  theta3(x_2,x_3,x_4)=-e_1-alpha then -e_1 elif x_1=c and
>  theta3(x_2,x_3,x_4)=-e_1+alpha then -e_1 elif x_1=d and
>  theta3(x_2,x_3,x_4)=-e_1+alpha then -e_1 else 0 end if;
>  vtheta4:=(x_1,x_2,x_3,x_4)->if x_1=a and theta3(x_2,x_3,x_4)=e_1+alpha
>  then 1 elif x_1=h and theta3(x_2,x_3,x_4)=e_1+alpha then 1 elif
x_1=c
>  and theta3(x_2,x_3,x_4)=e_1-alpha then 1 elif x_1=d and
>  theta3(x_2,x_3,x_4)=e_1-alpha then 1  elif x_1=a and
>  theta3(x_2,x_3,x_4)=-e_1-alpha then -1 elif x_1=h and
>  theta3(x_2,x_3,x_4)=-e_1-alpha then -1 elif x_1=c and
>  theta3(x_2,x_3,x_4)=-e_1+alpha then -1 elif x_1=d and
>  theta3(x_2,x_3,x_4)=-e_1+alpha then -1 else 0 end if; # define
a map
>  from G^4 to {0,1,-1}
```

$\theta 4 := (x\_1,\, x\_2,\, x\_3,\, x\_4) \rightarrow$
if $x\_1 = a$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 + \alpha$ **then** e_1
elif $x\_1 = h$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 + \alpha$ **then** e_1
elif $x\_1 = c$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 - \alpha$ **then** e_1
elif $x\_1 = d$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 - \alpha$ **then** e_1
elif $x\_1 = a$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 - \alpha$ **then** $-\, e\_1$
elif $x\_1 = h$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 - \alpha$ **then** $-\, e\_1$
elif $x\_1 = c$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 + \alpha$ **then** $-\, e\_1$
elif $x\_1 = d$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 + \alpha$ **then** $-\, e\_1$
else 0
end if

$vtheta4 := (x\_1,\, x\_2,\, x\_3,\, x\_4) \rightarrow$
if $x\_1 = a$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 + \alpha$ **then** 1
elif $x\_1 = h$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 + \alpha$ **then** 1
elif $x\_1 = c$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 - \alpha$ **then** 1
elif $x\_1 = d$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = e\_1 - \alpha$ **then** 1
elif $x\_1 = a$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 - \alpha$ **then** $-\, 1$
elif $x\_1 = h$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 - \alpha$ **then** $-\, 1$
elif $x\_1 = c$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 + \alpha$ **then** $-\, 1$
elif $x\_1 = d$ **and** $\theta 3(x\_2,\, x\_3,\, x\_4) = -e\_1 + \alpha$ **then** $-\, 1$
else 0
end if

```
>  cmu:=(x_1,x_2,x_3)->mu(mu(i(v(x_3)),i(v(x_2))),i(v(x_1))); #
define a
>  continued multiplication which will be used in the definition
of the
>  delta4
```

$$cmu := (x\_1,\, x\_2,\, x\_3) \to \mu(\mu(\mathrm{i}(\mathrm{v}(x\_3)),\, \mathrm{i}(\mathrm{v}(x\_2))),\, \mathrm{i}(\mathrm{v}(x\_1)))$$

```
>  cmu(a,c,-d);
```

$$b$$

```
>  delta4:=(x_1,x_2,x_3)->
>  -vtheta4(v(x_1),v(x_2),v(x_3),cmu(v(x_1),v(x_2),v(x_3)))+vtheta4(v(x_2
>  ),v(x_3),cmu(v(x_1),v(x_2),v(x_3)),v(x_1))-vtheta4(v(x_3),cmu(v(x_1),v
>  (x_2),v(x_3)),v(x_1),v(x_2))+vtheta4(cmu(v(x_1),v(x_2),v(x_3)),v(x_1),
>  v(x_2),v(x_3)); # define the delta-operator from H^4(G) to H^3(G)
```

$$\delta 4 := (x\_1,\, x\_2,\, x\_3) \to -\mathrm{vtheta4}(\mathrm{v}(x\_1),\, \mathrm{v}(x\_2),\, \mathrm{v}(x\_3),\, \mathrm{cmu}(\mathrm{v}(x\_1),\, \mathrm{v}(x\_2),\, \mathrm{v}(x\_3)))$$
$$+\,\mathrm{vtheta4}(\mathrm{v}(x\_2),\, \mathrm{v}(x\_3),\, \mathrm{cmu}(\mathrm{v}(x\_1),\, \mathrm{v}(x\_2),\, \mathrm{v}(x\_3)),\, \mathrm{v}(x\_1))$$
$$-\,\mathrm{vtheta4}(\mathrm{v}(x\_3),\, \mathrm{cmu}(\mathrm{v}(x\_1),\, \mathrm{v}(x\_2),\, \mathrm{v}(x\_3)),\, \mathrm{v}(x\_1),\, \mathrm{v}(x\_2))$$
$$+\,\mathrm{vtheta4}(\mathrm{cmu}(\mathrm{v}(x\_1),\, \mathrm{v}(x\_2),\, \mathrm{v}(x\_3)),\, \mathrm{v}(x\_1),\, \mathrm{v}(x\_2),\, \mathrm{v}(x\_3))$$

```
>  delta4(-a,d,h); # for test
```

$$-2$$

```
>  A4:={A,c,d,H}; # the terms of beta
```

$$A4 := \{A,\, H,\, c,\, d\}$$

```
>  B4:={a,b,c,d,h}; # the terms of alphabeta other than 1
```

$$B4 := \{a,\, b,\, c,\, d,\, h\}$$

```
>  C4:={A,C,d,h}; # the terms of alpha
```

$$C4 := \{A,\, C,\, d,\, h\}$$

```
>  mcarp:=proc()
>  local Z,U,k,x,y;
>  if nargs=0 then
>  Z:={};
>  elif nargs=1 then
>  Z:=args[1];
>  else Z:=args[1];
>  for k from 2 to nargs do
>  U:={};
>  for x in Z do
>  for y in args[k] do
>  U:= U union {[op(x),y]};
>  od;
>  od;
>  Z:=U;
```

```
> od;
> fi;
> return Z;
> end:
> mcarp(A4,B4,C4); # define the cartesian product of A4, B4 and
C4
```

$\{[A, a, A], [A, a, C], [A, a, d], [A, a, h], [A, b, A], [A, b, C], [A, b, d], [A, b, h], [A, c, A], [A, c, C], [A, c, d], [A, c, h], [A, d, A], [A, d, C], [A, d, d], [A, d, h], [A, h, A], [A, h, C], [A, h, d], [A, h, h], [H, a, A], [H, a, C], [H, a, d], [H, a, h], [H, b, A], [H, b, C], [H, b, d], [H, b, h], [H, c, A], [H, c, C], [H, c, d], [H, c, h], [H, d, A], [H, d, C], [H, d, d], [H, d, h], [H, h, A], [H, h, C], [H, h, d], [H, h, h], [c, a, A], [c, a, C], [c, a, d], [c, a, h], [c, b, A], [c, b, C], [c, b, d], [c, b, h], [c, c, A], [c, c, C], [c, c, d], [c, c, h], [c, d, A], [c, d, C], [c, d, d], [c, d, h], [c, h, A], [c, h, C], [c, h, d], [c, h, h], [d, a, A], [d, a, C], [d, a, d], [d, a, h], [d, b, A], [d, b, C], [d, b, d], [d, b, h], [d, c, A], [d, c, C], [d, c, d], [d, c, h], [d, d, A], [d, d, C], [d, d, d], [d, d, h], [d, h, A], [d, h, C], [d, h, d], [d, h, h]\}$

```
> con4:=i->op(i,mcarp(A4,B4,C4));f:=z->if z=a then a elif z=b then
b
> elif z=c then c elif z=d then d elif z=h then h elif z=A then
-a elif
> z=B then -b elif z=C then -c elif z=D then -d elif z=H then -h
end if;
> # take the i-th term of mcarp(A4,B4,C4)
> fin4:=i->delta4(v(f(op(1,con4(i)))),v(f(op(2,con4(i)))),v(f(op(3,con4
> (i))))); # take the value of delta-operator on the i-th term
of
> mcarp(A4,B4,C4)
```

$$con4 := i \to \mathrm{op}(i, \mathrm{mcarp}(A4, B4, C4))$$

$f := z \to$
if $z = a$ **then** a
elif $z = b$ **then** b
elif $z = c$ **then** c
elif $z = d$ **then** d
elif $z = h$ **then** h
elif $z = A$ **then** $-a$
elif $z = B$ **then** $-b$
elif $z = C$ **then** $-c$
elif $z = \mathrm{D}$ **then** $-d$
elif $z = H$ **then** $-h$
end if

$$fin4 := i \to \delta 4(\mathrm{v}(\mathrm{f}(\mathrm{op}(1, \mathrm{con4}(i)))), \mathrm{v}(\mathrm{f}(\mathrm{op}(2, \mathrm{con4}(i)))), \mathrm{v}(\mathrm{f}(\mathrm{op}(3, \mathrm{con4}(i)))))$$

```
> con4(54);op(1,con4(54));f(op(3,con4(54)));delta4(f(op(1,con4(54))),f(
> op(2,con4(54))),f(op(3,con4(54))));fin4(54);fin4(1); # for test
```

$$[c,\, d,\, C]$$
$$c$$
$$-c$$
$$2$$
$$2$$
$$0$$

```
> add(fin4(t),t=1..80); # take the value of delta-operator on
> beta\tensor alphabeta\tensor alpha
```

$$0$$

```
> theta5:=(x_1,x_2,x_3,x_4,x_5)->if x_1=a and
> theta4(x_2,x_3,x_4,x_5)=e_1 then -e_2-beta elif x_1=c and
> theta4(x_2,x_3,x_4,x_5)=e_1 then -e_2-beta elif x_1=d and
> theta4(x_2,x_3,x_4,x_5)=e_1 then e_2-beta elif x_1=h and
> theta4(x_2,x_3,x_4,x_5)=e_1 then e_2-beta elif x_1=a and
> theta4(x_2,x_3,x_4,x_5)=-e_1 then e_2+beta elif x_1=c and
> theta4(x_2,x_3,x_4,x_5)=-e_1 then e_2+beta elif x_1=d and
> theta4(x_2,x_3,x_4,x_5)=-e_1 then -e_2+beta elif x_1=h and
> theta4(x_2,x_3,x_4,x_5)=-e_1 then -e_2+beta else 0 end if; #
define a
> map from G^5 to {0,-e_2-beta,e_2+beta,e_2-beta,-e_2+beta}
```

$\theta5 := (x\_1,\, x\_2,\, x\_3,\, x\_4,\, x\_5) \rightarrow$
if $x\_1 = a$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = e\_1$ **then** $-\, e\_2 - \beta$
elif $x\_1 = c$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = e\_1$ **then** $-\, e\_2 - \beta$
elif $x\_1 = d$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = e\_1$ **then** $e\_2 - \beta$
elif $x\_1 = h$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = e\_1$ **then** $e\_2 - \beta$
elif $x\_1 = a$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = -e\_1$ **then** $e\_2 + \beta$
elif $x\_1 = c$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = -e\_1$ **then** $e\_2 + \beta$
elif $x\_1 = d$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = -e\_1$ **then** $-\, e\_2 + \beta$
elif $x\_1 = h$ **and** $\theta4(x\_2,\, x\_3,\, x\_4,\, x\_5) = -e\_1$ **then** $-\, e\_2 + \beta$
else 0
end if

```
> theta6:=(x_1,x_2,x_3,x_4,x_5,x_6)->if x_1=a and
> theta5(x_2,x_3,x_4,x_5,x_6)=e_2+beta then e_2 elif x_1=h and
> theta5(x_2,x_3,x_4,x_5,x_6)=e_2+beta then -e_2 elif x_1=c and
> theta5(x_2,x_3,x_4,x_5,x_6)=e_2-beta then -e_2 elif x_1=d and
> theta5(x_2,x_3,x_4,x_5,x_6)=e_2-beta then e_2 elif x_1=a and
> theta5(x_2,x_3,x_4,x_5,x_6)=-e_2-beta then -e_2 elif x_1=h and
> theta5(x_2,x_3,x_4,x_5,x_6)=-e_2-beta then e_2 elif x_1=c and
> theta5(x_2,x_3,x_4,x_5,x_6)=-e_2+beta then e_2 elif x_1=d and
> theta5(x_2,x_3,x_4,x_5,x_6)=-e_2+beta then -e_2 else 0 end if;
#
> define a map from G^6 to {0,-e_2,e_2}
```

$$\theta 6 := (x\_1,\, x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) \rightarrow$$
$$\textbf{if}\, x\_1 = a \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = e\_2 + \beta \,\textbf{then}\, e\_2$$
$$\textbf{elif}\, x\_1 = h \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = e\_2 + \beta \,\textbf{then} - e\_2$$
$$\textbf{elif}\, x\_1 = c \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = e\_2 - \beta \,\textbf{then} - e\_2$$
$$\textbf{elif}\, x\_1 = d \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = e\_2 - \beta \,\textbf{then}\, e\_2$$
$$\textbf{elif}\, x\_1 = a \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = -e\_2 - \beta \,\textbf{then} - e\_2$$
$$\textbf{elif}\, x\_1 = h \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = -e\_2 - \beta \,\textbf{then}\, e\_2$$
$$\textbf{elif}\, x\_1 = c \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = -e\_2 + \beta \,\textbf{then}\, e\_2$$
$$\textbf{elif}\, x\_1 = d \textbf{ and } \theta 5(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6) = -e\_2 + \beta \,\textbf{then} - e\_2$$
$$\textbf{else}\, 0$$
$$\textbf{end if}$$

```
>  theta7:=(x_1,x_2,x_3,x_4,x_5,x_6,x_7)->if x_1=a and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=e_2 then -e_1-alpha elif x_1=c
and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=e_2 then e_1+alpha elif x_1=d
and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=e_2 then e_1-alpha elif x_1=h
and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=e_2 then -e_1+alpha elif x_1=a
and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=-e_2 then e_1+alpha elif x_1=c
and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=-e_2 then -e_1-alpha elif x_1=d
and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=-e_2 then -e_1+alpha elif x_1=h
and
>  theta6(x_2,x_3,x_4,x_5,x_6,x_7)=-e_2 then e_1-alpha else 0 end
if; #
>  define a map from G^7 to
>  {0,-e_1-alpha,e_1+alpha,e_1-alpha,-e_1+alpha}
```

$$\theta 7 := (x\_1,\, x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) \rightarrow$$
$$\textbf{if}\, x\_1 = a \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = e\_2 \,\textbf{then} - e\_1 - \alpha$$
$$\textbf{elif}\, x\_1 = c \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = e\_2 \,\textbf{then}\, e\_1 + \alpha$$
$$\textbf{elif}\, x\_1 = d \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = e\_2 \,\textbf{then}\, e\_1 - \alpha$$
$$\textbf{elif}\, x\_1 = h \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = e\_2 \,\textbf{then} - e\_1 + \alpha$$
$$\textbf{elif}\, x\_1 = a \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = -e\_2 \,\textbf{then}\, e\_1 + \alpha$$
$$\textbf{elif}\, x\_1 = c \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = -e\_2 \,\textbf{then} - e\_1 - \alpha$$
$$\textbf{elif}\, x\_1 = d \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = -e\_2 \,\textbf{then} - e\_1 + \alpha$$
$$\textbf{elif}\, x\_1 = h \textbf{ and } \theta 6(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7) = -e\_2 \,\textbf{then}\, e\_1 - \alpha$$
$$\textbf{else}\, 0$$
$$\textbf{end if}$$

```
>  theta8:=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)->if x_1=a and
>  theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1+alpha then e_1 elif x_1=h
and
>  theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1+alpha then e_1 elif x_1=c
and
>  theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1-alpha then e_1 elif x_1=d
and
>  theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1-alpha then e_1 elif x_1=a
and
>  theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1-alpha then -e_1 elif
x_1=h
>  and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1-alpha then -e_1
elif
>  x_1=c and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1+alpha then
-e_1
>  elif x_1=d and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1+alpha
then
>  -e_1 else 0 end if; v2theta8:=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)->if
>  x_1=a and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1+alpha then
1 elif
>  x_1=h and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1+alpha then
1 elif
>  x_1=c and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1-alpha then
1 elif
>  x_1=d and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=e_1-alpha then
1 elif
>  x_1=a and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1-alpha then
-1 elif
>  x_1=h and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1-alpha then
-1 elif
>  x_1=c and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1+alpha then
-1 elif
>  x_1=d and theta7(x_2,x_3,x_4,x_5,x_6,x_7,x_8)=-e_1+alpha then
-1 else
>  0 end if;
```

$\theta 8 := (x\_1,\, x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) \rightarrow$
if $x\_1 = a$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = e\_1 + \alpha$ **then** e_1
elif $x\_1 = h$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = e\_1 + \alpha$ **then** e_1
elif $x\_1 = c$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = e\_1 - \alpha$ **then** e_1
elif $x\_1 = d$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = e\_1 - \alpha$ **then** e_1
elif $x\_1 = a$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = -e\_1 - \alpha$ **then** $-\, e\_1$
elif $x\_1 = h$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = -e\_1 - \alpha$ **then** $-\, e\_1$
elif $x\_1 = c$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = -e\_1 + \alpha$ **then** $-\, e\_1$
elif $x\_1 = d$ **and** $\theta 7(x\_2,\, x\_3,\, x\_4,\, x\_5,\, x\_6,\, x\_7,\, x\_8) = -e\_1 + \alpha$ **then** $-\, e\_1$
else 0
end if

$v2theta8 := (x\_1, x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) \rightarrow$

if $x\_1 = a$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = e\_1 + \alpha$ **then** 1

elif $x\_1 = h$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = e\_1 + \alpha$ **then** 1

elif $x\_1 = c$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = e\_1 - \alpha$ **then** 1

elif $x\_1 = d$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = e\_1 - \alpha$ **then** 1

elif $x\_1 = a$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = -e\_1 - \alpha$ **then** -1

elif $x\_1 = h$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = -e\_1 - \alpha$ **then** -1

elif $x\_1 = c$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = -e\_1 + \alpha$ **then** -1

elif $x\_1 = d$ **and** $\theta 7(x\_2, x\_3, x\_4, x\_5, x\_6, x\_7, x\_8) = -e\_1 + \alpha$ **then** -1

else 0

end if

```
> ccmu:=(x_1,x_2,x_3,x_4,x_5,x_6,x_7)->mu(mu(mu(mu(mu(mu(i(x_7),i(x_6))
> ,i(x_5)),i(x_4)),i(x_3)),i(x_2)),i(x_1)); # define a continued
> multiplication which will be used in the definition of the delta8
```

$ccmu := (x\_1, x\_2, x\_3, x\_4, x\_5, x\_6, x\_7) \rightarrow$

$\mu(\mu(\mu(\mu(\mu(\mu(\mathrm{i}(x\_7), \mathrm{i}(x\_6)), \mathrm{i}(x\_5)), \mathrm{i}(x\_4)), \mathrm{i}(x\_3)), \mathrm{i}(x\_2)), \mathrm{i}(x\_1))$

```
> delta8:=(x_1,x_2,x_3,x_4,x_5,x_6,x_7)->
> -v2theta8(v(x_1),v(x_2),v(x_3),v(x_4),v(x_5),v(x_6),v(x_7),ccmu(v(x_1)
> ,v(x_2),v(x_3),v(x_4),v(x_5),v(x_6),v(x_7)))+v2theta8(v(x_2),v(x_3),v(
> x_4),v(x_5),v(x_6),v(x_7),ccmu(v(x_1),v(x_2),v(x_3),v(x_4),v(x_5),v(x_
> 6),v(x_7)),v(x_1))-v2theta8(v(x_3),v(x_4),v(x_5),v(x_6),v(x_7),ccmu(v(
> x_1),v(x_2),v(x_3),v(x_4),v(x_5),v(x_6),v(x_7)),v(x_1),v(x_2))+v2theta
> 8(v(x_4),v(x_5),v(x_6),v(x_7),ccmu(v(x_1),v(x_2),v(x_3),v(x_4),v(x_5),
> v(x_6),v(x_7)),v(x_1),v(x_2),v(x_3))-v2theta8(v(x_5),v(x_6),v(x_7),ccm
> u(v(x_1),v(x_2),v(x_3),v(x_4),v(x_5),v(x_6),v(x_7)),v(x_1),v(x_2),v(x_
> 3),v(x_4))+v2theta8(v(x_6),v(x_7),ccmu(v(x_1),v(x_2),v(x_3),v(x_4),v(x
> _5),v(x_6),v(x_7)),v(x_1),v(x_2),v(x_3),v(x_4),v(x_5))-v2theta8(v(x_7)
> ,ccmu(v(x_1),v(x_2),v(x_3),v(x_4),v(x_5),v(x_6),v(x_7)),v(x_1),v(x_2),
> v(x_3),v(x_4),v(x_5),v(x_6))+v2theta8(ccmu(v(x_1),v(x_2),v(x_3),v(x_4)
> ,v(x_5),v(x_6),v(x_7)),v(x_1),v(x_2),v(x_3),v(x_4),v(x_5),v(x_6),v(x_7
> ));
```

$\delta8 := (x\_1, x\_2, x\_3, x\_4, x\_5, x\_6, x\_7) \to -\mathrm{v2theta8}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4),$
$\mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7),$
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7))) + \mathrm{v2theta8}(\mathrm{v}(x\_2),$
$\mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7),$
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7)), \mathrm{v}(x\_1)) - \mathrm{v2theta8}($
$\mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7),$
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7)), \mathrm{v}(x\_1), \mathrm{v}(x\_2))+$
$\mathrm{v2theta8}(\mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7),$
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7)), \mathrm{v}(x\_1), \mathrm{v}(x\_2),$
$\mathrm{v}(x\_3)) - \mathrm{v2theta8}(\mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7),$
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7)), \mathrm{v}(x\_1), \mathrm{v}(x\_2),$
$\mathrm{v}(x\_3), \mathrm{v}(x\_4)) + \mathrm{v2theta8}(\mathrm{v}(x\_6), \mathrm{v}(x\_7),$
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7)), \mathrm{v}(x\_1), \mathrm{v}(x\_2),$
$\mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5)) - \mathrm{v2theta8}(\mathrm{v}(x\_7),$
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7)), \mathrm{v}(x\_1), \mathrm{v}(x\_2),$
$\mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6)) + \mathrm{v2theta8}($
$\mathrm{ccmu}(\mathrm{v}(x\_1), \mathrm{v}(x\_2), \mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7)), \mathrm{v}(x\_1), \mathrm{v}(x\_2),$
$\mathrm{v}(x\_3), \mathrm{v}(x\_4), \mathrm{v}(x\_5), \mathrm{v}(x\_6), \mathrm{v}(x\_7))$

```
> ccmu(-a,c,d,h,a,c,d);delta8(-a,c,d,h,a,c,d); # for test
```

$$-c$$
$$0$$

```
> A8:={A,c,d,H}; # the terms of beta
```

$$A8 := \{A, H, c, d\}$$

```
> B8:={a,b,c,d,h}; # the terms of alphabeta other than 1
```

$$B8 := \{a, b, c, d, h\}$$

```
> C8:={A,C,d,h}; # the terms of alpha
```

$$C8 := \{A, C, d, h\}$$

```
> D8:={A,b,c,D,h}; # the terms of betaalpha other than 1
```

$$\mathrm{D8} := \{A, \mathrm{D}, b, c, h\}$$

```
> E8:={A,c,d,H}; # the terms of beta
```

$$E8 := \{A, H, c, d\}$$

```
> F8:={a,b,c,d,h}; # the terms of alphabeta other than 1
```

$$F8 := \{a, b, c, d, h\}$$

```
> G8:={A,C,d,h}; # the terms of alpha
```

$$G8 := \{A, C, d, h\}$$

```
> T8:=mcarp(A8,B8,C8,D8,E8,F8,G8):
> con8:=i->op(i,T8);
```

$$con8 := i \to \mathrm{op}(i,\ T8)$$

```
> con8(1);con8(32000);con8(14918); # for test
```

$$[A,\ a,\ A,\ A,\ A,\ a,\ A]$$

$$[d,\ h,\ h,\ h,\ d,\ h,\ h]$$

$$[H,\ h,\ C,\ \mathrm{D},\ H,\ h,\ C]$$

```
> fin8:=i->delta8(v(f(op(1,con8(i)))),v(f(op(2,con8(i)))),v(f(op(3,con8
> (i)))),v(f(op(4,con8(i)))),v(f(op(5,con8(i)))),v(f(op(6,con8(i)))),v(f
> (op(7,con8(i))))); # take the value of delta-operator on the
i-th term
> of T8
```

$$fin8 := i \to \delta 8(\mathrm{v}(\mathrm{f}(\mathrm{op}(1,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(2,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(3,\ \mathrm{con8}(i)))),$$
$$\mathrm{v}(\mathrm{f}(\mathrm{op}(4,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(5,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(6,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(7,\ \mathrm{con8}(i)))))$$

```
> fin8 := proc (i) options operator, arrow;
> delta8(v(f(op(1,con8(i)))),v(f(op(2,con8(i)))),v(f(op(3,con8(i)))),v(f
> (op(4,con8(i)))),v(f(op(5,con8(i)))),v(f(op(6,con8(i)))),v(f(op(7,con8
> (i))))) end proc;
```

$$fin8 := i \to \delta 8(\mathrm{v}(\mathrm{f}(\mathrm{op}(1,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(2,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(3,\ \mathrm{con8}(i)))),$$
$$\mathrm{v}(\mathrm{f}(\mathrm{op}(4,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(5,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(6,\ \mathrm{con8}(i)))),\ \mathrm{v}(\mathrm{f}(\mathrm{op}(7,\ \mathrm{con8}(i)))))$$

$$-a$$

$$0$$

$$0$$

$$4\, e\_1$$

```
> add(fin8(t),t=1..32000); # take the value of delta-operator on
> beta\tensor alphabeta\tensor alpha\tensor betaalpha\tensor
> beta\tensor alphabeta\tensor alpha0
```

$$0$$