Generalized parallel paths method for computing the first Hochschild cohomology groups with applications to Brauer graph algebras

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Abstract: We use algebraic Morse theory to generalize the parallel paths method for computing the first Hochschild cohomology groups. As an application, we describe and compare the Lie structures of the first Hochschild cohomology groups of Brauer graph algebras and their associated graded algebras.

1 Introduction

It is well-known that the Hochschild cohomology groups are important invariants of associative algebras. The computation of Hochschild cohomology groups are heavily based on a two-sided projective resolution of a given algebra. The smaller of the size of this projective resolution, the more efficient of the computation. For monomial algebras, Bardzell [3] constructed minimal two-sided projective resolutions. Based on the minimal two-sided projective resolution, Strametz [19] created the parallel paths method to compute the first Hochschild cohomology group of a monomial algebra.

One aim of the present paper is to generalize Strametz's parallel paths method on computing the first Hochschild cohomology groups from monomial algebras to general quiver algebras of the form kQ/I where Q is a finite quiver and I is an ideal of the path algebra kQ contained in $kQ_{\geq 2}$. Our idea is to use the two-sided Anick resolution, which is based on algebraic Morse theory, to replace Bardzell's minimal two-sided projective resolution.

About fifteen years ago, the algebraic Morse theory was developed by Kozlov [13], by Sköldberg [17], and by Jöllenbeck and Welker [12], independently. Since then, this theory has been widely used in algebra; for some further references on this direction, see the introduction of the recent paper [7]. In particular, Sköldberg [17] applied this theory to construct the so-called two-sided Anick resolution from the reduced bar resolution of an non-commutative polynomial algebra. Chen, Liu and Zhou [7] generalized the two-sided Anick resolution from non-commutative polynomial algebras to algebras given by quivers with relations.

For a monomial algebra kQ/I, the ideal I has a minimal generating set Z given by paths in Q, which is one of ingredients in Strametz's construction. For arbitrary quiver algebra kQ/I, we use the

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Gröbner basis of I to replace the above set Z. In order to generalize Strametz's construction, we use the two-sided Anick resolution (which is for an arbitrary quiver algebra) to replace Bardzell's minimal two-sided projective resolution (which is only for a monomial algebra). Similar to Strametz [19], we also describe the Lie algebra structure on the first Hochschild cohomology group. We implement all these ideas in Section 3. It should be noted that, Artenstein, Lanzilotta and Solotar [2] recently studied the Hochschild cohomology of toupie algebras and the cochain complex obtained in [2, Section 3] to compute HH^1 for toupie algebras coincides with the cochain complex in our Proposition 3.7.

In Section 4 we will apply our method to study the first Hochschild cohomology groups of Brauer graph algebras (or just BGAs); these algebras coincide with finite dimensional symmetric special biserial algebras. Under some mild characteristic condition (see Proposition 4.5), we can describe explicitly a set of generators of the first Hochschild cohomology group of a BGA, with a comparison in Section 5 to the first Hochschild cohomology group of the associated graded algebra. In particular, we will construct an injection i from $\mathrm{HH}^1(A)$ to $\mathrm{HH}^1(gr(A))$, where A is a BGA and $gr(A)$ its associated graded algebra. Actually, this map i is always a Lie algebra monomorphism with one exception. This injection also tells us that the difference between the dimension of $\mathrm{HH}^1(A)$ and of $\mathrm{HH}^1(gr(A))$ is equal to the difference between the rank of $\mathrm{Out}(A)^\circ$ and of $\mathrm{Out}(gr(A))^\circ$.

After we submitted this paper on arXiv, we noticed that Rubio y Degrassi, Schroll and Solotar have recently obtained similar results in [15]. However, our generalized parallel paths method on computing HH^1 is deduced from two-sided Anick resolutions using algebraic Morse theory, rather than directly uses the Chouhy-Solotar projective resolution which is constructed in [8]. Moreover, we described explicitly a set of generators of $\mathrm{HH}^1(A)$ for any Brauer graph algebra A, and our comparison study on the first Hochschild cohomology groups between BGAs and their associated graded algebras is also new. Finally, we noticed that there would exist counter-examples of Theorem 4.2 in [15] in positive characteristic, see our Example 4.13.

Outline. In Section 2, we make some preliminaries and give some notations which we need throughout this paper; in particular, we will review the Gröbner basis theory for path algebras and the two-sided Anick resolutions for quiver algebras based on algebraic Morse theory. In Section 3 we generalize the parallel paths method for computing HH^1 from monomial algebras to general quiver algebras. The main results in this section are Proposition 3.7 and Theorem 3.8. In Sections 4 and 5, we give the application of the generalized parallel paths method on BGAs and their associated graded algebras; the main results are Theorems 4.8, 5.8 and 5.11. One interesting consequence (Corollary 5.13, see also Corollary 5.18) gives a simple formula for the difference between the dimensions of the first Hochschild cohomology groups of a BGA and its associated graded algebra.

2 Preliminaries

Throughout this paper we will concentrate on quiver algebras of the form kQ/I, where k is a field, Q is a finite quiver, I is a two-sided ideal in the path algebra kQ. For each integer $n \geq 0$, we denote by Q_n the set of all paths of length n and by $Q_{\geq n}$ the set of all paths with length at least n. We shall assume that the ideal I is contained in $kQ_{\geq 2}$ so that kQ/I is not necessarily finite dimensional. We denote by $s(p)$ the source vertex of a path p and by $t(p)$ its terminus vertex. We will write paths from right to left, for example, $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ is a path with starting arrow α_1 and ending arrow α_n. The length of a path p will be denoted by $l(p)$. Two paths ε, γ of Q are called parallel if $s(\varepsilon) = s(\gamma)$ and $t(\varepsilon) = t(\gamma)$. If X and Y are sets of paths of Q, the set $X//Y$ of parallel paths is formed by the couples $(\varepsilon, \gamma) \in X \times Y$ such that ε and γ are parallel paths. For instance, $Q_0//Q_n$ is the set of oriented cycles of Q of length n. We denote by $k(X//Y)$ the k-vector space generated by the set $X//Y$; For a subset S of $k(X//Y)$, we denote by $\langle S \rangle$ the subspace of $k(X//Y)$ generated by S. By abuse of notation, for a subset S of the algebra kQ/I, we also use $\langle S \rangle$ to denote the ideal generated by S.

2.1 Gröbner bases of quiver algebras

Let $A = kQ/I$ be a quiver algebra where I is generated by a set of relations. In this subsection we recall from [10] the Gröbner basis theory for the ideal I. Let us first introduce a special kind of well-order on the basis $Q_{\geq 0}$ of the path algebra kQ. By [10, Section 2.2.2], a well-order $>$ on $Q_{\geq 0}$ is called admissible if it satisfies the following conditions where $p, q, r, s \in Q_{\geq 0}$:

- if $p < q$ then $pr < qr$ if both $pr \neq 0$ and $qr \neq 0$.
- if $p < q$ then $sp < sq$ if both $sp \neq 0$ and $sq \neq 0$.
- if $p = qr$, then $p \geq q$ and $p \geq r$.

Given a quiver Q as above, there are "natural" admissible orders on $Q_{\geq 0}$. Here is one example:

Example 2.1. *The (left) length-lexicographic order on $Q_{\geq 0}$:*
Order the vertices $Q_0 = \{v_1, \cdots, v_n\}$ and arrows $Q_1 = \{a_1, \cdots, a_m\}$ arbitrarily and set the vertices smaller than the arrows. Thus

$$v_1 < \cdots < v_n < a_1 < \cdots < a_m.$$

If p and q are paths of length at least 1, set $p < q$ if $l(p) < l(q)$ or if $p = b_1 \cdots b_r$ and $q = b'_1 \cdots b'_r$ with $b_1, \cdots, b_r, b'_1, \cdots, b'_r \in Q_1$ and for some $1 \leq i \leq r$, $b_j = b'_j$ for $j < i$ and $b_i < b'_i$.

We now fix an admissible well-order $>$ on $Q_{\geq 0}$. For any $a \in kQ$, we have $a = \sum_{p \in Q_{\geq 0},\, \lambda_p \in k} \lambda_p p$ and write $\mathrm{Supp}(a) = \{p \mid \lambda_p \neq 0\}$. We call $\mathrm{Tip}(a) = p$, if $p \in \mathrm{Supp}(a)$ and $p' \leq p$ for all $p' \in \mathrm{Supp}(a)$. Then we denote the tip of a set $W \subseteq kQ$ by $\mathrm{Tip}(W) = \{\mathrm{Tip}(w) | w \in W\}$ and write $\mathrm{NonTip}(W) := Q_{\geq 0} \backslash \mathrm{Tip}(W)$. We also denote the coefficient of the tip of a by $\mathrm{CTip}(a)$. In particular, we will use $\mathrm{Tip}(I)$ and $\mathrm{NonTip}(I)$ for the ideal I of kQ. By [10], there is a decomposition of vector spaces

$$kQ = I \oplus \mathrm{Span}_k(\mathrm{NonTip}(I)).$$

So $\mathrm{NonTip}(I)$ (modulo I) gives a "monomial" k-basis of the quotient algebra $A = kQ/I$.

Definition 2.2. ([*10, Definition* 2.4]) *With the notations as above, a subset $\mathcal{G} \subseteq I$ is a Gröbner basis for the ideal I with respect to the order $>$ if*

$$\langle \mathrm{Tip}(I) \rangle = \langle \mathrm{Tip}(\mathcal{G}) \rangle,$$

that is, $\mathrm{Tip}(I)$ *and* $\mathrm{Tip}(\mathcal{G})$ *generate the same ideal in kQ.*

Actually, in this case $I = \langle \mathcal{G} \rangle$. By the discussion in [10], we have a complete method to judge whether a set of generators of an ideal I in kQ is a Gröbner basis, which is called the Termination Theorem. The idea is to check whether some special elements of the ideal I are divisible by this basis, instead of to check all the elements in I.

Definition 2.3. ([*10, Definition* 2.7]) *Let kQ be a path algebra, $>$ an admissible order on $Q_{\geq 0}$ and $f, g \in kQ$. Suppose $b, c \in Q_{\geq 0}$, such that*

- $\mathrm{Tip}(f)c = b\mathrm{Tip}(g)$,
- $\mathrm{Tip}(f) \nmid b$, $\mathrm{Tip}(g) \nmid c$.

Then the overlap relation of f and g by b, c is

$$o(f, g, b, c) = (\mathrm{CTip}(f))^{-1} \cdot fc - (\mathrm{CTip}(g))^{-1} \cdot bg.$$

Note that $\mathrm{Tip}(o(f, g, b, c)) < \mathrm{Tip}(f)c = b\mathrm{Tip}(g)$.

Theorem 2.4. ([*10, Theorem* 2.3]) *Let kQ be a path algebra, $>$ an admissible order on $Q_{\geq 0}$, $\mathcal{G}$ a set of elements of kQ. Suppose for every overlap relation, we have*

$$o(g_1, g_2, p, q) \Rightarrow_{\mathcal{G}} 0,$$

which means that $o(g_1, g_2, p, q)$ can be divided by $\mathrm{Tip}(\mathcal{G})$, with $g_1, g_2 \in \mathcal{G}$ and $p, q \in Q_{\geq 0}$. Then $\mathcal{G}$ is a Gröbner basis of $\langle \mathcal{G} \rangle$, the ideal generated by $\mathcal{G}$.

Definition 2.5. ([*10, Definition* 2.8]) *A Gröbner basis $\mathcal{G}$ for the ideal I is reduced if the following three conditions are satisfied:*

- *$\mathcal{G}$ is tip-reduced: for $g, h \in \mathcal{G}$ with $g \neq h$, $\mathrm{Tip}(g) \nmid \mathrm{Tip}(h)$;*
- *$\mathcal{G}$ is monic: for every element $g \in \mathcal{G}$, $\mathrm{CTip}(g) = 1$;*
- *For any $g \in \mathcal{G}$, $g - \mathrm{Tip}(g) \in \mathrm{Span}_k(\mathrm{NonTip}(I))$.*

It is easy to see that under a given admissible order, I has a unique reduced Gröbner basis $\mathcal{G}$, and in this case $\mathrm{Tip}(\mathcal{G})$ is a minimal generator set of $\langle \mathrm{Tip}(I) \rangle$; moreover, $b \in Q_{\geq 0}$ lies in $\mathrm{NonTip}(I)$ if and only if b is not divided by any element of $\mathrm{Tip}(\mathcal{G})$. In the following, we always assume that $\mathcal{G}$ is a reduced Gröbner basis of I.

Note that when $\mathcal{G}$ is reduced, there is a one-to-one correspondence between $\mathcal{G}$ and $\mathrm{Tip}(\mathcal{G})$: for $g \in \mathcal{G}$, $\mathrm{Tip}(g) \in \mathrm{Tip}(\mathcal{G})$; conversely, for $w \in \mathrm{Tip}(\mathcal{G})$, there is a unique $g \in \mathcal{G}$ such that $w = \mathrm{Tip}(g)$. We shall denote the correspondence from $\mathcal{G}$ to $\mathrm{Tip}(\mathcal{G})$ by Tip and its inverse by Tip^{-1}.

2.2 The reduced bar resolution of quiver algebras

Now let $E :\simeq \oplus_{e \in Q_0} ke$ be the separable subalgebra of A generated by the classes modulo I of the vertices of Q, such that $A = E \oplus A_+$ as E^e-modules, where $E^e = E \otimes_k E$ and $A_+ := \mathrm{Span}_k\{\mathrm{NonTip}(I) \backslash Q_0\}$. Actually, there is an E^e-projection from A to A_+, denoted by p_A. The reduced bar resolution of the quiver algebra A can be written by the form of the following theorem in the sense of Cibils [5].

Theorem 2.6. *For the algebra $A = kQ/I$, the reduced bar resolution $B(A)$ is a two-sided projective resolution of A with $B_0(A) = A \otimes_E A$, $B_n(A) = A \otimes_E (A_+)^{\otimes_E n} \otimes_E A$ and the differential $d = (d_n)$ is*

$$d_n([a_1|\cdots|a_n]) = a_1[a_2|\cdots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\cdots|a_i a_{i+1}|\cdots|a_n] + (-1)^n[a_1|\cdots|a_{n-1}]a_n$$

with $[a_1|\cdots|a_n] := 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1$. By convention $B_{-1}(A) = A$, and $d_0 : A \otimes_E A \longrightarrow A$ is given by the multiplication μ_A in A.

Remark 2.7. *By the definition of A_+, $B_n(A)$ can be decomposed as*

$$B_n(A) = \bigoplus A[w_1|\cdots|w_n]A = \bigoplus A^e[w_1|\cdots|w_n],$$

where $A^e = A \otimes_k A^{op}$ is the enveloping algebra of A and the direct sum is taken over all the signs $[w_1|\cdots|w_n]$ such that all $w_i \in \mathrm{NonTip}(I) \backslash Q_0$ and $w_1 \cdots w_n$ is a path in Q.

Since the reduced bar resolution is a two-sided projective resolution of A, we can use it to compute the Hochschild cohomology groups $\mathrm{HH}^n(A)$ of A. More concretely, applying the functor $\mathrm{Hom}_{A^e}(-, A)$ to $B(A)$ we get a cochain complex $(C^*(A), \delta^*)$, where $C^0(A) \cong \mathrm{Hom}_{E^e}(E, A) \cong A^E = \{a \in A \mid sa =$

as for all $s \in E\}$, $C^n(A) = \mathrm{Hom}_{A^e}(B_n(A), A) \cong \mathrm{Hom}_{E^e}((A_+)^{\otimes_E n}, A)$ for $n \geq 1$ (cf. Lemma 3.3), $(\delta^0 a)(x) = ax - xa$ for $a \in A^E$ and $x \in A_+$, and

$$(\delta^n f)(x_1 \otimes \cdots \otimes x_{n+1}) = x_1 f(x_2 \otimes \cdots \otimes x_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) + (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n) x_{n+1}.$$

Then we have
$$\mathrm{HH}^n(A) = \mathrm{Ker}\delta^n / \mathrm{Im}\delta^{n-1}.$$

In particular, we have $\mathrm{HH}^1(A) = \mathrm{Ker}\delta^1/\mathrm{Im}\delta^0$ as k-spaces, where $\mathrm{Ker}\delta^1$ is the set of E^e-derivations of A_+ into A and the elements in $\mathrm{Im}\delta^0$ are inner E^e-derivations of A_+ into A. Note that we can identify $\mathrm{Der}_{E^e}(A_+, A)$ with $\mathrm{Der}_{E^e}(A, A)$ and $\mathrm{Ker}\delta^1 = \mathrm{Der}_{E^e}(A_+, A)$ has a Lie algebra structure under the Lie bracket
$$[f, g]_{HH} := f \circ p_A \circ g - g \circ p_A \circ f$$
for $f, g \in \mathrm{Der}_{E^e}(A_+, A)$, where p_A denotes the E^e-projection from A to A_+. Moreover, $\mathrm{Im}\delta^0$ is a Lie ideal of $\mathrm{Ker}\delta^1$, so that $\mathrm{HH}^1(A)$ is a Lie algebra. This structure was first defined by Gerstenhaber [9] using the standard bar resolution of A.

In next two subsections we will explain how to use the algebraic Morse theory to shrink the above reduced bar resolution of A to a "smaller" one, such that the homology of the two complexes coincides.

2.3 Algebraic Morse theory

The most general version of algebraic Morse theory was presented in Chen, Liu and Zhou [7]. For our purpose, we will adopt to a Morse matching condition defined in [7, Proposition 3.2].

Let R be an associative ring and $C_* = (C_n, \partial_n)_{n \in \mathbb{Z}}$ be a chain complex of left R-modules. We assume that each R-module C_n has a decomposition $C_n \simeq \oplus_{i \in I_n} C_{n,i}$ of R-modules, so we can regard the differentials ∂_n as a matrix $\partial_n = (\partial_{n,ji})$ with $i \in I_n$ and $j \in I_{n-1}$ and where $\partial_{n,ji} : C_{n,i} \to C_{n-1,j}$ is a homomorphism of R-modules.

Given the complex C_* as above, we construct a weighted quiver $G(C_*) := (V, E)$. The set V of vertices of $G(C_*)$ consists of the pairs (n, i) with $n \in \mathbb{Z}, i \in I_n$ and the set E of weighted arrows is given by the rule: if the map $\partial_{n,ji}$ does not vanish, draw an arrow in E from (n, i) to $(n-1, j)$ and denote the weight of this arrow by the map $\partial_{n,ji}$.

A full subquiver $\mathcal{M}$ of the weighted quiver $G(C_*)$ is called a partial matching if it satisfies the following two conditions:

- (*Matching*) Each vertex in V belongs to at most one arrow of $\mathcal{M}$.
- (*Invertibility*) Each arrow in $\mathcal{M}$ has its weight invertible as a R-homomorphism.

With respect to a partial matching $\mathcal{M}$, we can define a new weighted quiver $G_{\mathcal{M}}(C_*) = (V, E_{\mathcal{M}})$, where $E_{\mathcal{M}}$ is given by

- Keep everything for all arrows which are not in $\mathcal{M}$ and call them thick arrows.
- For an arrow in $\mathcal{M}$, replace it by a new dotted arrow in the reverse direction and the weight of this new arrow is the negative inverse of the weight of original arrow.

A path in $G_{\mathcal{M}}(C_*)$ is called a zigzag path if dotted arrows and thick arrows appear alternately.

Next, for convenience, we will introduce from Jöllenbeck and Welker [12] the notations related to the weighted quiver $G(C_*) = (V, E)$ with a partial matching $\mathcal{M}$ on it.

Definition 2.8. *(1) A vertex $(n,i) \in V$ is critical with respect to $\mathcal{M}$ if (n,i) does not lie in any arrow in $\mathcal{M}$. Let V_n denote all the vertices with the first number equal to n, we write*

$$V_n^{\mathcal{M}} := \{(n,i) \in V_n \mid (n,i) \text{ is critical}\}$$

for the set of all critical vertices of homological degree n.

(2) Write $(m,j) \leq (n,i)$ if there exists an arrow from (n,i) to (m,j) in $G(C_*)$.

(3) Denote by $P((n,i),(m,j))$ the set of all zigzag paths from (n,i) to (m,j) in $G_{\mathcal{M}}(C_*)$.

(4) The weight $w(p)$ of a path

$$p = ((n_1,i_1) \to (n_2,i_2) \to \cdots \to (n_r,i_r)) \in P((n_1,i_1),(n_r,i_r))$$

in $G_{\mathcal{M}}(C_*)$ is given by

$$w(p) := w((n_{r-1},i_{r-1}) \to (n_r,i_r)) \circ \cdots \circ w((n_1,i_1) \to (n_2,i_2)),$$

$$w((n,i) \to (m,j)) := \begin{cases} -\partial_{m,ij}^{-1} & , \quad (n,i) \leq (m,j), \\ \partial_{n,ji} & , \quad (m,j) \leq (n,i). \end{cases}$$

Then we write $\Gamma((n,i),(m,j)) = \sum_{p \in P((n,i),(m,j))} w(p)$ for the sum of weights of all zigzag paths from (n,i) to (m,j).

Following [7, Proposition 3.2], we call a partial matching $\mathcal{M}$ as above a Morse matching if any zigzag path starting from (n,i) is of finite length for each vertex (n,i) in $G_{\mathcal{M}}(C_*)$.

Now we can define a new complex $C_*^{\mathcal{M}}$, which we call the Morse complex of C_* with respect to $\mathcal{M}$. The complex $C_*^{\mathcal{M}} = (C_n^{\mathcal{M}}, \partial_n^{\mathcal{M}})_{n \in \mathbb{Z}}$ is defined by

$$C_n^{\mathcal{M}} := \oplus_{(n,i) \in V_n^{\mathcal{M}}} C_{n,i},$$

$$\partial_n^{\mathcal{M}} : \begin{cases} C_n^{\mathcal{M}} & \to \quad C_{n-1}^{\mathcal{M}} \\ x \in C_{n,i} & \mapsto \quad \sum_{(n-1,j) \in V_{n-1}^{\mathcal{M}}} \Gamma((n,i),(n-1,j))(x). \end{cases}$$

The main theorem of algebraic Morse theory can be stated as follows.

Theorem 2.9. *$C_*^{\mathcal{M}}$ is a complex of left R-modules which is homotopy equivalent to the original complex C_*. Moreover, the maps defined below are chain homotopies between C_* and $C_*^{\mathcal{M}}$:*

$$f : \begin{cases} C_n & \to \quad C_n^{\mathcal{M}} \\ x \in C_{n,i} & \mapsto \quad \sum_{(n,j) \in V_n^{\mathcal{M}}} \Gamma((n,i),(n,j))(x), \end{cases}$$

$$g : \begin{cases} C_n^{\mathcal{M}} & \to \quad C_n \\ x \in C_{n,i} & \mapsto \quad \sum_{(n,j) \in V_n} \Gamma((n,i),(n,j))(x). \end{cases}$$

2.4 Two-sided Anick resolution

Starting from the reduced bar resolution of an one-vertex quiver algebra A which is viewed as a chain complex of projective A^e-modules, Sköldberg [17] constructed a “smaller” A^e-projective resolution of A using algebraic Morse theory, which is called the two-sided Anick resolution of A. It was pointed out in [7] that Sköldberg’s construction generalizes to general quiver algebras.

Let $A = kQ/I$ be a quiver algebra, let $\mathcal{G}$ be a reduced Gröbner basis of the ideal I, and denote $W := \mathrm{Tip}(\mathcal{G})$. Denote by $B(A) = (B_*(A), d_*)$ the reduced bar resolution of A (cf. Section 2.2). Similar as in [7], we define a new quiver $Q_W = (V, E)$ with respect to W, which is called the Ufnarovskiĭ graph (or just Uf-graph).

Definition 2.10. *A Uf-graph $Q_W = (V, E)$ with respect to W of the algebra $A = kQ/I$ is given by*

$$V := Q_0 \cup Q_1 \cup \{u \in Q_{\geq 0} \mid u \text{ is a proper right factor of some } v \in W\}$$

$$E := \{e \to x \mid e \in Q_0,\ x = ex \in Q_1\} \ \cup$$
$$\{u \to v \mid uv \in \langle \mathrm{Tip}(\mathcal{G})\rangle,\ \text{but } w \notin \langle \mathrm{Tip}(\mathcal{G})\rangle \text{ for } uv = wp,\ l(p) \geq 1\}$$

Using Uf-graph Q_W one can define (for each $i \geq -1$) the i-chains, which form a subset of generators of $B_i(A) = \bigoplus A^e[w_1|\cdots|w_i]$ for $i \geq 0$, with $w_1, \cdots, w_i \in \mathrm{NonTip}(I)\backslash Q_0$ and $w_1 \cdots w_i \in Q_{\geq 0}$.

Definition 2.11.
- *The set $W^{(i)}$ of i-chains consists of all sequences $[w_1|\cdots|w_{i+1}]$ with each $w_k \in \mathrm{NonTip}(I)\backslash Q_0$, such that*

$$e \to w_1 \to w_2 \to \cdots \to w_{i+1}$$

is a path in Q_W. And define $W^{(-1)} := Q_0$.

- *For for all $p \in Q_{\geq 0}$, define*

$$V_{p,i}^{(n)} = \{[w_1|\cdots|w_n] \mid p = w_1 \cdots w_n,\ [w_1|\cdots|w_{i+1}] \in W^{(i)},\ [w_1|\cdots|w_{i+2}] \notin W^{(i+1)}\}$$

By using the definition above, we can define a partial matching $\mathcal{M}$ to be the set of arrows of the following form in the weighted quiver $G(B_*)$, where $B_* = B(A)$ is the reduced bar resolution of the algebra $A = kQ/I$:

$$[w_1|\cdots|w_{i+1}|w'_{i+2}|w''_{i+2}|w_{i+3}|\cdots|w_n] \overset{(-1)^{i+2}}{\longrightarrow} [w_1|\cdots|w_{i+2}|\cdots|w_n]$$

where

$$w = w_1 \cdots w_n = w_1 \cdots w'_{i+2} w''_{i+2} \cdots w_n, \quad w_{i+2} = w'_{i+2} w''_{i+2},$$

$$[w_1|\cdots|w_{i+1}|w'_{i+2}|w''_{i+2}|w_{i+3}|\cdots|w_n] \in V_{w,i+1}^{(n+1)}, \quad [w_1|\cdots|w_{i+2}|\cdots|w_n] \in V_{w,i}^{(n)}.$$

Theorem 2.12. ([7, *Theorem* 4.3]) *The partial matching $\mathcal{M}$ is a Morse matching of $G(B_*)$ with the set of critical vertices in n-th component is $W^{(n-1)}$.*

Therefore, by Theorem 2.12 and Theorem 2.9, $(B^{\mathcal{M}}(A), d^{\mathcal{M}})$ is also a A^e-projective resolution of A, but the projective A^e-modules in $B^{\mathcal{M}}(A)$ are usually much smaller than $B(A)$. The resolution $(B^{\mathcal{M}}(A), d^{\mathcal{M}})$ is called the two-sided Anick resolution of A in [17] and [7].

Note that for $n \geq 0$, the n-th component of the two-sided Anick resolution $B^{\mathcal{M}}(A)$ is $A \otimes_E kW^{(n-1)} \otimes_E A$. In particular, we have the following identifications:

$$W^{(-1)} = Q_0, \quad W^{(0)} = \{[w_1] \mid w_1 \in Q_1\} \cong Q_1,$$

$$W^{(1)} = \{[w_1|w_2] \mid \ w_1 \in Q_1,\ w_1 w_2 \in \mathrm{Tip}(\mathcal{G})\} \cong \mathrm{Tip}(\mathcal{G}) = W.$$

3 Generalized parallel paths method for computing the first Hochschild cohomology group

3.1 Parallel paths method of Strametz

Let $A = kQ/I$ be a quiver algebra and $E \simeq kQ_0$ be its separable subalgebra as in Section 2.2. Remind that we assume $I \subseteq kQ_{\geq 2}$. The following notations (most of them are taken from [19]) are useful.

Definition 3.1.
- *Let ε be a path in Q and $(\alpha, \gamma) \in Q_1//Q$. Denote by $\varepsilon^{(\alpha,\gamma)}$ the sum of all nonzero paths obtained by replacing one appearance of the arrow α in ε by path γ. If the path ε does not contain the arrow α, we set $\varepsilon^{(\alpha,\gamma)} = 0$.*
- *If we fix a Gröbner basis $\mathcal{G}$ of I in kQ, then there is a k-linear basis $\mathcal{B}$ of the algebra A with respect to $\mathcal{G}$. Indeed $\mathcal{B}$ is given by* $\mathrm{NonTip}(\mathcal{G})$ *modulo I and there is a bijection between the elements of $\mathcal{B}$ and the elements of* $\mathrm{NonTip}(\mathcal{G})$ *(cf. Section 2.1). By this reason, we often identify $\mathcal{B}$ with* $\mathrm{NonTip}(\mathcal{G})$.
- *Let the canonical projection be written as $\pi : kQ \to A$. For a path $p \in Q$, $\pi(p)$ can be uniquely written as a linear combination of the elements in the basis $\mathcal{B}$.*
- *If X is a set of paths of Q and e a vertex of Q, the set Xe is formed by the paths of X with source vertex e. In the same way eX denotes the set of all paths of X with terminus vertex e.*

We now give a brief review about Strametz's method in [19] for computing the first Hochschild cohomology groups of monomial algebras. Recall that $A = kQ/I$ is a monomial algebra means that the ideal I is generated by a set Z of paths in Q. We shall assume that Z is minimal, so that Z is a reduced Gröbner basis of I with $Z = \mathrm{Tip}(Z)$. Note that $\mathrm{NonTip}(Z)$ modulo I gives a k-basis of A, which we denote by $\mathcal{B}$.

Proposition 3.2. ([*19*, *Proposition* 2.6]) *Let A be a finite dimensional monomial algebra. Then the beginning of the cochain complex of the minimal A^e-projective resolution of A can be described by:*

$$0 \longrightarrow k(Q_0//\mathcal{B}) \xrightarrow{\psi_0} k(Q_1//\mathcal{B}) \xrightarrow{\psi_1} k(Z//\mathcal{B}) \longrightarrow \cdots$$

where the differentials are given by

$$\begin{array}{rcll} \psi_0 & : & k(Q_0//\mathcal{B}) & \to & k(Q_1//\mathcal{B}), \\ & & (e,\gamma) & \mapsto & \sum_{\alpha \in Q_1 e}(\alpha, \pi(\alpha\gamma)) - \sum_{\beta \in eQ_1}(\beta, \pi(\gamma\beta)); \end{array}$$

$$\begin{array}{rcll} \psi_1 & : & k(Q_1//\mathcal{B}) & \to & k(Z//\mathcal{B}), \\ & & (\alpha,\gamma) & \mapsto & \sum_{p \in Z}(p, \pi(p^{(\alpha,\gamma)})). \end{array}$$

In particular, we have $\mathrm{HH}^0(A) \cong \mathrm{Ker}\psi_0$, $\mathrm{HH}^1(A) \cong \mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$.

The proof of Proposition 3.2 uses the minimal A^e-projective resolution of the monomial algebra A given by Bardzell [3] and the following two general lemmas.

Lemma 3.3. ([*19*, *Lemma* 2.2]) *Let $A = kQ/I$ be a quiver algebra, $E \simeq kQ_0$ be the separable subalgebra of A, M be an E-bimodule and T be a A-bimodule. Then the vector space* $\mathrm{Hom}_{A^e}(A \otimes_E M \otimes_E A, T)$ *is isomorphic to* $\mathrm{Hom}_{E^e}(M, T)$.

Proof. It is easy to check that the k-linear maps given by

$$f \mapsto (m \mapsto f(1 \otimes m \otimes 1))$$

and

$$g \mapsto (1 \otimes m \otimes 1 \mapsto g(m))$$

are well-defined and inverse to each other. □

Lemma 3.4. ([19, *Lemma* 2.3]) *Let $A = kQ/I$ be a quiver algebra and $E \simeq kQ_0$ be its separable subalgebra. Let X and Y be the sets of paths of Q and let kX and kY be the corresponding E-bimodules. If X is a finite set, then the vector spaces $k(X//Y)$ and $\mathrm{Hom}_{E^e}(kX, kY)$ are isomorphic.*

Proof. It is easy to check that the k-linear maps given by

$$(x, y) \mapsto (x \mapsto y,\ x' \mapsto 0,\ \text{for } x' \neq x)$$

and

$$f \mapsto \sum_{x \in X} \sum_i \lambda_i (x, p_i)$$

with $f(x) = \sum_i \lambda_i p_i$, $\lambda_i \in k$, $p_i \in Y$, are well-defined and inverse to each other. □

Remark 3.5. *The lemma above is the intrinsic reason why the parallel paths method can not always be used for the quiver algebras with infinite dimension, since we need to restrict X to be finite to make the maps above well-defined. In particular, for $X = \mathrm{Tip}\mathcal{G}$, in order to use above lemma, we need it to be a finite set. When kQ/I is finite dimensional,* $\mathrm{Tip}\mathcal{G}$ *is always a finite set by [11, Proposition 2.10]. When kQ/I is infinite dimensional but I owns a finite Gröbner basis, we can still use above lemma for $X = \mathrm{Tip}\mathcal{G}$. For example, we can use the above lemma to $X = \mathrm{Tip}\mathcal{G}$ for the algebra $k\langle x, y\rangle/\langle x^2\rangle$, but not for the algebra $k\langle x, y\rangle/\langle xyx, xyyx, xyyyx, \cdots\rangle$.*

3.2 Generalized parallel paths method

In this subsection, we will extend parallel paths method for computing the first Hochschild cohomology groups from monomial algebras to general quiver algebras, which we call the generalized parallel paths method.

Let $A = kQ/I$ be a quiver algebra such that I has a finite reduced Gröbner basis $\mathcal{G}$. The following lemma can be seen as a generalization of the beginning of the two-sided minimal projective resolution of a monomial algebra given by Bardzell [3]. Our proof is a careful analysis of the beginning of the two-sided Anick resolution (cf. Section 2.4) of A.

Lemma 3.6. *The beginning of the two-sided Anick resolution $(B^{\mathcal{M}}(A), d^{\mathcal{M}})$ of A can be described by (for simplicity we just denote $d^{\mathcal{M}}$ by d):*

$$\cdots \longrightarrow A \otimes_E \mathrm{Tip}(\mathcal{G}) \otimes_E A \xrightarrow{d_2} A \otimes_E Q_1 \otimes_E A \xrightarrow{d_1} A \otimes_E Q_0 \otimes_E A \xrightarrow{d_0} A \longrightarrow 0,$$

where the differentials can be described as follows:

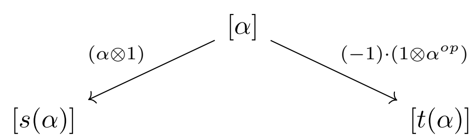
- *for all $1 \otimes e_i \otimes 1 \in A \otimes_E Q_0 \otimes_E A$, $d_0(1 \otimes e_i \otimes 1) = e_i$;*
- *for all $1 \otimes \alpha \otimes 1 \in A \otimes_E Q_1 \otimes_E A$, $d_1(1 \otimes \alpha \otimes 1) = \alpha \otimes s(\alpha) \otimes 1 - 1 \otimes t(\alpha) \otimes \alpha$;*

- *for all* $1 \otimes w \otimes 1 \in A \otimes_E \mathrm{Tip}(\mathcal{G}) \otimes_E A$,

$$d_2(1 \otimes w \otimes 1) = \sum_{\alpha_m \cdots \alpha_1 \in \mathrm{Supp}(g)} \sum_{i=1}^{m} c(\alpha_m \cdots \alpha_1)\alpha_m \cdots \alpha_{i+1} \otimes \alpha_i \otimes \alpha_{i-1} \cdots \alpha_1,$$

where $g \in \mathcal{G}$ *such that* $g = w + \sum_{p \in Q_{\geq 0};\ p \neq w} c(p)p$ *with* $\mathrm{Tip}(g) = w$ *and* $0 \neq c(p) \in k$. *(Note that* $c(w) = 1$ *and* $l(w) \geq 2$.*)*

Proof. By the identifications at the end of Section 2.4, we just need to check the differentials in this lemma. Obviously, $d_0 = \mu_A$ which inherits from the reduced bar resolution of A, is given by the multiplication of A. By the definition of the Morse matching $\mathcal{M}$ in the weighted quiver $G(B_*)$ (cf. Section 2.4), there are no dotted arrows starting from $W^{(-1)} = Q_0$ in the new weighted quiver $G_{\mathcal{M}}(B_*)$, where $B_* := B(A)$ is the reduced bar resolution of A. Thus for $[\alpha] \in W^{(0)}$ with $\alpha \in Q_1$, the zigzag paths from $W^{(0)} = Q_1$ to $W^{(-1)} = Q_0$ can be given by

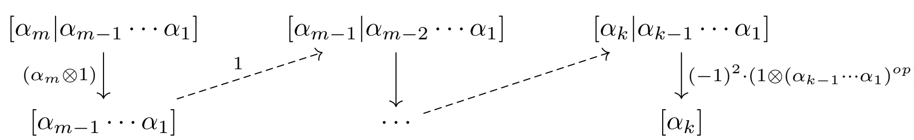


For $1 \otimes w \otimes 1 \in A \otimes_E \mathrm{Tip}(\mathcal{G}) \otimes_E A$, let $w = w_1 w_2$ with $w_1 \in Q_1$. We are going to find all zigzag paths from $[w_1|w_2]$ to some $[\alpha]$ in $G_{\mathcal{M}}(B_*)$ with $\alpha \in Q_1$. First of all, in the original reduced bar resolution B_*, the differential of $[w_1|w_2]$ is

$$w_1[w_2] + \sum_{p \in \mathrm{Supp}(g);\ p \neq w} c(p)[p] + [w_1]w_2,$$

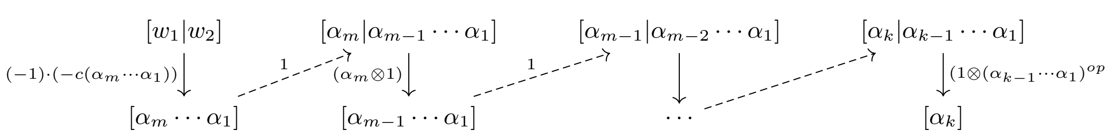
where $w = w_1 w_2 = -\sum_{p \in \mathrm{Supp}(g);\ p \neq w} c(p)p$ in A (modulo I). For some $\alpha_m \cdots \alpha_1 \in \mathrm{Supp}(g)$, there are two cases to be considered.

Case 1. If $\alpha_m \cdots \alpha_1 = w$, the only zigzag path from $[w_1|w_2]$ to $[\alpha_k]$ $(1 \leq k \leq m)$ is given by



Thus the coefficient of $[\alpha_k]$ is $(\alpha_m \cdots \alpha_{k+1}) \otimes (\alpha_{k-1} \cdots \alpha_1)^{op}$.

Case 2. If $\alpha_m \cdots \alpha_1 = p$ with $p \in \mathrm{Supp}(g)$ and $p \neq w$, the only zigzag path from $[w_1|w_2]$ to $[\alpha_k]$ $(1 \leq k \leq m)$ is given by



Thus the coefficient of $[\alpha_k]$ is $c(\alpha_m \cdots \alpha_1)(\alpha_m \cdots \alpha_{k+1}) \otimes (\alpha_{k-1} \cdots \alpha_1)^{op}$.

This finishes the proof of the lemma. $\square$

Now applying the functor $\mathrm{Hom}_{A^e}(-, A)$ to $B^{\mathcal{M}}(A)$ and using the identification in Lemma 3.3 yields the following cochain complex which we denote by $C_{\mathcal{M}}(A) = (C^*_{\mathcal{M}}, \delta^*)$:

$$0 \longrightarrow \mathrm{Hom}_{E^e}(kQ_0, A) \xrightarrow{\delta^0} \mathrm{Hom}_{E^e}(kQ_1, A) \xrightarrow{\delta^1} \mathrm{Hom}_{E^e}(k\mathrm{Tip}(\mathcal{G}), A) \longrightarrow \cdots$$

where the coboundaries δ^0 and δ^1 are given by

$$\delta^0(f)(\alpha) = \alpha \cdot f(s(\alpha)) - f(t(\alpha)) \cdot \alpha$$

$$\delta^1(g)(w) = \sum_{\alpha_m \cdots \alpha_1 \in \mathrm{Supp}(\mathrm{Tip}^{-1}(w))} \sum_{i=1}^{m} c(\alpha_m \cdots \alpha_1) \cdot \alpha_m \cdots \alpha_{i+1} g(\alpha_i) \alpha_{i-1} \cdots \alpha_1$$

where $f \in \mathrm{Hom}_{E^e}(kQ_0, A)$, $\alpha \in Q_1$; $g \in \mathrm{Hom}_{E^e}(kQ_1, A)$, $w \in \mathrm{Tip}(\mathcal{G})$.

If we carry out another identification suggested in Lemma 3.4, we can rewrite the coboundaries and obtain:

Proposition 3.7. *The beginning of the cochain complex $C_{\mathcal{M}}(A)$ can be characterized in the following way*

$$0 \longrightarrow k(Q_0//\mathcal{B}) \xrightarrow{\psi_0} k(Q_1//\mathcal{B}) \xrightarrow{\psi_1} k(\mathrm{Tip}(\mathcal{G})//\mathcal{B}) \longrightarrow \cdots$$

where the maps are given by

$$\begin{array}{rcll} \psi_0 & : & k(Q_0//\mathcal{B}) & \to & k(Q_1//\mathcal{B}), \\ & & (e,\gamma) & \mapsto & \sum_{\alpha \in Q_1 e} (\alpha, \pi(\alpha\gamma)) - \sum_{\beta \in eQ_1} (\beta, \pi(\gamma\beta)); \end{array}$$

$$\begin{array}{rcll} \psi_1 & : & k(Q_1//\mathcal{B}) & \to & k(\mathrm{Tip}(\mathcal{G})//\mathcal{B}), \\ & & (\alpha,\gamma) & \mapsto & \sum_{g \in \mathcal{G}} \sum_{p \in \mathrm{Supp}(g)} c_g(p) \cdot (\mathrm{Tip}(g), \pi(p^{(\alpha,\gamma)})). \end{array}$$

with $g = \sum_{p \in \mathrm{Supp}(g)} c_g(p) p$, $c_g(p) \in k$.

In particular, we have $\mathrm{HH}^0(A) \cong \mathrm{Ker}\psi_0$, $\mathrm{HH}^1(A) \cong \mathrm{Ker}\psi_1 / \mathrm{Im}\psi_0$.

Proof. The verifications are straightforward. □

We should mention that there is an analogous result in [15, Section 2.2] given by Rubio y Degrassi, Schroll and Solotar, whose method is based on the Chouhy-Solotar projective resolution.

Theorem 3.8. (*Compare to* [*19, Theorem* 2.7]) *The bracket*

$$[(\alpha,\gamma),(\beta,\varepsilon)] = (\beta, \pi(\varepsilon^{(\alpha,\gamma)})) - (\alpha, \pi(\gamma^{(\beta,\varepsilon)}))$$

for all $(\alpha,\gamma), (\beta,\varepsilon) \in Q_1//\mathcal{B}$ induces a Lie algebra structure on $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$, such that $\mathrm{HH}^1(A)$ and $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$ are isomorphic as Lie algebras.

Proof. As $B(A)$ and $B^{\mathcal{M}}(A)$ are projective resolutions of A^e-modules A, there exist, thanks to the Comparison Theorem, chain maps $w : B(A) \to B^{\mathcal{M}}(A)$ and $\xi : B^{\mathcal{M}}(A) \to B(A)$ such that they give inverse homotopy equivalences.

$$\begin{array}{lccccccc} B(A): & \cdots \longrightarrow & A \otimes_E A^+ \otimes_E A & \longrightarrow & A \otimes_E A & \longrightarrow & A & \longrightarrow 0 \\ & & w_1 \downarrow \uparrow \xi_1 & & w_0 \downarrow \uparrow \xi_0 & & \| & \\ B^{\mathcal{M}}(A): & \cdots \longrightarrow & A \otimes_E kQ_1 \otimes_E A & \longrightarrow & A \otimes_E kQ_0 \otimes_E A & \longrightarrow & A & \longrightarrow 0 \end{array}$$

Indeed, according to the algebraic Morse theory (Theorem 2.9), we can make the above chain maps explicitly as follows:

- $w_0 : A \otimes_E A \to A \otimes_E kQ_0 \otimes_E A$, $\quad \lambda \otimes \mu \mapsto \lambda \otimes e \otimes \mu$;
- $\xi_0 : A \otimes_E kQ_0 \otimes_E A \to A \otimes_E A$, $\quad \lambda \otimes e \otimes \mu \mapsto \lambda \otimes \mu$;

- $w_1 : A \otimes_E A^+ \otimes_E A \to A \otimes_E kQ_1 \otimes_E A$,

$$\lambda \otimes \alpha_n \cdots \alpha_1 \otimes \mu \mapsto \sum_{i=1}^{n} \lambda \alpha_n \cdots \alpha_{i+1} \otimes \alpha_i \otimes \alpha_{i-1} \cdots \alpha_1 \mu;$$

- $\xi_1 : A \otimes_E kQ_1 \otimes_E A \to A \otimes_E A^+ \otimes_E A, \quad \lambda \otimes \alpha \otimes \mu \mapsto \lambda \otimes \alpha \otimes \mu$,

where $\lambda, \mu \in A$, $\alpha, \alpha_1, \cdots, \alpha_n \in Q_1$ and $\alpha_n \cdots \alpha_1 \in \mathrm{NonTip}(I) \backslash Q_0$.

By the identifications in Lemma 3.3 and Lemma 3.4, there is an isomorphism of vector spaces from $k(Q_1//\mathcal{B})$ to $\mathrm{Hom}_{A^e}(A \otimes_E kQ_1 \otimes_E A, A)$ which we denote by i.

Using the above comparison morphisms, if we have defined some bilinear operation on $\mathrm{HH}^1(A)$, then we can define an induced operation on $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$. This method is widely used, see for example a precise introduction in Volkov [20, Section 3]. For simplicity, we often omit the sign $\circ$ and write the composition of morphisms directly. In this way, the Lie bracket on $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$ can be defined by

$$[(\alpha, \gamma), (\beta, \varepsilon)] := i^{-1}([i((\alpha, \gamma))w_1, i((\beta, \varepsilon))w_1]_{\mathrm{HH}} \cdot \xi_1),$$

for all $(\alpha, \gamma), (\beta, \varepsilon) \in Q_1//\mathcal{B}$. Actually,

$$\begin{aligned} [(\alpha, \gamma), (\beta, \varepsilon)] &= i^{-1}(i((\alpha, \gamma))w_1 p_A i((\beta, \varepsilon))w_1 \xi_1 - i((\beta, \varepsilon))w_1 p_A i((\alpha, \gamma))w_1 \xi_1) \\ &= i^{-1}(i((\alpha, \gamma))w_1 p_A i((\beta, \varepsilon)) - i((\beta, \varepsilon))w_1 p_A i((\alpha, \gamma))) \\ &= (\beta, \pi(\varepsilon^{(\alpha, \gamma)})) - (\alpha, \pi(\gamma^{(\beta, \varepsilon)})). \end{aligned}$$

The Lie algebra isomorphism from $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$ to $\mathrm{HH}^1(A)$ is given by $i(-) \circ w_1$, since the equation

$$i([(\alpha, \gamma), (\beta, \varepsilon)])w_1 = [i((\alpha, \gamma))w_1, i((\beta, \varepsilon))w_1]_{\mathrm{HH}}$$

naturally holds at the cohomology level. □

3.3 Graded structure of $\mathrm{HH}^1(A)$ when I is a homogeneous ideal

In this subsection, we discuss the graded structure of the Lie algebra $\mathrm{HH}^1(A)$ when I is a homogeneous ideal. Note that there is a similar discussion for this graded structure in [15, Section 2.3], however, our Lemma 3.10 and Proposition 3.11 below are new.

By Theorem 3.8, we can regard the Lie algebra structures on $\mathrm{HH}^1(A)$ and on $\mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$ as the same one without distinguishing. Let $A = kQ/I$ be a quiver algebra. Let $\mathcal{G}$ be a reduced Gröbner basis of I, $\mathcal{B} \subseteq Q_{\geq 0}$ be the k-linear basis of A with respect to $\mathcal{G}$, and denote $\mathcal{B}_n := \mathcal{B} \cap Q_n$. Assume that the ideal I is homogeneous, that is, under the length-lexicographic order, for all $g \in \mathcal{G}$, the summands of g are in same length. Therefore, if $(\alpha, \gamma) \in Q_1//\mathcal{B}_n$, $(\beta, \varepsilon) \in Q_1//\mathcal{B}_m$, then

$$[(\alpha, \gamma), (\beta, \varepsilon)] \in k(Q_1//\mathcal{B}_{n+m-1}).$$

Actually, $k(Q_1//\mathcal{B}) = \bigoplus_{i \in \mathbb{N}} k(Q_1//\mathcal{B}_i)$, $\mathrm{HH}^1(A) = \mathrm{Ker}\psi_1/\mathrm{Im}\psi_0$. Then, if we set

- $L_{-1} := k(Q_1//Q_0) \cap \mathrm{Ker}\psi_1$
- $L_0 := k(Q_1//Q_1) \cap \mathrm{Ker}\psi_1 \Big/ \langle \sum_{a \in Q_1 e}(a, a) - \sum_{b \in eQ_1}(b, b) | e \in Q_0 \rangle$ and
- $L_i := k(Q_1//\mathcal{B}_{i+1}) \cap \mathrm{Ker}\psi_1 \Big/ \langle \sum_{a \in Q_1 e}(a, \pi(a\gamma)) - \sum_{b \in eQ_1}(b, \pi(\gamma b)) | (e, \gamma) \in Q_0//\mathcal{B}_i \rangle$

for all $i \in \mathbb{N}, i \geq 1$, we obtain $\mathrm{HH}^1(A) = \bigoplus_{i \geq -1} L_i$, and $[L_i, L_j] \subseteq L_{i+j}$ for all $i, j \geq -1$, where $L_{-2} = 0$. The above discussion (following the idea from [19, Section 4]) shows that we have a graduation on the Lie algebra $\mathrm{HH}^1(A)$ if I is a homogeneous ideal of kQ.

Remark 3.9. *Note that the above notions L_{-1} and L_0 and the following lemma make sense even if the ideal I is not homogeneous and they will be used in later sections.*

If we restrict our discussion to the case where $A = kQ/I$ is finite dimensional, we can give some more precise description about the Gröbner basis of I and the L_{-1}.

Lemma 3.10. *Let $A = kQ/I$ be a finite dimensional quiver algebra (with $I \subseteq kQ_{\geq 2}$). Let a be a loop of the quiver Q. Then there exists an integer $m \geq 2$ such that $a^m \in \mathrm{Tip}(\mathcal{G})$ and $a^{m-1} \in \mathcal{B}$.*

Proof. Clearly there exists an integer $m \geq 2$ such that $a^m \in \mathrm{Tip}(\mathcal{G})$, since otherwise, the elements $a^2, a^3, \cdots$ ($\subseteq \mathrm{NonTip}(\mathcal{G})$) are k-linearly independent, contradicting the finite-dimensionality of A. Moreover, $a^{m-1} \in \mathcal{B}$ since $\mathcal{G}$ is tip-reduced. □

The next result is a proper generalization of some result for monomial algebras in [19, Proposition 4.2] and it may be useful when one deal with algebras defined by homogeneous ideals.

Proposition 3.11. *Let $A = kQ/I$ be a finite dimensional quiver algebra where I is a homogeneous ideal. Each of the following conditions implies $L_{-1} = 0$:*

(1) The quiver does not have a loop.

(2) For every loop $(a, e) \in Q_1//Q_0$ of Q, the characteristic of the field k does not divide the integer $m \geq 2$ for which $a^m \in \mathrm{Tip}(\mathcal{G})$, $a^{m-1} \in \mathcal{B}$.

(3) The characteristic of k is equal to 0.

Proof. (1) : Clear.

(2) : Let $(a, e) \in Q_1//Q_0$ be a loop of Q. By Lemma 3.10, there exists an integer $m \geq 2$ such that $p := a^m \in \mathrm{Tip}(\mathcal{G})$ and $a^{m-1} \in \mathcal{B}$. If the characteristic of k does not divide m, then $\pi(p^{(a,e)}) = ma^{m-1} \neq 0$. Consider $\mathrm{Tip}^{-1}(p) \in \mathcal{G}$, if there is another $p' \in \mathrm{Supp}(\mathrm{Tip}^{-1}(p))$ with $p' \neq p$, such that a^{m-1} is a summand of $\pi(p'^{(a,e)})$, so there exists a decomposition of p' such that $p' = p_1 a p_2$, $p_1 p_2$ is a summand of $p'^{(a,e)}$ and a^{m-1} is a summand of $\pi(p_1 p_2)$. However, since $\mathrm{Tip}(\mathrm{Tip}^{-1}(p)) = p \in \mathrm{Tip}(\mathcal{G})$, $p = a^m > p' = p_1 a p_2$ under the length-lexicographic order. This will lead to $p_1 p_2 < a^{m-1}$, which is contradictory to a^{m-1} is a summand of $\pi(p_1 p_2)$. Therefore, for $g := \mathrm{Tip}^{-1}(p) \in \mathcal{G}$, we have $\sum_{p \in \mathrm{Supp}(g)} c_g(p) \pi(p^{(a,e)}) \neq 0$.

If the (a, e) above is a summand of some nonzero element $t \in L_{-1}$, then there exists $(b_1, e_1) \in k(Q_1//Q_0)$ and is also a summand of t, such that for some $q_1 \in \mathrm{Supp}(g)$, $l(q_1) = m$ and a^{m-1} is a summand in $\pi(q_1^{(b_1,e_1)})$.

- If a^{m-1} is not a summand of $q_1^{(b_1,e_1)}$, since $a^m > q_1$, the only case is that $q_1 = a^{m_1} b_1 q_1'$ with $l(q_1') = m - m_1 - 1$, and so we get $q_1' > a^{m-m_1-1}$ and $a > b_1$.
- If a^{m-1} is a summand of $q_1^{(b_1,e_1)}$, then $q_1 = a^{k_1} b a^{k_2}$, with $k_1 + k_2 = m - 1$. By $a^m > a^{k_1} b a^{k_2}$, we also have $a > b_1$.

Now (b_1, e_1) is also a loop with $b_1^{n_1} \in \mathrm{Tip}\mathcal{G}$, $b_1^{n_1-1} \in \mathcal{B}$. Since (b_1, e_1) is also a summand of t, there exists some $(b_2, e_2) \in k(Q_1//Q_0)$ and is also a summand of t, such that for some $q_2 \in \mathrm{Supp}(\mathrm{Tip}^{-1}(b_1^{n_1}))$, $l(q_2) = n_1$ and $b_1^{n_1-1}$ is a summand in $\pi(q_2^{(b_2,e_2)})$. Same as before, we have $b_1 > b_2$.

Repeat this process, we get an infinite descending sequence of loops:

$$a > b_1 > b_2 > \cdots$$

This is contradictory to the fact that Q is a finite quiver. Thus there is no nonzero element t in L_{-1}.

(3) : Clear, because (3) implies (2).

□

Remark 3.12. *(i) The proof of [19, Proposition 4.2] uses [19, Lemma 4.1], but the latter one can not be generalized directly. Motivated by [19, Lemma 4.1] one might expect the following in our situation:*

L_{-1} equals 0 if and only if there exists for every loop $(a,e) \in Q_1//Q_0$, an element $g \in \mathcal{G}$, such that $\sum_{p\in \mathrm{Supp}(g)} c_g(p)\pi(p^{(a,e)}) \neq 0$.

However, the condition is not sufficient in general. For example, let $A = k\langle x, y\rangle/\langle x^3 + yx^2, xy + yx, y^2\rangle$ with $\mathrm{char}k = 2$, which is a finite dimensional symmetric algebra with $\langle x^3 + yx^2, xy+yx, y^2\rangle$ a homogeneous ideal of $k\langle x, y\rangle$. It can be checked that $\mathcal{G} = \{x^3+yx^2, xy+yx, y^2\}$ is a Gröbner basis of the ideal $\langle x^3 + yx^2, xy + yx, y^2\rangle$ under the length-lexicographic order with respect to $x > y$ and $\mathrm{Tip}\mathcal{G} = \{x^3, xy, y^2\}$. We have

$$\psi_1((x,1)) = 3(x^3, x^2) + 2(x^3, yx) + 2(xy, y) = (x^3, x^2) \neq 0,$$

$$\psi_1((y,1)) = (x^3, x^2) + 2(xy, x) + 2(y^2, y) = (x^3, x^2) \neq 0.$$

But $(x,1) + (y,1) \in L_{-1}$ and so $L_{-1} \neq 0$.

(ii) For Brauer graph algebras (whenever the ideal I is homogeneous or not), we can deduce a stronger conclusion from the condition (2) in Proposition 3.11, see Proposition 4.5.

4 Brauer graph algebras

Brauer graph algebras form an important class of finite dimensional tame algebras (see for example, Schroll [16]) and they beyond the scope of monomial algebras. In recent years, there has been a renewed interest in Brauer graph algebras. In this section, we will use our generalized parallel paths method to study the first Hochschild cohomology groups of Brauer graph algebras.

4.1 Review on Brauer graph algebras

Definition 4.1. ([*16, Definition* 2.1]) *A Brauer graph G is a tuple $G = (V, E, m, o)$ where*

- *(V, E) is a finite (unoriented) connected graph with vertex set V and edge set E.*
- *$m : V \to \mathbb{Z}_{>0}$ is a function, called the multiplicity or multiplicity function of G.*
- *o is called the orientation of G which is given, for every vertex $v \in V$, by a cyclic ordering of the edges incident with v such that if v is a vertex incident to a single edge i then if $m(v) = 1$, the cyclic ordering at v is given by i and if $m(v) > 1$ the cyclic ordering at v is given by $i < i$.*

We note that the Brauer graph $G = (V, E)$ may contain loops and multiple edges. Denote by $val(v)$ the valency of the vertex $v \in V$; It is defined to be the number of edges in G incident to v, with the convention that a loop is counted twice. We call the edge i with vertex v truncated at v if $m(v)val(v) = 1$.

Given a Brauer graph $G = (V, E, m, o)$, we can define a quiver $Q_G = (Q_0, Q_1)$ as follows:

$$Q_0 := E,$$

$$Q_1 := \{i \to j \mid i, j \in E, \text{ there exists } v \in V, \text{ such that } i < j \text{ belong to } o(v)\}.$$

For $i \in E$, if $v \in V$ is a vertex of i and i is not truncated at v, then there is a special i-cycle $C_v(\alpha)$ at v which is an oriented cycle given by $o(v)$ in Q_G with the starting arrow α (where the starting vertex of α in Q_G is i). Note that if i is a loop at v, there are exactly two special i-cycles at v; for an example, see [11, Example 2.3 (1)]. However, if we do not care about the starting arrows of the special cycles, then there is a unique special cycle (up to cyclic permutation) at each non-truncated vertex v and we will just denote it by C_v. The corresponding path algebra of the above quiver Q_G is denoted by kQ_G.

We now define an ideal I_G in kQ_G generated by three types of relations. For this recall that we identify the set of edges E of a Brauer graph G with the set of vertices Q_0 of the corresponding quiver Q_G and that we denote the set of vertices of the Brauer graph by V.

- Relations of type I
$$C_v(\alpha)^{m(v)} - C_{v'}(\alpha')^{m(v')}$$
for any $i \in Q_0$ and for any special i-cycles $C_v(\alpha)$ and $C_{v'}(\alpha')$ at v and v', respectively, such that both v and v' are not truncated.
- Relations of type II
$$\alpha C_v(\alpha)^{m(v)}$$
for any $i \in Q_0$, any $v \in V$ and where $C_v(\alpha)$ is a special i-cycle at v with starting arrow α.
- Relations of type III
$$\beta\alpha$$
for any $i \in Q_1$ such that $\beta\alpha$ is not a subpath of any special cycle except if $\beta = \alpha$ is a loop associated with a vertex v of valency one and multiplicity $m(v) > 1$.

The quotient algebra $A = kQ_G/I_G$ is called the Brauer graph algebra of the Brauer graph G. In fact, A is a finite dimensional symmetric algebra, that is, $A \cong \mathrm{Hom}_k(A, k)$ as A-A-bimodules. Moreover, A is also a special biserial algebra, which means that A satisfies the following conditions:

(1) At every vertex i in Q_G, there are at most two arrows starting at i and there are at most two arrows ending at i.

(2) For every arrow α in Q_G, there exists at most one arrow β such that $\beta\alpha \notin I_G$ and there exists at most one arrow γ such that $\alpha\gamma \notin I_G$.

We will also use the following notion on a Brauer graph G from Guo and Liu [11].

Definition 4.2. ([*11, Definition* 2.4]) *For each vertex v in a Brauer graph G, we define the graded degree $grd(v)$ as follows. If G is given by a single edge with both vertices v and v' of multiplicity* 1, *then $grd(v) = grd(v') = 1$; Otherwise*

$$grd(v) = \begin{cases} m(v)val(v), & \text{if } m(v)val(v) > 1; \\ grd(v'), & \text{if } m(v)val(v) = 1. \end{cases}$$

4.2 The Gröbner basis of the ideal I_G

In order to use the generalized parallel paths method for a given Brauer graph algebra $A = kQ_G/I_G$, we need to find a Gröbner basis of the ideal I_G in kQ_G. Throughout, we will use the left length-lexicographic order on the set of paths of Q_G. We first show that the three types of generating relations in I_G already give a Gröbner basis.

For convenience, for a quiver Q, $p, q \in Q_{\geq 0}$, we denote $p \mid q$ if p is a subpath of q.

Theorem 4.3. *Let G be a Brauer graph and Q_G be the quiver of G,*

$$R_i = \{\text{the } i\text{-th type of relations in } kQ_G\}.$$

Then the ideal $I_G = \langle R_1 \cup R_2 \cup R_3 \rangle$ of kQ_G has a Gröbner basis $R_1 \cup R_2 \cup R_3$.

Proof. By Theorem 2.4, it suffices to check all the relevant overlap relations. In the following proof, it is clear that for any relation f in R_1, we may without loss of generality assume that under the fixed left length-lexicographic order the coefficient of $\mathrm{Tip}(f)$ is 1.

- For all $f, g \in R_2 \cup R_3$, f, g are monomial relations, $o(f, g, *, *) = 0$;
- for all $f = C_v(\alpha)^{m(v)} - C_{v'}(\alpha')^{m(v')} \in R_1$, for all $g \in R_2 \cup R_3$. If there exists $b, c \in Q_{G\geq 0}$, such that $o(f, g, b, c) \neq 0$, then
$$o(f, g, b, c) = -C_{v'}(\alpha')^{m(v')} \cdot c;$$
- for all $f \in R_2 \cup R_3$, for all $f = C_v(\alpha)^{m(v)} - C_{v'}(\alpha')^{m(v')} \in R_1$. If there exists $b, c \in Q_{G\geq 0}$, such that $o(f, g, b, c) \neq 0$, then
$$o(f, g, b, c) = b \cdot C_{v'}(\alpha')^{m(v')};$$
- for all $f = C_v(\alpha)^{m(v)} - C_{v_1}(\alpha')^{m(v_1)}, g = C_{v_2}(\alpha)^{m(v_2)} - C_{v_3}(\alpha')^{m(v_3)} \in R_1$. If there exists $b, c \in Q_{G\geq 0}$, such that $o(f, g, b, c) \neq 0$, then
$$o(f, g, b, c) = b \cdot C_{v_3}(\alpha')^{m(v_3)} - C_{v_1}(\alpha')^{m(v_1)} \cdot c.$$

By the construction of I_G, for a path $p \in Q_{G\geq 0}$ that satisfies $p \mid C_v^{m(v)}$, if $p \cdot C_v(\alpha)^{m(v)} \neq 0$ in kQ_G (respectively, $C_v(\alpha)^{m(v)} \cdot p \neq 0$ in kQ_G), then $p \cdot C_v(\alpha)^{m(v)} \in \langle R_2 \rangle$ (respectively, $C_v(\alpha)^{m(v)} \cdot p \in \langle R_2 \rangle$); for a path $p \in Q_{G\geq 0}$ that satisfies $p \nmid C_v^{m(v)}$, if $p \cdot C_v(\alpha)^{m(v)} \neq 0$ in kQ_G (respectively, $C_v(\alpha)^{m(v)} \cdot p \neq 0$ in kQ_G), then $p \cdot C_v(\alpha)^{m(v)} \in \langle R_3 \rangle$ (respectively, $C_v(\alpha)^{m(v)} \cdot p \in \langle R_3 \rangle$).

Therefore, all the nonzero overlap relations with respect to $R_1 \cup R_2 \cup R_3$ can be divided by some monomial relations in $R_2 \cup R_3$. Hence $R_1 \cup R_2 \cup R_3$ is a Gröbner basis of I_G by Theorem 2.4. □

Remark 4.4. *Sometimes the above Gröbner basis of I_G is not tip-reduced. Some relations of type II can be reduced by the relations of type I and type III. However, with respect to an admissible order on kQ_G, we can reduce $R_1 \cup R_2 \cup R_3$ to a reduced Gröbner basis (which we denote by $\mathcal{G}$) consisting of R_1, R_3 and part of R_2.*

The following proposition is a strengthened and generalized version of Proposition 3.11 for Brauer graph algebras.

Proposition 4.5. *Let A be a Brauer graph algebra associated with a Brauer graph G. Suppose that for every loop $(a, e) \in Q_1//Q_0$ of $Q := Q_G$, the characteristic of the field k does not divide the integer $m \geq 2$ for which $a^m \in \mathrm{Tip}\mathcal{G}$, $a^{m-1} \in \mathcal{B}$. Then every nonzero element in $\mathrm{Ker}\psi_1$ will not have summands in $k(Q_1//Q_0)$.*

Proof. First we recall from Proposition 3.7 that the map $\psi_1 : k(Q_1//\mathcal{B}) \to k(\mathrm{Tip}(\mathcal{G})//\mathcal{B})$ is defined by

$$(\alpha, \gamma) \mapsto \sum_{g \in \mathcal{G}} \sum_{p \in \mathrm{Supp}(g)} c_g(p) \cdot (\mathrm{Tip}(g), \pi(p^{(\alpha,\gamma)}))$$

where $g = \sum_{p \in \mathrm{Supp}(g)} c_g(p) p$, $c_g(p) \in k$, and $\mathcal{B}$ is identified with $\mathrm{NonTip}(\mathcal{G})$ as before.

Now let $(a, e) \in Q_1//Q_0$ with a a loop of Q. Since A is finite dimensional, by Lemma 3.10, there exists an integer $m \geq 2$ such that $p := a^m \in \mathrm{Tip}(\mathcal{G})$ and $a^{m-1} \in \mathcal{B}$. If the characteristic of k does not divide m, then $\pi(p^{(a,e)}) = ma^{m-1}$ is different from 0. Now consider a special cycle C_v with $a \mid C_v$.

- If there is another arrow in C_v different from a, then according to the relations of the third type, $a^2 \in \mathcal{G}$, and $2(a^2, a)$ will not be reduced by other summands of the elements in $\mathrm{Im}\psi_1$.
- If a is the unique arrow in C_v, then by the definition of Brauer graph algebra and the property of the reduced Gröbner basis, exactly one of the following two cases holds: $a^{m(v)+1} \in \mathcal{G}$ or $a^{m(v)} \in \mathrm{Tip}\mathcal{G}$. In the first case, $(m(v)+1)(a^{m(v)+1}, a^{m(v)})$ will not be reduced by other summands of the elements in $\mathrm{Im}\psi_1$. In the second case, $\mathrm{Tip}^{-1}(a^{m(v)}) = a^{m(v)} - q$ where q is also a special cycle in A. Therefore, there does not exist some $(b, \varepsilon) \in k(Q_1//\mathcal{B})$, such that $m(v)(a^{m(v)}, a^{m(v)-1})$ becomes a summand of $\psi_1((b, \varepsilon))$.

□

Remark 4.6. *When I_G is a homogeneous ideal of kQ_G, the conclusion of Proposition 4.5 is equal to $L_{-1} = 0$, which can also be deduced from Proposition 3.11.*

4.3 Lie algebra structure on first Hochschild cohomology group of a Brauer graph algebra

In this subsection we assume that the characteristic of the ground field k is 0, unless otherwise stated. We often write simply Q for Q_G.

Let $A = kQ_G/I_G$ be a Brauer graph algebra, consider (for ψ_1, see Proposition 3.7)

$$L_0 := k(Q_1//Q_1) \cap \mathrm{Ker}\psi_1 \Big/ \langle \sum_{a \in Q_1 e} (a, a) - \sum_{b \in eQ_1} (b, b) | e \in Q_0 \rangle,$$

which is a Lie subalgebra of $\mathrm{HH}^1(A)$. Furthermore, we can take L_{00} which is given by

$$L_{00} := \langle (\alpha, \alpha) | \alpha \in Q_1 \rangle \cap \mathrm{Ker}\psi_1 \Big/ \langle \sum_{a \in Q_1 e} (a, a) - \sum_{b \in eQ_1} (b, b) | e \in Q_0 \rangle.$$

Actually, L_{00} is an abelian Lie subalgebra of $\mathrm{HH}^1(A)$ and $L_{00} \subseteq L_0$.

Lemma 4.7. *Let $G = (V, E)$ be a Brauer graph and $A = kQ_G/I_G$ be the corresponding Brauer graph algebra. Then*

$$\dim_k L_{00} = |E| - |V| + 2.$$

Proof. Let $g \in \mathcal{G}$ be an element in the reduced Gröbner basis of I_G, then g is a relation of type I or g is a monomial relation. Fix a $(\alpha, \alpha) \in Q_1//Q_1$, and consider the summand corresponding to g in $\psi_1((\alpha, \alpha))$. There are two cases to be considered.

Case 1. If g is a monomial relation, then $(g, \pi(g^{(\alpha,\alpha)})) = 0$.

Case 2. If g is a relation of type I, then $g = C_v(\beta)^{m(v)} - C_w(\gamma)^{m(w)}$ where $C_v(\beta), C_w(\gamma)$ are special cycles in the Brauer graph G such that both v and w are not truncated. Without loss of generality, let $\mathrm{Tip}(g) = C_v(\beta)^{m(v)}$.

- If $\alpha \mid C_v(\beta)$ and $\alpha \nmid C_w(\gamma)$,

$$\sum_{p \in \mathrm{Supp}(g)} c_g(p) \cdot (\mathrm{Tip}(g), \pi(p^{(\alpha,\alpha)})) = m(v)(C_v(\beta)^{m(v)}, C_w(\gamma)^{m(w)});$$

- if $\alpha \nmid C_v(\beta)$ and $\alpha \mid C_w(\gamma)$,

$$\sum_{p \in \mathrm{Supp}(g)} c_g(p) \cdot (\mathrm{Tip}(g), \pi(p^{(\alpha,\alpha)})) = -m(w)(C_v(\beta)^{m(v)}, C_w(\gamma)^{m(w)});$$

- if $\alpha \mid C_v(\beta)$ and $\alpha \mid C_w(\gamma)$, then $v = w$, and

$$\sum_{p \in \mathrm{Supp}(g)} c_g(p) \cdot (\mathrm{Tip}(g), \pi(p^{(\alpha,\alpha)})) = (m(v) - m(w))(C_v(\beta)^{m(v)}, C_w(\gamma)^{m(w)}) = 0.$$

Therefore, the following two kinds of typical elements lie in $\langle(\alpha,\alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1$:

- $(\alpha_j, \alpha_j) - (\alpha_0, \alpha_0)$ such that α_0, α_j are in the same special cycle C_v, where v is not truncated and $\alpha_j \neq \alpha_0$;
- $\sum_{v \in V^*} k_v(\alpha_v, \alpha_v)$, α_v is an arbitrary arrow in the special cycle at v, where

$$V^* = \{v \in V \mid v \text{ is not truncated}\} \text{ and } k_v = \prod_{w \in V^*, w \neq v} m(w).$$

For each $v \in V^*$, we have $(val(v) - 1)$ elements of the first kind; after reducing by the elements of the first kind above, we get exactly one representative element of the second kind. It is not hard to verify that these $(\sum_{v \in V^*}(val(v) - 1) + 1)$ elements are k-linearly independent and form a k-basis of $\langle(\alpha,\alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1$. Furthermore, we have

$$\begin{aligned}\dim_k(\langle(\alpha,\alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1) &= \textstyle\sum_{v \in V^*}(val(v) - 1) + 1 \\ &= 2|E| - (|V| - |V^*|) - |V^*| + 1 \\ &= 2|E| - |V| + 1.\end{aligned}$$

Since the quiver Q_G is connected, we can check that $\sum_{e \in Q_0} \psi_0(e, e) = 0$ and for a proper subset S_0 of Q_0, the elements $\psi_0(e', e')$ $(e' \in S_0)$ are linearly independent. (Actually, this is what the Lemma 3.5 in [14] is talking about, which is not necessary that A is a monomial algebra.) Therefore,

$$\dim_k(\langle \sum_{a \in Q_1 e}(a, a) - \sum_{b \in eQ_1}(b, b)|e \in Q_0\rangle) = |Q_0| - 1 = |E| - 1.$$

As a result, we get

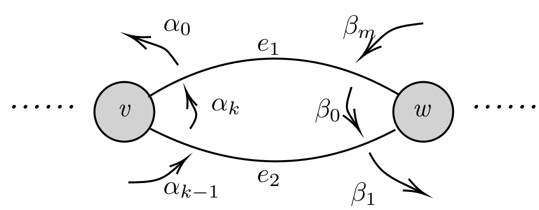
$$\begin{aligned}\dim_k L_{00} &= \dim_k(\langle(\alpha,\alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1) - \dim_k(\langle \textstyle\sum_{a \in Q_1 e}(a, a) - \sum_{b \in eQ_1}(b, b)|e \in Q_0\rangle) \\ &= (2|E| - |V| + 1) - (|E| - 1) \\ &= |E| - |V| + 2\end{aligned}$$

□

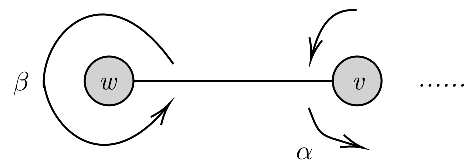
We now describe a set of generators of $\mathrm{HH}^1(A)$ for any Brauer graph algebra A as follows.

Theorem 4.8. *For a Brauer graph algebra A, there is a k-basis $\mathcal{B}_{L,Ker}$ of $\mathrm{Ker}\psi_1$ which consists of the following five subsets:*

- *S_1: the basis of $\langle(\alpha,\alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1$ described in Lemma 4.7.*
- *S_2: elements of the form $(\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1})$ with $\alpha_k \neq \beta_0$, $\alpha_k\beta_0, \beta_0\alpha_k \in R_3$, in the case as:*



- S_3: *elements of the form* $(\beta, C_v(\alpha)^{m(v)})$ *with* $C_w(\beta)^{m(w)} = \beta^{m(w)} > C_v(\alpha)^{m(v)}$*, in the case as:*



- S_4: *elements of the form* (α,p)*, such that* $l(p) > 1$*,* $\psi_1(\alpha, p) = 0$*, and there exist a special cycle* C_v*,* $\alpha \mid C_v$*,* $p \mid C_v^{m(v)}$*.*
- S_5: *the basis of the subspace of* $Im\psi_0$ *generated by all the elements* $(\alpha_1, \alpha_1 p) - (\alpha_0, p\alpha_0)$ *where* p *is a cycle in* Q *and* $\psi_1((\alpha_1, \alpha_1 p)) = \psi_1((\alpha_0, p\alpha_0)) \neq 0$*.*

Furthermore, $\mathrm{Im}\psi_0$ *is contained in* $\langle S_1 \cup S_4 \cup S_5 \rangle$*. The* k*-basis of* $\mathrm{HH}^1(A)$ *induced from* $\mathcal{B}_{L,Ker}$ *will be denoted by* $\mathcal{B}_L$*.*

Proof. By the definition of the elements above, the verification that $S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \subseteq \mathrm{Ker}\psi_1$ is straightforward; as a tip, we just verify that the elements in S_2 is contained in $\mathrm{Ker}\psi_1$. Indeed, assume (without loss of generality) that $C_v(\alpha_0)^{m(v)} > C_w(\beta_1)^{m(w)}$ and $C_v(\alpha_k)^{m(v)} > C_w(\beta_0)^{m(w)}$; then we have

$$\psi_1(\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) = (\alpha_k \beta_0, \pi(C_v(\alpha_0)^{m(v)})) + (\beta_0 \alpha_k, \pi(C_v(\alpha_k)^{m(v)}))$$

$$= (\alpha_k \beta_0, C_w(\beta_0)^{m(w)}) + (\beta_0 \alpha_k, C_w(\beta_1)^{m(w)}),$$

$$\psi_1(\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1}) = (\beta_0 \alpha_k, C_w(\beta_1)^{m(w)}) + (\alpha_k \beta_0, C_w(\beta_0)^{m(w)}).$$

This shows that $(\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1}) \in \mathrm{Ker}\psi_1$.

Now consider $t = \sum_{i \in I} c_i \cdot (\alpha_i, p_i) \in \mathrm{Ker}\psi_1$ with $c_i \in k$, and let $(\alpha, p) = (\alpha_i, p_i)$ be a summand of t. By Proposition 4.5, we can assume that $(\alpha, p) \in Q_1 // (\mathcal{B} \backslash Q_0)$.

First we suppose that $\psi_1(\alpha, p) \neq 0$. This implies that there exists some $g \in \mathcal{G}$,

$$\sum_{p' \in \mathrm{Supp}(g)} c_g(p') \cdot (\mathrm{Tip}(g), \pi(p'^{(\alpha,p)})) \neq 0.$$

Actually, $\psi_1(g, \pi(g^{(\alpha,p)})) = 0$ when $g \in R_2$. If there exist $g \in R_1 \subseteq \mathcal{G}$ and $p_0 \in \mathrm{Supp}(g)$, such that $(\mathrm{Tip}(g), \pi(p_0^{(\alpha,p)})) \neq 0$, which implies $l(p) = 1$, and it will ask for $\alpha = p$. By the form of $\psi_1((\alpha, \alpha))$, we know that (α, α) must appear in a summand t' of t with $t' \in kS_1$. If there exists $g \in R_3 \subseteq \mathcal{G}$, such that $(g, \pi(g^{(\alpha,p)})) \neq 0$, then there are two cases to to be considered. If $\pi(g^{(\alpha,p)}) \neq g^{(\alpha,p)}$, or if $\pi(g^{(\alpha,p)}) = g^{(\alpha,p)}$ and (α, p) is a summand of the elements in S_2, there will be an element in S_2 having a summand equals (α, p), therefore (α, p) must appear in a summand t'' of t with $t'' \in kS_2$. If $\pi(g^{(\alpha,p)}) = g^{(\alpha,p)}$ and (α, p) is not a summand of the elements in S_2, without loss of generality, we can just let $g = \beta\alpha$, so $(\beta\alpha, \pi(g^{(\alpha,p)}) = (\beta\alpha, \beta p)$. Then t must contain a summand of the form $c \cdot ((\beta, q) - (\alpha, p))$ with $\beta p = q\alpha$ and $c \in k$. This leads to $p = p_1 \alpha, q = \beta p_1$ with $p_1 \in Q_{\geq 1}$. Since A is a biserial algebra, $(\beta, q) - (\alpha, p) = \psi_0(e, p_1)$ for some $e \in Q_0$. So $((\beta, q) - (\alpha, p)) \in S_5$.

Next we suppose that $\psi_1(\alpha, p) = 0$. We will show that in this case $(\alpha, p) \in S_3 \cup S_4$. Suppose $(\alpha, p) \notin S_4$, and assume that $l(p) = 1$ with $p \mid C_v$. This means that p is a parallel arrow of α. Since A is a biserial algebra, let p' is the only arrow with $p'p \in \mathcal{B}$. Then we have $p'\alpha \in R_3 \subseteq \mathcal{G}$. However, $(p'\alpha, \pi((p'\alpha)^{(\alpha,p)})) = (p'\alpha, p'p) \neq 0$, which contradicts $\psi_1(\alpha, p) = 0$. Therefore, $l(p) > 1$.

Since $(\alpha, p) \notin S_4$, $p \nmid C_v^{m(v)}$. But in this case, if $(\alpha, p) \notin S_3$, there will exist a relation $g \in R_3 \subseteq \mathcal{G}$, such that $(g, \pi(g^{(\alpha,p)})) \neq 0$, which contradicts $\psi_1(\alpha, p) = 0$.

Summarizing the above discussion we know that $t \in k(S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5)$. □

The basis above can help us to compute the dimension of the first Hochschild cohomology group. In a special case, we have the following formula for this dimension.

Corollary 4.9. *Let A be a Brauer graph algebra whose corresponding Brauer graph $G = (V, E)$ does not have loops. Then* $\dim_k \mathrm{HH}^1(A) = |E| - 2|V| + \sum_{v \in V} m(v) + |S_2| + 2$.

Proof. We first identify the element in S_4. For $(\alpha, p) \in S_4$ with $v \in V$, since the Brauer graph G does not have loops, the starting arrow and the ending arrow of p are all α, that means $p = \alpha C_v(\alpha)^n$, with $s(C_v(\alpha)) = t(C_v(\alpha)) = e$ and $1 \leq n < m(v)$. On the other hand, when $val(v) = 1$, we have $C_v(\alpha) = \alpha$, so $(\alpha, \pi(\alpha C_v(\alpha)^n)) = (\alpha, \pi(\alpha^{n+1}))$ will also contain the elements in S_3. That means

$$S_3 \cup S_4 = \{(\alpha_i, \pi(\alpha_i C_v(\alpha_i)^n)) \mid 1 \leq i \leq val(v), 1 \leq n < m(v)\}.$$

Since these elements also appears in $\psi_0(e, C_v(\alpha)^k)$, all the elements in the form of $(\alpha_i, \pi(\alpha_i C_v(\alpha_i)^n))$ (where n is same and all α_i $(1 \leq i \leq val(v))$ are in same special cycle) will become a unique element in $\mathrm{HH}^1(A)$ reduced by the elements in $\mathrm{Im}\psi_0$. That means the elements in $S_3 \cup S_4$ is just relate to the vertex v and the integer n. Since the elements in other sets S_1, S_2 of the basis are quite clear, to sum them up, we have

$$\begin{aligned} \dim_k \mathrm{HH}^1(A) &= |S_1| + |S_2| + |S_3| + |S_4| + |S_5| - |\mathrm{Im}\psi_0| \\ &= \dim_k L_{00} + |S_2| + \textstyle\sum_{v \in V}(m(v) - 1) \\ &= |E| - 2|V| + \textstyle\sum_{v \in V} m(v) + |S_2| + 2. \end{aligned}$$

□

Another consequence of Theorem 4.8 is the following corollary, which has recently also obtained by Rubio y Degrassi, Schroll and Solotar in [15, Theorem 4.2] using different method (see also remarks before Example 4.13).

Corollary 4.10. *If A is a Brauer graph algebra over a field of characteristic zero such that the corresponding Brauer graph G is different from ($\bullet \!=\!=\! \bullet$) (here both vertices have multiplicity 1), then* $\mathrm{HH}^1(A)$ *is solvable.*

Proof. Let $L := \mathrm{HH}^1(A)$. Since there is a k-basis of L induced from $\mathcal{B}_{L,Ker}$ by Theorem 4.8 and since $S_5 \subseteq \mathrm{Im}\psi_0$, we only need to check the Lie brackets between the elements in $S_1 \cup S_2 \cup S_3 \cup S_4$.

- Since $(\alpha_i, \beta_j), (\beta_l, \alpha_k)$ will only appear in S_2 in the form of $(\alpha_i, \beta_j) - (\beta_l, \alpha_k)$, $\alpha_i, \alpha_k, \beta_j, \beta_l \in Q_1$, that means $grd(v) = grd(w) = 2$, in other words, $G =$ ($\bullet \!=\!=\! \bullet$) or ($\bullet_{[2]}$ —— $\bullet_{[2]}$) or ($\bullet\!\circlearrowright$). Then if $G \neq$ ($\bullet \!=\!=\! \bullet$), all the elements in S_1 will disappear in $L^{(1)}$, that is, $L^{(1)} \subseteq \langle S_2 \cup S_3 \cup S_4 \rangle$.
- For $x = (\beta_0, \hat{\alpha_k}) - (\alpha_k, \hat{\beta_0})$, $x = (\beta_0', \hat{\alpha_k}') - (\alpha_k', \hat{\beta_0}') \in S_2$, which the hats are the abbreviation for the product of arrows in the definition of S_2, for convenience. Suppose the corresponding special cycles are $C_v, C_w, C_{v'}, C_{w'}$, with $\alpha_k, \hat{\alpha_k} \mid C_v$, $\beta_0, \hat{\beta_0} \mid C_w$, $\alpha_k', \hat{\alpha_k}' \mid C_{v'}$, $\beta_0', \hat{\beta_0}' \mid C_{w'}$. Without loss of generality, assume $x \neq y$. If there exists, for example, $\pi(\hat{\beta_0}^{(\beta_0', \hat{\alpha_k})}) \neq 0$, then $C_w = C_{w'}$ and, $C_v = C_w$ or $l(\hat{\beta_0}) = 1$.
  - If $l(\hat{\beta_0}) = 1$, then $grd(w) = 2$. This leads the Brauer graph G to be ($\bullet \!=\!=\! \bullet$), contradict to the hypothesis in this corollary.

– If $C_v = C_w$, then $l(\hat{\beta_0}) = l(\hat{\alpha_k}) = grd(w) - 1$. When $l(\hat{\beta_0}) = 1$, same as the case above. When $l(\hat{\beta_0}) > 1$, then since $\pi(\hat{\beta_0}^{(\beta_0', \hat{\alpha_k})}) \neq 0$, that means $l(\hat{\beta_0}^{(\beta_0', \hat{\alpha_k})}) = l(\hat{\beta_0}) - 1 + l(\hat{\alpha_k}) = 2grd(w) - 3 \leq grd(w)$. Thus $grd(w) \leq 3$. Due to w is not truncated, if $grd(w) = 2$, same as the case above. If $grd(w) \leq 3$, however, $\beta_0, \alpha_k, \beta_0'$ are different arrows in special cycle C_w, with $s(\beta_0) = e(\alpha_k), s(\alpha_k) = e(\beta_0)$. This leads $val(w)$ bigger than 3. A contradiction.

Therefore, $[S_2, S_2] = 0$.

For $x = (\beta_0, \hat{\alpha_k}) - (\alpha_k, \hat{\beta_0}) \in S_2$, $y \in S_3$, if $\pi(\hat{\beta_0}^{y}) \neq 0$, then $l(\hat{\beta_0}) = 1$. That also means $\beta_0, \alpha_k, \hat{\beta_0}$ are loops. This lead G to be a Brauer graph of one edge. If $G = (\bullet\!\circlearrowright)$, the corresponding Brauer graph algebra is $A = k\langle x, y\rangle / \langle xy - yx, x^2, y^2\rangle$, and the basis of its first Hochschild cohomology group is $\{(x,x)+(y,y)\}$, hence $\mathrm{HH}^1(A)$ is solvable. If $G = (\bullet_{[m]} \text{——} \bullet_{[2]})$ with $m > 1$, the Brauer graph algebra $A = k\langle x, y\rangle / \langle x^m - y^2, x^{m+1}, y^3, xy, yx\rangle$. Thus $S_2 = \{(x, y) - (y, x^{m-1})\}$ and $S_3 = \{(x, y^2)\}$. When $m = 2$, $[S_2, S_3] \subseteq \langle S_3\rangle$. Otherwise, $[S_2, S_3] = 0$.

For $x = (\beta_0, \hat{\alpha_k}) - (\alpha_k, \hat{\beta_0}) \in S_2$, $y = (\alpha, p) \in S_4$. By the definition of S_4, $l(p) > 1$. Since (α, p) will change the order of the arrows in $\hat{\alpha_k}$ and $\hat{\beta_0}$. After this operation we will get some elements not have summands in S_2. That means $[S_2, S_4] \subseteq \langle S_3 \cup S_4\rangle$.

By the definition of S_3 and S_4, for all $(\beta, C_v^{m(v)}(\alpha))$, $l(C_v^{m(v)}(\alpha)) = grd(v)$. For all $(\alpha, p) \in S_4$, $l(p) > 1$. Thus, $[S_3, S_3] = [S_3, S_4] = 0$. Since the length of every summand of the non monomial relations is equal to graded degree of some vertex, we have $[S_4, S_4] \subseteq \langle S_3 \cup S_4\rangle$.

Therefore, by the discussion above, we have $L^{(2)} \subseteq \langle S_3 \cup S_4\rangle$.

- Let S be a subspace of $k(Q_1//\mathcal{B})$. Define

$$l(S) := min\{l(p) | (\alpha, p) \text{ is a summand of some elements in } S\}.$$

Denote $B := \langle S_3 \cup S_4\rangle \subseteq k(Q_1//\mathcal{B})$, then we have $l(B) > 1$. If $B^{(1)} \neq 0$, we can check that $l(B^{(1)}) > l(B)$. By induction, if for all $n \in \mathbb{N}$, $B^{(n)} \neq 0$, there exists $N \in \mathbb{N}$, such that $l(B^{(N)}) > M := max\{grd(v) + 1 | v \in V\}$. However, $Q_{\geq M} \subseteq I_G$. That leads $B^{(N)} = 0$.

Therefore, $L^{(N+2)} = 0$, which means that L is solvable. □

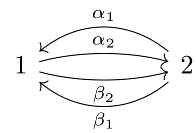
Remark 4.11. *Actually, although the discussion in Lemma 4.7 needs the condition that* char$k = 0$*, the conclusions of Theorem 4.8 and Corollary 4.10 only need that* char$k$ *fits the condition we discussed in Proposition 4.5. Under this characteristic condition, if A is a Brauer graph algebra such that the corresponding Brauer graph G is different from* ($\bullet = \bullet$) *(here both vertices have multiplicity* 1*), then* $\mathrm{HH}^1(A)$ *is solvable. In particular, when the Brauer graph G has trivial multiplicity, Corollary 4.10 is reduced to [6, Theorem 5.4].*

4.4 Some unsolvable examples of the first Hochschild cohomology groups of Brauer graph algebras

There is an exception in Corollary 4.10, which gives an example of unsolvable Lie algebra $\mathrm{HH}^1(A)$ in characteristic zero. Actually, this special case is also discussed in [6, Theorem 5.4], using a different method.

Example 4.12. *The Brauer graph algebra which is isomorphic to the trivial extension of the Kronecker algebra is given by*

$G:$ $\bullet[1]$ ═══ $\bullet[1]$ $\quad Q_G:$



The corresponding Brauer graph algebra of Brauer G is $A = kQ_G/I_G$, where I_G is generated by

- $R_1 = \{\alpha_1\alpha_2 - \beta_1\beta_2,\ \alpha_2\alpha_1 - \beta_2\beta_1\}$,
- $R_2 = \{\alpha_1\alpha_2\alpha_1,\ \alpha_2\alpha_1\alpha_2,\ \beta_1\beta_2\beta_1,\ \beta_2\beta_1\beta_2\}$,
- $R_3 = \{\alpha_1\beta_2,\ \beta_2\alpha_1,\ \alpha_2\beta_1,\ \beta_1\alpha_2\}$.

The length-lexicographic order is given by $\alpha_1 > \alpha_2 > \beta_1 > \beta_2 > e_1 > e_2$, then

$$\begin{array}{llllllll}
\psi_0 & : & (e_1,e_1) & \mapsto & (\alpha_2,\alpha_2)-(\alpha_1,\alpha_1)+(\beta_2,\beta_2)-(\beta_1,\beta_1) & , & (e_1,\alpha_1\alpha_2) & \mapsto 0 \\
 & & (e_2,e_2) & \mapsto & (\alpha_1,\alpha_1)-(\alpha_2,\alpha_2)+(\beta_1,\beta_1)-(\beta_2,\beta_2) & , & (e_2,\alpha_2\alpha_1) & \mapsto 0 \\
\psi_1 & : & (\alpha_1,\alpha_1),(\alpha_2,\alpha_2) & \mapsto & (\alpha_1\alpha_2,\beta_1\beta_2)+(\alpha_2\alpha_1,\beta_2\beta_1) & & & \\
 & & (\beta_1,\beta_1),(\beta_2,\beta_2) & \mapsto & -(\alpha_1\alpha_2,\beta_1\beta_2)-(\alpha_2\alpha_1,\beta_2\beta_1) & & & \\
 & & (\alpha_1,\beta_1),(\beta_2,\alpha_2) & \mapsto & (\alpha_1\beta_2,\beta_1\beta_2)+(\beta_2\alpha_1,\beta_2\beta_1) & & & \\
 & & (\beta_1,\alpha_1),(\alpha_2,\beta_2) & \mapsto & (\beta_1\alpha_2,\beta_1\beta_2)+(\alpha_2\beta_1,\beta_2\beta_1) & & &
\end{array}$$

Then we get

$$\mathrm{HH}^1(A) = k\{(\alpha_1,\alpha_1)-(\alpha_2,\alpha_2),\ (\alpha_1,\alpha_1)+(\beta_1,\beta_1),\ (\alpha_1,\beta_1)-(\beta_2,\alpha_2),\ (\alpha_2,\beta_2)-(\beta_1,\alpha_1)\}.$$

The Lie operations on $\mathrm{HH}^1(A)$ are always zero except

$$\begin{array}{lcl}
[(\alpha_1,\beta_1)-(\beta_2,\alpha_2),\ (\alpha_1,\alpha_1)-(\alpha_2,\alpha_2)] & = & (\alpha_1,\beta_1)-(\beta_2,\alpha_2) \\
[(\alpha_1,\alpha_1)-(\alpha_2,\alpha_2),\ (\alpha_2,\beta_2)-(\beta_1,\alpha_1)] & = & (\alpha_2,\beta_2)-(\beta_1,\alpha_1) \\
[(\alpha_1,\beta_1)-(\beta_2,\alpha_2),\ (\alpha_2,\beta_2)-(\beta_1,\alpha_1)] & = & -(\beta_1,\beta_1)-(\alpha_2,\alpha_2)+(\beta_2,\beta_2)+(\alpha_1,\alpha_1) \\
 & = & 2((\alpha_1,\alpha_1)-(\alpha_2,\alpha_2))
\end{array}$$

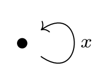
Therefore, $\mathrm{HH}^1(A) \cong k \oplus sl_2(k)$ which is unsolvable if and only if char$k \neq 2$.

Moreover, since there is a multiple edge in G and there are two elements $(\alpha_1,\beta_1)-(\beta_2,\alpha_2)$ and $(\alpha_2,\beta_2)-(\beta_1,\alpha_1)$ in S_2, we have $|S_2| = 2$. By Proposition 4.9, we can also compute the dimension of $\mathrm{HH}^1(A)$ by the formula that $\dim_k \mathrm{HH}^1(A) = 2 - 2\cdot 2 + (1+1) + 2 + 2 = 4$.

In general when the characteristic of the field is positive, things will be complicated. The following example shows that $\mathrm{HH}^1(A)$ may be unsolvable in this case. Note also that it is a counter-example of [15, Theorem 4.2] in positive characteristic. In fact, [15, Theorem 4.2] loses the following unsolvable case: $G = (\ \bullet_{[m]} \text{———} \bullet_{[1]}\ )$ with $m > 1$ and char$k|(m+1)$; This error is caused by missing to discuss the relations of the form $r = \alpha^{m+1}$ for $m \geq 2$.

Example 4.13. *Let* char$k = 3$.

$G:$ $\bullet[2]$ ——— $\bullet[1]$ $\quad Q_G:$



The corresponding Brauer graph algebra is $A = k\langle x\rangle/\langle x^3\rangle$.

$$\begin{array}{llllllllllll}
\psi_0 & : & (1,1) & \mapsto & 0 & , & (1,x) & \mapsto & 0 & , & (1,x^2) & \mapsto 0; \\
\psi_1 & : & (x,1) & \mapsto & 3(x^3,x^2)=0 & , & (x,x) & \mapsto & 0 & , & (x,x^2) & \mapsto 0.
\end{array}$$

Then we get $\mathrm{HH}^1(A) = k\{(x,1),(x,x),(x,x^2)\}$. *The Lie operations on* $\mathrm{HH}^1(A)$ *are*

$$\begin{array}{rcl} [(x,1),(x,x^2)] & = & (x,(x^2)^{(x,1)}) = 2(x,x) \\ [(x,1),(x,x)] & = & (x,1) \\ [(x,x^2),(x,x)] & = & (x,x^2) - (x,(x^2)^{(x,x)}) = -(x,x^2) \end{array}$$

Therefore, $\mathrm{HH}^1(A) \cong sl_2(k)$ *is a simple Lie algebra.*

5 The associated graded algebras of Brauer graph algebras

For a finite dimensional algebra A, one can construct a graded algebra $gr(A)$ associated with the radical filtration of A. In [11], Guo and Liu compared the representation types between a Brauer graph algebra and its associated graded algebra. In this section, we compare the first Hochschild cohomology groups between them.

5.1 The Lie algebra structure on $\mathrm{HH}^1(gr(A))$

Definition 5.1. *Let A be a finite dimensional algebra. Denote by $\mathfrak{r}$ the (Jacobson) radical $rad(A)$ of A. Then the graded algebra $gr(A)$ of A associated with the radical filtration is defined as follows. As a graded vector space,*

$$gr(A) = A/\mathfrak{r} \oplus \mathfrak{r}/\mathfrak{r}^2 \oplus \cdots \oplus \mathfrak{r}^t/\mathfrak{r}^{t+1} \oplus \cdots.$$

The multiplication of $gr(A)$ is given as follows. For any two homogeneous elements: $x + \mathfrak{r}^{m+1} \in \mathfrak{r}^m/\mathfrak{r}^{m+1}$, $y + \mathfrak{r}^{n+1} \in \mathfrak{r}^n/\mathfrak{r}^{n+1}$, we have

$$(x + \mathfrak{r}^{m+1}) \cdot (y + \mathfrak{r}^{n+1}) = xy + \mathfrak{r}^{m+n+1}.$$

We now specialize A to be a Brauer graph algebra associated with a Brauer graph $G = (V, E)$. Recall that the associated $gr(A)$ can be described as follows.

Lemma 5.2. ([*11, Lemma* 2.9]) *Let $A = kQ/I$ be a Brauer graph algebra. The generating relations of the second and the third types in I are given by paths, relation of the first type is of the form $\rho = p - q$, where p and q are two paths with $s(p) = t(p) = s(q) = t(q)$. For any relation $\rho = p - q$ of first type (suppose that the length of p is m and the length of q is n), we replace it by*

$$\rho' = \begin{cases} \rho & , \quad m = n, \\ q & , \quad m > n, \\ p & , \quad m < n. \end{cases}$$

Then, the associated graded algebra $gr(A)$ is isomorphic to kQ/I', where Q is the same quiver as above and I' is an admissible ideal whose generating relations are obtained from that of I by replacing each ρ by ρ'.

Remark 5.3. *It can be checked in the same way as in Theorem 4.3 that the generating set of I' above is a Gröbner basis of I' in kQ, which also may not be tip-reduced.*

The following notion of unbalanced edges in a Brauer graph is useful in our later discussions.

Definition 5.4. ([*11, Definition* 2.12]) *Let $G = (V, E)$ be a Brauer graph with graded degree function grd and $A = kQ/I$ the corresponding Brauer graph algebra. We identify Q_0 with E by the natural bijection between them.*

- *We call an edge $v_1 \overset{i}{-} v_2$ in G with $grd(v_1) \neq grd(v_2)$ an unbalanced edge, and denote the end points of i by $v_L^{(i)}, v_S^{(i)}$, with $grd(v_L^{(i)}) > grd(v_S^{(i)})$. Whenever, the context is clear, we will omit the superscript (i). Other edges which are not unbalanced will be called the balanced edges.*
- *For any unbalanced edge $v_S \overset{i}{-} v_L$ in G, there is a relation of the first type $\rho_i = p_i - q_i$ in I, where $p_i = C_{v_S}^{m(v_S)}, q_i = C_{v_L}^{m(v_L)}$ are two paths with lengths $grd(v_S), grd(v_L)$, respectively.*
- *We denote by $\mathbb{W}$ the set of unbalanced edges in G.*

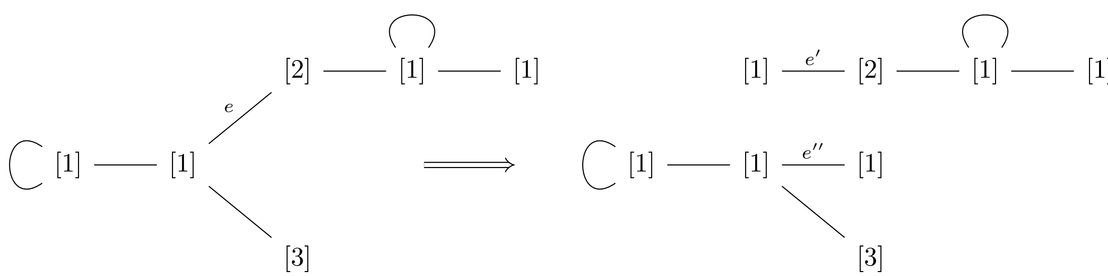
In order to deal with the problems about the Lie algebra structure on the first Hochschild cohomology group of $gr(A)$, we introduce the notion of the balanced components of a Brauer graph.

Definition 5.5. *Let $G = (V, E)$ be a Brauer graph with graded degree function grd. We define the balanced components of G by the following rules:*

- *retain the balanced edges in G;*
- *split the unbalanced edge into two edges by attaching two new truncated vertices.*

The connected components in G after remodeling by the rules above are the balanced components of G. Denote the set of the balanced components of G by Γ_G.

An example of a Brauer graph which has two balanced components (where the edge e splits into two edges e' and e''):



Note that when considered a balanced component of G as a Brauer graph, the corresponding Brauer graph algebra and its associated graded algebra are the same (cf. [11, Propostion 2.13]).

We are ready to study the Lie structure on $\mathrm{HH}^1(gr(A))$ under the assumption that the characteristic of the field k is 0. Recall the definition of the maps ψ_0, ψ_1 from Proposition 3.7 and denote these maps for $gr(A)$ by $\psi_0^{gr(A)}$, $\psi_1^{gr(A)}$ respectively. Let

$$L_0^{gr(A)} := k(Q_1//Q_1) \cap \mathrm{Ker}\psi_1^{gr(A)} \Big/ \langle \sum_{a \in Q_1 e} (a, a) - \sum_{b \in eQ_1} (b, b) | e \in Q_0 \rangle,$$

which is a Lie subalgebra of $\mathrm{HH}^1(gr(A))$. Furthermore, we can take $L_{00}^{gr(A)}$ which is given by

$$L_{00}^{gr(A)} := \langle (\alpha, \alpha) | \alpha \in Q_1 \rangle \cap \mathrm{Ker}\psi_1^{gr(A)} \Big/ \langle \sum_{a \in Q_1 e} (a, a) - \sum_{b \in eQ_1} (b, b) | e \in Q_0 \rangle.$$

Actually, $L_{00}^{gr(A)}$ is an abelian Lie subalgebra of $\mathrm{HH}^1(gr(A))$ and $L_{00}^{gr(A)} \subseteq L_0^{gr(A)}$.

Lemma 5.6. *Let A be a Brauer graph algebra associated with a Brauer graph $G = (V, E)$ and $gr(A)$ the associated graded algebra of A. Then*

$$\dim_k L_{00}^{gr(A)} = |E| - |V| + 1 + |\Gamma_G|.$$

Proof. In $L_{00}^{gr(A)}$, we only need to consider the linear combination of (α,α). For all $\alpha \in Q_1$, if α is not involved in some homogeneous relations of the first type, then $\psi_1^{gr(A)}((\alpha,\alpha)) = 0$. That means the monomial relations in the Gröbner basis $\mathcal{G}$ of I will not influence the basis of $L_{00}^{gr(A)}$. Therefore, it is enough to consider the balanced components of G as Brauer graphs. By Lemma 4.7, for $C \in \Gamma_G$, $\dim_k(\langle(\alpha,\alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1^C) = \sum_{v\in V_C}(val(v)-1)+1 = 2|E_C|-|V_C|+1$. In addition, $\dim_k(\langle\psi_0^{gr(A)}(e,e)|e \in Q_0\rangle) = |E|-1$ in the same way. Then

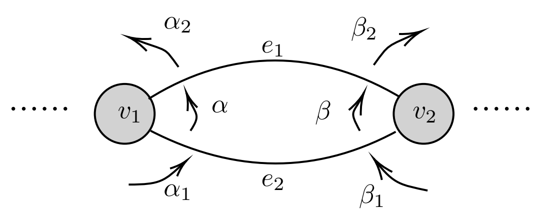
$$\begin{aligned}\dim_k L_{00}^{gr(A)} &= \textstyle\sum_{C\in\Gamma_G}(2|E_C|-|V_C|+1)-|E|+1\\ &= 2(|E|+|\mathbb{W}|)-(|V|+2|\mathbb{W}|)+|\Gamma_G|-|E|+1\\ &= |E|-|V|+1+|\Gamma_G|.\end{aligned}$$

□

We now check the solvability of $\mathrm{HH}^1(gr(A))$. Actually, it can be verified by using the same method as in Section 4. Besides, since I' is an homogeneous ideal of kQ_G, it is easier to verify the solvability of $\mathrm{HH}^1(gr(A))$ by using the graded structure of $\mathrm{HH}^1(gr(A))$ as we defined in Section 3.3.

Lemma 5.7. *If G does not contain ($\bullet = \bullet$) with an vertex v of the multiply edges has $grd(v)=2$, or $G \neq$ ($\bullet_{[2]}$ —— $\bullet_{[m]}$) with $m \geq 2$, then $L_0^{gr(A)} = L_{00}^{gr(A)}$.*

Proof. If $k(Q_1//Q_1) \cap \mathrm{Ker}\psi_1^{gr(A)} - \langle(\alpha,\alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1^{gr(A)} \neq \varnothing$, then there exists $(\alpha,\beta) \in k(Q_1,Q_1)$, such that $\alpha \neq \beta$. That means in the Brauer graph, there exists two edges e_1, e_2, such that they connect to the same vertices v_1, v_2.



If e_1,e_2 are different edges, then there is a multiple edge in Brauer graph. Since the relations of type III are contained in the reduced Gröbner basis $\mathcal{G}$, then $\beta\alpha_1, \alpha\beta_1, \beta_2\alpha, \alpha_2\beta \in \mathcal{G}$. Actually, without loss of generality, let $grd(v_1) \neq 2$. If $grd(v_2) \neq 2$, then

$$(\alpha\beta_1, \pi(\alpha\beta_1^{(\alpha,\beta)})) = (\alpha\beta_1, \pi(\beta\beta_1)) = (\alpha\beta_1, \beta\beta_1)$$

$$(\beta_2\alpha, \pi(\beta_2\alpha^{(\alpha,\beta)})) = (\beta_2\alpha, \pi(\beta_2\beta)) = (\beta_2\alpha, \beta_2\beta)$$

$$(\alpha_2\beta, \pi(\alpha_2\beta^{(\beta,\alpha)})) = (\alpha_2\beta, \pi(\alpha_2\alpha)) = (\alpha_2\beta, \alpha_2\alpha)$$

$$(\beta\alpha_1, \pi(\beta\alpha_1^{(\beta,\alpha)})) = (\beta\alpha_1, \pi(\alpha\alpha_1)) = (\beta\alpha_1, \alpha\alpha_1)$$

That means $(\alpha,\beta), (\beta,\alpha)$ will not appear in $k(Q_1//Q_1) \cap \mathrm{Ker}\psi_1^{gr(A)}$ since the parallel path pairs above will not be reduced by any $\psi_1^{gr(A)}((\alpha',\beta'))$ with $(\alpha',\beta') \in k(Q_1//Q_1)$. However, things will be different if $grd(v_2) = 2$:

$$(\alpha\beta_1, \pi(\alpha\beta_1^{(\alpha,\beta)})) = (\alpha\beta_1, \pi(\beta\beta_1)) = 0,$$

$$(\beta_2\alpha, \pi(\beta_2\alpha^{(\alpha,\beta)})) = (\beta_2\alpha, \pi(\beta_2\beta)) = 0,$$

since $\pi(\beta\beta_1) = \pi(\beta_2\beta) = 0$. That means $(\alpha,\beta) \in k(Q_1//Q_1) \cap \mathrm{Ker}\psi_1^{gr(A)}$.

If e_1 and e_2 are the same edges, then $G = (\bullet \!-\!\!-\!\!-\! \bullet)$ or $(\bullet \circlearrowright)$. By the hypothesis of proposition, if the vertices v_1, v_2 in the first case has the property that $m(v_1) > 2$ and $m(v_2) > 2$, or if the vertex v' in the second case has $m(v') > 1$, it can be checked (α, β) can not appear in $k(Q_1//Q_1) \cap \mathrm{Ker}\psi^{gr(A)}$ since there are some relations of the third type same as above. If the vertex v' in the second case has $m(v') = 1$, then the Brauer graph algebra is $k\langle x, y\rangle / \langle xy - yx, x^2, y^2\rangle$, the basis of its first Hochschild cohomology group is $\{(x, x) + (y, y)\}$, $L_0^{gr(A)} = L_{00}^{gr(A)}$. □

Since the associated graded algebra of a Brauer graph algebra is defined by homogeneous ideal, it is naturally to consider the graded structure on $\mathrm{HH}^1(gr(A))$. Same as the discussion in [19, Page 258], we have

$$\mathrm{HH}^1(gr(A)) \Big/ rad(\mathrm{HH}^1(gr(A))) \cong L_0^{gr(A)} \Big/ rad(L_0)^{gr(A)}.$$

Theorem 5.8. *Let A be a Brauer graph algebra over a field of characteristic zero such that the corresponding Brauer graph G is different from $(\bullet = \bullet)$ (here both vertices have multiplicity 1) and $gr(A)$ the associated graded algebra of A, then $\mathrm{HH}^1(gr(A))$ is solvable.*

Proof. By the discussion and the lemma above, we can only consider the solvability on $L_0^{gr(A)}$.

- If G does not contain $(\bullet = \bullet)$ with an vertex v of the multiply edges has $grd(v) = 2$, or $G \neq (\bullet_{[2]} \!-\!\!-\!\!-\! \bullet_{[m]})$ with $m \geq 2$, then $L_0^{gr(A)} = L_{00}^{gr(A)}$, and $(L_0^{gr(A)})^{(1)} = (L_{00}^{gr(A)})^{(1)} = 0$.
- Let $G = (v_{1[2]} \!-\!\!-\!\!-\! v_{2[m]})$ with $m \geq 2$. If $m = 2$, the case is same as the Brauer graph algebra A_G. So let $m > 2$ and denote the arrow around v_1 by α, the arrow around v_2 by β, then actually we have $L_0^{gr(A)} = L_{00}^{gr(A)} \cup \{(\beta, \alpha)\}$. Thus it can be checked that $(L_0^{gr(A)})^{(1)} = \{(\beta, \alpha)\}$, $(L_0^{gr(A)})^{(2)} = 0$.
- Let G contains $(v_i = v_i')$ with $grd(v_i) = 2$, $i = 1, \cdots, m$. Assume $grd(v_i') > 2$ and denote the parallel arrows in the multiple edges around v_i, v_i' by α_i, β_i respectively, then $L_0^{gr(A)} = L_{00}^{gr(A)} \cup \{(\beta_i, \alpha_i) | i = 1, \cdots, m\}$. Thus we can check that $(L_0^{gr(A)})^{(1)} = \{(\beta_i, \alpha_i) | i = 1, \cdots, m\}$, $(L_0^{gr(A)})^{(2)} = 0$.

To sum up, $\mathrm{HH}^1(gr(A))$ is solvable.

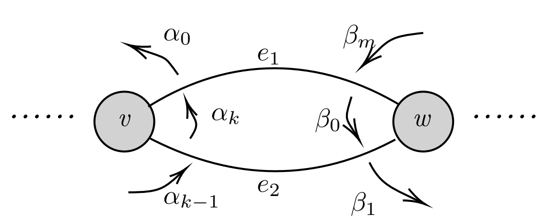
□

5.2 An injection from $\mathrm{HH}^1(A)$ to $\mathrm{HH}^1(gr(A))$

In this part, for a Brauer graph algebra A, we will construct a Lie algebra monomorphism from $\mathrm{HH}^1(A)$ to $\mathrm{HH}^1(gr(A))$, which will give us a specific comparison between them. Let us begin from constructing a set of generators of $\mathrm{HH}^1(gr(A))$.

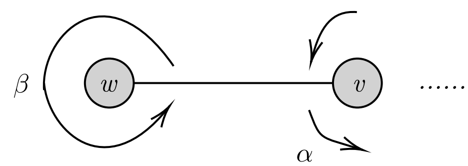
Lemma 5.9. *There is a k-basis $\mathcal{B}_{L,Ker}^{gr(A)}$ of $\mathrm{Ker}\psi_1^{gr(A)}$ which consists of the following five subsets:*

- S_1^{gr}*: the basis of $\langle(\alpha, \alpha) | \alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1^{gr(A)}$ described in Lemma 5.6;*
- S_2^{gr}*: in the case as:*



with $\alpha_k \neq \beta_0$*,* $\alpha_k\beta_0, \beta_0\alpha_k \in R_3$*. If* $gr(v) = gr(w)$*, then the corresponding element in* S_2^{gr} *is* $(\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1})$*; If* $gr(v) > gr(w)$*, then the corresponding element in* S_2^{gr} *is* $(\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1})$*; If* $gr(v) < gr(w)$*, then the corresponding element in* S_2^{gr} *is* $(\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1})$*.*

- S_3^{gr}*: elements of the form* $(\beta, C_v(\alpha)^{m(v)})$*, in the case as below, with* $l(C_v(\alpha)^{m(v)}) > m(w)$*, or* $l(C_v(\alpha)^{m(v)}) = m(w)$ *and* $\beta^{m(w)} > C_v(\alpha)^{m(v)}$*.*



- S_4^{gr}*: elements of the form* (α,p)*, such that* $l(p) > 1$*,* $\psi_1^{gr(A)}(\alpha, p) = 0$*, and there exist a special cycle* C_v*,* $\alpha \mid C_v$*,* $p \mid C_v^{m(v)}$*.*

- S_5^{gr}*: the basis of the subspace* $Im\psi_0^{gr(A)}$ *generated by all the elements* $(\alpha_1, \alpha_1 p) - (\alpha_0, p\alpha_0)$ *where* p *is a cycle in* Q *and* $\psi_1^{gr(A)}((\alpha_1, \alpha_1 p)) = \psi_1^{gr(A)}((\alpha_0, p\alpha_0)) \neq 0$*.*

Furthermore, $\mathrm{Im}\psi_0^{gr(A)}$ *is contained in* $\langle S_1^{gr} \cup S_4^{gr} \cup S_5^{gr}\rangle$*. The* k*-basis of* $\mathrm{HH}^1(gr(A))$ *induced from* $\mathcal{B}_{L,Ker}^{gr(A)}$ *will be denoted by* $\mathcal{B}_L^{gr(A)}$*.*

Proof. By the definition of the elements above, the verification that $S_1^{gr} \cup S_2^{gr} \cup S_3^{gr} \cup S_4^{gr} \cup S_5^{gr} \subseteq \mathrm{Ker}\psi_1^{gr(A)}$ is straightforward (cf. the proof of Theorem 4.8). Denote the Gröbner basis of $gr(A)$ by $\mathcal{G}^{gr(A)}$, the i-th type of relations in kQ_G by R_i, $i = 1, 2, 3$. We also denote the relations ρ' in I' induced by the first type by R_1' (cf. Lemma 5.2). Let $R_1 \cap R_1' = R_1^0$, $R_1' \backslash R_1 = R_1^1$, then $R_1^0 \cup R_1^1 \cup R_3 \subseteq \mathcal{G}^{gr(A)} \subseteq R_1^0 \cup R_1^1 \cup R_2 \cup R_3$.

Consider $t = \sum_{i \in I} k_i(\alpha_i, p_i) \in \mathrm{Ker}\psi_1^{gr(A)}$, and (α, p) is a summand of t. By the graded structure we discussed in Section 3 and the Proposition 3.11, we can assume $l(p) \geq 1$.

Same as the discussion in Theorem 4.8, first we suppose that $\psi_1(\alpha, p) \neq 0$. That means there exists some $g \in \mathcal{G}^{gr(A)}$,

$$\sum_{p' \in \mathrm{Supp}(g)} c_g(p') \cdot (\mathrm{Tip}(g), \pi(p'^{(\alpha,p)})) \neq 0.$$

- $g \in \mathcal{G}^{gr(A)} \cap R_2$, obviously, $(g, \pi(g^{(\alpha,p)})) = 0$.
- $g \in R_1^1 \subseteq \mathcal{G}^{gr(A)}$, then there exists $v \in V$, such that $l(g) = grd(v)$, thus $(g, \pi(g^{(\alpha,p)})) = 0$
- If there exists $g \in R_1^0 \subseteq \mathcal{G}^{gr(A)}$, such that there exists $p_0 \in \mathrm{Supp}(g)$, $(\mathrm{Tip}(g), \pi(p_0^{(\alpha,p)})) \neq 0$, that means $l(p) = 1$, and it will ask for $\alpha = p$. By the form of $\psi_1^{gr(A)}((\alpha, \alpha))$, we know that (α, α) must appears in a summand t' of t and $t' \in kS_1$.
- If there exists $g \in R_3 \subseteq \mathcal{G}^{gr(A)}$, such that $(g, \pi(g^{(\alpha,p)})) \neq 0$, then there are two cases to think about. If $\pi(g^{(\alpha,p)}) \neq g^{(\alpha,p)}$, or if $\pi(g^{(\alpha,p)}) = g^{(\alpha,p)}$ and (α, p) is a summand of the elements in S_2^{gr}, there will be an element in S_2^{gr} have a summand equals (α, p). (α, p) will induce a summand t'' of t and $t'' \in kS_2$. If $\pi(g^{(\alpha,p)}) = g^{(\alpha,p)}$ and (α, p) is not a summand of the elements in S_2^{gr}, we can just let $g = \beta\alpha$, $(\beta\alpha, \pi(g^{(\alpha,p)}) = (\beta\alpha, \beta p)$, then we need (β, q), such that $(\beta, q) - (\alpha, p)$ is summand of t. However, that means $\beta p = q\alpha$, and we will get $p = p_1\alpha$, $q = \beta p_1$, since $gr(A)$ is a biserial algebra, $(\beta, q) - (\alpha, p) = \psi_0^{gr(A)}(e, p_1)$ for some $e \in Q_0$. That means $(\beta, q) - (\alpha, p) \in S_5^{gr}$.

Next we assume $\psi_1(\alpha, p) = 0$. If $p \mid C_v^{m(v)}$ with C_v is the special cycle which contains α, same as Theorem 4.8, $l(p)$ must bigger than 1. That leads (α, p) is contained in S_4^{gr}. If $p \nmid C_v^{m(v)}$ with $p \mid C_w^{m(w)}$. Since there exists $\alpha\beta \in R_3$ with $\beta \mid C_w$, then $(\alpha\beta, \pi(\alpha\beta)^{(\alpha,p)}) = 0$, which means there exists $g \in R_2$ or $g \in R_1^1$, such that $g \mid p\beta$.

- If there exists $g \in R_2$, such that $g \mid p\beta$, then $l(p\beta) \geq grd(w) + 1$, that means p is a cycle and α is a loop. This element is contained in S_3^{gr}.
- If there exists $g \in R_1^1$, such that $g \mid p\beta$, then $grd(w) - 1 \leq l(p) \leq grd(w)$. When $l(p) = grd(w)$, this case is equal to the case above. When $l(p) = grd(w) - 1$, then p is in the form of $\beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1}$. Since $\alpha // p$, then (α, p) is contained in S_2^{gr}.

Therefore, each element $t \in Ker\psi_1^{gr(A)}$ can be represented by a linear combination of the elements in $\mathcal{S}^{gr}$. □

We also need a lemma to give some connection between A and $gr(A)$.

Lemma 5.10. *Let A be a Brauer graph algebra associated with a Brauer graph $G = (V, E)$ and $gr(A)$ the associated graded algebra of A. Then there is a canonical isomorphism between vector spaces from A to $gr(A)$, which also induces an isomorphism from $\mathrm{Im}\psi_0^A$ to $\mathrm{Im}\psi_0^{gr(A)}$.*

Proof. For $A = kQ/I$ and $gr(A) = kQ/I'$, by the property of Gröbner basis, under the same length-lexicographic order in $Q_{\geq 0}$, we can find a k-basis of A (respectively, of $gr(A)$) in $Q_{\geq 0}$ if we fix the natural Gröbner basis in I (respectively, in I'). By Lemma 5.2, define a map ϕ from the k-basis of A to the k-basis of $gr(A)$ by the following rules:

- If $v_L \overset{i}{-} v_S$ in G is an unbalanced edge, then $C_{v_L}(\alpha)^{m(v_L)} - C_{v_S}(\beta)^{m(v_S)} \in \mathcal{G}$. Define
$$\phi(C_{v_S}(\beta)^{m(v_S)}) = C_{v_L}(\alpha)^{m(v_L)};$$
- Otherwise, ϕ is identity morphism on the elements in the basis of A which are different from above case.

Thus ϕ gives an isomorphism from A to $gr(A)$.

Now define the map $\hat{\phi} : Q_1//\mathcal{B}^A \to Q_1//\mathcal{B}^{gr(A)}$, $(\alpha, p) \mapsto (\alpha, \phi(p))$. Obviously, $\hat{\phi}$ is also an isomorphism. Moreover, $\hat{\phi}|_{\mathrm{Im}\psi_0^A}$ and $\hat{\phi}^{-1}|_{\mathrm{Im}\psi_0^{gr(A)}}$ induce an isomorphism between $\mathrm{Im}\psi_0^A$ and $\mathrm{Im}\psi_0^{gr(A)}$. □

Now we prove the main result of this subsection.

Theorem 5.11. *Let A be a Brauer graph algbera associated with a Brauer graph $G = (V, E)$ and $gr(A)$ the associated graded algebra of A. If $G \neq (v_S - v_L)$ with $m(v_L) > m(v_S) \geq 2$, then there is a monomorphism $i : \mathrm{HH}^1(A) \to \mathrm{HH}^1(gr(A))$ as Lie algebras.*

Proof. By Theorem 4.8 and Lemma 5.9, let $\mathcal{B}_{L,Ker} = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, $\mathcal{B}_{L,Ker}^{gr(A)} = S_1^{gr} \cup S_2^{gr} \cup S_3^{gr} \cup S_4^{gr} \cup S_5^{gr}$ be the k-bases of $\mathrm{Ker}\psi_1$, $\mathrm{Ker}\psi_1^{gr(A)}$, respectively. By the correspondence in Lemma 5.10, we can choose S_5 and S_5^{gr} are corresponding to each other by the map $\hat{\phi}$. Denote the morphism i by:

- $i_1 : S_1 \to S_1^{gr}$, the natural embedding morphism, due to the basis of $\langle(\alpha, \alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1$ is the linear combinations of the basis of $\langle(\alpha, \alpha)|\alpha \in Q_1\rangle \cap \mathrm{Ker}\psi_1^{gr(A)}$.

- $i_2 : S_2 \to S_2^{gr}$, let $e = (\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1}) \in S_2$, then

$$e \mapsto \begin{cases} e & , \quad grd(v) = grd(w), \\ -(\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1}) & , \quad grd(v) > grd(w), \\ (\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) & , \quad grd(v) < grd(w). \end{cases}$$

- $i_3 : S_3 \cup S_4 \to S_3^{gr} \cup S_4^{gr}$, $(\alpha, p) \mapsto (\alpha, \phi(p))$, which ϕ is the one-to-one morphism between A and $gr(A)$ in Lemma 5.10.
- $i_4 := \hat{\phi}|_{S_5}$.

Then $i = i_1 \cup i_2 \cup i_3 \cup i_4$ is an injection from $\mathcal{B}_{L,Ker}$ to $\mathcal{B}_{L,Ker}^{gr(A)}$ by the definition of S_i and S_i^{gr}, $i = 1, 2, 3, 4, 5$. And actually, i_2, i_3, i_4 are bijections. By Lemma 5.10, $i(\mathrm{Im}\psi_0^A) = \mathrm{Im}\psi_0^{gr(A)}$. Therefore, i induces an injection from $\mathrm{HH}^1(A)$ to $\mathrm{HH}^1(gr(A))$. We will prove that i is a monomorphism between Lie algebras.

First of all, consider $\mathrm{ad}r := [r, -]$ with $r \in S_1$, then by the definition of i, $i(r) \in S_1^{gr}$. And since $i|_{S_1}$ is the natural embedding morphism, r and $i(r)$ are in same form in the linear combinations of some elements in $\{(\alpha, \alpha) | \alpha \in Q_1\}$. Since $L_{00}, L_{00}^{gr(A)}$ are solvable lie ideals of $A, gr(A)$, respectively, it is easy to check $\mathrm{ad}r|_{S_1}$ and $\mathrm{ad}i(r)|_{S_1^{gr}}$ are zero morphisms. Thus $[i(r_1), i(r_2)] = i([r_1, r_2]) = 0$, $r_1, r_2 \in S_1$. For $r' \in S_2$, let $r' = (\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1})$ with C_v and C_w are different special cycles. For simplicity, let $\hat{\beta_0} = (\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1})$, $\hat{\alpha_k} = (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1})$, then $r' = \hat{\beta_0} - \hat{\alpha_k}$. Actually, since $r \in S_1$, then $r = (\alpha_0, \alpha_0) - (\alpha_i, \alpha_i)$ or $r = \sum_{v \in V, \alpha_v | C_v} k_v(\alpha_v, \alpha_v)$ with $k_v = \prod_{v' \in V, v' \neq v} m(v')$ by Lemma 4.7. Moreover,

- If $r = (\alpha_0, \alpha_0) - (\alpha_i, \alpha_i)$, then

$$[r, r'] = \begin{cases} (m(v) - 1)(\hat{\beta_0} - \hat{\beta_0}) = 0 & , \quad i \neq k \\ \{m(v) - (m(v) - 1)\}\hat{\beta_0} - \hat{\alpha_k} = r' & , \quad i = k \end{cases}$$

  When $r' = i(r')$ as the same form, $i([r, r']) = [i(r), i(r')]$ obviously. When $r' \neq i(r')$, without loss of generality, let $i(r') = \hat{\beta_0}$, then

$$[i(r), i(r')] = \begin{cases} (m(v) - 1)(\hat{\beta_0} - \hat{\beta_0}) = 0 & , \quad i \neq k \\ \{m(v) - (m(v) - 1)\}\hat{\beta_0} = i(r') & , \quad i = k \end{cases}$$

  Thus $i([r, r']) = [i(r), i(r')]$.
- If $r = (\beta_0, \beta_0) - (\beta_j, \beta_j)$, similarly,

$$[r, r'] = \begin{cases} 0 & , \quad j \neq m \\ r' & , \quad j = m \end{cases}$$

  and it can be checked that $i([r, r']) = [i(r), i(r')]$ in the same way.
- If $r = \sum_{v \in V, \alpha_v | C_v} k_v(\alpha_v, \alpha_v)$ with $k_v = \prod_{v' \in V, v' \neq v} m(v')$, when β_0, α_k are not loops,

$$[r, r'] = \prod_{v \in V} m(v)\hat{\beta_0} - (\prod_{v \in V} m(v) - k_w)\hat{\alpha_k} - k_w\hat{\beta_0} = (\prod_{v \in V} m(v) - k_w)r'$$

  If α_k is a loop, by the definition of the elements in S_2, β_0 is also a loop, and

$$[r, r'] = (m(v) - 1)k_v\hat{\beta_0} - (m(w) - 1)k_w\hat{\alpha_k} - k_w\hat{\beta_0} + k_v\hat{\alpha_k} = (\prod_{v \in V} m(v) - k_v - k_w)r'$$

  then we got $i([r, r']) = [i(r), i(r')] = c \cdot i(r'), c \in k$.

- Else, if $r \in S_1$ and r is not in the forms above, then $[r, r'] = [i(r), i(r')] = 0$.

Therefore, $i([r, r']) = [i(r), i(r')]$ when $r \in S_1, r' \in S_2$.

If $r' \in S_3 \cup S_4$, then $r' = (\alpha, p)$ with $p \in C_w, \alpha \in C_v$, $v, w \in V$. By the structure of Brauer graph algebra, the number of times α_i appears in p is no more than one time different from the number of times α_0 appears in p. Since the one-to-one map ϕ only change the cycles around the unbalanced edges, we only need to consider the elements $r' = (\beta, C_v(\alpha)^{m(v)})$ with $i(r') = (\beta, C_w(\beta)^{m(w)})$. However, because of $\beta // C_v^{m(v)}$, β is a loop.

- If $r = (\alpha_0, \alpha_0) - (\alpha_i, \alpha_i)$, then $[r, r'] = m(v)(r' - r') = 0$, $[i(r), i(r')] = 0$.
- If $r = \sum_{v \in V, \alpha_v | C_v} k_v(\alpha_v, \alpha_v)$ with $k_v = \prod_{v' \in V, v' \neq v} m(v')$, then $[r, r'] = (\prod_{v \in V} m(v) - k_w) r'$, $[i(r), i(r')] = (\prod_{v \in V} m(v) - k_w) i(r')$.

Therefore, we have proved $i([r, r']) = [i(r), i(r')]$ when $r \in S_1, r' \in S$.

Secondly, consider adr with $r \in S_2$, then by the definition of i, $i(r) \in S_2^{gr}$. Let $r = (\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1}), r' = (\beta_t, \gamma_{n-1} \cdots \gamma_0 \cdot C_u(\gamma_0)^{m(u)-1}) - (\gamma_n, \beta_{t-1} \cdots \beta_{t+1} \cdot C_w(\beta_{t+1})^{m(w)-1}) \in S_2$. That means

$$[r, r'] = -(\alpha_k, \pi((\beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1})^{(\beta_t, \gamma_{n-1} \cdots \gamma_0 \cdot C_u(\gamma_0)^{m(u)-1})}))$$

$$+(\gamma_n, \pi((\beta_{t-1} \cdots \beta_{t+1} \cdot C_w(\beta_{t+1})^{m(w)-1})^{(\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1})}))$$

Since γ_n, β_t and β_0, α_k are corresponding two multiple edges in Brauer graph G, that makes $grd(w) > 2$ and $[r, r'] = 0$. It can be checked $[i(r), i(r')] = 0$ in the same way.

Let $r' \in S_3 \cup S_4$, it is also enough to discuss this question in following two cases:

- Let $r' = (\beta, C_v(\alpha)^{m(v)})$ with β is a loop and $\phi(C_v(\alpha)^{m(v)}) = \beta^{m(v)}$. Since $G \neq (v_S - v_L)$, then $r = (\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1})$ with C_v and C_w are not loops, thus $\beta \nmid C_v$ and $\beta \nmid C_w$. Thus $[r, r'] = 0$ and $[i(r), i(r')] = 0$.
- Let $r' = (\beta, \beta^{m(v)})$ with β is a loop and $\phi(\beta^{m(v)}) = C_v(\alpha)^{m(v)}$. Then $[r, r'] = 0$ and $[i(r), i(r')] = 0$ in the same reason.

Therefore, $i([r, r']) = [i(r), i(r')]$ when $r \in S_2, r' \in S$.

Now let us consider $r, r' \in S_3 \cup S_4$, then r, r' is in the form of (α, p) with $l(p) > 1$. It is also enough to consider there exists $r = (\beta, C_v(\alpha)^{m(v)})$ or $r = (\beta, \beta^{m(v)})$. then since $r' = (\alpha, p)$ with $l(p) > 1$, we have $[r, r'] = [i(r), i(r')] = 0$.

To sum it up, for all $r, r' \in \mathcal{S}$, $i([r, r']) = [i(r), i(r')]$. That means i is a monomorphism from $\mathrm{HH}^1(A)$ to $\mathrm{HH}^1(gr(A))$ as Lie algebras.

□

Remark 5.12. *This theorem also tells us for a Brauer graph algebra A with the corresponding Brauer graph $G \neq (v_S - v_L)$ with $m(v_L) > m(v_S) \geq 2$, if $HH^1(gr(A))$ is solvable, so is $HH^1(A)$.*

Although the injection above is not always a monomorphism between Lie algebras, it is enough to help us compute the difference between the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(gr(A))$.

Corollary 5.13. *Let A be a Brauer graph algebra associated with a Brauer graph $G = (V, E)$ and $gr(A)$ the associated graded algebra of A, then* $\dim_k \mathrm{HH}^1(gr(A)) - \dim_k \mathrm{HH}^1(A) = |\Gamma_G| - 1$.

Proof. By the proof of the theorem above, i is an injection from the basis of $\mathrm{Ker}\psi_1$ to the basis of $\mathrm{Ker}\psi_1^{gr(A)}$. Actually, i is a bijection between $S_2 \cup S_3 \cup S_4 \cup S_5$ and $S_2^{gr} \cup S_3^{gr} \cup S_4^{gr} \cup S_5^{gr}$. By Lemma 5.10, we have $i(\mathrm{Im}\psi_0^A) = \mathrm{Im}\psi_0^{gr(A)}$. Therefore, by Lemma 4.7 and Lemma 5.6, we have

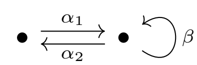
$$\begin{aligned} \dim_k \mathrm{HH}^1(gr(A)) - \dim_k \mathrm{HH}^1(A) &= \dim_k \mathrm{Ker}\psi_1^{gr(A)} - \dim_k \mathrm{Im}\psi_0^{gr(A)} - \dim_k \mathrm{Ker}\psi_1^A + \dim_k \mathrm{Im}\psi_0^A \\ &= |S_1^{gr}| - |S_1| \\ &= \dim_k L_{00}^{gr(A)} - \dim_k L_{00} \\ &= |\Gamma_G| - 1. \end{aligned}$$

□

Now let us check some examples to verify the results we obtained above.

Example 5.14. *Consider the Brauer graph G in following form.*

G : • —— • —— •[3] Q_G :



The corresponding Brauer graph algebra of Brauer G is

$$A = kQ_G / \langle \beta^3 - \alpha_1\alpha_2, \alpha_1\alpha_2\alpha_1, \alpha_2\alpha_1\alpha_2, \beta\alpha_1, \alpha_2\beta \rangle,$$

and the associated graded algebra of A is

$$gr(A) = kQ_G / \langle \beta^4, \alpha_1\alpha_2, \beta\alpha_1, \alpha_2\beta \rangle.$$

By the parallel paths method,

$$\mathrm{HH}^1(A) = k\{3(\alpha_1, \alpha_1) + (\beta, \beta),\ (\beta, \beta^2),\ (\beta, \alpha_1\alpha_2)\}$$

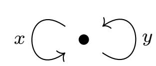
$$\mathrm{HH}^1(gr(A)) = k\{(\alpha_1, \alpha_1),\ (\beta, \beta),\ (\beta, \beta^2),\ (\beta, \beta^3)\}$$

Then $\dim_k \mathrm{HH}^1(gr(A)) - \dim_k \mathrm{HH}^1(A) = 1$. *And the monomorphism i is given by:*

$$\begin{aligned} i \ :\ 3(\alpha_1, \alpha_1) + (\beta, \beta) &\mapsto 3(\alpha_1, \alpha_1) + (\beta, \beta), \\ (\beta, \beta^2) &\mapsto (\beta, \beta^2), \\ (\beta, \alpha_1\alpha_2) &\mapsto (\beta, \beta^3). \end{aligned}$$

Example 5.15. *This is an exceptional example in Theorem 5.11:*

G : •[2] —— •[3] Q_G :



The corresponding Brauer graph algebra of Brauer G is

$$A = k\langle x, y \rangle / \langle y^3 - x^2, x^3, xy, yx \rangle,$$

and the associated graded algebra of A is

$$gr(A) = k\langle x, y \rangle / \langle y^4, x^2, xy, yx \rangle.$$

By the parallel paths method,

$$\mathrm{HH}^1(A) = k\{3(x, x) + 2(y, y),\ (x, x^2),\ (y, y^2),\ (y, x^2),\ (x, y^2) - (y, x)\}$$

$$\mathrm{HH}^1(gr(A)) = k\{(x,x),\ (y,y),\ (x,y^3),\ (y,y^2),\ (y,y^3),\ (y,x)\}$$

Then $\dim_k \mathrm{HH}^1(gr(A)) - \dim_k \mathrm{HH}^1(A) = 1$, *but since* $i((y,y^2)) = (y,y^2)$, $i((x,y^2)-(y,x)) = -(y,x)$, $i((x,x^2)) = (x,y^3)$, *we have*

$$i([(y,y^2),(x,y^2)-(y,x)]) = i(2(x,x^2)) = 2(x,y^3),$$

$$[i((y,y^2)), i((x,y^2)-(y,x))] = [(y,y^2), -(y,x)] = 0,$$

i is not a morphism between Lie algebras. Actually, if there is a monomorphism from $\mathrm{HH}^1(A)$ *to* $\mathrm{HH}^1(gr(A))$, *then* $\mathrm{HH}^1(A)$ *can be regard as a Lie subalgebra of* $\mathrm{HH}^1(gr(A))$. *That makes* $\mathrm{HH}^1(A)^{(2)}$ *is a subspace of* $\mathrm{HH}^1(gr(A))^{(2)}$. *However,*

$$\mathrm{HH}^1(A)^{(2)} = k\{(y,x^2),(x,x^2)\},$$

$$\mathrm{HH}^1(gr(A))^{(2)} = k\{(y,y^3)\}.$$

Thus $\dim_k(\mathrm{HH}^1(A)^{(2)}) = 2 > 1 = \dim_k(\mathrm{HH}^1(gr(A))^{(2)})$, *a contradiction.*

5.3 A discussion related to $\mathrm{Out}(A)^\circ$

In this subsection, we assume that k is an algebraically closed field with characteristic 0.

Recall the result we got in Lemma 4.7, for a Brauer graph algebra A with its corresponding Brauer graph $G = (V,E)$, $\dim_k L_{00} = |E| - |V| + 2$. It is interesting to note that Antipov and Zvonareva have got the same number in [1]. Denote by $T(A)$ the maximal torus of the identity component $\mathrm{Out}(A)^\circ$ of the group of the outer automorphisms for an algebra A. [1, Theorem 1.1] says that for a Brauer graph algebra A with its corresponding Brauer graph $G = (V,E)$, if G has at least two edges and is not a caterpillar (cf. [1, Section 3]), then the rank of $T(A)$ is $|E| - |V| + 2$.

Now we use a construction by Briggs and Rubio y Degrassi in [4] to discuss the connection between $T(A)$ and L_{00}. Recall some definitions.

Definition 5.16. ([*4, Definition* 2.2]) *Let A be a finite dimensional algebra over an algebraically closed field k. An element $f \in \mathrm{HH}^1(A)$ is called diagonalizable if it can be represented by a derivation $d \in \mathrm{Der}(A)$ which acts diagonalizably on A, with respect to some k-linear basis of A. More generally, we say that a subspace $S \subseteq \mathrm{HH}^1(A)$ is diagonalizable if its elements can be represented by derivations which are simultaneously diagonalizable on A. Note that S is automatically a Lie subalgebra of $\mathrm{HH}^1(A)$ since $[S,S] = 0$. The maximal diagonalizable subalgebras are by definition those diagonalizable subalgebras which are maximal with respect to inclusion.*

[19, Proposition 3.1] says that the Lie algebra of the identity component of the outer automorphism group of A is isomorphic to $\mathrm{HH}^1(A)$. Thus the rank of the maximal torus of $\mathrm{Out}(A)^\circ$ is the maximal toral rank of $\mathrm{HH}^1(A)$, and by [4, Proposition 2.3], it is equal to the dimension of the maximal diagonalizable subalgebra of $\mathrm{HH}^1(A)$. Obviously, L_{00} is a diagonalizable subalgebra of $\mathrm{HH}^1(A)$, we will prove that it is maximal for any Brauer graph algebra and its associated graded algebra.

Proposition 5.17. *Let A be a Brauer graph algebra and $gr(A)$ the associated graded algebra of A. Then L_{00} (respectively, $L_{00}^{gr(A)}$) is a maximal diagonalizable subalgebra of $\mathrm{HH}^1(A)$ (respectively, $\mathrm{HH}^1(gr(A))$).*

Proof. Since we have given a basis of $\mathrm{Ker}\psi_1$ in Theorem 4.8 (respectively, $\mathrm{Ker}\psi_1^{gr(A)}$ in Lemma 5.9), for every element in $\mathrm{HH}^1(A)$ (respectively, $\mathrm{HH}^1(gr(A))$), it can be presented by a representative element in $\langle S_1 \cup S_2 \cup S_3 \cup S_4\rangle$ (respectively, $\langle S_1^{gr} \cup S_2^{gr} \cup S_3^{gr} \cup S_4^{gr}\rangle$). Note that the elements in S_1 (respectively, S_1^{gr}) can be regarded as a generating set of L_{00} (respectively, $L_{00}^{gr(A)}$). Actually, for every element

$x \in S_i$ with $i = 2, 3, 4$ and $r \in kS_1$, we have $[x, r] = cx$, $c \in k$. Therefore, it is sufficient to show that for every element $x \in S_i$, $i = 2, 3, 4$, there exists an element $r \in kS_1$, such that $[x, r] \neq 0$. We only check the case of Brauer graph algebra A, $gr(A)$ can be checked in the same way.

Let $x = (\beta_0, \alpha_{k-1} \cdots \alpha_0 \cdot C_v(\alpha_0)^{m(v)-1}) - (\alpha_k, \beta_m \cdots \beta_1 \cdot C_w(\beta_1)^{m(w)-1}) \in S_2$. If C_v and C_w are different special cycles, same as the discussion in the proof of Theorem 5.11. If C_v and C_w are same special cycles, by the definition of S_2, $l(C_v) \geq 4$. Therefore, we can chose a nonzero element $r = (\beta_0, \beta_0) - (\beta, \beta) \in kS_1$ with $\beta \neq \alpha_k$, $[x, r] = x \neq 0$.

For $x \in S_3$, it can also be found in the proof in Theorem 5.11.

Let $x = (\alpha, p) \in S_4$ with $\alpha \mid C_v$, we have two cases to discuss.

Case 1. $\alpha \nmid p$, chose $r = \sum_{v \in V, \alpha_v \mid C_v} k_v(\alpha_v, \alpha_v)$ with $k_v = \prod_{v' \in V, v' \neq v} m(v')$ and $\alpha_v = \alpha$, then $r \in kS_1$ and $[x, r] = k_v r \neq 0$.

Case 2. $\alpha \mid p$.

- If p contains α more than one time, assume the times of α appear in p is n, $n \in \mathbb{N}$ and $n \geq 2$, chose $r = \sum_{v \in V, \alpha_v \mid C_v} k_v(\alpha_v, \alpha_v)$ with $k_v = \prod_{v' \in V, v' \neq v} m(v')$ and $\alpha_v = \alpha$, then $r \in kS_1$ and $[x, r] = (1 - n)k_v r \neq 0$.
- If p just contains α one time, there exist another $\alpha' \mid C_v$ and p contains α' n times. Chose $r = \sum_{v \in V, \alpha_v \mid C_v} k_v(\alpha_v, \alpha_v)$ with $k_v = \prod_{v' \in V, v' \neq v} m(v')$ and $\alpha_v = \alpha'$, then $r \in kS_1$ and $[x, r] = -nk_v r \neq 0$.

To sum up, for every element $x \in S_i$, $i = 2, 3, 4$, there exists an element $r \in kS_1$, such that $[x, r] \neq 0$. Thus L_{00} is the maximal diagonalizable subalgebra of $\mathrm{HH}^1(A)$. □

Since the dimension of a maximal torus of an algebraic group G is called the rank of G (see for example [18, Section 7.2.1]), we can rewrite Corollary 5.13 as follows.

Corollary 5.18. *The difference between the dimension of* $\mathrm{HH}^1(A)$ *and of* $\mathrm{HH}^1(gr(A))$ *is equal to the difference between the rank of* $\mathrm{Out}(A)^\circ$ *and of* $\mathrm{Out}(gr(A))^\circ$. *In particular, it is also equal to the difference between their maximal dimensions of the corresponding fundamental groups.*

Proof. By combining Corollary 5.13, Proposition 5.17 with the discussion in the beginning of this subsection, the first description is clear. The second description is also obvious by [4, Corollary 4.3], which tells us for any finite dimensional algebra A, the maximal torus rank of $\mathrm{HH}^1(A)$ is equal to the maximal dimension of the corresponding fundamental group of A. □

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