Trivial extensions of monomial algebras are symmetric fractional Brauer configuration algebras of type S

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Abstract: By giving some equivalent definitions of fractional Brauer configuration algebras of type S in some special cases, we construct a fractional Brauer configuration algebra from any monomial algebra. We show that this algebra is isomorphic to the trivial extension of the given monomial algebra. Moreover, we show that there exists a one-to-one correspondence between the isomorphism classes of monomial algebras and the equivalence classes of pairs consisting of a symmetric fractional Brauer configuration algebra of type S with trivial degree function and a given admissible cut over it.

1 Introduction

In the representation theory of algebras, monomial algebras form an important class of finite dimensional quiver algebras. They are nicely graded and often used to test some theories or conjectures. For example, it is known how to construct explicitly the minimal two-sided projective resolution of a given monomial algebra ([1]) and the finitistic dimension conjecture is true for monomial algebras ([6]). On the other hand, monomial algebras contain algebras of different global dimensions and different representation types, so they form a rich class of algebras from the representation theory point of view.

Symmetric algebras are another important class of finite dimensional algebras. Examples of symmetric algebras are given by groups algebras of finite groups and the trivial extensions of finite dimensional algebras. Trivial extension is a very common construction in representation theory and there are results linking trivial extension algebras with tilting theory ([12]). Until recently, a characterisation of trivial extension algebras in terms of quivers with relations is obtained in [5, Theorem 1.1].

In [16, Theorem 1.2], Schroll shows that trivial extensions of gentle algebras, which are some special monomial algebras, are Brauer graph algebras, a class of symmetric special biserial algebras. Moreover, she shows that there is a one-to-one correspondence between gentle algebras and Brauer graph algebras with multiplicity one associated with an admissible cut ([16, Theorem 1.3]). These results are generalized by Green and Schroll in [8]. They show that trivial extensions of almost gentle algebras, which are special monomial algebras containing gentle algebras, are Brauer configuration algebras, a class of symmetric special multiserial algebras containing Brauer graph algebras, and moreover, there is a one-to-one correspondence between almost gentle algebras and Brauer configuration algebras with multiplicity one associated with an admissible cut ([8, Theorem 5.5]).

Very recently, Li and Liu give a further generalization of Brauer configuration algebras in [10]. These algebras are defined by fractional Brauer configurations (abbr. f-BCs) and called fractional Brauer

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configuration algebras (abbr. f-BCAs), which may not be symmetric, may not be multiserial, even may not be finite-dimensional. However, if we assume that our f-BC is of type S, finite, and symmetric (see Section 2.2 for the meaning of these notions), then the corresponding f-BCA is symmetric and finite dimensional. We call such f-BCA a symmetric Brauer configuration algebra of type S (abbr. symmetric f_s-BCA).

In this paper, we generalize the results in [16] and [8]. We construct a fractional Brauer configuration of type S (abbr. f_s-BC) from any monomial algebra to get a corresponding f_s-BCA. We show that this algebra is isomorphic to the trivial extension of the given monomial algebra. Moreover, we show that there exists a one-to-one correspondence between the set of isomorphism classes of monomial algebras and equivalence classes of pairs consisting of a symmetric fractional Brauer configuration algebra of type S with trivial degree function and a given admissible cut over it.

Outline. In Section 2, we recall the definitions and basic properties on f-BCA and f_s-BCA. Moreover, we give some equivalent conditions about the defining relations of f-BCA in Lemma 2.3 and give an equivalent definition of f_s-BCA in Lemma 2.10 when its corresponding f_s-BC is symmetric. In Section 3, for each monomial algebra A we construct an f_s-BC E_A in Proposition 3.1. In section 4, we prove that the f_s-BCA of E_A is isomorphic to the trivial extension of the given monomial algebra A (Theorem 4.4). In section 5, we define admissible cuts of symmetric f_s-BCAs and show that there exists a one-to-one correspondence between the set of isomorphism classes of monomial algebras and the set of equivalence classes of pairs consisting of a symmetric f_s-BCA with trivial degree function and a given admissible cut over it (Corollary 5.6).

2 Basic knowledge about f_s-BCAs

2.1 Fractional Brauer configuration algebras

We recall some definitions of fractional Brauer configuration algebras in [10], which will play an important role in our following discussions. Furthermore, we give some equivalent conditions of the relations in fractional Brauer configuration algebras.

Definition 2.1. ([10, Definition 3.3]) *A fractional Brauer configuration (abbr. f-BC) is a quadruple $E = (E, P, L, d)$, where E is a G-set with $G = \langle g \rangle \cong (\mathbb{Z}, +)$, an infinite cyclic group, P and L are two partitions of E, and $d : E \to \mathbb{Z}_+$ is a function, such that the following conditions hold.*

(f1) $L(e) \subseteq P(e)$ and $P(e)$ is a finite set for each $e \in E$.

(f2) If $L(e_1) = L(e_2)$, then $P(g(e_1)) = P(g(e_2))$.

(f3) If e_1, e_2 belong to same $\langle g \rangle$-orbit, then $d(e_1) = d(e_2)$.

(f4) $P(e_1) = P(e_2)$ if and only if $P(g^{d(e_1)}(e_1)) = P(g^{d(e_2)}(e_2))$.

(f5) $L(e_1) = L(e_2)$ if and only if $L(g^{d(e_1)}(e_1)) = L(g^{d(e_2)}(e_2))$.

(f6) $L(g^{d(e)-1}(e)) \cdots L(g(e))L(e)$ is not a proper subsequence of $L(g^{d(h)-1}(h)) \cdots L(g(h))L(h)$.

For convenience, we recall the following notations in [10, Remark 3.4]. The elements in E are called angles. The $\langle g \rangle$-orbits of E are called vertices. The partitions P and L are called vertices partition and arrows partition, respectively. Moreover, the classes $P(e)$ of the partition P are called polygons. The arrows partition L is said to be trivial if $L(e) = e$ for each $e \in E$.

The function $d : E \to \mathbb{Z}_+$ is called degree function. Condition (f3) means that the degree function can be defined on vertices. Let v be a vertex such that v is a finite set, define the fractional-degree (abbr. f-degree) $d_f(v)$ of a vertex v to be the rational number $d(v)/|v|$. The degree function (respectively, the fractional-degree) is called trivial if each $d(e) = |v_e|$ (respectively, $d_f(v_e) = 1$) where v_e is the $\langle g \rangle$-orbit of each angle e.

Denote by σ the map $E \to E$, $e \mapsto g^{d(e)}(e)$, which is called the Nakayama automorphism of E. Moreover, we call an f-BC E *symmetric* if its Nakayama automorphism is identity. This equals to say that E has integral fractional-degree on each vertex.

Recall some basic notations about quiver algebras. Let k be a field and let A be a finite-dimensional k-algebra. Unless explicitly stated otherwise, all modules considered are finitely generated left modules. Furthermore, let $A = kQ/I$ be a quiver algebra, that means, k is a field, Q is a finite quiver and I is an ideal in kQ. (Note that in the present paper all the involved ideals I are lied in between the ideal $kQ_{\geq 1}$ generated by the arrows in Q and some power of $kQ_{\geq 1}$.) We denote by $s(p)$ the source vertex of a path p and by $t(p)$ its terminus vertex. We will write paths from right to left, for example, $p = \alpha_n \alpha_{n-1} \cdots \alpha_1$ is a path with starting arrow α_1 and ending arrow α_n. The length of a path p will be denoted by $l(p)$. Two paths ε, γ of Q are called parallel if $s(\varepsilon) = s(\gamma)$ and $t(\varepsilon) = t(\gamma)$. For convenience, $p, q \in Q$, we denote $p \mid q$ if p is a subpath of q. By abuse of notation we sometimes view an element in kQ as an element in kQ/I if no confusion can arise.

We will understand the concepts in Definition 2.1 easier through the quivers associated with the fractional Brauer configurations.

Definition 2.2. ([10, Definition 4.1]) *For an f-BC $E = (E, P, L, d)$, the quiver $Q_E = (Q_0, Q_1)$ associated with E defined as follow: $Q_0 = \{P(e) \mid e \in E\}$ and*

$$Q_1 = \{L(e) \mid e \in E,\ s(L(e)) = P(e) \text{ and } t(L(e)) = P(g(e))\}.$$

Therefore, the sequence $L(g^{d(e)-1}(e)) \cdots L(g(e))L(e)$ we considered in condition $(f6)$ in Definition 2.1 is actually a path in the quiver Q_E. We call the path of the form $L(g^{d(e)-1}(e)) \cdots L(g(e))L(e)$ the *special path* starting from $e \in E$.

Moreover, we can define the ideal I_E generated by the following three types of relations in [10, Definition 4.4]:

(R1) $L(g^{d(e)-1-k}(e)) \cdots L(g(e))L(e) - L(g^{d(h)-1-k}(h)) \cdots L(g(h))L(h)$, if $P(e) = P(h)$, $k \geq 0$ and $L(g^{d(e)-i}(e)) = L(g^{d(h)-i}(h))$ for $1 \leq i \leq k$.

(R2) $L(e_n) \cdots L(e_2)L(e_1)$, if $P(g(e_i)) = P(e_{i+1})$ for each $1 \leq i \leq n-1$ and $\bigcap_{i=1}^{n} g^{n-i}(L(e_i)) = \emptyset$.

(R3) $L(g^{n-1}(e)) \cdots L(g(e))L(e)$ for $n > d(e)$.

Call these three types of relations of I_E to be of type 1, type 2 and type 3 respectively. Moreover, the quiver algebra $A = kQ_E/I_E$ is called the fractional Brauer configuration algebra (abbr. f-BCA) associated with f-BC E.

In fact, the above relations are illustrated by using the concepts in the definition of fractional Brauer configurations. To make it easier to understand, let us restate the meaning of these relations in the language of the path algebra kQ_E. Consider the relations in kQ_E which are given by following forms.

(R1') $L(g^{d(e)-1-k}(e)) \cdots L(g(e))L(e) - L(g^{d(h)-1-k}(h)) \cdots L(g(h))L(h)$ with $P(e) = P(h)$ and $k \geq 0$, if there exists a nonzero path p in Q_E, such that

$$pL(g^{d(e)-1-k}(e)) \cdots L(g(e))L(e) \text{ and } pL(g^{d(h)-1-k}(h)) \cdots L(g(h))L(h)$$

are special paths starting from $P(e)$.

(R2') The nonzero path $L(e_n) \cdots L(e_2)L(e_1)$ which is not a subpath of the special path of an angle $e_1 \in E$, and each proper subpath of it is a subpath of some special path in kQ_E.

By Definition 2.1, it is easy to see that the sequences in the f-BC E fitting the condition (R1) is equivalent to the paths in Q_E fitting the condition (R1'). Moreover, we give the following lemma to show two descriptions of the relations in the ideal in kQ_E are equivalent.

Lemma 2.3. *The ideal generated by (R2') in kQ_E is equal to the ideal generated by (R2) and (R3).*

Proof. On the one hand, we prove each relation in (R2) and (R3) can be generated by (R2') in kQ_E. For all $L(g^{n-1}(e))\cdots L(g(e))L(e)$ which is a relation in (R3) with $n > d(e)$, it is obviously not a subpath of the special path $L(g^{d(e)-1}(e))\cdots L(g(e))L(e)$ of e, thus we can find some relation in (R2') that divides it exactly. If there exists a relation $L(e_n)\cdots L(e_2)L(e_1)$ is a subpath of some special path in kQ_E, without loss of generality, we can assume it is in the form of $L(g^{n-1}(e))\cdots L(g(e))L(e)$. However, that means $e_n \in \bigcap_{i=1}^{n} g^{n-i}(L(e_i)) \neq \emptyset$, contradict with the condition in (R2).

On the other hand, we prove each relation in (R2') can be generated by (R2) and (R3) in kQ_E. For each relation $L(e_n)\cdots L(e_2)L(e_1)$ in (R2'), If it does not contain $L(g^{d(e_1)-1}(e_1))\cdots L(g(e_1))L(e_1)$ as a proper subpath, consider the largest positive integer m such that $L(e_m) \neq L(g^m(e_1))$, then by Definition 2.1, in f-BC E, we have $L(e_m) \cap L(g^m(e_1)) = \emptyset$ since L is a partition of E. Therefore, $L(e_m)\cdots L(e_2)L(e_1)$ is a nonzero relation in (R2) which is a subpath of $L(e_n)\cdots L(e_2)L(e_1)$.

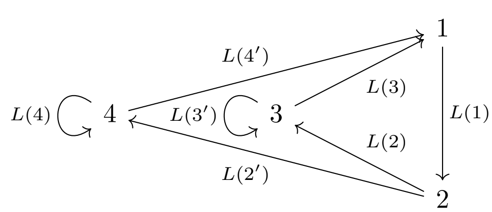
If $L(e_n)\cdots L(e_2)L(e_1)$ contains $L(g^{d(e_1)-1}(e_1))\cdots L(g(e_1))L(e_1)$ as a proper subpath, then there are two cases to consider about the arrow $L(e_{d(e_1)})$ in Q_E.

- If $L(e_{d(e_1)}) = L(g^{d(e)}(e))$, then it contains $L(g^{d(e)}(e))L(g^{d(e_1)-1}(e_1))\cdots L(g(e_1))L(e_1)$ which is a relation in (R3) as a subpath in Q_E.
- If $L(e_{d(e_1)}) \neq L(g^{d(e)}(e))$, then by Definition 2.1, in f-BC E, we have $L(e_{d(e_1)}) \cap L(g^{d(e)}(e)) = \emptyset$ since L is a partition of E. Therefore, $L(e_{d(e_1)})\cdots L(e_2)L(e_1)$ is a nonzero relation in (R2) which is a subpath of $L(e_n)\cdots L(e_2)L(e_1)$.

To sum up, the ideal generated by (R2') in kQ_E is equal to the ideal generated by (R2) and (R3). □

Recall some example in [10], and classify their relations according to the new conditions above.

Example 2.4. ([10, Example 3.6]) *Let $E = \{1, 1', 2, 2', 3, 3', 4, 4'\}$. Define the group action on E by $g(1) = 2$, $g(2) = 3$, $g(3) = 1$, $g(1') = 2'$, $g(2') = 4'$, $g(4') = 1'$, $g(3') = 3'$, $g(4) = 4$. Define $P(1) = \{1, 1'\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3, 3'\}$, $P(4) = \{4, 4'\}$, $L(1) = P(1)$ and $L(e) = \{e\}$ for $e \neq 1, 1'$. The degree function d of E is trivial. Then Q_E is the following quiver*



and for example, the special path of $1 \in E$ is given by

$$L(g^2(1))L(g(1))L(1) = L(3)L(2)L(1),$$

and the special path of $1' \in E$ is given by

$$L(g^2(1'))L(g(1'))L(1') = L(4')L(2')L(1).$$

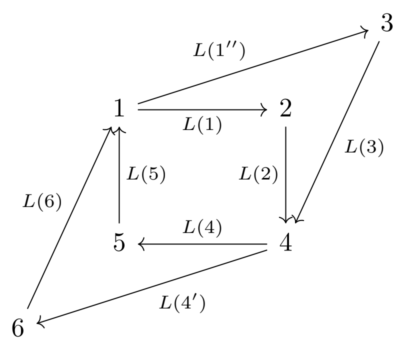
Moreover, the special paths of 4 and $3'$ are given by $L(4)$ and $L(3')$, respectively.

The ideal I_E is generated by

(R1') $L(3') - L(2)L(1)L(3)$, $L(4) - L(2')L(1)L(4')$, $L(3)L(2) - L(4')L(2')$.

(R2') $L(2)L(1)L(4')$, $L(2')L(1)L(3)$, $L(3)L(3')$, $L(2)L(3')$, $L(4')L(4)$, $L(4)L(2')$.

Example 2.5. ([10, Example 3.8]) *Let* $E = \{1, 1', 1'', 2, 2', 3, 4, 4', 4'', 5, 5', 6\}$. *Define the group action on* E *by* $g(1) = 2$, $g(2) = 4$, $g(4) = 5$, $g(5) = 1$, $g(1') = 2'$, $g(2') = 4'$, $g(4') = 6$, $g(6) = 1'$, $g(1'') = 3$, $g(3) = 4''$, $g(4'') = 5'$, $g(5') = 1''$. *Define* $P(1) = \{1, 1', 1''\}$, $P(2) = \{2, 2'\}$, $P(3) = \{3\}$, $P(4) = \{4, 4', 4''\}$, $P(5) = \{5, 5'\}$, $P(6) = \{6\}$, $L(1) = \{1, 1'\}$, $L(2) = \{2, 2'\}$, $L(4) = \{4, 4''\}$, $L(5) = \{5, 5'\}$ *and* $L(e) = \{e\}$ *for other* $e \in E$. *The degree function* d *of* E *is trivial. Then* Q_E *is the following quiver*



and for example, the special path of $1 \in E$ *is given by*

$$L(g^3(1))L(g^2(1))L(g(1))L(1) = L(5)L(4)L(2)L(1),$$

the special path of $1' \in E$ *is given by*

$$L(g^3(1'))L(g^2(1'))L(g(1'))L(1') = L(6)L(4')L(2)L(1),$$

and the special path of $1'' \in E$ *is given by*

$$L(g^3(1''))L(g^2(1''))L(g(1''))L(1'') = L(5)L(4)L(3)L(1'').$$

The ideal I_E *is generated by*

(R1') $L(5)L(4) - L(6)L(4')$, $L(2)L(1) - L(3)L(1'')$.

(R2') $L(1'')L(6)$, $L(4')L(3)$.

2.2 Fractional Brauer configuration algebras of type S

In this section, we recall a special class of f-BCAs, which are called the fractional Brauer configuration algebras of type S in [10]. We recall some basic definitions at first.

Definition 2.6. ([10, Definition 3.10]) *Let* E *be an f-BC, call a sequence* $p = (g^{n-1}(e), \cdots, g(e), e)$ *with* $e \in E$ *and* $0 \le n \le d(e)$ *a standard sequence of* E. *In particular, we define* $p = ()_e$ *when* $n = 0$ *which is called a trivial sequence in* E.

A standard sequence of the form $p = (g^{d(e)-1}(e), \cdots, g(e), e)$ *with* $e \in E$ *is called a full sequence of* E. *Actually, there is a bijective map between the full sequences* $p = (g^{d(e)-1}(e), \cdots, g(e), e)$ *of* E *and the special paths* $L(g^{d(e)-1}(e)) \cdots L(g(e))L(e)$ *in* kQ_E.

For a standard sequence $p = (g^{n-1}(e), \cdots, g(e), e)$, we can define two associated standard sequences

$$^\wedge p = \begin{cases} (g^{d(e)-1}(e), \cdots, g^{n+1}(e), g^n(e)) & , \text{ if } 0 < n < d(e); \\ ()_{g^{d(e)}(e)} & , \text{ if } n = d(e); \\ (g^{d(e)-1}(e), \cdots, g(e), e) & , \text{ if } n = 0 \text{ and } p = ()_e, \end{cases}$$

and

$$p^\wedge = \begin{cases} (g^{-1}(e), g^{-2}(e), \cdots, g^{n-d(e)}(e)) & , \text{ if } 0 < n < d(e); \\ ()_e & , \text{ if } n = d(e); \\ (g^{-1}(e), \cdots, g^{-d(e)}(e)) & , \text{ if } n = 0 \text{ and } p = ()_e. \end{cases}$$

Note that for a standard sequence p of E, $^\wedge pp$ and $pp^\wedge$ are full sequences of E.

For a standard sequence $p = (g^{n-1}(e), \cdots, g(e), e)$, define a formal sequence

$$L(p) = \begin{cases} L(g^{n-1}(e)) \cdots L(g(e))L(e), \cdots, g^{n-d(e)}(e)) & , \text{ if } 0 < n \leq d(e); \\ e_{P(e)} & , \ p = ()_e. \end{cases}$$

where $e_{P(e)}$ is the trivial path at vertex $P(e)$ in kQ_E. Actually, it is a subpath of the special path of $e \in E$. Moreover, for a set $\mathcal{X}$ of standard sequences, define $L(\mathcal{X}) = \{L(P) \mid p \in \mathcal{X}\}$. By the definition of f-BCA, for any standard sequence p in f-BC E, the formal sequence $L(p)$ corresponds to a nonzero path in the associated algebra $A = kQ_E/I_E$. By abuse of notation we sometimes view a standard sequence p in E as a path in kQ_E if no confusion can arise.

Definition 2.7. ([10, Definition 3.11]) *Let E be an f-BC, p and q be two standard sequences of E, define $p \equiv q$ if $L(p) = L(q)$. In the case we say p and q are identical.*

Actually, for standard sequences p, q, $^\wedge p \equiv ^\wedge q$ if and only if $p^\wedge \equiv q^\wedge$. For a set $\mathcal{X}$ of standard sequences, denote $[\mathcal{X}] = \{\text{standard sequence } q \mid q \text{ is identical to some } p \in \mathcal{X}\}$, denote $^\wedge\mathcal{X} = \{^\wedge p \mid p \in \mathcal{X}\}$ (resp. $\mathcal{X}^\wedge = \{p^\wedge \mid p \in \mathcal{X}\}$).

Definition 2.8. ([10, Definition 3.13]) *An f-BC E is said to be of type S (or E is an f_s-BC for short) if it satisfies additionally the following condition.*

(f7) For standard sequences $p \equiv q$, we have $[[^\wedge p]^\wedge] = [[^\wedge q]^\wedge]$.

The algebra $A = kQ_E/I_E$ is called the fractional Brauer configuration algebra of type S (abbr. f_s-BCA) associated with an f_s-BC E.

Remark 2.9. *For a standard sequence p, we have $[^\wedge [p^\wedge]] = [^\wedge [(^\wedge p)^{\wedge\wedge}]] = [^\wedge ([^\wedge p]^{\wedge\wedge})] = [[^\wedge p]^\wedge]$ whenever $(f7)$ holds or not.*

Since the definition of f_s-BC is given by the notations of some f-BC E, we also transform it to a concept corresponding to the paths in kQ_E.

From now on, let the f-BC E be symmetric, which means the Nakayama automorphism of E is identity. In this case, we have that all special paths in Q_E are cycles and $^\wedge p = p^\wedge$ for all standard sequence p in E. Given a new condition as following.

(sf7) For two nonzero relations $p - q$, $p' - q'$ of type 1 in I_E, if pp', qp' and pq' are some special paths in Q_E at the same time, then so is qq'.

Lemma 2.10. *The condition (f7) implies (sf7). Moreover, if the f-BC E is symmetric, then the condition (sf7) also implies (f7).*

Proof. On the one hand, we prove $(f7)$ implies $(sf7)$. If $(sf7)$ does not hold, then there exist two nonzero relations $p-q$, $p'-q'$ of type 1 in I_E, pp', qp' and pq' are different special paths in Q_E at the same time, but qq' is not. Therefore, denote $l(p')=n$, we can find different angles $e,h\in E$, such that

$$p'=L(g^{n-1}(e))\cdots L(g(e))L(e)=L(g^{n-1}(h))\cdots L(g(h))L(h).$$

with $p=L(g^{d(e)-1}(e))\cdots L(g^n(e))$ and $q=L(g^{d(h)-1}(h))\cdots L(g^n(h))$.

Denote the standard sequences corresponding to p' by $p_1=(g^{n-1}(e),\cdots,e)$ and $p_2=(g^{n-1}(h),\cdots,h)$, thus $p_1\equiv p_2$. Since pq' is a special path in E but qq' is not, we have $q'\in L([[^\wedge p_1]^\wedge])$ but $q'\notin L([[^\wedge p_2]^\wedge])$. Therefore, $(f7)$ is also not true at the same time.

On the other hand, we prove $(sf7)$ implies $(f7)$ when E is symmetric. If not, there exist standard sequences $p\equiv q$ and $p\neq q$, such that $[[^\wedge p]^\wedge]\neq[[^\wedge q]^\wedge]$, which also means $L([^\wedge p]^\wedge)\neq L([^\wedge q]^\wedge)$.

To be more specific, we may assume that there exist a path p_0 in Q_E, such that $L(^\wedge p)p_0$ is a special path in Q_E, but $L(^\wedge q)p_0$ is not. In this case, we have $L(^\wedge p)\neq L(^\wedge q)$. Actually, by the definition of standard sequences and $L(p)=L(q)$, we have $L(^\wedge p)L(p)$ and $L(^\wedge q)L(p)$ are special paths in Q_E. Since E is symmetric, all special paths are cycles in Q_E. Therefore, $L(p)L(^\wedge p)$ and $L(p)L(^\wedge q)$ are different special paths in Q_E at the same vertex, that means $L(^\wedge p)-L(^\wedge q)$ is a relation of type 1. In the same reason, we have $L(p)-p_0$ is also a relation of type 1. However, $L(^\wedge p)-L(^\wedge q)$ and $L(p)-p_0$ do not fit the condition $(sf7)$, a contradiction!

□

It is easier to check that the f-BC in Example 2.4 is an f_s-BC but the f-BC in Example 2.5 is not by using the lemma above than using the definition of f_s-BC .

Proposition 2.11. ([10, Proposition 5.2, Proposition 5.4]) *If E is a f_s-BC with a finite angle set, then the corresponding f_s-BCA $A=kQ_E/I_E$ is a finite-dimensional Frobenius algebra with the Nakayama automorphism of A induced by the inverse of the Nakayama automorphism of the f_s-BC E.*

Therefore, if the f_s-BC with a finite angle set is symmetric, then $A=kQ_E/I_E$ is symmetric, that is, $A\cong \mathrm{Hom}_k(A,k)$ as A-A-bimodules (More details on equivalent definitions of symmetric algebras can be found, for example, in [13, Theorem 3.1]). These algebras are called symmetric fractional Brauer configuration algebras of type S (abbr. symmetric f_s-BCA).

3 The f_s-BC associated to a monomial algebra

In this section, we construct an f_s-BC from a given monomial algebra. Actually, it is a generalization of the graph of a gentle algebra in [16, Section 3.1].

Let $A=kQ/I$ be a finite-dimensional monomial algebra, that means, k is a field, Q is a finite quiver and I is an ideal in kQ which is generated by paths. Moreover, consider the set $\mathcal{M}=\{p_1,\cdots,p_m\}$ of maximal paths in A (for all $p\in\mathcal{M}$ and $\alpha\in Q_1$, $\alpha p=0=p\alpha$ in A).

Define a quadruple $E_A=(E,P,L,d)$ of the monomial algebra $A=kQ/I$ as follows.

- $E=\bigcup_{p\in\mathcal{M}}\{(e_i,p)\mid p=(e_1\to e_2\to\cdots\to e_n)\}$;
- $P((e_i,p))=\{(e_i',p')\in E\mid e_i=e_i'\in Q_0\}$;
- $L((e_i,p))=\{(e_i',p')\mid$ the arrow starting at e_i in p is same as the arrow starting at e_i' in $p'\}$;
- $d((e_i,p))=l(p)+1$;
- if $p=(e_1\to e_2\to\cdots\to e_n)$, then $g((e_i,p))=(e_{i+1},p), i=1,\cdots,n-1$ and $g((e_n,p))=(e_1,p)$.

We need to state that we treat (e_i, p) and (e_j, p) with $i \neq j$ as distinct angles in E_A, even if $e_i = e_j$ in Q_0. It is obviously to find that E_A has trivial degree function, P, L are partitions of E and each $\langle g \rangle$-orbit corresponds to a unique maximal path in A. Actually, by the definition above, for all $p \in \mathcal{M}$, $L((t(p), p)) = \{(t(p), p)\}$.

Proposition 3.1. *E_A is a symmetric f_s-BC.*

Proof. We check the conditions in Definition 2.1 step by step.

(f1). Since A is finite-dimensional and $\mathcal{M}$ is contained in a k-basis of A, by the definition of E_A, the angle set E is finite. Moreover, each $P((e_i, p)) \subseteq E$ is finite. By the definition of the partitions P and L in E_A, for all $(e'_i, p') \in L((e_i, p))$, there exists an common subarrow $\alpha \in Q_1$ of p and p', such that $s(\alpha) = e_i = e'_i$, thus $(e'_i, p') \in P((e_i, p))$, that means $L((e_i, p)) \subseteq P((e_i, p))$ for all $(e_i, p) \in E$.

(f2). If $L((e_i, p)) = L((e'_i, p'))$, then there exists an arrow α in Q with $\alpha \mid p$ and $\alpha \mid p'$, such that $s(\alpha) = e_i = e'_i \in Q_0$. Denote the terminus vertex of α in p and p' by e_j and e'_j, respectively. Then $e_j = e'_j \in Q_0$. Therefore,

$$P(g(e_i, p)) = P((e_j, p)) = P((e'_j, p')) = P(g(e'_i, p')).$$

(f3). For all $(e_i, p) \in E$, each $\langle g \rangle$-orbit of (e_i, p) is defined by the maximal path $p \in \mathcal{M}$. Therefore, by the definition of degree function, we have $d((e_i, p)) = d(g^k(e_i, p))$, $k \in \mathbb{Z}$.

(f4) and *(f5)*. For all $e \in E$ which is an angle of E_A, we have $g^{d(e)}(e) = e$. Therefore, the conditions *(f4)* and *(f5)* are automatically established.

(f6). If $L(g^{d(e)-1}(e)) \cdots L(g(e))L(e)$ is a proper subsequence of $L(g^{d(h)-1}(h)) \cdots L(g(h))L(h)$ for some $e, h \in E$, without loss of generality, we can assume that there exist a positive integer $n < d(h)$, such that

$$L(g^{d(e)-1}(e)) \cdots L(g(e))L(e) = L(g^{n-1}(h)) \cdots L(g(h))L(h).$$

To be more specific, let $e = (e_i, p)$ and $h = (e_j, q)$. By the definition of the partition L of E_A, there exist a non-trivial path p' in Q, such that $q = p'p$. However, that means p is a proper subpath of q, contradict to $p \in \mathcal{M}$ which is a maximal path in A.

To sum up, by Definition 2.1, we have E_A is a f-BC . Moreover, since for all $e \in E$ which is an angle of E_A, we have $g^{d(e)}(e) = e$. Therefore, the Nakayama automorphism σ of E_A is identity. That means, f-BC E_A is symmetric.

By using Lemma 2.10, we show that the f-BC E_A fits the condition *(sf7)*. If not, consider the quiver Q_{E_A} associated with E_A, there exist $p_i := L(g^{n_i-1}(h_i)) \cdots L(g(h_i))L(h_i)$ with $i = 1, 2, 3, 4$, and two nonzero path q_1, q_2 in Q_{E_A} such that q_1p_1, q_1p_2, q_2p_3, q_2p_4 are special paths in Q_{E_A}. In other words, $p_1 - p_2$, $p_3 - p_4$ are relations of type 2 in E_A. Moreover, p_1p_3, p_2p_3, p_1p_4 are special paths in E_A, but p_2p_4 is not.

However, if $p_1 \neq p_2$ and q_1p_1, q_1p_2 are special paths in Q_E, we have the corresponding elements in $\mathcal{M}$ is given by M_1, M_2. Since $L((t(M_1), M_1))$ is trivial, its corresponding arrow in Q_{E_A} can only appear in exactly one special path (under cyclic permutation of cycles in Q_{E_A}) in Q_{E_A}. Therefore, $L((t(M_1), M_1)) \mid p_1$. Moreover, if $p_3 \neq p_4$ and p_1p_3, p_1p_4 are special paths in Q_E, without loss of generality, we have $p_4 \neq q_1$. Therefore, the arrow $L((t(M_1), M_1))$ appears in distinct special paths q_1p_1 and p_1p_4 in Q_{E_A}. However, $L((t(M_1), M_1))$ is a trivial angle set, which can only be involved in exactly one $\langle g \rangle$-orbit, a contradiction!

In conclusion, E_A fits *(sf7)*. By Lemma 2.10, E_A fits *(f7)* since E_A is symmetric. Therefore, E_A is a symmetric f_s-BC .

□

Let $A = kQ/I$ be a finite-dimensional monomial algebra and $E_A = (E, P, L, m)$ be the f_s-BC associated with A. Denote the f_s-BCA of E_A by A_E. Since the angle set E is finite, A_E is finite-dimensional. By Proposition 2.11 and the discussion that follows it, A_E is a finite-dimensional symmetric algebra. We call A_E the symmetric f_s-BCA associated with the monomial algebra A.

4 Trivial extensions of monomial algebras

Let $A = kQ/I$ be a finite-dimensional k-algebra and let $D(A) = \mathrm{Hom}_k(A, k)$ be its k-linear dual. Recall the trivial extension $T(A) = A \rtimes D(A)$ is an algebra defined as the vector space $A \oplus D(A)$ and with multiplication given by $(a, f)(b, g) = (ab, ag + bf)$, for any $a, b \in A$ and $f, g \in D(A)$. Note that $D(A)$ is an A-A-bimodule via the following. If $a, b \in A$ and $f \in D(A)$, then $afb : A \to k$ by $(afb)(x) = f(bxa)$. It is well-known that the trivial extension algebra $T(A)$ is a symmetric algebra, see for example in [15, Proposition 6.5]. It is proven in [3, Proposition 2.2] that the vertices of quiver $Q_{T(A)}$ of $T(A)$ correspond to the vertices of quiver Q of A and that the number of arrows from a vertex i to a vertex j in $T(A)$ is equal to the number of arrows from i to j in Q plus the dimension of the k-vector space $e_i(\mathrm{soc}_{A^e} A)e_j$.

Let Q be the set of finite directed paths and suppose that I is generate by paths, that means, A is a monomial algebra. Consider $\mathcal{B} = \{p \in Q \mid p \notin I\}$. The set $\pi(\mathcal{B})$ with the canonical surjection $\pi : kQ \to A$ forms a k-basis of A. We abuse the notation and view $\mathcal{B}$ as a k-basis of A. Then by [3, Proposition 2.2], the set $\mathcal{M}$ of maximal paths of A is a subset of $\mathcal{B}$ and forms a k-basis of $\mathrm{soc}_{A^e}(A)$. Therefore, denote the arrow set of the quiver $Q_{T(A)}$ of $T(A)$ by $(Q_{T(A)})_1$ and the arrow set of the quiver Q of the monomial algebra $A = kQ/I$ by Q_1, then we have

$$|(Q_{T(A)})_1| = |Q_1| + |\mathcal{M}|.$$

Lemma 4.1. *Let $A = kQ/I$ be a finite-dimensional monomial algebra. Denote the f_s-BCA associated with A by $A_E = kQ_E/I_E$ and the trivial extension of A by $T(A) = kQ_{T(A)}/I_{T(A)}$. Then the quiver Q_E is isomorphic to $Q_{T(A)}$.*

Proof. Denote the f_s-BC associated with A by $E_A = (E, P, L, m)$. Since the vertices of Q_E are corresponding to the partition P of E and for all $(e_i, p) \in E$, the angles in $P((e_i, p))$ have a common first coordinate in Q_0, the vertices in Q_E are corresponding to the vertices in Q. Thus the vertices in Q_E are corresponding to the vertices in $Q_{T(A)}$.

Now consider the arrows in Q_E from a vertex i to a vertex j in Q_E. Denote the trivial path corresponding to i and j by e_i and e_j in Q. For each arrow α in e_jQe_i, it can be extended to a maximal path p (may not unique) in A. Moreover, by definition of the f_s-BC E_A, $g(e_i, p) = (e_j, p)$. Therefore, by definition of the quiver of f-BC , there is an arrow in Q_E corresponding to α. If we choose a different maximal path p' containing α, then by definition of the f_s-BC E_A, $L((e_i, p)) = L((e_i, p'))$. Thus this correspondence is a bijection between the arrows in $Q_{T(A)}$ corresponding to an arrow $\alpha \in e_jQe_i$ in Q and the arrows $L((e_i, p))$ in Q_E with p a maximal path in A containing α. Moreover, for all maximal path $p \in \mathcal{M}$ from j to i (Note that $p \in e_i(\mathrm{soc}_{A^e} A)e_j$), we have that $L((e_i, p))$ is trivial and $g(e_i, p) = (e_j, p)$. Therefore, there exist a unique arrow $\alpha_p := L((e_i, p))$ in Q_E from i to j corresponding to the maximal path p.

In conclusion, the quiver Q_E is isomorphic to $Q_{T(A)}$.

□

Recall some basic properties in $T(A)$. The dual basis $\mathcal{B}^\vee = \{p^\vee \mid p \in \mathcal{B}\}$ is a k-basis of $D(A)$ where, if $p \in \mathcal{B}$, $p^\vee \in D(A)$ is the element in $D(A)$ defined by $p^\vee(q) = \delta_{p,q}$ for $q \in \mathcal{B}$.

Lemma 4.2. ([8, Lemma 4.1]) *Let A be a finite dimensional monomial algebra with k-basis $\mathcal{B}$ as above. Then, for $p, q, r \in \mathcal{B}$, the following holds in $T(A)$.*

(1) $(p, 0)(0, r^\vee) = \begin{cases} (0, s^\vee) & , \text{ if there is some } s \in \mathcal{B} \text{ with } sp = r \\ 0 & , \text{ otherwise.} \end{cases}$

(2) $(0, r^\vee)(q, 0) = \begin{cases} (0, s^\vee) & , \text{ if there is some } s \in \mathcal{B} \text{ with } qs = r \\ 0 & , \text{ otherwise.} \end{cases}$

(3) $(0, p^\vee)(q, 0)(0, r^\vee) = 0$.

(4) *If* $prq \in \mathcal{B}$*, then* $(q, 0)(0, (prq)^\vee)(p, 0) = (0, r^\vee)$.

Proposition 4.3. ([8, Proposition 4.2]) *Let A be a finite-dimensional monomial algebra. Then $T(A)$ is generated by $\{(\alpha, 0) \mid \alpha \in Q_1\} \cup \{(0, m^\vee) \mid m \in \mathcal{M}\}$.*

We now prove the main result of this section.

Theorem 4.4. *Let $A = kQ/I$ be a finite-dimensional monomial algebra, A_E the symmetric f_s-BCA of E_A, and $T(A)$ the trivial extension of A by $D(A)$. Then A_E is isomorphic to $T(A)$.*

Proof. By Lemma 4.1, we can divide arrows in Q_E into two parts. Denote the arrow in Q_E corresponding to some arrow α in Q by α, and the arrow in Q_E corresponding to some arrow induced by some $m \in \mathcal{M}$ by α_m.

We first prove $\dim_k A_E = 2 \dim_k A$. Actually, by definition of the f_s-BC E_A and the quiver associated with E_A, we can regard the monomial algebra $A = kQ/I$ as a subalgebra of A_E. Therefore, we can embedding the k-basis $\mathcal{B}$ of A to nonzero paths in A_E without involving arrows induced by maximal paths in A. To be more specific, it is an injection $i_1 : A \to A_E$ of k-vector space given by $p \mapsto p$ for all nonzero path $p \in \mathcal{B}$ in A.

Recall each nonzero path in A_E is induced by a standard sequence in E_A. Thus, we can define the second map $i_2 : A \to A_E$ of k-vector space which is given by $p \mapsto {}^\wedge p$. By [10, Lemma 4.18], this is also an injection. Moreover, all nonzero paths in $\mathrm{Im} i_2$ have a subpath which is an arrow that induced by maximal paths in A. Therefore, $\mathrm{Im} i_1 \cap \mathrm{Im} i_2 = \emptyset$. Moreover, we prove $\mathrm{Span}_k(\mathrm{Im} i_1 \cup \mathrm{Im} i_2) = A_E$. It is obviously to find that $\mathrm{Span}_k(\mathrm{Im} i_1 \cup \mathrm{Im} i_2) \subseteq A_E$. For all non zero path q in A, if q does not contain a subarrow induced by some maximal path in A, then $q \in \mathrm{Im} i_1$. If there exist an arrow induced by some maximal path in A that is a subpath of q, then by definition of f_s-BCA A_E, ${}^\wedge q \in \mathrm{Im} i_1$. Therefore, $q = {}^{\wedge\wedge} q \in \mathrm{Im} i_2$. To sum up, we have $\dim_k A_E = 2 \dim_k A = \dim_k T(A)$.

We construct a surjection $\psi : kQ_E \to T(A)$ of k-algebras which is given by $\alpha \in Q_1 \mapsto (\alpha, 0)$, $\alpha_m \mapsto (0, m^\vee)$ with $m \in \mathcal{M}$. It is straightforward to see that it is a surjection by Proposition 4.3. Now we prove it can induce a surjection from A_E to $T(A)$, which means for all relations $\rho \in I_E$, $\psi(\rho) = 0$.

By discussion in Section 2.1, I_E can be generated by relations fitting the condition (R1') or the condition (R2').

Let $\rho = \alpha_n \cdots \alpha_1$ is a relation fitting the condition (R2'). There are two cases to consider as follows.

Case 1. If ρ does not contain arrows induced by maximal paths in A, then by definition of E_A, it is actually a relation in the ideal I in A. Then $\psi(\rho) = (\rho, 0) = 0$ in $T(A)$.

Case 2. If ρ contains an arrow α_i induced by a maximal path $m \in \mathcal{M}$ in A, then we can write $m = \beta_n \cdots \beta_1$. If ρ have some subarrow which is not in $\{\beta_1, \cdots, \beta_n\}$, by Lemma 4.2, $\psi(\rho) = 0$. If all arrows in ρ is in $\{\beta_1, \cdots, \beta_n\}$, then without loss of generality, we can assume $\rho = \beta_n \cdots \beta_1 \alpha_i \beta_n$ or $\rho = \alpha_i \beta_n \cdots \beta_1 \alpha_i$. Thus

$$\psi(\beta_n \cdots \beta_1 \alpha_i \beta_n) = (\beta_n \cdots \beta_1, 0)(0, (\beta_n \cdots \beta_1)^\vee)(\beta_n, 0) = (0, e^\vee_{t(\beta_n)})(\beta_n, 0) = 0;$$

$$\psi(\alpha_i \beta_n \cdots \beta_1 \alpha_i) = (0, (\beta_n \cdots \beta_1)^\vee)(\beta_n \cdots \beta_1, 0)(0, (\beta_n \cdots \beta_1)^\vee) = (0, e^\vee_{t(\beta_n)})(0, (\beta_n \cdots \beta_1)^\vee) = 0.$$

Let $\rho = \alpha_n \cdots \alpha_1 - \beta_m \cdots \beta_1$ is a relation fitting the condition (R1'). Then there exist a nonzero path p in A_E, such that $p\alpha_n \cdots \alpha_1$ and $p\beta_m \cdots \beta_1$ are special paths in A_E. By the proof in Proposition 3.1, there are arrows α_i and β_j induced by maximal paths m_1 and m_2 in A which are subarrows of $\alpha_n \cdots \alpha_1$ and $\beta_m \cdots \beta_1$ respectively. Moreover, $m_1 = \alpha_{i-1} \cdots \alpha_1 p \alpha_n \cdots \alpha_{i+1}$ and $m_2 = \beta_{j-1} \cdots \beta_1 p \beta_m \cdots \beta_{j+1}$. Therefore,

$$\psi(\rho) = (0, p^\vee) - (0, p^\vee) = 0.$$

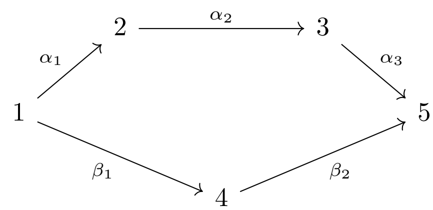
In conclusion, the surjection ψ induces a surjection of k-algebras from A_E to $T(A)$. Moreover, since $\dim_k A_E = \dim_k T(A)$, A_E is isomorphic to $T(A)$.

□

Actually, Theorem 4.4 is a generalization of [16, Theorem 1.2] and [8, Theorem 4.3]. In these two papers the authors proved that the trivial extensions of gentle algebras are Brauer graph algebras and the trivial extensions of almost gentle algebras are Brauer configuration algebras. In particular, gentle algebras and almost gentle algebras are monomial algebras and Brauer graph algebras and Brauer configuration algebras are symmetric fractional Brauer configuration algebras of type S.

We give some examples of the trivial extensions of monomial algebras. These monomial algebras are far from being gentle or almost gentle, since the defining relations are not quadratic.

Example 4.5. *Consider $A = kQ/I$, where Q is given by the following quiver*



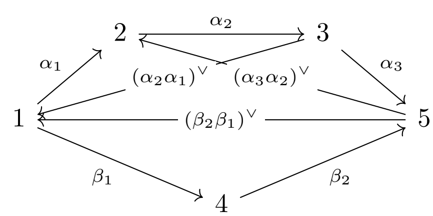
and $I = \langle \alpha_3\alpha_2\alpha_1 \rangle$.

Then we have $\mathcal{M} = \{p_1 := \alpha_2\alpha_1, p_2 := \alpha_3\alpha_2, p_3 := \beta_2\beta_1\}$. Moreover, consider the f_s-BC E_A of A:

$$E = \{(e_1,p_1),(e_2,p_1),(e_3,p_1),(e_2,p_2),(e_3,p_2),(e_5,p_2),(e_1,p_3),(e_4,p_3),(e_5,p_3)\};$$

$$P((e_1,p_1)) = \{(e_1,p_1),(e_1,p_3)\};$$
$$P((e_2,p_1)) = \{(e_2,p_1),(e_2,p_2)\};$$
$$P((e_3,p_1)) = \{(e_3,p_1),(e_3,p_2)\};$$
$$P((e_4,p_3)) = \{(e_4,p_3)\};$$
$$P((e_5,p_2)) = \{(e_5,p_2),(e_5,p_3)\};$$
$$L((e_2,p_1)) = \{(e_2,p_1),(e_2,p_2)\},$$

and $L(e) = \{e\}$, otherwise. The degree function in E_A is equal to 3 for all angles in E. Thus the quiver correspond to E_A is given by:



It is easy to check that $\dim_k A_E = 2\dim_k A = 26$ and $T(A) \cong A_E$.

Example 4.6. *Consider $A = k[x]/\langle x^3 \rangle$. The unique maximal path in A is given by x^2. Let $x^2 = (e_1 \to e_2 \to e_3)$ with $e_1 = e_2 = e_3 = 1$ in A. Therefore, consider the f_s-BC E_A of A:*

$$E = \{(e_1,x^2),(e_2,x^2),(e_3,x^2)\};$$

$$P((e_1,x^2)) = E;$$
$$L((e_1,x^2)) = \{(e_1,x^2),(e_2,x^2)\};$$

$$L((e_3, x^2)) = \{(e_3, x^2)\}.$$

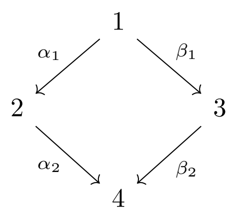
The degree function in E_A is equal to 3 for all angles in E. Thus the quiver correspond to E_A is given by:

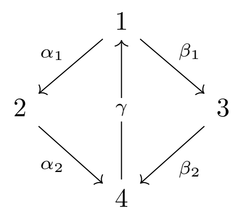
Actually, $A_E \cong k[x,y]/\langle x^3, y^2\rangle$. It is easy to check that $\dim_k A_E = 2\dim_k A = 6$ and $T(A) \cong A_E$.

It is proven in [11, 14, 17] that A is gentle if and only if $T(A)$ is special biserial. That also means $T(A)$ is a Brauer graph algebra in this case by [16, Theorem 1.1]. In [8, Question 4.5], Green and Schroll left a question about whether an algebra A is almost gentle if $T(A)$ is a Brauer configuration algebra, which is still unsolved.

However, in our case we have an example to show that $T(A)$ is symmetric f_s-BCA does not imply that A is a monomial algebra.

Example 4.7. *Consider $A = kQ_A/I_A$ and $B = Q_B/I_B$ with Q_A and Q_B are given by the following quivers:*

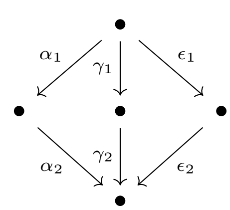
Q_A : 

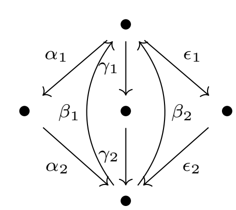
Q_B : 

and $I_A = \langle \alpha_2\alpha_1 - \beta_2\beta_1\rangle$, $I_B = \langle \alpha_2\alpha_1 - \beta_2\beta_1, \beta_1\gamma\alpha_2, \alpha_1\gamma\beta_2, \gamma\alpha_2\alpha_1\gamma\rangle$. By [3, Theorem 3.9], B is the trivial extension of A.

Let $E = \{1, 1', 2, 3, 4, 4'\}$. Define the group action on E by $g(1) = 2$, $g(2) = 4$, $g(4) = 1$, $g(1') = 3$, $g(3) = 4'$, $g(4') = 1'$. Define $P(1) = \{1, 1'\}$, $P(2) = \{2\}$, $P(3) = \{3\}$, $P(4) = \{4, 4'\}$, $L(4) = \{4, 4''\}$ and $L(e) = \{e\}$ for other $e \in E$. The degree function m of E is trivial. Therefore, E is an f_s-BC and the f_s-BCA associated with E is isomorphic to B.

It is shown in [10, Corollary 7.15] that the class of finite dimensional representation-finite f_s-BCAs is closed under derived equivalence. Our result in this section provides a potential way to test whether the same is true or not for representation-infinite f_s-BCAs.

Example 4.8. Q : 

$Q_{T(A)}$: 

Consider the canonical algebra A of type $(2,2,2)$ with the quiver given by Q. To be more specific, $A = kQ/\langle \alpha_2\alpha_1 + \gamma_2\gamma_1 + \epsilon_2\epsilon_1\rangle$. Then the quiver of the trivial extension of A can be given by $Q_{T(A)}$ which has a specific description in [5, Example 2.5]. Actually, by using [5, Theorem 1.1], $T(A) = kQ_{T(A)}/I$ with I generated by following relations:

- $\alpha_2\alpha_1 + \gamma_2\gamma_1 + \epsilon_2\epsilon_1$;
- $\beta_2\gamma_2$, $\gamma_1\beta_2$, $\beta_1\epsilon_2$, $\epsilon_1\beta_1$, $\gamma_1\beta_1\alpha_2$, $\alpha_1\beta_1\gamma_2$, $\epsilon_1\beta_2\gamma_2$, $\alpha_1\beta_2\epsilon_2$;
- $\gamma_1\beta_1\gamma_2\gamma_1$, $\gamma_2\gamma_1\beta_1\gamma_2$, $\beta_1\gamma_2\gamma_1\beta_1$, $\epsilon_1\beta_2\epsilon_2\epsilon_1$, $\epsilon_2\epsilon_1\beta_2\epsilon_2$, $\beta_2\epsilon_2\epsilon_1\beta_2$;

- $\alpha_1\beta_i\alpha_2\alpha_1$, $\alpha_2\alpha_1\beta_i\alpha_2$, $\beta_i\alpha_2\alpha_1\beta_i$, *with* $i = 1, 2$*;*
- $\beta_1\gamma_2\gamma_1 - \beta_2\epsilon_2\epsilon_1$, $\gamma_2\gamma_1\beta_1 - \epsilon_2\epsilon_1\beta_2$, $\beta_1\gamma_2\gamma_1 + \beta_2\alpha_2\alpha_1$, $\gamma_2\gamma_1\beta_1 + \alpha_2\alpha_1\beta_2$, $\beta_1\alpha_2 - \beta_2\alpha_2$, $\alpha_1\beta_1 - \alpha_1\beta_2$.

It is well-known that A is derived equivalent to the path algebra B of type $\widetilde{D_4}$, see for example in [9, Theorem 3.5]. Then $T(A)$ and $T(B)$ are also derived equivalent ([12, Theorem 3.1]). By Theorem 4.4, $T(B)$ is a symmetric f_s-BCA. It would be interesting to know whether the above $T(A)$ is isomorphic to some f_s-BCA or not, this seems not easy to judge because there is a relation $\alpha_2\alpha_1 + \gamma_2\gamma_1 + \epsilon_2\epsilon_1$ and many anti-commutative relations in the generating set of I.

5 Admissible cut on symmetric f_s-BCA

One way of constructing new algebras by deleting arrows in quivers is to use the notion of admissible cuts of finite dimensional algebras, which has been studied for example in [2, 4, 8, 16]. The aim of this section is to generalize the main result in [8] from BCAs to symmetric f_s-BCAs.

Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a symmetric f_s-BCA with trivial degree function. We assume that the associated angle set E is finite, this is equal to say that the algebra Λ is finite dimensional. All special paths (the paths corresponding to $L(g^{d(e)-1}(e))\cdots L(g(e))L(e)$ in f_s-BC E) which is actually some cycles in Q_Λ since its associated f_s-BC is symmetric. We call them *special cycles* in this case. Denote the set of all special cycles under cyclic permutation by $\mathcal{S}$. Let $\{C_1, \cdots, C_t\}$ be a set of representatives of equivalence classes under cyclic permutation of all special cycles.

Definition 5.1. (Compare to [4, Definition 3.2] and [8, Definition 5.1]) *A cutting set D of Q_Λ is a subset of arrows in Q_Λ consisting of exactly one arrow in each special cycle corresponding to an equivalence class representative C_i for $i = 1, \cdots, t$. We call $kQ_\Lambda/\langle I_\Lambda \cup D\rangle$ the cut algebra associated with D where $\langle I_\Lambda \cup D\rangle$ is the ideal generated by $I_\Lambda \cup D$. Moreover, we call a cutting set D is an admissible cut if $|D| = t$.*

Actually, this definition is a generalized version of the admissible cut in [16, Section 4] and [7, Definition 5.1]. To be more specific, since different special cycles of a given Brauer graph algebra (resp. Brauer configuration algebra) do not have common arrows in general, thus in these cases, $|D| = t$ is always true. However, if there exists a non-trivial L-partition in the f_s-BC , different special cycles in Λ may have some common arrow. Therefore, we need to assume $|D| = t$ if we want to ensure the cut algebra of Λ to be monomial.

Proposition 5.2. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be symmetric f_s-BCA with trivial degree function and let D be an admissible cut of Q_Λ. Set Q to be the quiver given by $Q_0 = (Q_\Lambda)_0$ and $Q_1 = (Q_\Lambda)_1\backslash D$. Then the cut algebra $kQ_\Lambda/\langle I_\Lambda \cup D\rangle$ associated with D is isomorphic to $kQ/\langle I_\Lambda \cap kQ\rangle$. Moreover, $kQ/\langle I_\Lambda \cap kQ\rangle$ is a monomial algebra.*

Proof. The inclusion of quivers $Q \subset Q_\Lambda$ induces a k-algebra homomorphism $f : kQ \to kQ_\Lambda/\langle I_\Lambda \cup D\rangle$, and f is surjective since each path p' in a nonzero element which is a linear combination of paths in $kQ_\Lambda/\langle I_\Lambda \cup D\rangle$ can not contain an arrow in D. Therefore, we can find the corresponding path p in Q, such that $f(p) = p'$. By the first isomorphism theorem, we have $kQ_\Lambda/\langle I_\Lambda \cup D\rangle \cong kQ/\langle I_\Lambda \cap kQ\rangle$.

Moreover, $kQ/\langle I_\Lambda \cap kQ\rangle$ is monomial since all the relations of type 1 in I_Λ do not intersect with kQ. □

The next result shows that if one starts with a monomial algebra and takes the appropriate admissible cut in the trivial extension of the monomial algebra, then the monomial algebra is isomorphic to the cut algebra.

Theorem 5.3. *Let $A = kQ/I$ be a monomial algebra with set of maximal paths $\mathcal{M}$ and let $T(A) = kQ_{T(A)}/I_{T(A)}$ be the trivial extension of A by $D(A)$ where the set of new arrows of $Q_{T(A)}$ is given by $D = \{\alpha_m \mid m \in \mathcal{M}\}$. Then D is an admissible cut of $Q_{T(A)}$ and the cut algebra associated with D is isomorphic to A.*

Proof. By Theorem 4.4, we can regard $T(A)$ as an f_s-BCA with $\mathcal{S} = \{C_1, \cdots, C|t\}$. Moreover, each special cycle in $T(A)$ corresponding to a unique maximal path in A. It follows from the construction of $T(A)$ that there exists exactly one arrow from D in any special cycle corresponding to an equivalence class representative C_i for $i = 1, \cdots, t$. Hence D is an admissible cut of $T(A)$. Moreover, by the proof in Theorem 4.4, the cut algebra associated with D is isomorphic to $T(A)$. □

The next result shows that if one starts with a symmetric f_s-BCA with trivial degree function and an admissible cut D, then the cut algebra associated with D, trivially extended by its dual, is isomorphic to the original symmetric f_s-BCA .

Theorem 5.4. *Let $\Lambda = kQ_\Lambda/I_\Lambda$ be a symmetric f_s-BCA with trivial degree function. Let D be an admissible cut of Q_Λ. Denote by $A = kQ/I$ the cut algebra associated with D. Then $T(A)$ is isomorphic to Λ.*

Proof. By the definition of f_s-BCA , it is easy to see that I is generated by paths. Actually, the special cycles in Q_Λ are of the form $C = p\alpha q$ for $\alpha \in D$. Since C is a special cycle, $qp\alpha$ and αqp are also special cycles. Thus $qp \notin I_\Lambda$ and hence $qp \notin I_\Lambda \cap kQ$. Since Λ is an f_s-BCA , if there exist an arrow β and an arrow γ in Q_Λ, such that βqp and $qp\gamma$ is not zero in Λ, then $\gamma = \beta = \alpha$. That means for all arrow β' in Q, $\beta' qp = qp\beta' = 0$ in A. Therefore, qp is a maximal path in A. By the definition of the f_s-BC associated with A and Theorem 4.4, we have $T(A) \cong \Lambda$. □

Remark 5.5. *If we do not request the cutting set D to be admissible, then the cut algebra may not be a monomial algebra. For example, in Example 4.7, if we choose the cutting set D as $\{\gamma\}$ in Q_B, then the cut algebra of B associated with D is isomorphic to A.*

Consider the set of pairs (Λ, D) such that $\Lambda = kQ_\Lambda/I_\Lambda$ is a symmetric f_s-BCA with trivial degree function and D is an admissible cut of Q_Λ. We say that (Λ, D) and (Λ', D') are equivalent if there exists a k-algebra isomorphism from Λ to Λ' sending D to D'. Denote by $\mathcal{Y}$ the equivalent classes. It is obviously to see that all corresponding cut algebras in a same equivalent class are isomorphic. Combining previous two theorems, we get the following main result of this section.

Corollary 5.6. *There is a bijection $\phi : \mathcal{A} \to \mathcal{Y}$ from the set $\mathcal{A}$ of isomorphism classes of monomial algebras to the set of equivalence classes of pairs consisting of a symmetric f_s-BCA and an admissible cut as defined above. The isomorphism is given, for $A \in \mathcal{A}$, by $\phi(A) = (T(A), D)$ where $D = \{\alpha_m \mid m \text{ is a maximal path in } A\}$. Moreover, for (Λ, D), we have $\phi^{-1}((\Lambda, D)) = A$ where A is the isomorphism class of the cut algebra associated with the admissible cut D.*

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