THE BATALIN-VILKOVISKY STRUCTURE ON THE TATE-HOCHSCHILD COHOMOLOGY RING OF A GROUP ALGEBRA

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Abstract. We determine the Batalin-Vilkovisky structure on the Tate-Hochschild cohomology of the group algebra kG of a finite group G in terms of the additive decomposition. In particular, we show that the Tate cohomology of G is a Batalin-Vilkovisky subalgebra of the Tate-Hochschild cohomology of the group algebra kG, and that the Tate cochain complex of G is a cyclic A_∞-subalgebra of the Tate-Hochschild cochain complex of kG.

Introduction

For any associative algebra A, Hochschild introduced in 1945 a graded group $\mathrm{HH}^*(A,A)$ defined as the cohomology of certain cochain complex $C^*(A,A)$, where $C^n(A,A)$ is the space of linear maps from $A^{\otimes n}$ to A. In the 1960's, when studying the deformation theory of associative algebras, Gerstenhaber observed that there is a rich algebraic structure on $\mathrm{HH}^*(A,A)$, called a Gerstenhaber algebra, consisting of the following date:

(i) $\mathrm{HH}^*(A,A)$ is a graded-commutative associative algebra via the cup product;
(ii) $\mathrm{HH}^*(A,A)$ is endowed with a graded Lie bracket (nowadays called *Gerstenhaber bracket*) of degree -1 so that it becomes a graded Lie algebra (of degree -1);
(iii) The Gerstenhaber bracket is compatible with the cup product via the graded Leibniz rule.

During the past few decades, a new structure (the so-called *Batalin-Vilkovisky structure*) has been extensively studied in topology and mathematical physics, and recently it was introduced into algebra. Roughly speaking, a Batalin-Vilkovisky (aka. BV) structure on Hochschild cohomology is a square-zero operator (called *BV-operator*) of degree -1 such that the Gerstenhaber bracket is the obstruction of the BV-operator being a derivation with respect to the cup product. A typical example of a BV-algebra was given by Tradler [29] and Menichi [24] motivated from string topology. Namely, the Hochschild cohomology ring of a finite dimensional symmetric algebra (e.g. the group algebra of a finite group) is a BV-algebra.

From the point of view of derived categories, the i-th Hochschild cohomology group $\mathrm{HH}^i(A,A)$ of an algebra A over a field k is isomorphic to the space of morphisms from A to $A[i]$ in $\mathcal{D}^b(A\otimes_k A^{\mathrm{op}})$, the bounded derived category of A-A-bimodules, where $[i]$ is the i-th shift functor in $\mathcal{D}^b(A\otimes_k A^{\mathrm{op}})$. As a generalization of the Hochschild cohomology, the Tate-Hochschild cohomology group $\widehat{\mathrm{HH}}^i(A,A)$ ($i \in \mathbb{Z}$) is defined as the space of morphisms from A to $A[i]$ in the singularity category $\mathcal{D}_{\mathrm{sg}}(A\otimes_k A^{\mathrm{op}})$ of A-A-bimodules, where $[i]$ is the i-th shift functor in $\mathcal{D}_{\mathrm{sg}}(A\otimes_k A^{\mathrm{op}})$. Recall that $\mathcal{D}_{\mathrm{sg}}(A\otimes_k A^{\mathrm{op}})$ is the Verdier quotient of the bounded derived category $\mathcal{D}^b(A\otimes_k A^{\mathrm{op}})$ by the full subcategory consisting of bounded complexes of projective A-A-bimodules, which was introduced by Buchweitz [9] and later independently by Orlov [26]. As a particular example that will be relevant in this paper, the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(kG,kG)$ of a finite abelian group G is isomorphic to $kG \otimes \widehat{\mathrm{H}}^*(G,k)$ as graded associative algebras, where $\widehat{\mathrm{H}}^*(G,k)$ is the Tate cohomology of G (cf. Section 3.2). The notion of Tate-Hochschild cohomology has been studied in the literature, such as [9, 3, 25].

As already mentioned, the Hochschild cohomology $\mathrm{HH}^*(A,A)$ may be computed by the Hochschild cochain complex $C^*(A,A)$. In [31] the second named author constructed a complex, the so-called "singular Hochschild cochain complex" to compute the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(A,A)$. Via

this complex, the author in loc. cit. showed that there is a Gerstenhaber structure on $\widehat{\mathrm{HH}}^*(A, A)$ extending the classical Gerstenhaber structure on $\mathrm{HH}^*(A, A)$ (cf. Theorem 1.2 below). Generalizing the result of Tradler and Menichi, the author proved that the Gerstenhaber structure of the Tate-Hochschild cohomology of a symmetric algebra extends to the BV structure on the Hochschild cohomology (cf. Theorem 1.8 below).

In a later work [27], the authors studied the Tate-Hochschild cohomology of finite dimensional differential graded (dg) symmetric algebras, from the point of view of string topology. Generalizing the classical Tate cochain complex of a finite group (cf. Section 3.2), the authors constructed an analogous complex $\mathcal{D}^*(A, A)$ (called *Tate-Hochschild cochain complex*) computing the Tate-Hochschild cohomology of a symmetric algebra A. The negative part of $\mathcal{D}^*(A, A)$ is the Hochschild chain complex $C_*(A, A)$ with $\mathcal{D}^{-m-1}(A, A) = C_m(A, A)$ $(m \geq 0)$; the non-negative part of $\mathcal{D}^*(A, A)$ is the Hochschild cochain complex $C^*(A, A)$ with $\mathcal{D}^m(A, A) = C^m(A, A)$ $(m \geq 0)$; and the differential $\tau : C_0(A, A) \to C^0(A, A)$ in degree -1 is given by $a \mapsto \sum_i e_i a f_i$, where $\sum_i e_i \otimes f_i$ is the Casimir element of A (cf. Section 1.2). It is shown in loc. cit. that there is a cyclic A_∞-algebra structure $(m_1 = \partial, m_2, m_3, \cdots)$ and an L_∞-algebra structure $(l_1 = \partial, l_2, l_3, \cdots)$ on $\mathcal{D}^*(A, A)$ such that $m_i = 0$ for $i > 3$. Moreover, the restrictions of m_2 and l_2 to the non-negative part $\mathcal{D}^{\geq 0}(A, A)$ are respectively, the usual cup product and the Gerstenhaber bracket on $C^*(A, A)$.

The aim of the present article is to describe explicit complex level formulas for the BV structure on $\widehat{\mathrm{HH}}^*(A, A)$ in a special case where $A = kG$ is the group algebra of a finite group G over a field k.

It is well-known that the Hochschild (co)homology of the group algebra of a finite group admits a decomposition as vector spaces into a direct sum of group (co)homology spaces of centralizers of elements. More precisely, let k be a field and G a finite group. Then we have the following additive decompositions (cf. e.g. [2, Theorem 2.11.2]):

$$\begin{aligned} \mathrm{HH}^*(kG, kG) &\simeq \bigoplus_{x \in X} \mathrm{H}^*(C_G(x), k), \\ \mathrm{HH}_*(kG, kG) &\simeq \bigoplus_{x \in X} \mathrm{H}_*(C_G(x), k) \end{aligned}$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer of $x \in X$. Siegel and Witherspoon in [28] gave a formula for the cup product of the Hochschild cohomology $\mathrm{HH}^*(kG, kG)$ in terms of the above additive decomposition, and later, Bouc in [6] gave a quick proof of this formula using Green functors. In [22], the first and the third named authors lifted the additive decomposition of $\mathrm{HH}^*(kG, kG)$ to the complex level. More concretely, the authors lifted the above isomorphism on $\mathrm{HH}^*(kG, kG)$ to a chain homotopy equivalence given by two maps of complexes (cf. Theorem 4.3 below)

$$\tag{1} C^*(kG, kG) \underset{\rho^*}{\overset{\iota^*}{\rightleftarrows}} \bigoplus_{x \in X} C^*(C_G(x), k)$$

such that $\iota^* \circ \rho^* = \mathrm{id}$ and $\mathrm{id} - \rho^* \circ \iota^*$ is homotopic to the zero map, where $C^*(kG, kG)$ denotes the Hochschild cochain complex (see Section 2) and $C^*(C_G(x), k)$ the group cohomology complex (see Section 3.1). As a result, they described explicitly how to transfer the cup product, the Lie bracket and the BV-operator from the left-hand side to the right-hand side at the complex level. In the present article, we construct an explicit homotopy s^* between id and $\rho^* \circ \iota^*$, namely $\delta^* \circ s^* + s^* \circ \delta^* = \mathrm{id} - \rho^* \circ \iota^*$ (cf. Theorem 4.3). Such a triple (ρ^*, ι^*, s^*) is called a *homotopy deformation retract*.

Combining the above two additive decompositions, we obtain an additive decomposition of the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(kG, kG)$. That is,

$$\widehat{\mathrm{HH}}^*(kG, kG) \simeq \bigoplus_{x \in X} \widehat{\mathrm{H}}^*(C_G(x), k),$$

where we fix X to be a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of $x \in G$, and where $\widehat{\mathrm{H}}^*(C_G(x), k)$ is the Tate cohomology of $C_G(x)$ (see Section 3.2). In [25], Nguyen gave the cup product formula for the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(kG, kG)$ in terms of the above additive decomposition. In the present article, we shall describe

the cup product and the BV-operator on $\widehat{\mathrm{HH}}^*(kG, kG)$ in terms of the additive decomposition at the complex level. By this, we mean that we will give an explicit formula for the cup product and the BV-operator on the complex $\mathcal{D}^*(kG, kG)$ and also give some explicit calculations in terms of the additive decomposition at the complex level. To achieve this, we extend the chain homotopy equivalence (1) to the following homotopy deformation retract (cf. Remark 4.7),

$$\widehat{s} \,\circlearrowright\quad \mathcal{D}^*(kG, kG) \underset{\widehat{\iota}}{\overset{\widehat{\rho}}{\rightleftarrows}} \bigoplus_{x \in X} \widehat{C}^*(C_G(x), k),$$

namely $\widehat{\rho} \circ \widehat{\iota} = \mathrm{id}$ and $\mathrm{id} - \widehat{\iota} \circ \widehat{\rho} = \partial \circ \widehat{s} + \widehat{s} \circ \partial$, where ∂ is the differential of $\mathcal{D}^*(kG, kG)$ and $\widehat{C}^*(C_G(x), k)$ is the Tate cochain complex of $C_G(x)$ (see Section 3.2). Via this homotopy deformation retract, we may transfer the cup product on the left hand side to the right hand side and thus obtain a cup product formula at the complex level (cf. Remark 5.7), as doing so for $\mathrm{HH}^*(kG, kG)$ in [22, Section 7]. The BV-operator preserves each summand of the additive decomposition (cf. Section 6). As a consequence, we obtain that $\widehat{\mathrm{H}}^*(G, k)$ is a BV subalgebra of $\widehat{\mathrm{HH}}^*(kG, kG)$ (cf. Corollary 6.5).

We have an explicit formula for the cup product $\cup$ on $\mathcal{D}^*(kG, kG)$ (see Definition 2.1). We observe that the restriction of $\cup$ to the negative part $\mathcal{D}^{<0}(kG, kG)$ (i.e. the Hochschild chain complex $C_*(kG, kG)$) is in general not compatible with the differential of $C_*(kG, kG)$ (cf. Remark 6.8). For this reason, $\cup$ is not well-defined in the whole Hochschild homology $\mathrm{HH}_*(kG, kG)$. To deal with this issue, we shall consider the following truncated subcomplex of $\mathcal{D}^*(kG, kG)$,

$$\widetilde{C}_*(kG, kG)): \quad \cdots \xrightarrow{\partial_p} C_{p-1}(kG, kG) \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_2} C_1(kG, kG) \xrightarrow{\partial_1} \mathrm{Ker}(\tau) \to 0.$$

It is clear that the cohomology of $\widetilde{C}_*(kG, kG)$ is isomorphic to the negative part $\widehat{\mathrm{HH}}^{<0}(kG, kG)$ of $\widehat{\mathrm{HH}}^*(kG, kG)$. Actually, it also coincides with the stable Hochschild homology $\mathrm{HH}_*^{st}(kG, kG)$ studied in [14, 23] (cf. Remark 1.4 below). Similarly, we denote by $\mathrm{H}_*^{st}(G, k)$ the negative part of the Tate cohomology $\widehat{\mathrm{H}}^*(G, k)$, namely $\mathrm{H}_m^{st}(G, k) = \widehat{\mathrm{H}}^{-m-1}(G, k)$ $(m \geq 0)$. The additive decomposition of $\widehat{\mathrm{HH}}^*(kG, kG)$ yields an additive decomposition

$$\mathrm{HH}_*^{st}(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}_*^{st}(C_G(x), k).$$

We prove that $\mathrm{HH}_{-*-1}^{st}(kG, kG)$ is a BV-algebra (without unit) (cf. Theorem 6.9). It is well-known that there is an isomorphism between the Hochschild homology $\mathrm{HH}_*(kG, kG)$ and the singular homology $\mathrm{H}_*(LBG, k)$ of the free loop space LBG of the classifying space BG (cf. [21, 7.3.13 Corollary]). We denote by $\mathrm{H}_*^{st}(LBG, k)$ the subspace of $\mathrm{H}_*(LBG, k)$ corresponding to $\mathrm{HH}_*^{st}(kG, kG)$ under the above isomorphism. Then we obtain that $\mathrm{H}_{-*-1}^{st}(LBG, k)$, equipped with the S^1-action and the product transferred from the cup product on $\mathrm{HH}_*^{st}(kG, kG)$, is a BV-algebra (see Corollary 6.10).

To the best of the authors' knowledge, it is still an open question whether there is a BV_∞-algebra structure (cf. [17, 30]) on $\mathcal{D}^*(kG, kG)$. What we have done in the present article is only the first step toward understanding this higher algebraic structure on $\mathcal{D}^*(kG, kG)$ and its behavior in terms of the additive decomposition. By one of our results, namely that $\widehat{\mathrm{H}}^*(G, k)$ is a BV subalgebra of $\widehat{\mathrm{HH}}^*(kG, kG)$, it is interesting to ask whether the Tate cochain complex $\widehat{C}^*(G, k)$ is a BV_∞-subalgebra of $\mathcal{D}^*(kG, kG)$ (cf. Remark 6.7). With this ultimate goal, we could give a partial result in the present article. We show that the Tate cochain complex $\widehat{C}^*(G, k)$ is a cyclic A_∞-subalgebra of the Tate-Hochschild cochain complex $D^*(kG, kG)$ (cf. Theorem 4.10). In particular, we obtain an isomorphism of cyclic A_∞-algebras between $\mathcal{D}^*(kG, kG)$ and $kG \otimes \widehat{C}^*(G, k)$ when G is a finite abelian group (cf. Corollary 4.11). Further problems along this direction will be explored in future research.

This paper is organized as follows. In Section 1, we recall some notions and algebraic structures on Hochschild (co)homology and Tate-Hochschild cohomology. In Section 2, we study algebraic structures on the Tate-Hochschild cochain complex $\mathcal{D}^*(kG, kG)$ for a finite group G, including the explicit description of the cyclic A_∞-algebra structure. In Section 3, we recall the notions of cohomology

and Tate cohomology of finite groups, including the Tate cochain complex $\widehat{C}^*(G,k)$ (computing the Tate cohomology). In Section 4, we lift explicitly the additive decomposition of the Tate-Hochschild cohomology of a group algebra to the complex level. We also prove that the Tate cochain complex $\widehat{C}^*(G,k)$ is a cyclic A_∞-subalgebra of $\mathcal{D}^*(kG,kG)$. We explain in Section 5 the cup product formula in $\widehat{\mathrm{HH}}^*(kG,kG)$ which appeared in [25] and give a new proof for it using Green functors, following Bouc. In Section 6, we deal with the BV-operator $\widehat{\Delta}$ and the Lie bracket in $\widehat{\mathrm{HH}}^*(kG,kG)$. In particular, we show that the operator $\widehat{\Delta}$ preserves each summand of the additive decomposition, and that $\widehat{\mathrm{H}}^*(G,k)$ is indeed a BV subalgebra of $\widehat{\mathrm{HH}}^*(kG,kG)$. In Section 7, we use our results to compute the BV structure of the Tate-Hochschild cohomology for symmetric group of degree 3 over a field k of characteristic 3. In Appendix A, we provide a proof scattered in literature of the fact that the Connes' B-operator is trivial in the group homology $\mathrm{H}_*(G,k)$.

1. REMINDER ON HOCHSCHILD (CO)HOMOLOGY AND TATE-HOCHSCHILD COHOMOLOGY

Throughout this paper, we fix a field k. All group algebras denoted by kG or kH, and all algebras denoted by A in the sequel, and their modules we considers as such, will be assumed to be finite dimensional. We shall write $\otimes$ for $\otimes_k$, the tensor product over the field k, for two k-vector spaces V and W, write $\mathrm{Hom}(V,W)$ for $\mathrm{Hom}_k(V,W)$, the set of k-linear maps from V to W.

In this section we recall the definition and algebraic structure of the Tate-Hochschild cohomology of an associative k-algebra. For more details, we refer the reader to [31, 22] and the references therein.

1.1. **Hochschild (co)homology of algebras.** Let A be a finite dimensional k-algebra. Denote the enveloping algebra $A \otimes_k A^{\mathrm{op}}$ of A by A^e.

Let us first recall the construction of the Hochschild (co)chain complexes, using the normalized bar resolution of A. Denote by $\overline{A}$ the quotient k-vector space $A/(k{\cdot}1)$. The normalized bar resolution $(\mathrm{Bar}_*(A), d_*)$ of A is a free resolution of A as A-A-bimodules, where

$$\mathrm{Bar}_{-1}(A) = A, \quad \text{and for } n \geq 0, \quad \mathrm{Bar}_n(A) = A \otimes \overline{A}^{\otimes n} \otimes A,$$

and the differential is defined as follows: for $n \geq 1$,

$$d_n : \mathrm{Bar}_n(A) \to \mathrm{Bar}_{n-1}(A)$$

sends $a_0 \otimes \overline{a_{1,n}} \otimes a_{n+1}$ to

$$a_0 a_1 \otimes \overline{a_{2,n}} \otimes a_{n+1} + \sum_{i=1}^{n-1}(-1)^i a_0 \otimes \overline{a_{1,i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+1,n}} \otimes a_{n+1} + (-1)^n a_0 \otimes \overline{a_{1,n-1}} \otimes a_n a_{n+1},$$

and for $n = 0$,

$$d_0 : \mathrm{Bar}_0(A) = A \otimes A \to A, \quad a_0 \otimes a_1 \mapsto a_0 a_1.$$

Here for simplicity we write $\overline{a_{i,j}} := \overline{a_i} \otimes \overline{a_{i+1}} \otimes \cdots \otimes \overline{a_j}$ ($i \leq j$), and when $n = 0$, $\overline{A}^{\otimes n} := k$. This complex is exact as there exists a contracting homotopy: for $p \geq 0$

$$s_p : \mathrm{Bar}_p(A) \to \mathrm{Bar}_p(A), a_0 \otimes \overline{a_{1,p}} \otimes a_{p+1} \mapsto 1 \otimes \overline{a_{0,p}} \otimes a_{p+1}$$

and $s_{-1} : A \to \mathrm{Bar}_0(A), a \mapsto 1 \otimes a$. Note that each s_p ($p \geq -1$) is a morphism of right A-modules.

Recall that the *Hochschild cochain complex* $(C^*(A,A), \delta^*)$ is defined as follows:

$$C^n(A,A) = \mathrm{Hom}_{A^e}(\mathrm{Bar}_n(A), A) \simeq \mathrm{Hom}(\overline{A}^{\otimes n}, A), \quad \text{for } n \geq 0,$$

and the differential is given by

$$\delta^n : C^n(A, A) \to C^{n+1}(A, A), \quad \varphi \mapsto \delta^n(\varphi),$$

where $\delta^n(\varphi)$ sends $\overline{a_{1,n+1}} \in \overline{A}^{\otimes(n+1)}$ to

$$a_1 \varphi(\overline{a_{2,n+1}}) + \sum_{i=1}^{n} (-1)^i \varphi(\overline{a_{1,i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2,n+1}}) + (-1)^{n+1} \varphi(\overline{a_{1,n}}) a_{n+1}.$$

In degree zero, the differential map $\delta^0 : A \to \mathrm{Hom}(\overline{A}, A)$ is given by

$$\delta^0(x)(\overline{a}) = ax - xa \quad (\text{for } x \in A \text{ and } \overline{a} \in \overline{A}).$$

For any $n \geq 0$, the n-th *Hochschild cohomology group* of A is defined to be the cohomology group

$$\mathrm{HH}^n(A, A) = H^n(C^*(A, A)) \simeq \mathrm{Ext}^n_{A^e}(A, A).$$

Recall that the *Hochschild chain complex* $(C_*(A, A), \partial_*)$ is defined as follows:

$$C_n(A, A) = A \otimes_{A^e} \mathrm{Bar}_n(A) \simeq A \otimes \overline{A}^{\otimes n}, \quad \text{for } n \geq 0,$$

and, for $n \geq 2$, the differential $\partial_n : A \otimes \overline{A}^{\otimes n} \to A \otimes \overline{A}^{\otimes(n-1)}$ sends $a_0 \otimes \overline{a_{1,n}}$ to

$$a_0 a_1 \otimes \overline{a_{2,n}} + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \overline{a_{1,i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2,n}} + (-1)^n a_n a_0 \otimes \overline{a_{1,n-1}},$$

and in degree one, the differential $\partial_1 : A \otimes \overline{A} \to A$ is given by

$$\partial_1(a_0 \otimes \overline{a_1}) = a_0 a_1 - a_1 a_0 \quad (\text{for } a_0 \in A \text{ and } \overline{a_1} \in \overline{A}).$$

For all $n \geq 0$, the n-th *Hochschild homology group* of A is defined to be the homology group

$$\mathrm{HH}_n(A, A) = H_n(C_*(A, A)) \simeq \mathrm{Tor}_n^{A^e}(A, A).$$

Recall that the bounded derived category $\mathcal{D}^b(A)$ is the triangulated category obtained from the homotopy category of bounded complexes of finitely generated A-modules by inverting all quasi-isomorphisms. The Hochschild cohomology groups $\mathrm{HH}^*(A)$ can be realized as

$$\mathrm{HH}^n(A, A) = \mathrm{Hom}_{\mathcal{D}^b(A^e)}(A, A[n]) \quad n \geq 0,$$

where $\mathcal{D}^b(A^e)$ is the bounded derived category of A^e and $[n]$ denotes the n-th shift functor in $\mathcal{D}^b(A^e)$ (cf. e.g. [32]). We end this subsection with a remark.

Remark 1.1. Let M be a left A-module. Then $\mathrm{Bar}_*(A) \otimes_A M$ is a free resolution of M. In fact, this complex is exact with the contracting homotopy $\{s_p \otimes_A \mathrm{id}, \ p \geq -1\}$ since $\{s_p, \ p \geq -1\}$ are homomorphisms of right A-modules.

The similar result holds for right A-modules.

1.2. **Tate-Hochschild cohomology.** Let A be a finite dimensional k-algebra. The singularity category $\mathcal{D}_{\mathrm{sg}}(A)$ of A is defined to be the Verdier quotient

$$\mathcal{D}_{\mathrm{sg}}(A) = \mathcal{D}^b(A) / \mathrm{per} A,$$

where $\mathrm{per} A$ is the the bounded homotopy category of finitely generated projective A-modules.

Recall that the i-th ($i \in \mathbb{Z}$) Tate-Hochschild cohomology group $\widehat{\mathrm{HH}}^i(A, A)$ of A is defined as the space $\mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^e)}(A, A[i])$, where $[i]$ denotes the i-th shift functor in $\mathcal{D}_{\mathrm{sg}}(A^e)$. Clearly, the quotient functor from $\mathcal{D}^b(A^e)$ to $\mathcal{D}_{\mathrm{sg}}(A^e)$ induces a natural morphism

$$\rho : \mathrm{HH}^*(A, A) \to \widehat{\mathrm{HH}}^*(A, A).$$

To compute the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(A, A)$, the second named author constructed a complex $C^*_{\mathrm{sg}}(A, A)$ (called *singular Hochschild cochain complex*) in [31, Section 3.1]. Roughly speaking, it is a colimit of the inductive system consisting of Hochschild cochain complexes with coefficients in the bimodules of noncommutative differential forms. On $C^*_{\mathrm{sg}}(A, A)$, the author constructed a cup product $\cup$ and a Lie bracket $[\cdot, \cdot]$, which leads to the following result.

Theorem 1.2. ([31, Corollary 5.3]) *The Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(A, A)$, equipped with the cup product $\cup$ and the Lie bracket $[\cdot, \cdot]$, is a Gerstenhaber algebra. Moreover, the above map $\rho : \mathrm{HH}^*(A, A) \to \widehat{\mathrm{HH}}^*(A, A)$ is a morphism of Gerstenhaber algebras.*

In the case of a self-injective algebra A over a field k, the Tate-Hochschild cohomology agrees with the Tate cohomology defined in [9]. We have the following descriptions of the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(A, A)$.

Proposition 1.3. ([9, Corollary 6.4.1]) *Let A be a self-injective algebra over a field k. Denote $\mathrm{Hom}_{A^e}(A, A^e)$ by $A^\vee$. Then*

  (i) $\widehat{\mathrm{HH}}^n(A, A) \simeq \mathrm{HH}^n(A, A)$ for all $n > 0$,

  (ii) $\widehat{\mathrm{HH}}^n(A, A) \simeq \mathrm{HH}_{-n-1}(A^\vee, A)$ for all $n < -1$,

  (iii) $\widehat{\mathrm{HH}}^0(A, A) \simeq \underline{\mathrm{Hom}}_{A^e}(A, A)$, $\widehat{\mathrm{HH}}^{-1}(A, A) \simeq \underline{\mathrm{Hom}}_{A^e}(A, \Omega_{A^e}(A))$, and there is an exact sequence

$$0 \to \widehat{\mathrm{HH}}^{-1}(A, A) \to A^\vee \otimes_{A^e} A \xrightarrow{\sigma} \mathrm{Hom}_{A^e}(A, A) \to \widehat{\mathrm{HH}}^0(A, A) \to 0,$$

  where the map σ is given by $\sigma(f \otimes a)(a') = f(a') \cdot a$ for $a, a' \in A$ and $f \in A^\vee$. Here $\underline{\mathrm{Hom}}_{A^e}(-, -)$ denotes the homomorphism space in the stable category A^e-$\underline{\mathrm{Mod}}$ and Ω_{A^e} is the syzygy functor over A^e-$\underline{\mathrm{Mod}}$.

Now we specialize A to be a symmetric algebra. Symmetric algebras are self-injective and include group algebras of finite groups. Recall that a symmetric algebra is a finite dimensional k-algebra A such that there is a symmetric non-degenerate associative bilinear form $\langle \cdot, \cdot \rangle : A \times A \to k$ (called the symmetrizing form), or equivalently, $A \simeq A^* = \mathrm{Hom}_k(A, k)$ as A-A-bimodules. Note that we can choose an A-A-bimodule isomorphism (denoted by t) as follows: $t(a) = \langle a, \cdot \rangle$ for $a \in A$. This isomorphism t induces the following isomorphism

$$t \otimes \mathrm{id} : A \otimes_k A \to A^* \otimes_k A \simeq \mathrm{End}_k(A)$$

$$a \otimes b \mapsto t(a) \otimes b \mapsto (x \mapsto t(a)(x)b).$$

Following Broué (see [7]), we call the element $(t \otimes \mathrm{id})^{-1}(\mathrm{id}) := \sum_i e_i \otimes f_i \in A \otimes_k A$ the Casimir element of A. It follows from [7, Proposition 3.3] that the Casimir element induces an isomorphism

$$A \simeq A^\vee = \mathrm{Hom}_{A^e}(A, A^e), \quad a \mapsto \sum_i e_i a \otimes f_i$$

as A-A-bimodules, where we identify $\mathrm{Hom}_{A^e}(A, A^e)$ as

$$(A \otimes A)^A := \{\sum_i a_i \otimes b_i \in A \otimes_k A \mid \sum_i a a_i \otimes b_i = \sum_i a_i \otimes b_i a \text{ for any } a \in A\}.$$

Hence, if A is a symmetric algebra and $n < -1$, then, by Proposition 1.3 (ii), the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^n(A, A)$ is isomorphic to the usual Hochschild homology:

$$\widehat{\mathrm{HH}}^n(A, A) \simeq \mathrm{Tor}_{-n-1}^{A^e}(A^\vee, A) \simeq \mathrm{Tor}_{-n-1}^{A^e}(A, A) = \mathrm{HH}_{-n-1}(A, A).$$

Moreover, for $n = -1, 0$ we have the following interesting observations.

Remark 1.4. By Proposition 1.3 (iii), the 0-th Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^0(A, A)$ is a quotient of the 0-th Hochschild cohomology $\mathrm{HH}^0(A, A)$ and coincides with the stable center $Z^{st}(A) = Z(A)/Z^{pr}(A)$ (cf. [23, Section 2]). Moreover, the map $\sigma : A^\vee \otimes_{A^e} A \to \mathrm{Hom}_{A^e}(A, A)$ in Proposition 1.3 (iii) is identified with the trace map

$$\tau : \mathrm{HH}_0(A, A) = A/[A, A] \to \mathrm{HH}^0(A, A) = Z(A), \quad a + [A, A] \mapsto \sum_i e_i a f_i,$$

where $\mathrm{Ker}(\tau) = Z^{pr}(A)^\perp/[A, A]$ is equal to the so-called 0-th stable Hochschild homology $\mathrm{HH}_0^{st}(A)$ (cf. [14, Section 4], [23, Section 2 and 3]). Thus, in this case, the -1-th Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^{-1}(A, A)$ is a subspace of the 0-th Hochschild homology $\mathrm{HH}_0(A, A)$ and coincides with the 0-th stable Hochschild homology $\mathrm{HH}_0^{st}(A)$ (cf. [14, Section 4], [23, Section 2 and 3]).

Therefore, $\widehat{\mathrm{HH}}^*(A, A)$ is a "combination" of the Hochschild cohomology $\mathrm{HH}^*(A, A)$ and the Hochschild homology $\mathrm{HH}_*(A, A)$. We can summarize the above results by means of the following diagram:

In [31, Section 6.4], the author constructed a complex (called *Tate-Hochschild cochain complex*)

$$\mathcal{D}^*(A, A) := (\cdots \xrightarrow{\partial_2} C_1(A, A) \xrightarrow{\partial_1} C_0(A, A) \xrightarrow{\tau} C^0(A, A) \xrightarrow{\delta^0} C^1(A, A) \xrightarrow{\delta^1} \cdots),$$

to compute $\widehat{\mathrm{HH}}^*(A, A)$ for a symmetric algebra A, where ∂_* (resp. δ^*) is the differential of $C_*(A, A)$ (resp. $C^*(A, A)$) (see Section 1.1); and $\tau(x) = \sum_i e_i x f_i$. Here $\sum_i (e_i \otimes f_i)$ is the Casimir element. Note that the bilinear form $\langle \cdot, \cdot \rangle$ on A defines a non-degenerate bilinear form (still denoted by $\langle \cdot, \cdot \rangle$)

$$\langle \cdot, \cdot \rangle : \mathcal{D}^*(A, A) \times \mathcal{D}^*(A, A) \to k$$

on $\mathcal{D}^*(A, A)$: For $\alpha \in C^m(A, A)$ and $\beta = a_0 \otimes \overline{a_{1,n}} \in C_n(A, A)$, we define

$$\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle := \begin{cases} \langle \alpha(\overline{a_{1,n}}), a_0 \rangle & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.5. In fact, this bilinear form $\langle \cdot, \cdot \rangle$ is induced by the duality between $C_*(A, A)$ and $C^*(A, A)$ defined in [19, Lemma 2.9]. Note that $\langle \cdot, \cdot \rangle$ descends to $\widehat{\mathrm{HH}}^*(A, A)$ since it is compatible with the differential of $\mathcal{D}^*(A, A)$ (cf. Lemma 2.3 below). In particular, we have a non-degenerate bilinear form between $\widehat{\mathrm{HH}}^0(A, A) \simeq Z^{st}(A)$ and $\widehat{\mathrm{HH}}^{-1}(A, A) \simeq \mathrm{HH}_0^{st}(A)$ (cf. [19, Theorem 2.15 (3)]).

The following result shows that $\mathcal{D}^*(A, A)$ has a rich algebraic structure.

Theorem 1.6. ([27, Theorem 6.3 and Proposition 6.5]) *Let A be a symmetric k-algebra. Then there is a cyclic A_∞-algebra structure $(m_1 = \partial, m_2, m_3, \cdots)$ and an L_∞-algebra structure $(l_1 = \partial, l_2, l_3, \cdots)$ on $\mathcal{D}^*(A, A)$ such that $m_i = 0$ for $i > 3$, where the restrictions of m_2 and l_2 to the nonnegative part $\mathcal{D}^{\geq 0}(A, A)$ are respectively, the usual cup product and Gerstenhaber bracket on $C^*(A, A)$.*

Remark 1.7. We have simple and explicit formulas for the A_∞-products since $m_i = 0$ for $i > 3$. But the formulas for the L_∞-brackets l_i are in general very complicated and messy. In Section 2, we write down the explicit formulas for the A_∞-products m_i on $\mathcal{D}^*(kG, kG)$. In Theorem 4.10 below, we prove that the Tate cochain complex $\widehat{C}^*(G, k)$ is a cyclic A_∞-subalgebra of $\mathcal{D}^*(kG, kG)$.

Recall that the Connes' B-operator on the Hochschild chain complex $C_*(A, A)$ is defined as

$$B(a_0 \otimes \overline{a}_{1,m}) = \sum_{i=0}^{m} (-1)^{mi} 1 \otimes \overline{a_{i,m}} \otimes \overline{a_0} \otimes \overline{a_{1,i-1}}.$$

Tradler in [29] and Menichi [24] showed that the Hochschild cohomology $\mathrm{HH}^*(A, A)$ of a symmetric algebra A is a BV-algebra whose BV-operator Δ is the dual of the Connes' B-operator with respect to the bilinear form $\langle \cdot, \cdot \rangle$. That is,

$$\langle \Delta(f)(\overline{a_{1,m}}), a_0 \rangle = \langle B(a_0 \otimes \overline{a_{1,m}}), f \rangle.$$

Generalizing the above result, we have the following result.

Theorem 1.8. ([31, Theorem 6.17][27, Corollary 6.7]) *Let A be a symmetric k-algebra. Then the Gerstenhaber algebra $(\widehat{\mathrm{HH}}^*(A, A), \cup, [\cdot, \cdot])$ is a BV-algebra whose BV-operator $\widehat{\Delta}$ is given by*

$$\widehat{\Delta}^i := \begin{cases} \Delta^i & \text{for } i > 0, \\ 0 & \text{for } i = 0, \\ B_{-i-1} & \text{for } i \leq -1. \end{cases}$$

2. TATE-HOCHSCHILD COHOMOLOGY OF A GROUP ALGEBRA

Let k be a field, G a finite group and kG the group algebra. Recall that kG is a symmetric algebra with the symmetrizing form:

$$\langle g, h \rangle = 1 \text{ if } gh = 1 \text{ and } \langle g, h \rangle = 0 \text{ otherwise}$$

for all $g, h \in G$. In particular, $\sum_{g \in G} g^{-1} \otimes g$ is a Casimir element of kG. Thus from Section 1.2, we have that the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(kG, kG)$ is a "combination" of the Hochschild cohomology $\mathrm{HH}^*(kG, kG)$ and the Hochschild homology $\mathrm{HH}_*(kG, kG)$.

The Hochschild (co)chain complexes of kG have the following simple descriptions. For a set X, we denote by $k[X]$ the k-vector space spanned by the elements in X. In particular, we have $kG = k[G]$. Note that $\overline{kG}$ can be identified with the k-vector space $k[\overline{G}]$, where $\overline{G} = G - \{1\}$. When $n = 0$, $\overline{G}^{\times n}$ denotes a one-point set and $k[\overline{G}^{\times n}] := k$. For simplicity, we write $(g_{1,n})$ for $(g_1, g_2, \cdots, g_n) \in G^{\times n}$.

The normalized bar resolution $(\mathrm{Bar}_*(kG), d_*)$ of kG has the form (throughout we just write all the maps on the base elements)

$$\mathrm{Bar}_{-1}(kG) = kG, \text{ and for } n \geq 0, \quad \mathrm{Bar}_n(kG) = k[G \times \overline{G}^{\times n} \times G],$$

$$d_0 : \mathrm{Bar}_0(kG) = k[G \times G] \to kG, \quad (g_0, g_1) \mapsto g_0 g_1, \quad \text{and for } n \geq 1,$$

$$d_n : \mathrm{Bar}_n(kG) \to \mathrm{Bar}_{n-1}(kG), \quad (g_0, \overline{g_{1,n}}, g_{n+1}) \mapsto \sum_{i=0}^{n} (-1)^i (g_0, \overline{g_1}, \cdots, \overline{g_i g_{i+1}}, \cdots, \overline{g_n}, g_{n+1}).$$

Here $k[G \times \overline{G}^{\times n} \times G]$ denotes the k-vector space spanned by the elements in the Cartesian product $G \times \overline{G}^{\times n} \times G$. We always use the normalized bar resolution (except in Appendix A) since it greatly simplifies the computations. From now on, we just write g for its image $\overline{g}$ in $\overline{G}$.

Recall that the Hochschild cochain complex $(C^*(kG, kG), \delta^*)$ is defined as follows:

$$C^n(kG, kG) = \mathrm{Hom}_{(kG)^e}(\mathrm{Bar}_n(kG), kG) \simeq \mathrm{Hom}_k(k[\overline{G}^{\times n}], kG) \simeq \mathrm{Map}(\overline{G}^{\times n}, kG), \quad \text{for } n \geq 0,$$

where $\mathrm{Map}(\overline{G}^{\times n}, kG)$ denotes the set of maps from $\overline{G}^{\times n}$ to kG, and the differential is given by

$$\delta^n : \mathrm{Map}(\overline{G}^{\times n}, kG) \to \mathrm{Map}(\overline{G}^{\times (n+1)}, kG), \quad \varphi \mapsto \delta^n(\varphi),$$

where $\delta^n(\varphi)$ sends $g_{1,n+1} \in \overline{G}^{(n+1)}$ to

$$g_1 \varphi(g_{2,n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(g_{1,i-1}, g_i g_{i+1}, g_{i+2,n+1}) + (-1)^{n+1} \varphi(g_{1,n}) g_{n+1}.$$

In degree zero, the differential map $\delta^0 : kG \to \mathrm{Map}(\overline{G}, kG)$ is given by

$$\delta^0(x)(g) = gx - xg \quad (\text{for } x \in kG \text{ and } g \in \overline{G}).$$

Recall that the Hochschild chain complex $(C_*(kG, kG), \partial_*)$ is defined as follows:

$$C_n(kG, kG) = kG \otimes_{(kG)^e} \mathrm{Bar}_n(kG) \simeq k[G \times \overline{G}^{\times n}], \quad \text{for } n \geq 0,$$

where $k[G \times \overline{G}^{\times n}]$ denotes the k-vector space spanned by the elements in $G \times \overline{G}^{\times n}$, and the differential is given by

$$\partial_n : k[G \times \overline{G}^{\times n}] \to k[G \times \overline{G}^{\times (n-1)}],$$

$$(g_0, g_{1,n}) \mapsto (g_0 g_1, g_{2,n}) + \sum_{i=1}^{n-1} (-1)^i (g_0, g_{1,i-1}, g_i g_{i+1}, g_{i+2,n}) + (-1)^n (g_n g_0, g_{1,n-1}).$$

In degree one, the differential map $\partial_1 : k[G \times \overline{G}] \to kG$ is given by

$$\partial_1(g_0, g_1) = g_0 g_1 - g_1 g_0 \quad (\text{for } g_0 \in G \text{ and } g_1 \in \overline{G}).$$

From Section 1.2, the Tate-Hochschild cohomology $\widehat{\mathrm{HH}}^*(kG, kG)$ can be computed by the following Tate-Hochschild cochain complex $\mathcal{D}^*(kG, kG)$:

$$(2) \qquad \cdots \xrightarrow{\partial_2} k[G \times \overline{G}] \xrightarrow{\partial_1} kG \xrightarrow{\tau} kG \xrightarrow{\delta_0} \mathrm{Map}(\overline{G}, kG) \xrightarrow{\delta_1} \cdots,$$

where the differential τ (from degree -1 component to degree 0 component) is defined to be the trace map $x \mapsto \sum_{g \in G} g x g^{-1}$. Notice that $\sum_{g \in G} g \otimes g^{-1}$ is a Casimir element of kG.

Since we have an algebra isomorphism $(kG)^e \simeq k(G \times G)$ given by $g_1 \otimes g_2 \mapsto (g_1, g_2^{-1})$, we can identify each kG-kG-bimodule M as a left $k(G \times G)$-module by the action $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$, or as a right $k(G \times G)$-module by the action $x \cdot (g_1, g_2) = g_2^{-1} x g_1$. In the following, we shall view the bar resolution, the Hochschild co(chain) complexes for the group algebra kG in terms of $k(G \times G)$-modules. Consequently,

$$\mathrm{HH}^n(kG, kG) \simeq \mathrm{Ext}^n_{k(G \times G)}(kG, kG),$$

where the $k(G \times G)$-module structure on both kG by the action $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$, and

$$\mathrm{HH}_n(kG, kG) \simeq \mathrm{Tor}^{k(G \times G)}_n(kG, kG),$$

where the first kG has a right $k(G \times G)$-module structure by the action $x \cdot (g_1, g_2) = g_2^{-1} x g_1$, and the second kG has a left $k(G \times G)$-module structure by the action $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$.

Now we recall from [31] the (generalized) cup product on $\mathcal{D}^*(kG, kG)$.

Definition 2.1. Let $\alpha \in \mathcal{D}^n(kG, kG)$ and $\beta \in \mathcal{D}^m(kG, kG)$. Then the (generalized) cup product $\alpha \cup \beta$ is defined by the following six cases:

Case 1. $n \geq 0, m \geq 0$. Then $\alpha \in C^n(kG, kG)$, $\beta \in C^m(kG, kG)$, and the cup product $\alpha \cup \beta \in C^{n+m}(kG, kG) = \mathcal{D}^{n+m}(kG, kG)$ is the same as the usual cup product on $C^*(kG, kG)$:

$$\alpha \cup \beta : \overline{G}^{\times n+m} \to kG, \quad g_{1,n+m} \mapsto \alpha(g_{1,n})\beta(g_{n+1,n+m}).$$

Case 2. $n \leq -1, m \leq -1$. Then $\alpha = (g_0, g_{1,s}) \in C_s(kG, kG)$ with $s = -n-1 \geq 0$, $\beta = (h_0, h_{1,t}) \in C_t(kG, kG)$ with $t = -m-1 \geq 0$, and the cup product $\alpha \cup \beta \in C_{s+t+1}(kG, kG) = \mathcal{D}^{n+m}(kG, kG)$ is defined by

$$\alpha \cup \beta = \sum_{g \in G}(gh_0, h_{1,t}, g^{-1}g_0, g_{1,s}) \in k[G \times \overline{G}^{\times s+t+1}].$$

This product in $C_*(kG, kG)$ is originally defined in [1, Theorem 6.1] inspired from string topology.

Case 3. $n \geq 0, m \leq -1$ and $n + m \leq -1$. Then $\alpha \in C^n(kG, kG)$, $\beta = (h_0, h_{1,t}) \in C_t(kG, kG)$ with $t = -m-1 \geq 0$, and the cup product $\alpha \cup \beta \in C_{t-n}(kG, kG) = \mathcal{D}^{n+m}(kG, kG)$ is the same as the usual cap product $\cap$ (which induces an action of Hochschild cohomology on Hochschild homology):

$$\alpha \cup \beta = (\alpha(h_{t-n+1,t})h_0, h_{1,t-n}) \in k[G \times \overline{G}^{\times t-n}].$$

Case 4. $n \geq 0, m \leq -1$ and $n + m \geq 0$. Then $\alpha \in C^n(kG, kG)$, $\beta = (g_0, g_{1,t}) \in C_t(kG, kG)$ with $t = -m-1 \geq 0$, and the cup product $\alpha \cup \beta \in C^{n-t-1}(kG, kG) = \mathcal{D}^{n+m}(kG, kG)$ is defined as the following generalized cap product:

$$\alpha \cup \beta : \overline{G}^{\times n-t-1} \to kG, \quad h_{1,n-t-1} \mapsto \sum_{g \in G}\alpha(h_{1,n-t-1}, g^{-1}, g_{1,t})g_0g.$$

Case 5. $n \leq -1, m \geq 0$ and $n + m \leq -1$. Then $\alpha = (g_0, g_{1,s}) \in C_s(kG, kG)$ with $s = -n-1 \geq 0$, $\beta \in C^m(kG, kG)$, and the cup product $\alpha \cup \beta \in C_{s-m}(kG, kG) = \mathcal{D}^{n+m}(kG, kG)$ is the following cap product $\cap$ from the right side:

$$\alpha \cup \beta = (g_0\beta(g_{1,m}), g_{m+1,s}) \in k[G \times \overline{G}^{\times s-m}].$$

Case 6. $n \leq -1, m \geq 0$ and $n + m \geq 0$. Then $\alpha = (g_0, g_{1,s}) \in C_s(kG, kG)$ with $s = -n-1 \geq 0$, $\beta \in C^m(kG, kG)$, and the cup product $\alpha \cup \beta \in C^{m-s-1}(kG, kG) = \mathcal{D}^{n+m}(kG, kG)$ is defined as the following generalized cap product from the right side:

$$\alpha \cup \beta : \overline{G}^{\times m-s-1} \to kG, \quad h_{1,m-s-1} \mapsto \sum_{g \in G}gg_0\beta(g_{1,s}, g^{-1}, h_{1,m-s-1}).$$

Remark 2.2. Since the definition of the cup product $\cup$ in this paper is different from that in [31], in order to make the following identity still hold in $\mathcal{D}^*(kG, kG)$ (cf. Lemma 2.3),

$$\partial(\alpha \cup \beta) = \partial(\alpha) \cup \beta + (-1)^m \alpha \cup \partial(\beta), \quad \text{for } \alpha \in \mathcal{D}^m(kG, kG) \text{ and } \beta \in \mathcal{D}^n(kG, kG),$$

we have to change the signs of the differential in the negative part $\mathcal{D}^{<0}(kG, kG)$. That is, the new differential ∂' on $\mathcal{D}^*(kG, kG)$ is given as follows:

$$\partial'_m(\alpha) = \begin{cases} (-1)^m \partial_m(\alpha) & \text{for } \alpha \in \mathcal{D}^m(kG, kG) \text{ and } m < 0, \\ \delta(\alpha) & \text{for } \alpha \in \mathcal{D}^m(kG, kG) \text{ and } m \geq 0. \end{cases}$$

By Section 1.2, there is a non-degenerate bilinear form on $\mathcal{D}^*(kG, kG)$ (induced by the symmetrizing form $\langle \cdot, \cdot \rangle$ on kG)

$$\langle \cdot, \cdot \rangle : \mathcal{D}^*(kG, kG) \times \mathcal{D}^*(kG, kG) \to k$$

For $\alpha \in C^m(kG, kG)$ and $\beta = (g_0, g_{1,n}) \in C_n(kG, kG)$, we define

$$\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle := \begin{cases} \langle \alpha(g_{1,n}), g_0 \rangle & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

As usual, we call an element $\alpha \in \mathcal{D}^n(kG, kG)$ homogeneous of degree n, and its degree will be denoted by $|\alpha|$. In particular, $|\alpha| = -m - 1$ for $\alpha \in C_m(kG, kG) = \mathcal{D}^{-m-1}(kG, kG)$.

Lemma 2.3. *The following identities hold in the complex* $(\mathcal{D}^*(kG, kG), \partial')$

$$\langle \partial'(\alpha), \beta \rangle = (-1)^{|\alpha|+1} \langle \alpha, \partial'(\beta) \rangle, \quad \langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta \cup \gamma \rangle,$$

$$\partial'(\alpha \cup \beta) = \partial'(\alpha) \cup \beta + (-1)^{|\alpha|} \alpha \cup \partial'(\beta)$$

for homogeneous elements $\alpha, \beta, \gamma \in \mathcal{D}^*(kG, kG)$.

Proof. The first equality follows from a straightforward computation. Let us verify the second identity. We have the following two cases.

(i) For $\phi \in C^m(kG, kG), \psi \in C^n(kG, kG)$ and $\alpha := (g_0, g_{1,m+n}) \in C_{m+n}(kG, kG)$, we have

$$\langle \phi \cup \psi, \alpha \rangle = \langle g_0, \phi(g_{1,m})\psi(g_{m+1,m+n}) \rangle = \langle \phi, (\psi(g_{m+1,m+n})g_0, g_{1,m}) \rangle = \langle \phi, \psi \cup \alpha \rangle$$

$$\langle \phi \cup \psi, \alpha \rangle = \langle g_0, \phi(g_{1,m})\psi(g_{m+1,m+n}) \rangle = \langle (g_0\phi(g_{1,m}), g_{m+1,m+n}), \psi \rangle = \langle \alpha \cup \phi, \psi \rangle.$$

This implies that $\langle \phi \cup \psi, \alpha \rangle = \langle \phi, \psi \cup \alpha \rangle = \langle \alpha \cup \phi, \psi \rangle$.

(ii) For $\alpha = (g_0, g_{1,r}) \in C_r(kG, kG), \beta = (h_0, h_{1,t}) \in C_t(kG, kG)$ and $\phi \in C^{r+t+1}(kG, kG)$, we have

$$\langle \alpha \cup \beta, \phi \rangle = \sum_{g \in G} \langle gh_0, \phi(h_{1,t}, g^{-1}g_0, g_{1,r}) \rangle = \sum_{g \in G} \langle (g_0, g_{1,r}), gh_0\phi(h_{1,t}, g^{-1}, ?) \rangle = \langle \alpha, \beta \cup \phi \rangle$$

$$\langle \alpha \cup \beta, \phi \rangle = \sum_{g \in G} \langle gh_0, \phi(h_{1,t}, g^{-1}g_0, g_{1,r}) \rangle = \sum_{g \in G} \langle (h_0, h_{1,t}), \phi(?, g^{-1}g_0, g_{1,r})g \rangle = \langle \beta, \phi \cup \alpha \rangle$$

where we need to use the identity $\sum_{g \in G}(gg_0, g^{-1}) = \sum_{g \in G}(g, g_0g^{-1})$ in $k[G \times \overline{G}]$ for $g_0 \in G$.

This verifies the second identity. From the first two identities, to verify the third identity, it is sufficient to consider the following two cases.

(i) For $\phi, \psi \in C^*(kG, kG)$, it is well-known that

$$\delta(\phi \cup \psi) = \delta(\phi) \cup \psi + (-1)^{|\phi|} \phi \cup \delta(\psi).$$

In this case, the third identity holds since $\partial' = \delta$ for $C^*(kG, kG)$.

(ii) For $\alpha = (g_0, g_{1,s}) \in C_s(kG, kG)$ and $\beta = (h_0, h_{1,t}) \in C_t(kG, kG)$, we have

$$\partial(\alpha \cup \beta) = \sum_{g \in G} \partial((gh_0, h_{1,t}, g^{-1}g_0, g_{1,s}))$$

$$= \alpha \cup \partial(\beta) + (-1)^{|\beta|} \partial(\alpha) \cup \beta.$$

Thus $\partial'(\alpha \cup \beta) = (-1)^{|\alpha|} \alpha \cup \partial'(\beta) + \partial'(\alpha) \cup \beta$ since $\partial'(\alpha) = (-1)^{|\alpha|} \partial(\alpha)$.

This proves the lemma. $\qquad \square$

As a consequence, the (generalized) cup product $\cup$ on $\mathcal{D}^*(kG, kG)$ induces a graded-commutative associative product (still denoted by $\cup$) over $\widehat{\mathrm{HH}}^*(kG, kG)$, which coincides with the Yoneda product in the singularity category $\mathcal{D}_{\mathrm{sg}}((kG)^e)$ (cf. [31]). We remind that, contrary to the Hochschild cochain complex case, the above cup product $\cup$ on the Tate-Hochschild cochain complex is not associative, but it is associative up to homotopy (cf. [27]). From [27, Theorem 6.3], it follows that the cup product extends to an A_∞-algebra structure $(m_1, m_2, m_3, \cdots)$ on $\mathcal{D}^*(kG, kG)$ with $m_1 = \partial'$, $m_2 = \cup$ and $m_i = 0$ for $i > 3$ (cf. Theorem 1.6). The formula for m_3 is described as follows.

(i) If either $\phi, \varphi, \psi \in C^*(kG, kG)$ or $\phi, \varphi, \psi \in C_*(kG, kG)$, then $m_3(\phi, \varphi, \psi) = 0$.

(ii) If $\alpha, \beta \in C_*(kG, kG)$ and $\phi \in C^*(kG, kG)$, then $m_3(\alpha, \beta, \phi) = 0 = m_3(\phi, \alpha, \beta)$.

(iii) If $\alpha \in C_*(kG, kG)$ and $\phi, \varphi \in C^*(kG, kG)$, then $m_3(\phi, \varphi, \alpha) = 0 = m_3(\alpha, \phi, \varphi)$.

(iv) For $\phi \in C^m(kG, kG)$, $\varphi \in C^n(kG, kG)$ and $\alpha = (g_0, g_1, \cdots, g_r) \in C_r(kG, kG)$,
   - if $r + 2 \leq m + n$, then $m_3(\phi, \alpha, \varphi) \in C^{m-r+n-2}(kG, kG)$ is defined by

$$m_3(\phi, \alpha, \varphi)(h_1, \cdots, h_{m-r+n-2}) =$$
$$\sum_{g \in G} \sum_{j=1}^{\min\{n, r+1\}} (-1)^{m+r+j-1} \phi(h_{1, m-r+j-2}, g, g_{j,r}) g_0 \varphi(g_{1, j-1}, g^{-1}, h_{m-r+j-1, m-r+n-2}),$$

   - if $r + 2 > m + n$, then $m_3(\phi, \alpha, \varphi) = 0$.

(v) For $\alpha = (g_0, g_{1,r}) \in C_r(kG, kG)$, $\beta = (h_0, h_{1,s}) \in C_s(kG, kG)$ and $\phi \in C^m(kG, kG)$,
   - if $m - 1 \leq r + s$, then

$$m_3(\alpha, \phi, \beta) = \sum_{g \in G} \sum_{j=0}^{s} (-1)^{n-j} (g_0 \phi(g_{1, m-s+j-1}, g, h_{j+1, s}) h_0, h_{1,j}, g^{-1}, g_{m-s+j, r}),$$

   - if $m - 1 > r + s$, then $m_3(\alpha, \phi, \beta) = 0$.

Therefore we have the following identity, for $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{D}^*(kG, kG)$,

$$\alpha_1 \cup (\alpha_2 \cup \alpha_3) - (\alpha_1 \cup \alpha_2) \cup \alpha_3 = \partial'(m_3(\alpha_1, \alpha_2, \alpha_3)) + m_3(\partial'(\alpha_1), \alpha_2, \alpha_3)$$
$$+ (-1)^{|\alpha_1|} m_3(\alpha_1, \partial'(\alpha_2), \alpha_3) + (-1)^{|\alpha_1| + |\alpha_2|} m_3(\alpha_1, \alpha_2, \partial'(\alpha_3)).$$

From [27, Proposition 6.5], it follows that the A_∞-algebra structure is compatible with the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ in the following sense:

$$(3) \qquad \langle \alpha_0, m_k(\alpha_1, \cdots, \alpha_k) \rangle = (-1)^{|\alpha_0|(2-k)+k} \langle m_k(\alpha_0, \cdots, \alpha_{k-1}), \alpha_k \rangle$$

for any $\alpha_i \in \mathcal{D}^*(kG, kG)$, $0 \leq i \leq k$. Such A_∞-algebra is called *cyclic*. In particular, Formula (iv) is dual to Formula (v) in the sense of Equation (3). In other words, we may get one from the other by Equation (3).

Moreover, one can define a Lie bracket $[\cdot, \cdot]$ on $\widehat{\mathrm{HH}}^*(kG, kG)$ such that $(\widehat{\mathrm{HH}}^*(kG, kG), \cup, [\cdot, \cdot])$ becomes a Gerstenhaber algebra, that is, for homogeneous elements α, β, γ in $\widehat{\mathrm{HH}}^*(kG, kG)$, the following three conditions hold:

   - $(\widehat{\mathrm{HH}}^*(kG, kG), \cup)$ is an associative algebra and it is graded commutative, that is, the cup product $\cup$ is an associative multiplication and satisfies $\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$;
   - $(\widehat{\mathrm{HH}}^*(kG, kG), [\cdot, \cdot])$ is a graded Lie algebra of degree -1, that is, the Lie bracket $[\cdot, \cdot]$ satisfies $[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]$ and the graded Jacobi identity

$$(-1)^{(|\alpha|-1)(|\gamma|-1)}[[\alpha, \beta], \gamma] + (-1)^{(|\beta|-1)(|\alpha|-1)}[[\beta, \gamma], \alpha] + (-1)^{(|\gamma|-1)(|\beta|-1)}[[\gamma, \alpha], \beta] = 0;$$

   - Poisson rule: $[\alpha \cup \beta, \gamma] = [\alpha, \gamma] \cup \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \cup [\beta, \gamma]$.

The Lie bracket $[\cdot, \cdot]$ is a generalization of the Gerstenhaber bracket $[\cdot, \cdot]$ in Hochschild cohomology, and we can write it down explicitly at the complex level. Since $[\cdot, \cdot]$ is determined by the cup product $\cup$ and the BV-operator $\widehat{\Delta}$ in $\widehat{\mathrm{HH}}^*(kG, kG)$ (see below), we refrain from giving a formula at the complex level. The interested reader can refer the details to the paper [31].

Let us now define the BV-operator $\widehat{\Delta}$ in $\widehat{\mathrm{HH}}^*(kG, kG)$. At the complex level, $\widehat{\Delta}$ is the Connes' B-operator B for the negative part $\mathcal{D}^{<0}(kG, kG) = C_*(kG, kG)$, is the operator Δ for the positive

part $\mathcal{D}^{>0}(kG, kG) = C^{>0}(kG, kG)$, and $\widehat{\Delta} : \mathcal{D}^0(kG, kG) \to \mathcal{D}^{-1}(kG, kG)$ is zero. More precisely, if $n \leq -1$, then $\widehat{\Delta} = B : k[G \times \overline{G}^{\times s}] \to k[G \times \overline{G}^{\times s+1}]$ (let $s = -n - 1$) is given by

$$\alpha = (g_0, g_{1,s}) \mapsto \widehat{\Delta}(\alpha) = \sum_{i=0}^{s} (-1)^{is}(1, g_{i,s}, g_{0,i-1});$$

if $n \geq 1$, then $\widehat{\Delta} = \Delta : \mathrm{Map}(\overline{G}^{\times n}, kG) \to \mathrm{Map}(\overline{G}^{\times (n-1)}, kG)$ maps any $\alpha : \overline{G}^{\times n} \to kG$ to $\Delta(\alpha) : \overline{G}^{\times (n-1)} \to kG$ such that

$$\Delta(\alpha)(g_{1,n-1}) = \sum_{g_n \in G} \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \alpha(g_{i,n-1}, g_n, g_{1,i-1}), 1 \rangle g_n^{-1},$$

where $\langle \cdot, \cdot \rangle$ is the symmetrizing form on kG (cf. [22, Section 8]). It is easy to verify that the operator $\widehat{\Delta} : \mathcal{D}^*(kG, kG) \to \mathcal{D}^{*-1}(kG, kG)$ is a chain map (with $\widehat{\Delta}^2 = 0$) and therefore induces an operation (still denoted by $\widehat{\Delta}$) in $\widehat{\mathrm{HH}}^*(kG, kG)$. It turns out that the Gerstenhaber algebra $(\widehat{\mathrm{HH}}^*(kG, kG), \cup, [\cdot, \cdot])$ together with the operator $\widehat{\Delta}$ is a Batalin-Vilkovsky algebra (BV-algebra), that is, in addition to be a Gerstenhaber algebra, $(\widehat{\mathrm{HH}}^*(kG, kG), \widehat{\Delta})$ is a complex and

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)|\beta|}(\widehat{\Delta}(\alpha \cup \beta) - \widehat{\Delta}(\alpha) \cup \beta - (-1)^{|\alpha|} \alpha \cup \widehat{\Delta}(\beta))$$

for all homogeneous elements $\alpha, \beta \in \widehat{\mathrm{HH}}^*(kG, kG)$ (cf. [31]).

Remark 2.4. The signs in the definition of a BV-algebra depend on the choice of the definitions of cup product and Lie bracket. If we define $\alpha \cup' \beta = (-1)^{|\alpha||\beta|} \alpha \cup \beta$ and $\Delta'(\alpha) = (-1)^{(|\alpha|-1)} \Delta(\alpha)$, then we get

$$[\alpha, \beta] = (-1)^{|\alpha|}(\Delta'(\alpha \cup' \beta) - \Delta'(\alpha) \cup' \beta - (-1)^{|\alpha|} \alpha \cup' \Delta'(\beta)),$$

which is the equality in the usual definition of a BV-algebra (see, for example [13, 24]). We choose the sign convention from [29] because of our convention of the definitions of cup product and Connes' B-operator in the Hochschild (co)homology theory.

3. REMINDER ON COHOMOLOGY AND TATE COHOMOLOGY OF FINITE GROUPS

In this section we recall some notions on Tate cohomology of finite groups. For the details, we refer the reader to [8, Chapter VI].

3.1. **Group (co)homology.** Let k be a field, G a finite group and kG the group algebra. Let M be a left kG-module. Then the cohomology of G with coefficients in M is defined to be

$$\mathrm{H}^p(G, M) = \mathrm{Ext}_{kG}^p(k, M), \ p \geq 0,$$

and the homology of G with coefficients in M is defined to be

$$\mathrm{H}_p(G, M) = \mathrm{Tor}_p^{kG}(k, M), \ p \geq 0,$$

where k is the left trivial kG-module in $\mathrm{Ext}_{kG}^p(k, M)$ and is the right trivial kG-module in $\mathrm{Tor}_p^{kG}(k, M)$.

By Remark 1.1, the complex $P_* := \mathrm{Bar}_*(kG) \otimes_{kG} k$ is the *standard resolution* of the trivial kG-module k. So there exist canonical complexes computing group (co)homology.

Recall that the *group cohomology complex* $(C^*(G, M), \delta^*)$ is defined as follows:

$$C^n(G, M) = \mathrm{Hom}_{kG}(\mathrm{Bar}_n(kG) \otimes_{kG} k, M) \simeq \mathrm{Hom}_{kG}(k[\overline{G}^{\times n}], M) \simeq \mathrm{Map}(\overline{G}^{\times n}, MG), \quad \text{for } n \geq 0,$$

and the differential is given by

$$\delta^n : \mathrm{Map}(\overline{G}^{\times n}, M) \to \mathrm{Map}(\overline{G}^{\times (n+1)}, M), \quad \varphi \mapsto \delta^n(\varphi),$$

where $\delta^n(\varphi)$ sends $g_{1,n+1} \in \overline{G}^{n+1}$ to

$$g_1 \varphi(g_{2,n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(g_{1,i-1}, g_i g_{i+1}, g_{i+2,n+1}) + (-1)^{n+1} \varphi(g_{1,n}).$$

In degree zero, the differential map $\delta^0 : M \to \mathrm{Map}(\overline{G}, M)$ is given by

$$\delta^0(x)(g) = gx - x \quad \text{(for } x \in M \text{ and } g \in \overline{G}).$$

We can consider M as a right kG-module via $x \cdot g = g^{-1}x$, $x \in M, g \in G$. Then $\operatorname{Tor}_*^{kG}(k, M) \cong \operatorname{Tor}_*^{kG}(M, k)$, where we use the right kG-module M in $\operatorname{Tor}_*^{kG}(M, k)$. Notice that $\operatorname{Tor}_*^{kG}(M, k)$ can by computed by the *group homology complex* $(C_*(G, M), \partial_*)$, which is defined as follows:

$$C_n(G, M) = M \otimes_{kG} \operatorname{Bar}_n(kG) \otimes_{kG} k \simeq M \otimes k[\overline{G}^{\times n}], \quad \text{for } n \geq 0,$$

and the differential $\partial_n : M \otimes k[\overline{G}^{\times n}] \to M \otimes k[\overline{G}^{\times(n-1)}], n \geq 2$ is given by

$$x \otimes g_{1,n} \mapsto x \cdot g_1 \otimes (g_{2,n}) + \sum_{i=1}^{n-1}(-1)^i x \otimes (g_{1,i-1}, g_i g_{i+1}, g_{i+2,n}) + (-1)^n x \otimes (g_{1,n-1}),$$

and in degree one, the differential map $\partial_1 : M \otimes k[\overline{G}] \to M$ is given by

$$\partial_1(x \otimes g_1) = x \cdot g_1 - x \quad \text{(for } x \in M \text{ and } g_1 \in \overline{G}).$$

Let us explain conjugation maps, restriction maps and corestriction maps on group (co)homology, as we shall need them in Section 5. For more details, we refer the reader to the textbook [11].

Let G be a finite group and M a kG-module. Let Q_* be a projective resolution of k as a kG-module.

(1) For any $g \in G$ and $H \leq G$, write ${}^gH = gHg^{-1}$. The conjugation map $g^* : \mathrm{H}^*(H, M) \longrightarrow \mathrm{H}^*({}^gH, M)$ is induced by the map

$$g^* : \operatorname{Hom}_{kH}(Q_*, M) \to \operatorname{Hom}_{kH}(Q_*, M), \quad \varphi \mapsto {}^g\varphi,$$

where ${}^g\varphi(x) = g\varphi(g^{-1}x), x \in Q_*$;

(2) For $H \leq G$, the restriction map $res_H^G : \mathrm{H}^*(G, M) \longrightarrow \mathrm{H}^*(H, M)$ is induced by the natural inclusion map $\operatorname{Hom}_{kG}(Q_*, M) \to \operatorname{Hom}_{kH}(Q_*, M)$, as homomorphisms of kG-modules are necessarily homomorphisms of kH-modules;

(3) For $H \leq G$ the corestriction map $cor_H^G : \mathrm{H}^*(H, M) \longrightarrow \mathrm{H}^*(G, M)$ is induced from the map

$$\operatorname{Hom}_{kH}(Q_*, M) \to \operatorname{Hom}_{kG}(Q_*, M), \quad \varphi \mapsto \sum_{t \in T} {}^t\varphi,$$

where T is a complete set of representatives of the left cosets of the subgroup H in G;

(1') For any $g \in G$, the conjugation map $g_* : \mathrm{H}_*(H, M) \longrightarrow \mathrm{H}_*({}^gH, M)$ is induced by the map

$$g_* : M \otimes_{kH} Q_* \to M \otimes_{k^gH} Q_*, \quad x \otimes_{kH} y \mapsto xg^{-1} \otimes_{k^gH} gy;$$

(2') For $H \leq G$, the restriction map $res_H^G : \mathrm{H}_*(G, M) \longrightarrow \mathrm{H}_*(H, M)$ is is induced by the map

$$M \otimes_{kG} Q_* \to M \otimes_{kH} Q_*, \quad x \otimes_{kG} y \mapsto \sum_{t \in T} xt \otimes_{kH} t^{-1}x,$$

where T is a complete set of representatives of the left cosets of the subgroup H in G;

(3') For $H \leq G$, the corestriction map $cor_H^G : \mathrm{H}_*(H, M) \longrightarrow \mathrm{H}_*(G, M)$ is induced from the natural quotient map $M \otimes_{kH} Q_* \to M \otimes_{kG} Q_*$.

3.2. **Tate cohomology of groups.** Applying the duality functor $()^* = \operatorname{Hom}_k(-, k)$ to the standard resolution $P_* = \operatorname{Bar}_*(kG) \otimes_{kG} k$, we get a "backwards projective resolution" $\operatorname{Hom}(\operatorname{Bar}_*(kG) \otimes_{kG} k, k)$ of k. By splicing together $\operatorname{Bar}_*(kG) \otimes_{kG} k$ and $\operatorname{Hom}(\operatorname{Bar}_*(kG) \otimes_{kG} k), k)$ we get a complete resolution of the trivial module k:

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_0{}^* \longrightarrow P_1{}^* \longrightarrow \cdots,$$

where the (left) kG-module structure over $P_n^* = \operatorname{Hom}_k(P_n, k)$ is given by $(g\varphi)(x) = \varphi(g^{-1}x)$ for any $\varphi \in P_n^*$ and $x \in P_n$. We denote this complete resolution by F_*, where $F_n = P_n$ for $n \geq 0$ and $F_n = P_{-n-1}^*$ for $n \leq -1$. Let U be any (left) kG-module. Applying the functor $\operatorname{Hom}_{kG}(-, U)$ to F_*, we get a cochain complex, denoted by $\widehat{C}^*(G, U)$ (called *Tate cochain complex* of G):

(1) The nonnegative part $\widehat{C}^{\geq 0}(G, U)$ is exactly the group cohomology complex $C^*(G, U)$ with coefficients in U.

(2) For each $n \leq -1$ (let $s = -n - 1 \geq 0$), we notice that there is a natural isomorphism

$$U \otimes_{kG} P_s \simeq \mathrm{Hom}_{kG}(P_s^*, U), \quad u \otimes_{kG} x \mapsto (\alpha \mapsto \sum_{g \in G} \alpha(gx)gu),$$

where in $U \otimes_{kG} P_s$ we consider U as a right kG-module by the action $ug = g^{-1}u$, and where P_s^* is viewed as a left kG-module. It follows that, for $n \leq -1$ (let $s = -n - 1 \geq 0$),

$$\widehat{C}^n(G, U) = \mathrm{Hom}_{kG}(P_s^*, U) \simeq U \otimes_{kG} P_s \simeq U \otimes_{kG} (kG \otimes k(\overline{G}^{\times s})) \simeq U \otimes k[\overline{G}^{\times s}]$$

and the differential is given by

$$\partial_s(x, g_{1,s}) = (g_1^{-1}x, g_{2,s}) + \sum_{i=1}^{s-1}(-1)^i(x, g_{1,i-1}, g_i g_{i+1}, g_{i+2,s}) + (-1)^s(x, g_{1,s-1})$$

for all $s \geq 1$, $x \in U$ and $g_1, \cdots, g_s \in \overline{G}$. Therefore the negative part $C^{<0}(G, U)$ is exactly the group homology complex $C_*(G, U)$ with coefficients in U (Here we consider U as a right kG-module).

(3) For $n = -1$ (or $s = 0$), the differential $\delta_{-1} : C_0(G, U) = U \to U = C^0(G, U)$ is given by $u \mapsto (\sum_{g \in G} g)u$ for $u \in U$.

The Tate cohomology of G with coefficients in U is defined to be the (co)homology group

$$\widehat{H}^n(G, U) = H^n(\widehat{C}^*(G, U)) = H^n(\mathrm{Hom}_{kG}(F_*, U)).$$

We have the following descriptions of the Tate cohomology $\widehat{H}^n(G, U)$:

(i) $\widehat{H}^n(G, U) \simeq H^n(G, U) := \mathrm{Ext}_{kG}^n(k, U)$ for all $n > 0$,
(ii) $\widehat{H}^n(G, U) \simeq H_{-n-1}(G, U) := \mathrm{Tor}_{-n-1}^{kG}(U, k)$ for all $n < -1$,
(iii) there is an exact sequence

$$0 \to \widehat{H}^{-1}(G, U) \to H_0(G, U) \xrightarrow{\alpha} H^0(G, U) \to \widehat{H}^0(G, U) \to 0.$$

Denote the sum $\sum_{g \in G} g$ by N. Then the map α is the so-called norm map:

$$H_0(G, U) = U_G \to U^G = H^0(G, U), \quad u \mapsto Nu.$$

Therefore, $\widehat{H}^*(G, U)$ is a "combination" of the group cohomology $H^*(G, U)$ and the group homology $H_*(G, U)$. We can summarize the above results by means of the following diagram (cf. [8, VI. 4]):

Of particular interest to us is the case when $U = k$, the trivial kG-module. From now on, we always refer to Tate cohomology of a group algebra kG as $\widehat{H}^*(G, k)$, unless stated otherwise.

Remark 3.1. If the characteristic of k divides the order of G, then the map $\alpha : H_0(G, k) \to H^0(G, k)$ is zero and we have that $\widehat{H}^{-1}(G, k) \simeq H_0(G, k)$ and $\widehat{H}^0(G, k) \simeq H^0(G, k)$. Otherwise, the map $\alpha : H_0(G, k) \to H^0(G, k)$ is an isomorphism and we have that $\widehat{H}^p(G, k) = 0$ for all $p \in \mathbb{Z}$.

4. Lifting the additive decomposition to the complex level

Let k be a field and G a finite group. Then the Tate-Hochschild cohomology $\widehat{HH}^*(kG, kG)$ admits an additive decomposition (as k-vector spaces):

$$\widehat{HH}^*(kG, kG) \simeq \bigoplus_{x \in X} \widehat{H}^*(C_G(x), k),$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer subgroup of $x \in G$, and where $\widehat{H}^*(C_G(x), k)$ is the Tate cohomology of $C_G(x)$ (cf. [25, Section 5]).

In this section, we give an explicit construction of the additive decomposition at the complex level. We deal with this task in three cases:

The first case: $n > 0$. In this case,

$$\widehat{\mathrm{HH}}^n(kG, kG) \simeq \mathrm{HH}^n(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}^n(C_G(x), k) = \bigoplus_{x \in X} \mathrm{Ext}^n_{kC_G(x)}(k, k);$$

The second case: $n < -1$ (let $s = -n - 1 > 0$). In this case,

$$\widehat{\mathrm{HH}}^n(kG, kG) \simeq \mathrm{HH}_s(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}_s(C_G(x), k) = \bigoplus_{x \in X} \mathrm{Tor}^{kC_G(x)}_s(k, k);$$

The third case: $n = 0, -1$. In this case,

$$\widehat{\mathrm{HH}}^0(kG, kG) \simeq \bigoplus_{x \in X} \widehat{\mathrm{H}}^0(C_G(x), k), \quad \widehat{\mathrm{HH}}^{-1}(kG, kG) \simeq \bigoplus_{x \in X} \widehat{\mathrm{H}}^{-1}(C_G(x), k).$$

The first case. This is just the usual additive decomposition of the Hochschild cohomology $\mathrm{HH}^*(kG, kG)$ (except in degree zero component) and its lifting (to the complex level) has been done by the first named author and the third named author in [22]. The idea is as follows. Cibils and Solotar [10] constructed a subcomplex of the Hochschild cochain complex $C^*(kG, kG)$ for each conjugacy class, and then they showed that for a finite abelian group, the subcomplex is isomorphic to the group cohomology complex (cf. Section 3.1). This was generalized to any finite group in [22]: for each conjugacy class, this complex computes the group cohomology of the corresponding centralizer subgroup. Let us briefly recall the construction there. For simplicity, we denote by $\mathcal{H}_*$ and $\mathcal{H}^*$ the Hochschild chain complex $C_*(kG, kG)$ and the Hochschild cochain complex $C^*(kG, kG)$ respectively.

Recall from Section 2 that the Hochschild cohomology $\mathrm{HH}^*(kG, kG)$ of the group algebra kG can be computed by the Hochschild cochain complex $C^*(kG, kG)$:

$$(\mathcal{H}^*) \qquad 0 \longrightarrow kG \xrightarrow{\delta^0} \mathrm{Map}(\overline{G}, kG) \xrightarrow{\delta^1} \cdots \longrightarrow \mathrm{Map}(\overline{G}^{\times n}, kG) \xrightarrow{\delta^n} \cdots,$$

where the differential is given by

$$\delta^0(x)(g) = gx - xg \quad (\text{for } x \in kG \text{ and } g \in \overline{G})$$

and (for $\varphi : \overline{G}^{\times n} \longrightarrow kG$ and $g_1, \cdots, g_{n+1} \in \overline{G}$)

$$\delta^n(\varphi)(g_{1,n+1}) = g_1 \varphi(g_{2,n+1}) + \sum_{i=1}^n (-1)^i \varphi(g_{1,i-1}, g_i g_{i+1}, g_{i+2,n+1}) + (-1)^{n+1} \varphi(g_{1,n}) g_{n+1}.$$

Let X be a complete set of representatives of the conjugacy classes in the finite group G. For $x \in X$, $C_x = \{gxg^{-1} | g \in G\}$ is the conjugacy class corresponding to x and $C_G(x) = \{g \in G | gxg^{-1} = x\}$ is the centralizer subgroup. Now take a conjugacy class C_x and define

$$\mathcal{H}_x^0 = k[C_x], \text{ and for } n \geq 1,$$

$$\mathcal{H}_x^n = \{\varphi : \overline{G}^{\times n} \longrightarrow kG \mid \varphi(g_1, \cdots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \cdots, g_n \in \overline{G}\},$$

where $g_1 \cdots g_n C_x$ denotes the subset of G obtained by multiplying $g_1 \cdots g_n$ on C_x and $k[g_1 \cdots g_n C_x]$ is the k-subspace of kG spanned by this set. Note that we have $g_1 \cdots g_n C_x = C_x g_1 \cdots g_n$ and $k[g_1 \cdots g_n C_x] = k[C_x g_1 \cdots g_n]$. Let $\mathcal{H}_x^* = \bigoplus_{n \geq 0} \mathcal{H}_x^n$. Cibils and Solotar observed that $\mathcal{H}_x^*$ is a subcomplex of $\mathcal{H}^*$ and $\mathcal{H}^* = \bigoplus_{x \in X} \mathcal{H}_x^*$ (see [10, Page 20, Proof of the theorem]).

Lemma 4.1. *The complex $\mathcal{H}_x^*$ is isomorphic to the complex*

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k),$$

which computes the group cohomology $\mathrm{H}^*(C_G(x), k)$ of $C_G(x)$. More concretely, there is an isomorphism of complexes:

$$\begin{array}{rcl}
\mathcal{H}_x^* & \to & \mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k), \\
(\varphi_x : \overline{G}^{\times n} \to kG) & \mapsto & (\widehat{\varphi}_x : S_x \times \overline{G}^{\times n} \to k), \quad \widehat{\varphi}_x(\gamma_{i,x}, g_{1,n}) = a_{i,x}
\end{array}$$

where we write $\varphi_x(g_{1,n})g_n^{-1}\cdots g_1^{-1} = \sum_{i=1}^{n_x} a_{i,x}x_i \in kC_x$; *we fix a right coset decomposition of* $C_G(x)$
in G:

$$G = C_G(x)\gamma_{1,x} \cup \cdots \cup C_G(x)\gamma_{n_x,x}$$

and thus $C_x = \{\gamma_{1,x}^{-1}x\gamma_{1,x}, \cdots, \gamma_{n_x,x}^{-1}x\gamma_{n_x,x}\}$. *Here we write* $x_i = \gamma_{i,x}^{-1}x\gamma_{i,x}$ *and* $S_x = \{\gamma_{1,x}, \cdots, \gamma_{n_x,x}\}$,
and we take $\gamma_{1,x} = 1$ *and* $x_1 = x$. *The inverse is given by*

$$\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k) \quad \rightarrow \quad \mathcal{H}_x^*,$$
$$(\widehat{\varphi}_x : S_x \times \overline{G}^{\times n} \to k) \quad \mapsto \quad (\varphi_x : \overline{G}^{\times n} \to kG)), \ \varphi_x(g_{1,n}) = \sum_{i=1}^{n_x} \widehat{\varphi}_x(\gamma_{i,x}, g_{1,n})x_i g_1 \cdots g_n.$$

Passing to the cohomology, we have $H^*(\mathcal{H}_x^*) \simeq \mathrm{H}^*(C_G(x), k)$.

Proof. This follows from the first five steps in [22, Page 9]. $\qquad\qquad\qquad\qquad\square$

To compare the two complexes $\mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k)$ and

$$C^*(C_G(x), k) = \mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k, k),$$

we need the following comparison maps defined in [22, Page 16]. The comparison map

$$\iota : \mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k \to \mathrm{Bar}_*(kG) \otimes_{kG} k$$

is just defined as the inclusion map $\iota : k[C_G(x) \times \overline{C_G(x)}^{\times n}] \hookrightarrow k[G \times \overline{G}^{\times n}]$. The comparison map

$$\rho : \mathrm{Bar}_*(kG) \otimes_{kG} k \longrightarrow \mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k$$

is defined as follows:

$$\begin{aligned}
\rho_{-1} : k &\longrightarrow k, \quad 1 \longmapsto 1, \\
\rho_0 : kG &\longrightarrow kC_G(x), \quad h\gamma_{i,x} \longmapsto h, \text{ for } h \in C_G(x), \\
\rho_1 : k[G \times \overline{G}] &\longrightarrow k[C_G(x) \times \overline{C_G(x)}], \quad (h\gamma_{i,x}, g_1) \longmapsto (h, h_{i_1}),
\end{aligned}$$

where $h \in C_G(x)$ and $\gamma_{i,x}g_1 = h_{i_1}\gamma_{s_i^1,x}$ for $h_{i_1} \in \overline{C_G(x)}$,

$$\rho_n : k[G \times \overline{G}^{\times n}] \longrightarrow k[C_G(x) \times \overline{C_G(x)}^{\times n}], \quad (h\gamma_{i,x}, g_{1,n}) \longmapsto (h, h_{i_1}, \cdots, h_{i_n}),$$

where $h_{i_1}, \cdots, h_{i_n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \cdots, g_n\}$ as follows:

$$\gamma_{i,x}g_1 = h_{i_1}\gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x}g_2 = h_{i_2}\gamma_{s_i^2,x}, \quad \cdots, \quad \gamma_{s_i^{n-1},x}g_n = h_{i_n}\gamma_{s_i^n,x}.$$

Remark 4.2. Notice that $\rho \circ \iota = \mathrm{id}$. There is a homotopy $s : \mathrm{Bar}_*(kG) \otimes_{kG} k \to \mathrm{Bar}_*(kG) \otimes_{kG} k$
between id and $\iota \circ \rho$. For $(h\gamma_{i,x}, g_{1,n}) \in k[G \times \overline{G}^{\times n}]$, we define

$$s(h\gamma_{i,x}, g_{1,n}) := (h, \gamma_{i,x}, g_{1,n}) + \sum_{j=1}^{n} (-1)^j (h, h_{i_1}, \cdots, h_{i_j}, \gamma_{s_i^j,x}, g_{j+1}, \cdots, g_n),$$

where $h_{i_1}, \cdots, h_{i_n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \cdots, g_n\}$ as follows:

$$\gamma_{i,x}g_1 = h_{i_1}\gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x}g_2 = h_{i_2}\gamma_{s_i^2,x}, \quad \cdots, \quad \gamma_{s_i^{n-1},x}g_n = h_{i_n}\gamma_{s_i^n,x}.$$

By a straightforward computation, we get that $\mathrm{id} - \iota \circ \rho = (d \otimes_{kG} k) \circ s + s \circ (d \otimes_{kG} k)$, where d is the
differential of $\mathrm{Bar}_*(kG)$ (cf. Section 2). As a consequence, we get a homotopy deformation retract
of complexes of (left) $kC_G(x)$-modules

$$s \ \circlearrowleft \ \mathrm{Bar}_*(kG) \otimes_{kG} k \ \underset{\iota}{\overset{\rho}{\rightleftarrows}} \ \bigoplus_{x \in X} \mathrm{Bar}_*(kC_G(x)) \otimes_{kC_G(x)} k.$$

That is, we have $\rho \circ \iota = \mathrm{id}$ and $\mathrm{id} - \iota \circ \rho = (d \otimes_{kG} k) \circ s + s \circ (d \otimes_{kG} k)$. We remark that our construction
of the homotopy s is inspired from [15, Definition 3.4].

Applying the functor $\mathrm{Hom}_{kC_G(x)}(-, k)$ to the above homotopy deformation retract in Remark
4.2 and then composing with the isomorphism in Lemma 4.1, we obtain the following homotopy
deformation retract of complexes for any $x \in X$,

$$s^x \ \circlearrowleft \ \mathcal{H}_x^* \ \underset{\rho^x}{\overset{\iota^x}{\rightleftarrows}} \ C^*(C_G(x), k).$$

Here the surjection ι^x is given by

$$\iota^x : \mathcal{H}_x^* \to C^*(C_G(x), k), \quad [\varphi_x : \overline{G}^{\times n} \to kG] \mapsto [\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \to k],$$

$$\text{with} \quad \widehat{\varphi}_x(h_{1,n}) = a_{1,x}, \text{ where } \varphi_x(h_{1,n})h_n^{-1}\cdots h_1^{-1} = \sum_{i=1}^{n_x} a_{i,x}x_i \in kC_x.$$

In other words, $\widehat{\varphi}_x(h_{1,n})$ is just the coefficient of x in $\varphi_x(h_{1,n})h_n^{-1}\cdots h_1^{-1} \in kC_x$. The map ρ^x is given by

$$\rho^x : C^*(C_G(x), k) \to \mathcal{H}_x^*, \quad [\widehat{\varphi}_x : \overline{C_G(x)}^{\times n} \to k] \mapsto [\varphi_x : \overline{G}^{\times n} \to kG],$$

$$\text{with} \quad \varphi_x \in \mathcal{H}_x^n, \quad \text{and} \quad \varphi_x(g_{1,n}) = \sum_{i=1}^{n_x} \widehat{\varphi}_x(h_{i_1}, \cdots, h_{i_n})x_ig_1\cdots g_n,$$

where $h_{i_1}, \cdots, h_{i_n} \in \overline{C_G(x)}$ are determined by the sequence $\{g_1, \cdots, g_n\}$ as follows:

$$\gamma_{i,x}g_1 = h_{i_1}\gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x}g_2 = h_{i_2}\gamma_{s_i^2,x}, \quad \cdots, \quad \gamma_{s_i^{n-1},x}g_n = h_{i_n}\gamma_{s_i^n,x}.$$

The homotopy s^x is given by: For $(\varphi_x : \overline{G}^{\times n} \to kG) \in \mathcal{H}_x^n$, we define $s^x(\varphi_x) \in \mathcal{H}_x^{n-1}$ as

$$s^x(\varphi_x)(g_{1,n-1}) = \sum_{j=0}^{n-1}\sum_{i=1}^{n_x}(-1)^j a_{i,j}^1 x_ig_1\cdots g_{n-1},$$

where the coefficients $a_{i,j}^1$ are determined by the following identity (when $j = 0$, we set $\gamma_{s_i^0,x} = \gamma_{i,x}$)

$$\varphi_x(h_{i_1}, \cdots, h_{i_j}, \gamma_{s_i^j,x}, g_{j+1}, \cdots, g_{n-1})g_{n-1}^{-1}\cdots g_1^{-1}\gamma_{i,x}^{-1} = \sum_{k=1}^{n_x} a_{i,j}^k x_k$$

since we have $h_{i_1}h_{i_2}\cdots h_{i_j}\gamma_{s_i^j,x}g_{j+1}\cdots g_{n-1} = \gamma_{i,x}g_1\cdots g_{n-1}$ for any $0 \le j \le n-1$.

Therefore, we get a lifting of the additive decomposition of $\mathrm{HH}^*(kG, kG)$ at the complex level.

Theorem 4.3. (cf. [22, Theorem 6.3]) *Let k be a field and G a finite group. Consider the additive decomposition of Hochschild cohomology algebra of the group algebra kG:*

$$\mathrm{HH}^*(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}^*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer of G. Then the above additive decomposition lifts to a homotopy deformation retract of complexes

$$s^* \circlearrowright \ C^*(kG, kG) \underset{\rho^*}{\overset{\iota^*}{\rightleftarrows}} \bigoplus_{x \in X} C^*(C_G(x), k),$$

where $\iota^* = \sum_{x \in X} \iota^x$, $\rho^* = \sum_{x \in X} \rho^x$, and $s^* = \sum_{x \in X} s^x$.

Notice that the homotopy s^* in the above theorem is induced from the homotopy s in Remark 4.2, which is not contained in [22, Theorem 6.3].

The second case. This is just the usual additive decomposition of the Hochschild homology $\mathrm{HH}_*(kG, kG)$ (except in degree zero component). We use a similar idea as in the first case to give a lifting of $\mathrm{HH}_*(kG, kG)$ to the complex level.

Recall from Section 2 that the Hochschild homology $\mathrm{HH}_*(kG, kG)$ of the group algebra kG can be computed by the Hochschild chain complex $C_*(kG, kG)$:

$$(\mathcal{H}_*) \qquad \cdots \longrightarrow k[G \times \overline{G}^{\times s}] \overset{\partial_s}{\longrightarrow} \cdots \longrightarrow k[G \times \overline{G}] \overset{\partial_1}{\longrightarrow} kG \longrightarrow 0,$$

where the differential is given by

$$\partial_s(g_0, g_{1,s}) = (g_0g_1, g_{2,s}) + \sum_{i=1}^{s-1}(-1)^i(g_0, g_{1,i-1}, g_ig_{i+1}, g_{i+2,s}) + (-1)^s(g_sg_0, g_{1,s-1}),$$

$$\partial_1(g_0, g_1) = g_0g_1 - g_1g_0 \quad (\text{for } g_0 \in G \text{ and } g_1 \in \overline{G}).$$

We denote
$$\mathcal{H}_{x,0} = k[C_x], \text{ and for } s \geq 1,$$
$$\mathcal{H}_{x,s} = k[(g_s^{-1}\cdots g_1^{-1}u, g_{1,s}) \mid u \in C_x, g_1, \cdots, g_s \in \overline{G}].$$
Let $\mathcal{H}_{x,*} = \bigoplus_{s \geq 0} \mathcal{H}_{x,s}$. It is easy to verify that $\mathcal{H}_{x,*}$ is a subcomplex of $\mathcal{H}_*$ and $\mathcal{H}_* = \bigoplus_{x \in X} \mathcal{H}_{x,*}$.

Remark 4.4. We obtain this decomposition of $C_*(kG, kG) = \mathcal{H}_*$ motivated from Cibils-Solatar's decomposition of $\mathcal{H}^*$, but this decomposition has already appeared in [21, 7.4.4 Proposition]. In fact, the complex $k[\Gamma_*(G, x)]$ in [21, 7.4.4 Proposition], which is constructed as certain subcyclic set of the cyclic bar construction, coincides with $\mathcal{H}_{x,*}$. We thank an anonymous referee for pointing out this to us.

Lemma 4.5. *The complex $\mathcal{H}_{x,*}$ is isomorphic to $k \otimes_{kC_G(x)} \mathrm{Bar}_*(kG) \otimes_{kG} k$, which computes the group homology $\mathrm{H}_*(C_G(x), k)$ of $C_G(x)$. More concretely, there is an isomorphism of complexes:*

$$\mathcal{H}_{x,s} \longrightarrow k \otimes_{kC_G(x)} \mathrm{Bar}_s(kG) \otimes_{kG} k \simeq k \otimes_{kC_G(x)} k[G \times \overline{G}^{\times s}],$$
$$(g_s^{-1}\cdots g_1^{-1}g_0^{-1}xg_0, g_{1,s}) \longmapsto 1 \otimes_{kC_G(x)} g_{0,s},$$

and its inverse is given by

$$k \otimes_{kC_G(x)} k[G \times \overline{G}^{\times s}] \longrightarrow \mathcal{H}_{x,s},$$
$$1 \otimes_{kC_G(x)} g_{0,s} \longmapsto (g_s^{-1}\cdots g_1^{-1}g_0^{-1}xg_0, g_{1,s}).$$

Proof. The differential in the complex $\mathcal{H}_{x,*}$ is induced from $\mathcal{H}_*$, while the differential in the complex $k \otimes_{kC_G(x)} k[G \times \overline{G}^{\times s}]$ is given by

$$1 \otimes_{kC_G(x)} g_{0,s} \longmapsto 1 \otimes_{kC_G(x)} \left((g_0 g_1, g_{2,s}) + \sum_{i=1}^{s-1}(-1)^i(g_0, g_{1,i-1}, g_i g_{i+1}, g_{i+2,s}) + (-1)^s g_{0,s-1}\right).$$

It is straightforward to check that the given maps commute with the above differentials. Passing to the homology, we have $H_*(\mathcal{H}_{x,*}) \simeq \mathrm{H}_*(C_G(x), k) = \mathrm{Tor}_*^{kC_G(x)}(k, k)$ since $\mathrm{Bar}_*(kG) \otimes_{kG} k$ is a projective resolution of k as $kC_G(x)$-modules. $\square$

Applying the functor $k \otimes_{kC_G(x)} -$ to the homotopy deformation retract in Remark 4.2 and then composing with the isomorphism in Lemma 4.5, we obtain the following homotopy deformation retract for any $x \in X$,

$$s_x \circlearrowright \quad \mathcal{H}_{x,*} \underset{\iota_x}{\overset{\rho_x}{\rightleftarrows}} C_*(C_G(x), k).$$

Here the injection ι_x is given by

$$\iota_x : C_*(C_G(x), k) \overset{\sim}{\longrightarrow} \mathcal{H}_{x,*},$$
$$[\widehat{\alpha}_x = (h_1, \cdots, h_n) \in k[\overline{C_G(x)}^{\times n}] \longmapsto [\alpha_x = (h_n^{-1}\cdots h_1^{-1}x, h_{1,n}) \in \mathcal{H}_{x,n}].$$

and the surjection ρ_x is given by

$$\rho_x : \mathcal{H}_{x,*} \to C_*(C_G(x), k)$$

$$[\alpha_x = (g_n^{-1}\cdots g_1^{-1}g_0 x g_0^{-1}, g_{1,n}) \in \mathcal{H}_{x,n}] \longmapsto [\widehat{\alpha}_x = (h_{i_1}, \cdots, h_{i_n}) \in k[\overline{C_G(x)}^{\times n}]]$$

where $h_{i_1}, \cdots, h_{i_n} \in \overline{C_G(x)}$ are determined by the following sequence:

$$g_0 = h\gamma_{i,x}, \quad \gamma_{i,x}g_1 = h_{i_1}\gamma_{s_i^1,x}, \quad \gamma_{s_i^1,x}g_2 = h_{i_2}\gamma_{s_i^2,x}, \quad \cdots, \quad \gamma_{s_i^{n-1},x}g_n = h_{i_n}\gamma_{s_i^n,x}.$$

The homotopy s_x is given as follows: For $\alpha_x = (g_n^{-1}\cdots g_1^{-1}g_0^{-1}xg_0, g_{1,n}) \in \mathcal{H}_{x,n}$,

$$s_x(\alpha_x) = \sum_{j=0}^{n}(-1)^j(g_n^{-1}\cdots g_1^{-1}g_0^{-1}xh, h_{i_1}, \cdots, h_{i_j}, \gamma_{s_i^j,x}, g_{j+1}, \cdots, g_n) \in \mathcal{H}_{x,n+1},$$

when $j = 0$, we set $\gamma_{s_i^0,x} = \gamma_{i,x}$.

Therefore, we get a lifting of the additive decomposition of $\mathrm{HH}_*(kG, kG)$ at the complex level.

Theorem 4.6. *Let k be a field and G a finite group. Consider the additive decomposition of Hochschild homology of the group algebra kG:*

$$\mathrm{HH}_*(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}_*(C_G(x), k)$$

where X is a set of representatives of conjugacy classes of elements of G and $C_G(x)$ is the centralizer of G. Then, the above additive decomposition lifts to a homotopy deformation retract of complexes

$$s_* \circlearrowleft \quad C_*(kG, kG) = \bigoplus_{x \in X} \mathcal{H}_{x,*} \xrightleftharpoons[\iota_*]{\rho_*} \bigoplus_{x \in X} C_*(C_G(x), k).$$

where $\iota_* = \sum_{x \in X} \iota_x$, $\rho_* = \sum_{x \in X} \rho_x$ and $s_* = \sum_{x \in X} s_x$.

The third case. Recall from Proposition 1.3 that we have the following exact sequence

$$0 \to \widehat{\mathrm{HH}}^{-1}(kG, kG) \to \mathrm{HH}_0(kG, kG) \xrightarrow{\tau} \mathrm{HH}^0(kG, kG) \to \widehat{\mathrm{HH}}^0(kG, kG) \to 0.$$

By Remark 1.4, the map τ is lifted to

$$\tau : C_0(kG, kG) = kG \to C^0(kG, kG) = kG, \quad h \mapsto \sum_{g \in G} g h g^{-1}.$$

From the above two cases, it follows that the additive decompositions:

$$\mathrm{HH}_0(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}_0(C_G(x), k), \quad \mathrm{HH}^0(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}^0(C_G(x), k)$$

are lifted to

$$(4) \qquad \begin{array}{rcl} kG = C_0(kG, kG) \simeq \bigoplus_{x \in X} k[C_x] & \to & \bigoplus_{x \in X} C_0(C_G(x), k) = \bigoplus_{x \in X} k_x, \\ x_i \in k[C_x] & \mapsto & 1_x \\ x & \hookleftarrow & 1_x \end{array}$$

and

$$(5) \qquad \begin{array}{rcl} kG = C^0(kG, kG) = \bigoplus_{x \in X} k[C_x] & \to & \bigoplus_{x \in X} C^0(C_G(x), k) = \bigoplus_{x \in X} k_x. \\ x \in k[C_x] & \mapsto & 1_x \in k_x \\ x_i \in k[C_x] & \mapsto & 0 \ \text{ if } x_i \neq x \\ \sum_{i=1}^{n_x} x_i & \hookleftarrow & 1_x \in k_x \end{array}$$

where k_x is the one-dimensional vector space indicated by x and 1_x the unit of k_x. For each $x \in X$, we also have an exact sequence

$$0 \to \widehat{\mathrm{H}}^{-1}(C_G(x), k) \to \mathrm{H}_0(C_G(x), k) \xrightarrow{\alpha_x} \mathrm{H}^0(C_G(x), k) \to \widehat{\mathrm{H}}^0(C_G(x), k) \to 0,$$

where the map α_x is lifted to the map

$$\alpha_x : C_0(C_G(x), k) = k \to C^0(C_G(x), k) = k, \quad 1 \mapsto |C_G(x)|.$$

Note that we have the following commutative diagram

$$\begin{array}{ccc} C_0(C_G(x), k) & \xrightarrow{\alpha_x} & C^0(C_G(x), k) \\ \downarrow & & \uparrow \\ C_0(kG, kG) & \xrightarrow{\tau} & C^0(kG, kG) \end{array}$$

where the vertical injections are defined in (4) and (5), respectively. This implies that the restriction of the trace map τ to $C_0(C_G(x), k)$ is α_x for any $x \in X$. Thus we have the additive decompositions

$$\widehat{\mathrm{HH}}^{-1}(kG, kG) \simeq \bigoplus_{x \in X} \widehat{\mathrm{H}}^{-1}(C_G(x), k), \quad \widehat{\mathrm{HH}}^0(kG, kG) \simeq \bigoplus_{x \in X} \widehat{\mathrm{H}}^0(C_G(x), k).$$

As a conclusion of the above three cases, we get the following additive decomposition

$$(6) \qquad \widehat{\mathrm{HH}}^*(kG, kG) \simeq \bigoplus_{x \in X} \widehat{\mathrm{H}}^*(C_G(x), k).$$

Since we note that the trace map τ restricts to $\tau_x : \mathcal{H}_{x,0} \to \mathcal{H}_x^0$ for any $x \in X$, we get a subcomplex:

$$\widehat{\mathcal{H}}_x^* : \quad \cdots \to \mathcal{H}_{x,1} \to \mathcal{H}_{x,0} \xrightarrow{\tau_x} \mathcal{H}_x^0 \to \mathcal{H}_x^1 \to \cdots$$

of $\mathcal{D}^*(kG, kG)$. It is clear that $\mathcal{D}^*(kG, kG) = \bigoplus_{x \in X} \widehat{\mathcal{H}}_x^*$ as complexes.

Remark 4.7. By Theorems 4.3 and 4.6, we obtain a homotopy deformation retract

$$\widehat{s} \, \circlearrowleft \quad \mathcal{D}^*(kG, kG) \underset{\widehat{\iota}}{\overset{\widehat{\rho}}{\rightleftarrows}} \bigoplus_{x \in X} \widehat{C}^*(C_G(x), k),$$

where, for $m \geq 0$, we have

$$\widehat{\iota}^m = \rho^m, \ \widehat{\iota}^{-m-1} = \iota_m; \quad \widehat{\rho}^m = \iota^m, \ \widehat{\rho}^{-m-1} = \rho_m; \quad \widehat{s}^m = s^m, \ \widehat{s}^{-m-1} = s_m.$$

This homotopy deformation retract should play a crucial role in the future study of the behavior of the higher algebraic structures on $\mathcal{D}^*(kG, kG)$ in terms of the additive decomposition.

Taking $x = 1$, we obtain a split inclusion of complexes $\widehat{\iota}_{x=1} : \widehat{C}^*(G, k) \hookrightarrow \mathcal{D}^*(kG, kG)$ given as follows:

$$\begin{aligned}
C^*(G, k) &\hookrightarrow C^*(kG, kG), \\
(\varphi : \overline{G}^{\times n} \longrightarrow k) &\longmapsto (\psi : \overline{G}^{\times n} \longrightarrow kG), \quad \psi(g_{1,n}) = \varphi(g_{1,n}) g_1 \cdots g_n; \\
C_*(G, k) &\longrightarrow C_*(kG, kG), \\
(g_{1,n}) &\longmapsto (g_n^{-1} \cdots g_1^{-1} x, g_{1,n}).
\end{aligned}$$

In particular, this induces an inclusion $\widehat{\iota}_{x=1} : \widehat{H}^*(G, k) \hookrightarrow \widehat{HH}^*(kG, kG)$.

We define a left kG-module $_ckG$ as follows. As a vector space $_ckG = kG$ and the action of G on kG is given by conjugation: $g \cdot x = gxg^{-1}$ for any $g \in G$ and $x \in kG$. Note that we have a kG-module decomposition $_ckG = \bigoplus_{x \in X} kC_x$, where C_x denotes the conjugacy class of x.

Proposition 4.8. *We have an isomorphism of complexes $\rho : \mathcal{D}^*(kG, kG) \to \widehat{C}^*(G, {}_ckG)$. As a result, we can present the isomorphisms in the additive decomposition as follows:*

$$\widehat{HH}^*(kG, kG) \simeq \widehat{H}^*(G, {}_ckG) \simeq \bigoplus_{x \in X} \widehat{H}^*(G, kC_x) \simeq \bigoplus_{x \in X} \widehat{H}^*(C_G(x), k).$$

Proof. (Compare to [22, Remark 6.2]) Let us construct the morphism of complexes $\rho : \mathcal{D}^*(kG, kG) \to \widehat{C}^*(G, {}_ckG)$ as follows. For $m \geq 0$ and $\phi \in \mathcal{D}^m(kG, kG) \simeq \mathrm{Map}(\overline{G}^{\times m}, kG)$, we define

$$\rho(\phi) \in \widehat{C}^m(G, {}_ckG) \simeq \mathrm{Map}(\overline{G}^{\times m}, kG), \quad (g_{1,m}) \mapsto \phi(g_{1,m}) g_m^{-1} \cdots g_1^{-1}.$$

In fact, for each $x \in X$, ρ restricts to an isomorphism $\rho_x : \mathcal{H}_x^m \xrightarrow{\simeq} \mathrm{Map}(\overline{G}^{\times m}, kC_x)$. Similarly, for $m \geq 0$ and $(h, g_{1,m}) \in \mathcal{D}^{-m-1}(kG, kG) \simeq k[G \times \overline{G}^{\times m}]$, we define

$$\rho((h, g_{1,m})) = (hg_1 \cdots g_m, g_{1,m}) \in \widehat{C}^{-m-1}(G, {}_ckG) \simeq k[G \times \overline{G}^{\times m}].$$

In fact, for each $x \in X$, ρ restricts to an isomorphism $\rho_x : \mathcal{H}_{x,m} \xrightarrow{\simeq} k[C_x \times \overline{G}^{\times m}]$. It is easy to check that ρ is a morphism of complexes. Note that ρ is an isomorphism with inverse ρ^{-1} given by

$$\rho^{-1}(\psi)(g_{1,m}) = \psi(g_{1,m}) g_1 \cdots g_m, \quad \text{for any } \psi \in \widehat{C}^m(G, {}_ckG);$$

$$\rho^{-1}((h, g_{1,m})) = (hg_m^{-1} \cdots g_1^{-1}, g_{1,m}), \quad \text{for any } (h, g_{1,m}) \in \widehat{C}^{-m-1}(G, {}_ckG)$$

for $m \geq 0$. Thus the isomorphism ρ induces the first two isomorphisms in this proposition. The third isomorphism in the proposition follows from the following isomorphisms of complexes

$$kC_x \otimes_{kG} \mathrm{Bar}_*(kG) \otimes_{kG} k \simeq k \otimes_{kC_G(x)} \mathrm{Bar}_*(kG) \otimes_{kG} k,$$

$$\mathrm{Hom}_{kG}(\mathrm{Bar}_*(kG) \otimes_{kG} k, kC_x) \simeq \mathrm{Hom}_{kC_G(x)}(\mathrm{Bar}_*(kG) \otimes_{kG} k, k).$$

We remark that the above two isomorphisms are induced by

$$\mathrm{Hom}_{kC_G(x)}(kG, k) \simeq kC_x \simeq k \otimes_{kC_G(x)} kG, \quad (\gamma_{j,x} \mapsto \delta_{i,j}) \leftarrow \gamma_{i,x}^{-1} x \gamma_{i,x} \mapsto 1 \otimes_{kC_G(x)} \gamma_{i,x}.$$

Here we fix a right coset decomposition of $C_G(x)$ in G: $G = C_G(x)\gamma_{1,x} \cup \cdots \cup C_G(x)\gamma_{n_x,x}$ and thus $C_x = \{\gamma_{1,x}^{-1} x \gamma_{1,x}, \cdots, \gamma_{n_x,x}^{-1} x \gamma_{n_x,x}\}$. This proves the proposition. $\square$

Remark 4.9. Note that the non-degenerate bilinear pairing on $\mathcal{D}^*(kG, kG)$ (cf. Remark 1.5) induces non-degenerate bilinear pairings (for each $n \in \mathbb{Z}$):

$$\langle \cdot, \cdot \rangle : \widehat{\mathcal{H}}_x^n \times \widehat{\mathcal{H}}_y^{-n-1} \to k \quad \text{for any } x, y \in X \text{ such that } x^{-1} \in C_y.$$

In particular, we have $\langle \alpha, \beta \rangle = 0$ for any $\alpha \in \widehat{\mathcal{H}}_x^*$ and $\beta \in \widehat{\mathcal{H}}_y^*$ $(x, y \in X)$ such that $x^{-1} \notin C_y$.

Recall that we have a cyclic A_∞-algebra structure $(\langle \cdot, \cdot \rangle, \partial', \cup, m_3, m_i = 0 \ (i > 3))$ on $\mathcal{D}^*(kG, kG)$ (see Section 2). Notice that the Tate cochain complex $\widehat{C}^*(G, k)$ can be seen as a subcomplex of $\mathcal{D}^*(kG, kG)$ under the inclusion $\widehat{\iota}_{x=1} : \widehat{C}^*(G, k) \hookrightarrow \mathcal{D}^*(kG, kG)$ (cf. Remark 4.7).

Theorem 4.10. *The Tate cochain complex $\widehat{C}^*(G, k)$ is a cyclic A_∞-subalgebra of $\mathcal{D}^*(kG, kG)$, and moreover $\widehat{\mathcal{H}}_x^*$ is an A_∞-module of $\widehat{C}^*(G, k)$ for each $x \in X$, under the decomposition $\mathcal{D}^*(kG, kG) = \bigoplus_{x \in X} \widehat{\mathcal{H}}_x^*$.*

Proof. By Proposition 4.8, we have an isomorphism of complexes

$$\widehat{\iota}_{x=1} = \rho^{-1}|_{x=1} : \widehat{C}^*(G, k) \xrightarrow{\simeq} \widehat{\mathcal{H}}_1^* \ (\subset \mathcal{D}^*(kG, kG)).$$

For any $\alpha \in \widehat{\mathcal{H}}_1^m$ and $\beta \in \widehat{\mathcal{H}}_x^n$ $(x \in X)$, by the definition of $\cup$ in Section 2, it is easy to check that

$$\alpha \cup \beta, \ \beta \cup \alpha \in \widehat{\mathcal{H}}_x^{m+n}.$$

This shows that $\cup$ on $\mathcal{D}^*(kG, kG)$ restricts to $\widehat{\mathcal{H}}_1^*$ and $\widehat{\mathcal{H}}_x^*$ is a module of $\widehat{C}^*(G, k)$. It remains to verify that $m_3(\alpha, \beta, \gamma) \in \widehat{\mathcal{H}}_x^*$ for $\alpha, \beta \in \widehat{\mathcal{H}}_1^*$ and $\gamma \in \widehat{\mathcal{H}}_x^*$. Recall from Section 2, we only need to consider the two nontrivial cases for m_3. The first case is as follows. Let $\phi \in \mathcal{H}_1^m, \alpha = (g_0, g_{1,r}) \in \mathcal{H}_{1,r}$ and $\varphi \in \mathcal{H}_x^n$ for $r + 2 \leq m + n$. We have

$$m_3(\phi, \alpha, \varphi)(h_1, \cdots, h_{m-r+n-2}) =$$

$$\sum_{g \in G} \sum_{j=1}^{\min\{m,n,r\}} (-1)^{m+r+j-1} \phi(h_{1,m-r+j-2}, g, g_{j,r}) g_0 \varphi(g_{1,j-1}, g^{-1}, h_{m-r+j-1, m-r+n-2}).$$

Since we have that $\phi(y_1, \cdots, y_m) \in k[y_1 y_2 \cdots y_m]$ for any $y_i \in G$ $(1 \leq i \leq m)$, $g_0 g_1 \cdots g_r = 1$, and $\varphi(y_1, \cdots, y_n) \in k[y_1 y_2 \cdots y_n C_x]$ for any $y_i \in G$ $(1 \leq i \leq n)$,

$$\phi(h_{1,m-r+j}, g, g_{j,r}) g_0 \varphi(g_{1,j-1}, g^{-1}, h_{m-r+j+1, m-r+n}) \in k[h_1 \cdots h_{m-r+n} C_x].$$

Thus $m_3(\phi, \alpha, \psi) \in \mathcal{H}_x^{m-r+n}$. The second case can be verified in a similar way. $\square$

Corollary 4.11. *Let G be a finite abelian group. Then we have an isomorphism*

$$\mathcal{D}^*(kG, kG) \simeq kG \otimes \widehat{C}^*(G, k)$$

as cyclic A_∞-algebras.

Remark 4.12. The cyclic A_∞-algebra structure $(\langle \cdot, \cdot \rangle', m_1', m_2', \cdots)$ on $kG \otimes \widehat{C}^*(G, k)$ is defined as follows.

$$\langle g_1 \otimes \alpha_1, g_2 \otimes \alpha_2 \rangle' = \langle g_1, g_2 \rangle \langle \alpha_1, \alpha_2 \rangle,$$

$$m_p'((g_1 \otimes \alpha_1), \cdots, (g_p \otimes \alpha_p)) = g_1 \cdots g_p \otimes m_p(\alpha_1, \cdots, \alpha_p)$$

for any $g_i \otimes \alpha_i \in kG \otimes \widehat{C}^*(G, k)$ $(i = 1, \cdots, p)$.

Proof of Corollary 4.11. From Proposition 4.8, we have an isomorphism $\rho : \mathcal{D}^*(kG, kG) \xrightarrow{\simeq} kG \otimes \widehat{C}^*(G, k)$, which induces an isomorphism $\rho_x : \widehat{\mathcal{H}}_x^* \xrightarrow{\simeq} kx \otimes \widehat{C}^*(G, k)$ for any $x \in G$. It is easy to check that ρ respects the cup product. Let us prove that ρ respects the product m_3 as well. We need to check the following two cases. For the first case, let $\phi \in \mathcal{H}_x^m$, $\psi \in \mathcal{H}_y^n$ and $\alpha = (g_0, g_1, \cdots, g_r) \in \mathcal{H}_{z,r}$ for $x, y, z \in G$ and $m + n \geq r$. By the above isomorphisms ρ_x $(x \in X)$, we have that $\phi = x\overline{\phi}, \ \psi = y\overline{\psi}$ for some $\overline{\phi} \in \mathcal{H}_1^m, \overline{\psi} \in \mathcal{H}_1^n$, and $g_0 \cdots g_r = z$. Thus

$$m_3(\phi, \alpha, \psi)(h_1, \cdots, h_{m-r+n}) =$$

$$\sum_{g \in G} \sum_{j=1}^{\min\{m,n,r\}} (-1)^{\kappa_{ij}} \phi(h_{1,m-r+j}, g, g_{j,r}) g_0 \psi(g_{1,j-1}, g^{-1}, h_{m-r+j+1, m-r+n}) \in k[h_1 \cdots h_{m-r+n} C_{xyz}].$$

This implies that $m_3(\phi, \alpha, \psi) \in \mathcal{H}_{xyz}^{m-r+n}$. For the second case, let $\alpha \in \mathcal{H}_{x,r}$, $\beta \in \mathcal{H}_{y,s}$ and $\phi \in \mathcal{H}_z^m$ for $m-1 \leq r+s$. By Equation (3), we have $\langle m_3(\alpha, \phi, \beta), \psi \rangle = \langle \alpha, m_3(\phi, \beta, \psi) \rangle$ for any $\psi \in \widehat{\mathcal{H}}_w^*$ ($w \in X$). It follows from Remark 4.9 that $m_3(\alpha, \phi, \beta) \in \widehat{\mathcal{H}}_{xyz}^*$ since we have $m_3(\phi, \beta, \psi) \in \widehat{\mathcal{H}}_{yzw}^*$ by the first case. This proves the corollary. $\qquad\square$

5. The cup product formula and a new proof via Green functors

In this section, we describe Nguyen's cup product formula for the Tate-Hochschild cohomology algebra $\widehat{\mathrm{HH}}^*(kG, kG)$ in terms of the additive decomposition and provide a new proof via Green functors, following Bouc.

5.1. **The cup product formula.** We shall state the cup product formula at the cohomology level analogous to the result of Siegel-Witherspoon [28]. In fact, this has been done by Nguyen in [25].

Let $X = \{g_1 = 1, g_2, \cdots, g_r\}$ be a complete set of representatives of conjugacy classes of elements of G and $H_i := C_G(g_i)$ is the centralizer subgroup of G. Let $\gamma_i : \widehat{\mathrm{H}}^*(H_i, k) \to \widehat{\mathrm{HH}}^*(kG, kG)$ be the split injection appearing in the additive decomposition in Proposition 4.8

$$\widehat{\mathrm{HH}}^*(kG, kG) \simeq \widehat{\mathrm{H}}^*(G, {}_ckG) \simeq \bigoplus_{g_i \in X} \widehat{\mathrm{H}}^*(H_i, k).$$

We now state the cup product formula in terms of the above additive decomposition of $\widehat{\mathrm{HH}}^*(kG, kG)$. For $i, j \in \{1, \cdots, r\}$, let D be a set of double coset representatives for $H_i \backslash G / H_j$. Recall that for each $x \in D$, there is a unique $k = k(x)$ such that $g_k = {}^yg_i{}^{yx}g_j$ for some $y \in G$.

Theorem 5.1. ([25, Theorem 5.5]) *Let* $\alpha \in \widehat{\mathrm{H}}^*(H_i, k)$, $\beta \in \widehat{\mathrm{H}}^*(H_j, k)$. *Then*

$$\gamma_i(\alpha) \cup \gamma_j(\beta) = \sum_{x \in D} \gamma_k(cor_W^{H_k}(res_W^{{}^yH_i}y^*(\alpha) \cup res_W^{{}^{yx}H_j}(yx)^*(\beta))),$$

where $W = W(x) = {}^{yx}H_j \bigcap {}^yH_i$.

Note that conjugation maps, restriction maps and corestriction maps appeared in the above formula have been recalled in Section 3.1. The cup product formula for Hochschild cohomology $\mathrm{HH}^*(kG, kG)$ was given by Siegel and Witherspoon in [28].

From the above cup product formula it is clear that the Tate cohomology algebra $\widehat{\mathrm{H}}^*(G, k)$ can be seen as a graded subalgebra of $\widehat{\mathrm{HH}}^*(kG, kG)$ and

$$\widehat{\mathrm{HH}}^*(kG, kG) \simeq \bigoplus_{x \in X} \widehat{\mathrm{H}}^*(C_G(x), k) = \widehat{\mathrm{H}}^*(G, k) \oplus (\bigoplus_{x \in X-\{1\}} \widehat{\mathrm{H}}^*(C_G(x), k))$$

is an isomorphism of graded $\widehat{\mathrm{H}}^*(G, k)$-modules (see also Theorem 4.10).

5.2. **Mackey functors.** There are at least three different (but equivalent) points of view for Mackey functors and Green functors. We shall recall two of them; for more details we refer the reader to [4, 5, 6]. We will see that the cup product formula from the previous section follows essentially from the equivalence between two definitions of Green functors.

Let k be a field and G be a finite group.

Definition 5.2. *A Mackey functor for* G *over* k *is given by the following data:*

For an arbitrary subgroup H *of* G, *we are given a* k-*module* $M(H)$ *and homomorphisms of* k-*modules*

$$t_H^K : M(H) \to M(K),$$
$$r_H^K : M(K) \to M(H),$$
$$c_{g,H} : M(H) \to M({}^gH),$$

for $H \leq K \leq G, g \in G$, *where* ${}^gH = gHg^{-1}$. *They satisfy the following conditions:*

(1) *(transitivity) If $H \subseteq K \subseteq L$ are subgroups of G, then*
$$t_K^L t_H^K = t_H^L, r_H^K r_K^L = r_H^L;$$

if $g, g' \in G, H \leq G$, then $c_{g', {}^g H} c_{g, H} = c_{g'g, H}$.

(2) *(compatibility) If $H \subseteq K$ are subgroups of G and $g \in G$, then*
$$c_{g, K} t_H^K = t_{{}^g H}^{{}^g K} c_{g, H},$$
$$c_{g, H} r_H^K = r_{{}^g H}^{{}^g K} c_{g, K}.$$

(3) *(triviality) If H is a subgroup of G, then*
$$t_H^H = r_H^H = \mathrm{Id}_{M(H)};$$

moreover, if $g \in H$, $c_{g, H} = \mathrm{Id}_{M(H)}$.

(4) *(Mackey axiom) If $H \subseteq K \supseteq L$ are subgroups of G, then*
$$r_H^K t_L^K = \sum_{u \in [H \backslash K / L]} t_{H \cap {}^u L}^H c_{u, H^u \cap L} r_{H^u \cap L}^L,$$

where $[H \backslash K / L]$ is a set of representatives of the double cosets of K modulo H and L and $H^u = u^{-1} H u$.

The maps t_H^K are called transfers or traces and the maps r_H^K are called restrictions.

The cohomology of finite groups is a Mackey functor. In fact, fix a finite group G, for $H \leq G$, define $M(H) = \mathrm{H}^*(H, k)$; for $H \leq K \leq G, g \in G$, the maps $t_H^K, r_H^K, c_{g, H}$ are respectively the maps cor_H^K, res_H^K, g^* recalled in Section 3.1. The Tate cohomology of finite groups is also a Mackey functor with similarly defined maps.

Let us introduce the second definition of Mackey functors and this definition uses the category G-set of finite left G-sets. Recall that a *bivariant functor* from G-set to the category k-Mod of k-modules is a pair of functors $(\tilde{M}_*, \tilde{M}^*)$ from G-set to k-Mod, where $\tilde{M}_*$ is covariant and $\tilde{M}^*$ is contravariant, such that the pair of functors coincide on objects. That is, for any G-set X, the two k-modules $\tilde{M}_*(X)$ and $\tilde{M}^*(X)$ are the same (denoted by $\tilde{M}(X)$).

Definition 5.3. *A Mackey functor for G over k is a bivariant functor $(\tilde{M}_*, \tilde{M}^*)$ from G-set to k-Mod satisfying the following conditions:*

(1) *(additivity) If X and Y are finite G-sets, i_X and i_Y are respectively the inclusion map of X and Y to the disjoint union $X \sqcup Y$, then the maps*
$$\tilde{M}(X) \oplus \tilde{M}(Y) \xrightarrow{(\tilde{M}_*(i_X), \tilde{M}_*(i_Y))} \tilde{M}(X \sqcup Y)$$
and
$$\tilde{M}(X \sqcup Y) \xrightarrow{\begin{pmatrix} \tilde{M}^*(i_X) \\ \tilde{M}^*(i_Y) \end{pmatrix}} \tilde{M}(X) \oplus \tilde{M}(Y)$$
are isomorphisms inverse one to each other.

(2) *(cartesian squares) If*
$$\begin{array}{ccc} X & \xrightarrow{a} & Y \\ \downarrow b & & \downarrow c \\ T & \xrightarrow{d} & Z \end{array}$$

is a pullback (or equivalently, cartesian square) of finite G-sets, then we have the equality
$$\tilde{M}_*(b) \tilde{M}^*(a) = \tilde{M}^*(d) \tilde{M}_*(c).$$

Notice that the property for cartesian squares in the second definition (i.e. Definition 5.2) implies the Mackey formula in the first definition (i.e. Definition 5.3).

Morphisms between Mackey functors are natural transformations between bivariant functors and compositions of morphisms are just compositions of natural transformations.

The equivalence between the two definitions can be explained as follows.

Give a Mackey functor M in the sense of the first definition, for a finite G-set X and each $x \in X$, write $G_x \leq G$ its stabilizer. Then define $\tilde{M}(X) = (\oplus_{x \in X} M(G_x))^G$, where the G-action on

$\oplus_{x \in X} M(G_x)$ is given by $g \cdot \alpha = c_{g,G_x}(\alpha) \in M(^g G_x) = M(G_{gx})$ for $x \in X, g \in G, \alpha \in M(G_x)$. It is not difficult to verify that $\tilde{M}$ is a Mackey functor in the sense of the second definition. Usually we take $[G \backslash X]$ a set of representatives of the orbits of G in X, and write $\tilde{M}(X) = \oplus_{x \in [G \backslash X]} M(G_x)$.

Conversely, given a Mackey functor $\tilde{M}$ in the sense of the second definition, then for a subgroup $H \leq G$, the set of left cosets G/H can be considered as a transitive G-set with left multiplication as G-action, then define $M(H) = \tilde{M}(G/H)$, and this is a Mackey functor in the sense of the first definition.

In the following, we will not distinguish between M and $\tilde{M}$, and write only M.

Return to the example of the Mackey functor given by Tate cohomology of finite groups. For a finite G-set X, define $M(X) = \widehat{\mathrm{H}}^*(G, k[X])$; for a morphism of G-sets $f : X \to Y$, the map $M_*(f) : M(X) \to M(Y)$ is the usual map $\widehat{\mathrm{H}}^*(G, k[X]) \to \widehat{\mathrm{H}}^*(G, k[Y])$, since Tate cohomology is covariant in the coefficients. Obviously for $H \leq G$, $M(G/H) = \widehat{\mathrm{H}}^*(G, k[G/H]) = \widehat{\mathrm{H}}^*(G, kG \otimes_{kH} k) \simeq \widehat{\mathrm{H}}^*(H, k)$.

Let Γ be a finite G-set. For a Mackey functor M for G over k, the *Dress construction* gives a new Mackey functor M_Γ defined as follows: for any G-set X, $M_\Gamma(X) = M(X \times \Gamma)$. It is not difficult to see that M_Γ is a Mackey functor ([4, 1.2]).

5.3. Green functors.

A Green functor A is a Mackey functor "with a compatible ring structure". There is also a definition of Green functors in terms of G-sets ([4, 2.2]).

Definition 5.4. *A Green functor for a finite group G over k is a Mackey functor A, such that for each subgroup $H \leq G$, $A(H)$ has a structure of k-algebra. We ask that the Mackey functor structure and the algebra structure satisfy the following conditions:*

(1) *If $H \subseteq K$ are subgroups of G, for arbitrary $g \in G$, the k-module homomorphisms r_H^K and $c_{g,H}$ are homomorphisms of k-algebras.*

(2) *(Frobenius identity) If $H \subseteq K$ are subgroups of G, for arbitrary elements $a \in A(H), b \in A(K)$, we have*
$$b \cdot (t_H^K a) = t_H^K((r_H^K b) \cdot a),$$
$$(t_H^K a) \cdot b = t_H^K(a \cdot (r_H^K b)).$$

Let A, B be two Green functors, and let $f : A \longrightarrow B$ be a morphism between Mackey functors. If for each subgroup $H \leq G$, the map $f_H : A(H) \longrightarrow B(H)$ is a homomorphism of k-algebras, then f is a morphism of Green functors.

Similarly, Green functors have a definition via G-sets.

Definition 5.5. *A Green functor over k is a Mackey functor A, such that for each pair of finite G-sets X, Y, there is a homomorphism of k-modules $A(X) \times A(Y) \to A(X \times Y), (a, b) \longrightarrow a \times b$ satisfying the following conditions:*

(1) *(bifunctoriality) If $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are homomorphisms of finite G-sets, then there exist the following commutative diagrams*

$$
\begin{array}{ccc}
A(X) \times A(Y) & \overset{\times}{\to} & A(X \times Y) \\
\downarrow{A_*(f) \times A_*(g)} & & \downarrow{A_*(f \times g)} \\
A(X') \times A(Y') & \overset{\times}{\to} & A(X' \times Y')
\end{array}
$$

and

$$
\begin{array}{ccc}
A(X) \times A(Y) & \overset{\times}{\to} & A(X \times Y) \\
\uparrow{A^*(f) \times A^*(g)} & & \uparrow{A^*(f \times g)} \\
A(X') \times A(Y') & \overset{\times}{\to} & A(X' \times Y')
\end{array}
$$

(2) *(associativity) If X, Y, Z are finite G-sets, we have the following commutative diagram:*

$$
\begin{array}{ccc}
A(X) \times A(Y) \times A(Z) & \overset{Id_{A(X)} \times (\times)}{\to} & A(X) \times A(Y \times Z) \\
\downarrow{(\times) \times Id_{A(Z)}} & & \downarrow{\times} \\
A(X \times Y) \times A(Z) & \overset{\times}{\to} & A(X \times Y \times Z)
\end{array}
$$

where $((X \times Y) \times Z) \simeq X \times Y \times Z \simeq X \times (Y \times Z)$.

(3) *Let • be the G-set with one element. Then there exists an element $\varepsilon_A \in A(\bullet)$ such that for each finite G-set X and arbitrary element $a \in A(X)$, we have*

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a),$$

where p_X (resp. q_X) is the projection from $X \times \bullet$ (resp. $\bullet \times X$) to X.

Let A, B be Green functors for the group G over k. Then a morphism between them is a morphism of Mackey functors $f : A \longrightarrow B$ such that for any finite G-sets X, Y, the following diagram is commutative:

$$\begin{array}{ccc} A(X) \times A(Y) & \stackrel{\times}{\to} & A(X \times Y) \\ \downarrow f_X \times f_Y & & \downarrow f_{X \times Y} \\ B(X) \times B(Y) & \stackrel{\times}{\to} & B(X \times Y). \end{array}$$

Let $_cG$ be the G-set G with conjugation action. Recall that a *crossed G-monoid* Γ is a G-monoid with a G-monoid map from Γ to $_cG$. Let A be a Green functor over k for G and Γ a crossed G-monoid. Then Bouc proved that the Dress construction A_Γ is a Green functor ([6, Theorem 5.1]). Notice that in this case, $A(\Gamma) = A_\Gamma(\bullet)$, the evaluation at the trivial G-set • of A_Γ, has a new k-algebra structure: for $a, b \in A(\Gamma)$, define their product $a \times_\Gamma b$ to be $A_*(\mu_\Gamma)(a \times b)$, where $\mu_\Gamma : \Gamma \times \Gamma \to \Gamma$ is the multiplication of the G-monoid Γ.

5.4. **A new proof of the cup product formula.** Now we explain how the result in [6] gives a quick proof of the main result of [25], that is, the cup product formula for Tate-Hochschild cohomology.

Let us recall Bouc's result. In the following statement, we write the action of G on Γ as $^g\gamma$.

Theorem 5.6. [6, Theorem 6.1 and Corollary 6.2] *Let A be a Green functor for G over k and Γ a crossed G-monoid. Then*

$$A_\Gamma(\bullet) = A(\Gamma) = \Big(\bigoplus_{\gamma \in \Gamma} A(G_\gamma) \Big)^G$$

and the γ-component of the product of $a, b \in A(\Gamma)$ is

$$(a \times_\Gamma b)_\gamma = \sum_{(\alpha, \beta) \in G_\gamma \backslash (\Gamma \times \Gamma), \alpha\beta = \gamma} t^{G_\gamma}_{G_{(\alpha,\beta)}} \big((r^{G_\alpha}_{G_{(\alpha,\beta)}} a_\alpha) \times (r^{G_\beta}_{G_{(\alpha,\beta)}} b_\beta) \big)$$

Taking a set of orbit representatives $[G \backslash \Gamma]$, there is an isomorphism of k-modules

$$A(\Gamma) \simeq \bigoplus_{\gamma \in [G \backslash \Gamma]} A(G_\gamma)$$

where $[G \backslash \Gamma]$ is a set of representatives of the orbits of G in Γ. With this notation, the product of $a \in A(G_\gamma)$ and $b \in A(G_\delta)$ is equal to

$$\bigoplus_{\epsilon \in [G \backslash \Gamma]} \bigoplus_{\omega \in [G_\gamma \backslash G / G_\delta]} t^{G_\epsilon}_{G_{g(\omega,\epsilon)_\gamma} \cap G_{g(\omega,\epsilon)\omega_\delta}} c_{g(\omega,\epsilon), G_\gamma \cap G_{\omega\delta}} \big(r^{G_\gamma}_{G_\gamma \cap G_{\omega\delta}} a \cdot r^{G_{\omega\delta}}_{G_\gamma \cap G_{\omega\delta}} b \big)$$

where $g(\omega, \epsilon) = g$ is an element of the unique class $G_\epsilon g$ in $G_\epsilon \backslash G$ such that $^g(\gamma^\omega\delta) = \epsilon$.

Let k be a field and G a finite group. Then G acts by conjugation on itself and denote by $_cG$ this G-set. Then it is well known that the Tate cohomology $A = \widehat{H}^*(G, k[?])$ sending a finite G-set X to $\widehat{H}^*(G, k[X])$ is a Green functor. Consider the crossed G-monoid $\Gamma = (_cG, u)$ where $u : _cG \to _cG$ is the trivial homomorphism of G-monoids sending each element to the unit in $_cG$. Then one verifies easily that for the Dress construction we get $A_\Gamma = \widehat{H}^*(G, k[? \times _cG])$. Remark that by [25, Section 4], $A(\Gamma) = A_\Gamma(\bullet) = \widehat{H}^*(G, k[_cG])$, together with the k-algebra structure $a \times_\Gamma b$ defined above, is isomorphic to the Tate-Hochschild cohomology ring $\widehat{HH}^*(kG, kG)$.

Nguyen [25] considered the additive decomposition for the Tate-Hochschild cohomology ring of a group algebra. Her proof is similar to that of [28]. Notice that Bouc's above result also applies to this situation and yields a new proof of [25, Theorem 5.5] just as is explained in [6, Page 421]. Let us explain the details.

As a G-set, the G-orbits in $_cG$ are just conjugacy classes. Let $X = \{g_1 = 1, g_2, \cdots, g_r\}$ be a complete set of representatives of conjugacy classes of elements of G and denote the centralizer subgroup $C_G(g_i)$ of G by H_i for $1 \leq i \leq r$. As G-sets, $_cG$ is isomorphic to $\coprod_{i=1}^r G/H_i$, and

$$A(G/H_i) = \widehat{H}^*(G, k[G/H_i]) = \widehat{H}^*(G, kG \otimes_{kH_i} k]) \simeq \widehat{H}^*(H_i).$$

So we have an isomorphism of graded vector spaces

$$\widehat{\mathrm{HH}}^*(kG) \simeq \widehat{H}^*(G, k[_cG]) \simeq \bigoplus_{i=1}^r \widehat{H}^*(H_i).$$

Fix $i, j \in \{1, \cdots, r\}$. Let D be a set of double coset representatives for $H_i \backslash G / H_j$. Recall that for each $x \in D$, there is a unique $k = k(x)$ such that $g_k = {}^y(g_i{}^x g_j)$ for some $y \in G$. Now Theorem 5.6 gives the cup product formula:

$$\begin{aligned}
\gamma_i(\alpha) \cup \gamma_j(\beta) &= \sum_{x \in D} \gamma_k(cor^{H_k}_{{}^yH_i \cap {}^{yx}H_j} y^* (res^{H_i}_{H_i \cap {}^xH_j} \alpha \cup res^{{}^xH_j}_{H_i \cap {}^xH_j}(\beta))) \\
&= \sum_{x \in D} \gamma_k(cor^{H_k}_W (res^{{}^yH_i}_W y^*(\alpha) \cup res^{{}^{yx}H_j}_W (yx)^*(\beta))),
\end{aligned}$$

where $W = W(x) = {}^{yx}H_j \bigcap {}^yH_i$. This is exactly [25, Theorem 5.5].

Remark 5.7. The above cup product formula on $\widehat{\mathrm{HH}}^*(kG, kG)$ can be lifted to a cup product formula at the complex level by the homotopy deformation retract in Remark 4.7[a]. Recall that the cup product formula in the nonnegative part $\mathcal{D}^{\geq 0}(kG, kG)$ in terms of the additive decomposition at the complex level was obtained in [22, Section 7]. Now, we describe the cup product formula in the negative part $\mathcal{D}^{<0}(kG, kG)$ as follows. Let X be the fixed set of representatives of conjugacy classes of elements of G. Recall that for any $z \in X$, we have fixed a right coset decomposition of $C_G(z)$ in G: $G = C_G(z)\gamma_{1,z} \cup \cdots \cup C_G(z)\gamma_{n_z,z}$. Let $\alpha_x := (g_{1,s}) \in k[\overline{C_G(x)}^{\times s}] = C_s(C_G(x), k)$ and $\alpha_y := (h_{1,t}) \in k[\overline{C_G(y)}^{\times t}] = C_t(C_G(y), k)$ for $x, y \in X$. By Theorem 4.6, we get that

$$\iota_*(\alpha_x) = (g_s^{-1} \cdots g_1^{-1} x, g_{1,s}) \in C_*(kG, kG);$$
$$\iota_*(\alpha_y) = (h_t^{-1} \cdots h_1^{-1} y, h_{1,t}) \in C_*(kG, kG).$$

This yields

$$\iota_*(\alpha_x) \cup \iota_*(\alpha_y) = \sum_{g \in G}(gh_t^{-1} \cdots h_1^{-1} y, h_{1,t}, g^{-1}g_s^{-1} \cdots g_1^{-1} x, g_{1,s}) \in C_{s+t+1}(kG, kG).$$

Therefore, we have the following cup product formula:

$$\rho_*(\iota_*(\alpha_x) \cup \iota_*(\alpha_y)) = \sum_{z \in X}(\alpha_x \cup \alpha_y)_z \in \bigoplus_{z \in X} C_{s+t+1}(C_G(z), k),$$

where for a fixed $z \in X$,

$$(\alpha_x \cup \alpha_y)_z = \sum_{g \in I_z}(k_{i_1}, \cdots, k_{i_{s+t+1}}) \in C_{s+t+1}(C_G(z), k)$$

where

$$I_z := \{g \in G | h_1 \cdots h_t g^{-1} x g h_t^{-1} \cdots h_1^{-1} y = \phi(g) z \phi(g)^{-1} \text{ for some } \phi(g) \in G\}$$

and $k_{i_1}, \cdots, k_{i_{s+t+1}} \in \overline{C_G(z)}$ are uniquely determined by the following equations

$$\phi(g) \in C_G(z)\gamma_{i,z}, \quad \gamma_{i,z}h_1 = k_{i_1}\gamma_{s_i^1,z}, \quad \gamma_{s_i^1,z}h_2 = k_{i_2}\gamma_{s_i^2,z}, \quad \cdots, \quad \gamma_{s_i^{t-1},z}h_t = k_{i_t}\gamma_{s_i^t,z},$$

$$\gamma_{s_i^t,z}g^{-1}g_s^{-1} \cdots g_1^{-1} x = k_{i_{t+1}}\gamma_{s_i^{t+1},z}, \quad \gamma_{s_i^{t+1},z}g_1 = k_{i_{t+2}}\gamma_{s_i^{t+2},z}, \quad \cdots, \quad \gamma_{s_i^{t+s},z}g_s = k_{i_{t+s+1}}\gamma_{s_i^{t+s+1},z}.$$

By Remark 4.9 and Equation (3), it is not difficult to obtain the cup product formula for the other cases between $\mathcal{D}^{<0}(kG, kG)$ and $\mathcal{D}^{\geq 0}(kG, kG)$. The details are left to the reader.

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[a]There should have an A_∞-product formula at the complex level using the Homotopy Transfer Theorem. Here we only consider the product formula for m_2. The higher product formulas will be explored in future research.

6. The $\widehat{\Delta}$-operator formula

We have defined the BV-operator $\widehat{\Delta}$ in $\widehat{\mathrm{HH}}^*(kG, kG)$ at the complex level (cf. Section 2). In this section, we determine the behavior of the operator $\widehat{\Delta}$ under the additive decomposition. There are three cases to be considered:

The first case: $\widehat{\Delta}$ in $\widehat{\mathrm{HH}}^{>0}(kG, kG)$. At the complex level, for any $n > 0$,

$$\widehat{\Delta} : \mathrm{Map}(\overline{G}^{\times n}, kG) \to \mathrm{Map}(\overline{G}^{\times (n-1)}, kG)$$

maps any $\alpha : \overline{G}^{\times n} \to kG$ to $\widehat{\Delta}(\alpha) : \overline{G}^{\times (n-1)} \to kG$ such that

$$\widehat{\Delta}(\alpha)(g_{1,n-1}) = \sum_{g_n \in G} \sum_{i=1}^{n} (-1)^{i(n-1)} \langle \alpha(g_{i,n}, g_{1,i-1}), 1 \rangle g_n^{-1}.$$

The second case: $\widehat{\Delta}$ in $\widehat{\mathrm{HH}}^{\leq -1}(kG, kG)$. At the complex level, for any $n \leq -1$ (let $s = -n-1 \geq 0$),

$$\widehat{\Delta} : k[G \times \overline{G}^{\times s}] \to k[G \times \overline{G}^{\times s+1}]$$

is given by

$$\alpha = (g_0, g_{1,s}) \mapsto \widehat{\Delta}(\alpha) = \sum_{i=0}^{s} (-1)^{is} (1, g_{i,s}, g_{0,i-1}).$$

The third case: $\widehat{\Delta} : \widehat{\mathrm{HH}}^0(kG, kG) \to \widehat{\mathrm{HH}}^{-1}(kG, kG)$. In this case, $\widehat{\Delta} : kG \to kG$ is zero.

Since the last case is trivial, we deal with the first two cases. In the first case, $\widehat{\Delta}$ is the BV-operator Δ in the Hochschild cohomology $\mathrm{HH}^*(kG, kG)$, and its behavior under the additive decomposition has been determined in [22]. Let us briefly recall the results there. As in Section 4, we fix a complete set X of representatives of the conjugacy classes in the finite group G. For $x \in X$, $C_x = \{gxg^{-1} | g \in G\}$ is the conjugacy class corresponding to x and $C_G(x) = \{g \in G | gxg^{-1} = x\}$ is the centralizer subgroup. For each $x \in X$, $\mathcal{H}_x^* = \bigoplus_{n \geq 0} \mathcal{H}_x^n$ is a subcomplex of the Hochschild cochain complex $C^*(kG, kG) = \mathcal{H}^*$, where

$$\mathcal{H}_x^n = \{\varphi : \overline{G}^{\times n} \longrightarrow kG | \varphi(g_1, \cdots, g_n) \in k[g_1 \cdots g_n C_x] \subset kG, \forall g_1, \cdots, g_n \in \overline{G}\}.$$

Lemma 6.1. ([22, Lemma 8.1]) *For any $x \in X$ and $n \geq 1$, the BV-operator $\Delta : \mathcal{H}^n \longrightarrow \mathcal{H}^{n-1}$ restricts to $\Delta_x : \mathcal{H}_x^n \longrightarrow \mathcal{H}_x^{n-1}$.*

We can define an operator $\widetilde{\Delta}_x$ by the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^n(\mathcal{H}_x^*) & \xrightarrow{\Delta_x} & \mathrm{H}^{n-1}(\mathcal{H}_x^*) \\ \wr \downarrow & & \wr \downarrow \\ \mathrm{H}^n(C_G(x), k) & \xrightarrow{\widetilde{\Delta}_x} & \mathrm{H}^{n-1}(C_G(x), k), \end{array}$$

where the vertical isomorphisms are given in Theorem 4.3.

Theorem 6.2. ([22, Theorem 8.2]) *Let $\widetilde{\Delta}_x : \mathrm{H}^n(C_G(x), k) \longrightarrow \mathrm{H}^{n-1}(C_G(x), k)$ be the map induced by the operator $\Delta : \mathrm{HH}^n(kG, kG) \longrightarrow \mathrm{HH}^{n-1}(kG, kG)$. Then, at the complex level, $\widetilde{\Delta}_x$ is defined as follows:*

$$\widetilde{\Delta}_x(\psi)(h_{1,n-1}) = \sum_{i=1}^{n} (-1)^{i(n-1)} \psi(h_{i,n-1}, h_{n-1}^{-1} \cdots h_1^{-1} x^{-1}, h_{1,i-1})$$

for $\psi : \overline{C_G(x)}^{\times n} \longrightarrow k$ and for $h_1, \cdots, h_{n-1} \in \overline{C_G(x)}$.

In the second case, $\widehat{\Delta}$ is the Connes' B-operator in the Hochschild homology $\mathrm{HH}_*(kG, kG)$. Recall that for each $x \in X$, $\mathcal{H}_{x,*} = \bigoplus_{s \geq 0} \mathcal{H}_{x,s}$ is a subcomplex of the Hochschild chain complex $C_*(kG, kG) = \mathcal{H}_*$, where

$$\mathcal{H}_{x,s} = k[(g_s^{-1} \cdots g_1^{-1} u, g_{1,s}) | u \in C_x, g_1, \cdots, g_s \in \overline{G}].$$

Lemma 6.3. *For $x \in X$ and $s \geq 0$, the operator $B : \mathcal{H}_s \longrightarrow \mathcal{H}_{s+1}$ restricts to $B_x : \mathcal{H}_{x,s} \longrightarrow \mathcal{H}_{x,s+1}$.*

Proof. We need to show that $B(\alpha) \in \mathcal{H}_{x,s+1}$ for each $\alpha = (g_s^{-1} \cdots g_1^{-1} g_0^{-1} x g_0, g_{1,s}) \in \mathcal{H}_{x,s}$, where $g_0 \in G$, $g_1, \cdots, g_s \in \overline{G}$. This follows from the definition of the operator B:

$$B(g_s^{-1} \cdots g_1^{-1} g_0^{-1} x g_0, g_{1,s}) = (1, g_s^{-1} \cdots g_1^{-1} g_0^{-1} x g_0, g_{1,s}) + \sum_{i=1}^{s} (-1)^{is}(1, g_{i,s}, g_s^{-1} \cdots g_1^{-1} g_0^{-1} x g_0, g_{1,i-1})$$

and for each $0 \le i \le s$,

$$g_{i-1}^{-1} \cdots g_1^{-1} (g_s^{-1} \cdots g_1^{-1} g_0^{-1} x g_0)^{-1} g_s^{-1} \cdots g_i^{-1} \cdot (g_0 \cdots g_{i-1})^{-1} x (g_0 \cdots g_{i-1}) = 1.$$

$\square$

We can define an operator $\widetilde{B}_x$ by the following commutative diagram

$$
\begin{array}{ccc}
H_s(\mathcal{H}_{x,*}) & \xrightarrow{B_x} & H_{s+1}(\mathcal{H}_{x,*}) \\
\wr \downarrow & & \wr \downarrow \\
\mathrm{H}_s(C_G(x), k) & \xrightarrow{\widetilde{B}_x} & \mathrm{H}_{s+1}(C_G(x), k),
\end{array}
$$

where the vertical isomorphisms are given in Theorem 4.6.

Theorem 6.4. *Let* $\widetilde{B}_x : \mathrm{H}_s(C_G(x), k) \longrightarrow \mathrm{H}_{s+1}(C_G(x), k)$ *be the map induced by the operator* $B : \mathrm{HH}_s(kG, kG) \longrightarrow \mathrm{HH}_{s+1}(kG, kG)$. *Then, at the complex level,* $\widetilde{B}_x$ *is defined as follows:*

$$\widetilde{B}_x(\gamma) = (1, h_s^{-1} \cdots h_1^{-1} x, h_{1,s}) + \sum_{i=1}^{s}(-1)^{is}(1, h_{i,s}, h_s^{-1} \cdots h_1^{-1} x, h_{1,i-1})$$

for $\gamma = (h_{1,s}) \in k[\overline{C_G(x)}^{\times s}]$.

Proof. By Theorem 4.6, this is straightforward by chasing the above commutative diagram. $\square$

Theorem 4.10 shows that the natural inclusion $\widehat{\iota}_{x=1} : \widehat{\mathrm{H}}^*(G, k) \hookrightarrow \widehat{\mathrm{HH}}^*(kG, kG)$ (cf. Remark 4.7) is an inclusion of graded algebras. We now further prove that it is an inclusion of BV-algebras.

Corollary 6.5. *Let* k *be a field and* G *a finite group. Then* $\widehat{\iota}_{x=1} : \widehat{\mathrm{H}}^*(G, k) \hookrightarrow \widehat{\mathrm{HH}}^*(kG, kG)$ *is an (unitary) embedding of BV-algebras.*

Proof. It follows from Theorem 4.10 that the inclusion is an embedding of graded algebras. Theorems 6.2 and 6.4 show that this inclusion preserves the operator $\widehat{\Delta}$. Since this operator together with the cup product $\cup$ generates the Lie bracket $[\cdot, \cdot]$ on $\widehat{\mathrm{HH}}^*(kG, kG)$, we deduce that the Lie bracket $[\cdot, \cdot]$ restricts to $\widehat{\mathrm{H}}^*(G, k) = \widehat{\mathrm{H}}^*(C_G(1), k)$. This proves the corollary. $\square$

Remark 6.6. We remark that $\widehat{\Delta}$ restricts to zero on $\widehat{\mathrm{H}}^*(G, k)$ due to the fact that the Connes' B-operator is trivial in the group homology $\mathrm{H}_*(G, k)$. In Appendix A, we shall provide a proof of this non-trivial result, which is known and only implicit in the literature.

Let G be a finite abelian group. By Corollary 4.11, we have an isomorphism of graded algebras

$$\widehat{\mathrm{HH}}^*(kG, kG) \simeq kG \otimes_k \widehat{\mathrm{H}}^*(G, k).$$

Since the Lie bracket on $\widehat{\mathrm{HH}}^*(kG, kG)$ is in general nontrivial (see e.g. [20, Corollary 4.2]) and the Lie bracket on $\widehat{\mathrm{H}}^*(G, k)$ is always trivial, the above isomorphism is not an isomorphism of BV-algebras.

Remark 6.7. Notice that the restrictions of $\widehat{\Delta}$ to other summands $\widehat{\mathrm{H}}^*(C_G(x), k)$ (where $x \neq 1$) in the additive decomposition are non-trivial in general. Notice also that although the $\widehat{\Delta}$-operator is trivial on the Tate cohomology $\widehat{\mathrm{H}}^*(G, k)$, it is not trivial at the complex level. We conjecture that the Tate-Hochschild cochain complex $\mathcal{D}^*(kG, kG)$ is a BV_∞-algebra and the Tate cochain complex $\widehat{C}^*(G, k)$ is a BV_∞ subalgebra. Equivalently, we conjecture that the operad of the frame little 2-discs acts on $\mathcal{D}^*(kG, kG)$ and this action restricts to the subcomplex $\widehat{C}^*(G, k)$.

Let us consider the stable Hochschild homology $\mathrm{HH}^{st}_*(kG, kG)$ which has been studied in [14] [23]. From Remark 1.5, we have that $\mathrm{HH}^{st}_m(kG, kG) \cong \widehat{\mathrm{HH}}^{-m-1}(kG, kG)$ for $m \geq 0$. Hence $\mathrm{HH}^{st}_*(kG, kG)$ is computed by the following truncated (at degree -1) complex of $\mathcal{D}^*(kG, kG)$,

$$\widetilde{C}_*(kG, kG): \quad \cdots \xrightarrow{\partial'_p} C_{p-1}(kG, kG) \xrightarrow{\partial'_{p-1}} \cdots \xrightarrow{\partial'_2} C_1(kG, kG) \xrightarrow{\partial'_1} \mathrm{Ker}(\tau) \to 0$$

where $\widetilde{C}_{-1}(kG, kG) = \mathrm{Ker}(\tau)$ and $\widetilde{C}_{-p-1}(kG, kG) = C_p(kG, kG)$ for $p > 0$; and we recall that $\partial'_p = (-1)^{-p-1}\partial_p$ (cf. Remark 2.2). Note that the restriction of the cup product $\cup$ to $\widetilde{C}_*(kG, kG)$ is strictly associative (since $m_3 = 0$ when restricted to $\mathcal{D}^{<0}(kG, kG)$) and compatible with the differential ∂'.

Remark 6.8. In general, the restriction of $\cup$ to the whole negative part $\mathcal{D}^{<0}(kG, kG) = C_*(kG, kG)$, is not compatible with ∂': For $g_0, h_0 \in D^{-1}(kG, kG) = C_0(kG, kG)$, we have that $g_0 \cup h_0 \in C_1(kG, kG)$ and

$$\partial'_1(g_0 \cup h_0) = \sum_{g \in G} \partial_1((gh_0, g^{-1}g_0)) = \sum_{g \in G}(gh_0 g^{-1}g_0 - g^{-1}g_0 gh_0),$$

which is not zero in general, but $\partial'_0(g_0) = 0 = \partial'_0(h_0)$ in $C_*(kG, kG)$. Hence $\cup$ is not well-defined on the whole $H^{-*-1}(\mathcal{D}^{<0}(kG, kG)) = \mathrm{HH}_*(kG, kG)$, but it is well-defined on the subspace $\mathrm{HH}^{st}_*(kG, kG) \subset \mathrm{HH}_*(kG, kG)$.

Analogously, let us denote by $\widetilde{C}_*(G, k)$ the truncated (at degree -1) complex of the Tate cochain complex $\widehat{C}^*(G, k)$. The cohomology of this complex is denoted by $\mathrm{H}^{st}_{-*-1}(G, k)$, namely $\mathrm{H}^{st}_m(G, k) = H^{-m-1}(\widetilde{C}_*(G, k))$ for $m \geq 0$. Then the homotopy deformation retract in Remark 4.7 induces the following additive decomposition

$$\mathrm{HH}^{st}_*(kG, kG) \simeq \bigoplus_{x \in X} \mathrm{H}^{st}_*(C_G(x), k).$$

As a consequence, we have the following result.

Theorem 6.9. *The stable Hochschild homology* $\mathrm{HH}^{st}_{-*-1}(kG, kG)$, *equipped with the Connes' B-operator and the cup product* $\cup$, *is a BV-algebra (without unit). Moreover,* $\mathrm{H}^{st}_{-*-1}(G, k)$ *is a BV subalgebra of* $\mathrm{HH}^{st}_{-*-1}(kG, kG)$.

Proof. This follows from Corollary 6.5. $\qquad \square$

Denote by BG the classifying space of a finite group G. There is a well-known isomorphism between the Hochschild homology $\mathrm{HH}_*(kG, kG)$ and the singular homology $\mathrm{H}_*(LBG, k)$ of the free loop space $LBG := \mathrm{Map}(S^1, BG)$ of BG (cf. [21, 7.3.13 Corollary]). Under this isomorphism, the Connes' B-operator on $\mathrm{HH}_*(kG, kG)$ corresponds to the S^1-action on $\mathrm{H}_*(LBG, k)$ (cf. [21]). We denote by $\mathrm{H}^{st}_*(LBG, k)$ the subspace of $\mathrm{H}_*(LBG, k)$ corresponding to $\mathrm{HH}^{st}_*(kG, kG)$ under the above isomorphism. Transferring the cup product on $\mathrm{HH}^{st}_*(kG, kG)$ to $\mathrm{H}^{st}_*(LBG, k)$, we obtain the following result.

Corollary 6.10. *Let G be a finite group and k be a field. Then* $\mathrm{H}^{st}_{-*-1}(LBG, k)$ *equipped with the S^1-action and the transferred product, is a BV-algebra (without unit).*

Proof. This follows from Theorem 6.9 and the above analysis. $\qquad \square$

Remark 6.11. Clearly, $\mathrm{H}^{st}_m(LBG, k) = \mathrm{H}_m(LBG, k)$ for $m > 0$ and $\mathrm{HH}^{st}_0(kG, kG) \cong \mathrm{H}^{st}_0(LBG, k) \subset \mathrm{H}_0(LBG, k)$. It would be interesting to give a topological construction of the transferred product on $\mathrm{H}^{st}_{-*-1}(LBG, k)$.

7. THE SYMMETRIC GROUP OF DEGREE 3

In this section, we use our results to compute the BV structure of the Tate-Hochschild cohomology for symmetric group of degree 3 over a field k of characteristic 3. For convenience, we write the BV-operator $\widehat{\Delta}$ in $\widehat{\mathrm{HH}}^*(kG, kG)$ as Δ in this section.

Recall that in a BV-algebra, there is the following equation (see [13]; here we have changed the original equation according to the sign convention in Remark 2.4 and we write $\delta\gamma$ instead of $\delta \cup \gamma$):

$$\Delta(\alpha\beta\gamma) = (-1)^{|\alpha||\beta||\gamma|}[(-1)^{|\gamma|}\Delta(\alpha\beta)\gamma + \alpha\Delta(\beta\gamma) + (-1)^{|\alpha||\beta|}\beta\Delta(\alpha\gamma)$$

$$-(-1)^{|\alpha|}\Delta(\alpha)\beta\gamma - (-1)^{|\alpha|+|\beta|-|\alpha||\gamma|}\alpha(\Delta(\beta))\gamma - (-1)^{|\alpha|+|\beta|+|\gamma|}\alpha\beta\Delta(\gamma)],$$

where α, β, γ are homogeneous elements. So in order to compute the Δ-operator in $\widehat{\mathrm{HH}}^*(kG, kG)$, it suffices to find the value of Δ on each generator and the value of Δ on the cup product of every two generators. Also recall that we can use the cup product formula, the Δ-operator formula and the following formulas to compute the Lie bracket in a BV-algebra:

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)|\beta|}(\Delta(\alpha \cup \beta) - \Delta(\alpha) \cup \beta - (-1)^{|\alpha|}\alpha \cup \Delta(\beta)),$$

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha].$$

Notice that the associative algebra structure of the positive part $\widehat{\mathrm{HH}}^{\geq 0}(kS_3, kS_3)$ has been determined by Siegel and Witherspoon in [28] and the associative algebra structure of the whole algebra $\widehat{\mathrm{HH}}^*(kG, kG)$ has been determined by Nguyen in [25]. Moreover, the Δ operator and the Lie bracket of the positive part $\widehat{\mathrm{HH}}^{\geq 0}(kS_3, kS_3)$ has been computed by the first and the third named authors in [22].

Let $G = S_3 = \langle a, b | a^3 = 1 = b^2, bab = a^{-1} \rangle$. Choose the conjugacy class representatives as $1, a, b$. The corresponding centralizers are $H_1 = G, H_2 = \langle a \rangle$ and $H_3 = \langle b \rangle$. So $\widehat{\mathrm{HH}}^*(kS_3) \simeq \widehat{H}^*(S_3) \oplus \widehat{H}^*(\langle a \rangle) \oplus \widehat{H}^*(\langle b \rangle)$. The algebra structures of $\widehat{H}^*(S_3)$, of $\widehat{H}^*(\langle a \rangle)$, and of $\widehat{H}^*(\langle b \rangle)$ are known (see [25]). $\widehat{H}^*(S_3)$ is of the form $k[x]/(x^2) \otimes_k k[z, z^{-1}]$, where x, z, z^{-1} are of degrees $3, 4, -4$, respectively, subject to the graded-commutative relations. $\mathrm{H}^*(\langle a \rangle)$ is of the form $k[w_1]/(w_1{}^2) \otimes_k k[w_2, w_2^{-1}]$, where w_1, w_2, w_2^{-1} are of degrees $1, 2, -2$, respectively, subject to the graded-commutative relations. $\widehat{H}^*(\langle b \rangle) = 0$, since $k\langle b \rangle$ is semisimple. Identify the elements x, z with their images under γ_1 in $\widehat{\mathrm{HH}}^*(kG, kG)$ and denote by $W_1, W_2, W_1^{-1}, W_2^{-1}$ the images of the elements (resp.) $w_1, w_2, w_1^{-1}, w_2^{-1}$ under γ_2, and put $E_i := \gamma_i(1)$ $(i = 1, 2)$ and $C := E_2 + E_1 = E_2 + 1$. Then Nguyen proved in [25] the following presentation for the Tate-Hochschild cohomology algebra $\widehat{\mathrm{HH}}^*(kG, kG)$: it is generated as an algebra by elements $x, z, z^{-1}, C, W_1, W_2$, and W_2^{-1} of degrees (resp.) $3, 4, -4, 0, 1, 2$, and -2, subject to the relations

$$xW_1 = 0, \quad xW_2 = zW_1, \quad z^{-1}W_1 = (xz^{-1})W_2^{-1},$$

$$C^2 = CW_2^{-1} = CW_i = 0 \quad (i = 1, 2),$$

$$W_2^2 = zC, \quad W_2^{-2} = z^{-1}C, \quad W_1W_2 = xC, \quad W_1W_2^{-1} = xz^{-1}C,$$

together with the graded commutative relations. Observe that although both w_2^3 and w_2^{-3} are nonzero in $\widehat{H}^*(\langle a \rangle)$, we have that $W_2^3 = W_2^2W_2 = zCW_2 = 0$ and $W_2^{-3} = W_2^{-2}W_2^{-1} = z^{-1}CW_2^{-1} = 0$ in $\widehat{\mathrm{HH}}^*(kG, kG)$. Moreover, $w_2w_2^{-1} = 1$ in $\widehat{H}^*(\langle a \rangle)$ but $W_2W_2^{-1} = C \neq 1$ in $\widehat{\mathrm{HH}}^*(kG, kG)$.

By Section 7, the operator $\Delta : \widehat{\mathrm{HH}}^n(kS_3) \longrightarrow \widehat{\mathrm{HH}}^{n-1}(kS_3)$ restricts to the operators $\widehat{\Delta}_b : \widehat{H}^n(\langle b \rangle) \longrightarrow \widehat{H}^{n-1}(\langle b \rangle)$, $\widehat{\Delta}_a : \widehat{H}^n(\langle a \rangle) \longrightarrow \widehat{H}^{n-1}(\langle a \rangle)$, and $\widehat{\Delta}_1 : \widehat{H}^n(S_3) \longrightarrow \widehat{H}^{n-1}(S_3)$. Both $\widehat{\Delta}_1$ and $\widehat{\Delta}_b$ are zero maps and we only need to consider $\widehat{\Delta}_a : \widehat{H}^n(\langle a \rangle) \longrightarrow \widehat{H}^{n-1}(\langle a \rangle)$. In [22], we have computed $\widehat{\Delta}_a$ for the positive part $\widehat{H}^{>0}(\langle a \rangle)$ up to degree 4: $\widehat{\Delta}_a(w_2^2) = 0$, $\widehat{\Delta}_a(w_1w_2) = -w_2$, $\widehat{\Delta}_a(w_2) = 0$, $\widehat{\Delta}_a(w_1) = -1$. By the duality mentioned in Remark 1.5, we get the values of $\widehat{\Delta}_a$ for the negative part $\widehat{H}^{<0}(\langle a \rangle)$ up to degree -4: $\widehat{\Delta}_a(w_1w_2^{-1}) = -w_2^{-1}$, $\widehat{\Delta}_a(w_2^{-1}) = 0$, $\widehat{\Delta}_a(w_1w_2^{-2}) = -w_2^{-2}$, $\widehat{\Delta}_a(w_2^{-2}) = 0$. Moreover, in degree 0, we have $\widehat{\Delta}_a(1) = 0$. From these results we can compute the values of Δ on the elements of degrees between 4 and -4 in $\widehat{\mathrm{HH}}^*(kS_3)$. For example, $\Delta(x) = 0$ since $x \in \widehat{H}^*(S_3)$ and $\widehat{\Delta}_1$ is trivial; $\Delta(W_1W_2) = -W_2$, the reason is as follows: under the additive decomposition, W_1W_2 corresponds to the element $x + w_1w_2$, $\widehat{\Delta}_1(x) = 0$, $\widehat{\Delta}_a(w_1w_2) = -w_2$; $\Delta(W_2^{-1}) = 0$, the reason is as follows: W_2^{-1} is an element of degree -2, under the additive decomposition, it corresponds to an element λw_2^{-1} with some $\lambda \in k$, but $\widehat{\Delta}_a(w_2^{-1}) = 0$; $\Delta(W_1) = -E_2 = 1 - C$ (here $1 = E_1$ denotes the unit element of $\widehat{\mathrm{HH}}^*(kS_3)$) since $\widehat{\Delta}_a(w_1) = -1$ (here 1 denotes the unit element of $\widehat{H}^*(\langle a \rangle)$); etc.

We now compute the Lie brackets. Since we have the following Poisson rule: $[\alpha \cup \beta, \gamma] = [\alpha, \gamma] \cup \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \cup [\beta, \gamma]$, it suffices to write down the Lie brackets between generators in $\widehat{\mathrm{HH}}^*(kS_3)$. There are 49 cases, we list them explicitly. In the following computations, we shall freely use the three formulas mentioned at the beginning of this section.

(1) $[x, x] = 0$. (2) $[x, z] = 0$. (3) $[z, x] = 0$. (4) $[x, z^{-1}] = 0$. (5) $[z^{-1}, x] = 0$.

The cases (1) to (5) can be seen from the facts that $\widehat{\Delta}_1$ and therefore the Lie brackets are trivial over $\widehat{\mathrm{H}}^*(S_3)$.

(6) $[x, C] = -(\Delta(xC) - \Delta(x)C + x\Delta(C)) = -\Delta(W_1W_2) = W_2$, since $\Delta(x) = \Delta(C) = 0$.

(7) $[C, x] = -[x, C] = -W_2$.

(8) $[x, W_1] = -(\Delta(xW_1) - \Delta(x)W_1 + x\Delta(W_1)) = -x(-E_2) = x(C - 1)$, since $xW_1 = 0, \Delta(x) = 0$ and $\Delta(W_1) = -E_2$.

(9) $[W_1, x] = -[x, W_1] = x(1 - C)$.

(10) $[x, W_2] = -(\Delta(xW_2) - \Delta(x)W_2 + x\Delta(W_2)) = -\Delta(xW_2) = 0$. In the last step, we use the fact that $\Delta(xW_2) = 0$. The reason is as follows: xW_2 is an element of degree 5, under the additive decomposition, it corresponds to an element in $\mathrm{H}^*(\langle a \rangle)$ and has the form $\lambda w_1 w_2{}^2$ with some $\lambda \in k$. Using the formula for Δ-operator it is easy to show that $\widehat{\Delta}_a(w_1 w_2{}^2) = 0$.

(11) $[W_2, x] = 0$.

(12) $[x, W_2^{-1}] = -(\Delta(xW_2^{-1}) - \Delta(x)W_2^{-1} + x\Delta(W_2^{-1})) = -\Delta(xW_2^{-1}) = -\Delta(W_1) = C - 1$, since $xW_2^{-1} = xzx^{-1}z^{-1}W_1 = zxx^{-1}z^{-1}W_1 = W_1$.

(13) $[W_2^{-1}, x] = -[x, W_2^{-1}] = 1 - C$.

(14) $[x, x] = 0$. (15) $[x, z] = 0$. (16) $[z, x] = 0$. (Since the Lie brackets are trivial over $\widehat{\mathrm{H}}^*(S_3)$.)

(17) $[z, C] = -(\Delta(zC) - \Delta(z)C - z\Delta(C)) = -\Delta(zC) = -\Delta(W_2^2) = 0$.

(18) $[C, z] = 0$.

(19) $[z, W_1] = \Delta(zW_1) - \Delta(z)W_1 - z\Delta(W_1) = \Delta(xW_2) - z(1 - C) = -z(1 - C) = z(C - 1)$, since in the above we have computed that $\Delta(xW_2) = 0$.

(20) $[W_1, z] = -[z, W_1] = z(1 - C)$.

(21) $[z, W_2] = -(\Delta(zW_2) - \Delta(z)W_2 - z\Delta(W_2)) = -\Delta(zW_2) = 0$. The reason for the last step is as follows: zW_2 is an element of degree 6, under the additive decomposition, it corresponds to an element in $\mathrm{H}^*(\langle a \rangle)$ and has the form $\lambda w_2{}^3$ with some $\lambda \in k$. Using the formula for Δ-operator it is easy to show that $\widehat{\Delta}_a(w_2{}^3) = 0$.

(22) $[W_2, z] = 0$.

(23) $[z, W_2^{-1}] = -(\Delta(zW_2^{-1}) - \Delta(z)W_2^{-1} - z\Delta(W_2^{-1})) = 0$. Notice that zW_2^{-1} is an element of degree 2, under the additive decomposition, it corresponds to an element in $\mathrm{H}^*(\langle a \rangle)$ and has the form λw_2 with some $\lambda \in k$. However, $\widehat{\Delta}_a(w_2) = 0$.

(24) $[W_2^{-1}, z] = 0$.

(25) $[z^{-1}, z^{-1}] = 0$.

(26) $[z^{-1}, C] = -(\Delta(z^{-1}C) - \Delta(z^{-1})C - z^{-1}\Delta(C)) = 0$. The reason is as follows: $z^{-1}C = W_2^{-2}$ is an element of degree -4, under the additive decomposition, it corresponds to an element $z^{-1} + \lambda w_2^{-2}$ with some $\lambda \in k$, $\widehat{\Delta}_1(z^{-1}) = 0, \widehat{\Delta}_a(w_2^{-2}) = 0$.

(27) $[C, z^{-1}] = 0$.

(28) $[z^{-1}, W_1] = \Delta(z^{-1}W_1) - \Delta(z^{-1})W_1 - z^{-1}\Delta(W_1) = -z^{-1}\Delta(W_1) = z^{-1}(C - 1)$. Notice that $\Delta(z^{-1}W_1) = 0$, since $z^{-1}W_1$ is an element of degree -3, under the additive decomposition, it corresponds to the element $w_1 w_2^{-2}$ and $\widehat{\Delta}_a(w_1 w_2^{-2}) = 0$.

(29) $[W_1, z^{-1}] = -[z^{-1}, W_1] = z^{-1}(1 - C)$.

(30) $[z^{-1}, W_2] = -(\Delta(z^{-1}W_2) - \Delta(z^{-1})W_2 - z^{-1}\Delta(W_2)) = -\Delta(z^{-1}W_2) = 0$. The reason for the last step is as follows: $z^{-1}W_2$ is an element of degree -2, under the additive decomposition, it corresponds to an element λw_2^{-1} with some $\lambda \in k$, but $\widehat{\Delta}_a(w_2^{-1}) = 0$.

(31) $[W_2, z^{-1}] = 0$.

(32) $[z^{-1}, W_2^{-1}] = -(\Delta(z^{-1}W_2^{-1}) - \Delta(z^{-1})W_2^{-1} - z^{-1}\Delta(W_2^{-1})) = -\Delta(z^{-1}W_2^{-1}) = 0$. The reason for the last step is as follows: $z^{-1}W_2^{-1}$ is an element of degree -6, under the additive decomposition, it corresponds to an element λw_2^{-3} with some $\lambda \in k$, but $\widehat{\Delta}_a(w_2^{-3}) = 0$.

(33) $[W_2, z^{-1}] = 0$.

(34) $[C, C] = -(\Delta(C^2) - \Delta(C)C - C\Delta(C)) = 0$.

(35) $[C, W_1] = -(\Delta(CW_1) - \Delta(C)W_1 - C\Delta(W_1)) = C\Delta(W_1) = C(1 - C) = C$.

(36) $[W_1, C] = [C, W_1] = -C$.

(37) $[C, W_2] = -(\Delta(CW_2) - \Delta(C)W_2 - C\Delta(W_2)) = 0$.

(38) $[W_2, C] = 0$.

(39) $[C, W_2^{-1}] = -(\Delta(CW_2^{-1}) - \Delta(C)W_2^{-1} - C\Delta(W_2^{-1})) = 0$.

(40) $[W_2^{-1}, C] = 0$.

(41) $[W_1, W_1] = -[W_1, W_1] = 0$.

(42) $[W_1, W_2] = -(\Delta(W_1W_2) - \Delta(W_1)W_2 + W_1\Delta(W_2)) = W_2 + (1 - C)W_2 = -W_2$.

(43) $[W_2, W_1] = -[W_1, W_2] = W_2$.

(44) $[W_1, W_2^{-1}] = -(\Delta(W_1W_2^{-1}) - \Delta(W_1)W_2^{-1} + W_1\Delta(W_2^{-1})) = -\Delta(z^{-1}x) + (1-C)W_2^{-1} = W_2^{-1}$.

(45) $[W_2^{-1}, W_1] = -[W_1, W_2^{-1}] = -W_2^{-1}$.

(46) $[W_2, W_2] = -(\Delta(W_2^2) - \Delta(W_2)W_2 - W_2\Delta(W_2)) = 0$. The reason that $\Delta(W_2^2) = 0$ is as follows: $W_2^2 = zC$ is an element of degree 4, under the additive decomposition, it corresponds to an element $z + \lambda w_2^2$ with some $\lambda \in k$, $\widehat{\Delta}_1(z) = 0$, $\widehat{\Delta}_a(w_2^2) = 0$.

(47) $[W_2, W_2^{-1}] = -(\Delta(W_2W_2^{-1}) - \Delta(W_2)W_2^{-1} - W_2\Delta(W_2^{-1})) = -\Delta(W_2W_2^{-1}) = 0$, since $W_2W_2^{-1}$ is an element of degree 0.

(48) $[W_2^{-1}, W_2] = 0$.

(49) $[W_2^{-1}, W_2^{-1}] = -(\Delta(W_2^{-2}) - \Delta(W_2^{-1})W_2^{-1} - W_2^{-1}\Delta(W_2^{-1})) = 0$. The reason that $\Delta(W_2^{-2}) = 0$ is as follows: $W_2^{-2} = z^{-1}C$ is an element of degree -4, under the additive decomposition, it corresponds to an element $z^{-1} + \lambda w_2^{-2}$ with some $\lambda \in k$, $\widehat{\Delta}_1(z^{-1}) = 0$, $\widehat{\Delta}_a(w_2^{-2}) = 0$.

Remark 7.1. In [22], our computations for Lie brackets contain some minor errors, which are caused by the same reason: The equality $\widehat{\Delta}_a(w_1) = -1$ on $\mathrm{H}^*(\langle a \rangle)$ should correspond to the equality $\Delta(X_1) = -E_2 = 1 - C_1$ in $\mathrm{HH}^*(kS_3)$, but we used $\Delta(X_1) = -1$ in [22]. We list all the corrections in [22] as follows:

$$[u, X_1] = u(C_1 - 1) = -[X_1, u], \quad [v, X_1] = v(1 - C_1) = -[X_1, v];$$

$$[C_1, X_1] = C_1(C_1 - 1) = -[X_1, C_1], \quad [C_2, X_1] = C_2(C_1 - 1) = -[X_1, C_2].$$

But the following phenomena in our examples is still true both in Hochschild cohomology and in Tate-Hochschild cohomology: The Lie bracket of any two generators in even degrees vanishes.

APPENDIX A. A PROOF OF $B = 0$ IN $\mathrm{H}_*(G, k)$

In this appendix, we denote kG by A and use the unnormalized bar resolution $\mathrm{Bar}_*(A)$ of A. Recall that the cyclic bicomplex $\{CC_{p,q}(A) \mid p, q \geq 0\}$ is the double complex (cf. [21, Section 2.1]):

$$
\begin{array}{ccccccc}
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} &
\end{array}
$$

where

- $CC_{p,q}(A) = A^{\otimes (q+1)}$,
- $b'(a_0 \otimes \cdots \otimes a_p) = \sum_{i=0}^{p-1}(-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p$,
- $b(a_0 \otimes \cdots \otimes a_p) = b'(a_0 \otimes \cdots \otimes a_p) + (-1)^p a_p a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1}$,
- $t_p(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p \otimes a_0 \otimes \cdots \otimes a_{p-1}$,
- $N_p = \sum_{i=0}^{p} t^i$, where t^i denotes the i-th power of the map t.

It is well known (cf., e.g., [32, Section 2.1]) that the cyclic homology $\mathrm{HC}_*(A)$ of A is defined as the homology of the total complex of $CC_{*,*}(A)$. Note that the odd columns of $CC_{*,*}(A)$ are exact, as

they are exactly $\mathrm{Bar}_*(A)[1]$ (the shift $[1]$ of the bar resolution $\mathrm{Bar}_*(A)$), and the even columns are the Hochschild chain complex $C_*(A, A)$. We define, for $p \geq 0$, a map $s : A^{\otimes(p+1)} \to A^{\otimes(p+2)}$ by

$$s(a_0 \otimes \cdots \otimes a_p) = 1 \otimes a_0 \otimes \cdots \otimes a_p.$$

Now the Connes' B-operator is defined to be $B = (1 - t)sN : A^{\otimes(p+1)} \to A^{\otimes(p+2)}$. Since it can be shown that $B^2 = Bb + bB = 0$, B induces a map $\mathrm{HH}_p(A) \to \mathrm{HH}_{p+1}(A)$, still denoted by B. Note that if we use the normalized Hochschild chain complex as in the main text of the present paper, then B is equal to sN at the complex level. Consider the following short exact sequence of double complexes:

$$0 \to CC_{<2,*}(A) \xrightarrow{I} CC_{*,*}(A) \xrightarrow{S} CC_{*,*}(A)[2] \to 0$$

where $CC_{<2,*}(A)$ is the double complex formed by the first two columns of $CC_{*,*}(A)$, I is the natural embedding, S is the quotient map, and where for the double complex $CC_{*,*}(A)[2]$,

$$(CC_{*,*}(A)[2])_{p,q} = CC_{p-2,q}(A).$$

By a standard fact (Killing contractible complexes) from homological algebra (see [21, 2.1.6 Lemma]), $C_{<2,*}(A)$ has a total complex which is quasi-isomorphic to the Hochschild chain complex $C_*(A, A)$. Thus we get a long exact sequence

$$\cdots \to \mathrm{HH}_n(A, A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A, A) \to \cdots.$$

Note that the composition of the maps (still denoted by B)

$$\mathrm{HH}_n(A, A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{B} \mathrm{HH}_{n+1}(A, A)$$

is exactly the Connes' B-operator (cf. [32, Exercise 9.8.2]).

Now we consider another double complex $\{C_{p,q}(G) \mid p, q \geq 0\}$:

$$
\begin{array}{ccccccc}
\downarrow \partial & & \downarrow -\partial' & & \downarrow \partial & & \\
A^{\otimes 2} & \xleftarrow{1-t'} & A^{\otimes 2} & \xleftarrow{N'} & A^{\otimes 2} & \xleftarrow{1-t'} & \\
\downarrow \partial & & \downarrow -\partial' & & \downarrow \partial & & \\
A & \xleftarrow{1-t'} & A & \xleftarrow{N'} & A & \xleftarrow{1-t'} & \\
\downarrow \partial & & \downarrow -\partial' & & \downarrow \partial & & \\
k & \xleftarrow{1-t'} & k & \xleftarrow{N} & k & \xleftarrow{1-t'} & ,
\end{array}
$$

where

- $C_{p,q}(G) = A^{\otimes q}$,
- $\partial'(g_1 \otimes \cdots \otimes g_n) = g_2 \otimes g_{3,n} + \sum_{i=1}^{n-1} (-1)^i g_{1,i-1} \otimes g_i g_{i+1} \otimes g_{i+2,n}$,
- $\partial(g_1 \otimes \cdots \otimes g_n) = \partial'(g_1 \otimes \cdots \otimes g_n) + (-1)^n g_{1,n-1}$,
- $t'_n(g_1 \otimes \cdots \otimes g_{n-1}) = (-1)^{n-1} g_{n-1}^{-1} \cdots g_1^{-1} \otimes g_1 \otimes \cdots \otimes g_{n-2}$,
- $N'_n = \sum_{i=1}^n t'^i$.

The odd columns of $C_{*,*}(G)$ are exact since they are isomorphic to $(k \otimes_{kG} \mathrm{Bar}_*(kG))[1]$, the projective resolution (shifted by $[1]$) of the trivial module k as right A-modules. The even columns are obtained by shift $[1]$ on the group homology chain complex $C_*(G, k)$. Here we adapt the notation from Karoubi [18] and denote by $\mathrm{HC}_*(G)$ the homology of the total complex of $C_{*,*}(G)$. Similarly, the double complex $C_{<2,*}(G)$ has a total complex whose n-th homology is naturally isomorphic to the group homology $\mathrm{H}_n(G, k)$. Observe that there is a split injection (which is clearly compatible with the additive decomposition map in Theorem 4.6):

$$
\begin{array}{rcl}
\iota : C_{*,*}(G) & \hookrightarrow & CC_{*,*}(A), \\
A^{\otimes n} & \longrightarrow & A \otimes A^{\otimes n}, \\
g_{1,n} & \longmapsto & (g_n^{-1} \cdots g_1^{-1}, g_{1,n}),
\end{array}
$$

whose retraction is given by

$$
\begin{array}{rcl}
r : CC_{*,*}(A) & \twoheadrightarrow & C_{*,*}(G), \\
A \otimes A^{\otimes n} & \longrightarrow & A^{\otimes n}, \\
g_{0,n} & \longmapsto & g_{1,n} \ \text{if} \ g_0 g_1 \cdots g_n = 1, \\
\text{or} \ g_{0,n} & \longmapsto & 0 \quad \text{otherwise.}
\end{array}
$$

Thus we have the following commutative diagram between long exact sequences:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathrm{HH}_n(A,A) & \xrightarrow{I} & \mathrm{HC}_n(A) & \xrightarrow{S} & \mathrm{HC}_{n-2}(A) & \xrightarrow{B} & \mathrm{HH}_{n-1}(A,A) & \longrightarrow \cdots \\
& \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow & \\
\cdots \longrightarrow & \mathrm{H}_n(G,k) & \xrightarrow{I} & \mathrm{HC}_n(G) & \xrightarrow{S} & \mathrm{HC}_{n-2}(G) & \xrightarrow{B} & \mathrm{H}_{n-1}(G,k) & \longrightarrow \cdots
\end{array}
$$

Note that the restriction of the Connes' B-operator to $\mathrm{H}_*(G,k)$ is the composition of maps

$$
\mathrm{H}_n(G,k) \xrightarrow{I} \mathrm{HC}_n(G) \xrightarrow{B} \mathrm{H}_{n+1}(G,k).
$$

So in order to prove that the restriction of the Connes' B-operator vanishes, it is sufficient to prove that $B : \mathrm{HC}_n(G) \to \mathrm{H}_{n+1}(G,k)$ vanishes for $n \geq 0$. The following result is due to Karoubi.

Theorem A.1 ([18]). *We have*

$$
\mathrm{HC}_n(G) \simeq \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathrm{H}_{n-2i}(G,k)
$$

and the map $S : \mathrm{HC}_n(G) \to \mathrm{HC}_{n-2}(G)$ *is a projection with kernel* $\mathrm{H}_n(G,k)$.

As a consequence, the long exact sequence in the second row of the above commutative diagram splits as short exact sequences

$$
0 \to \mathrm{H}_n(G,k) \xrightarrow{I} \mathrm{HC}_n(G) \xrightarrow{S} \mathrm{HC}_{n-2}(G) \to 0.
$$

In particular, the map $B : \mathrm{HC}_n(G) \to \mathrm{H}_{n+1}(G,k)$ is zero.

For the reader's convenience, we include a proof of Theorem A.1.

Proof of Theorem A.1. Note that $C_{*,*}(G) \simeq k \otimes_{kG} \widetilde{C}_{*,*}(G)$, where the double complex $\widetilde{C}_{*,*}(G)$ is defined as follows,

$$
\begin{array}{ccccccc}
\downarrow{\widetilde{b}} & & \downarrow{-b'} & & \downarrow{\widetilde{b}} & & \\
A^{\otimes 3} & \xleftarrow{1-\widetilde{t}} & A^{\otimes 3} & \xleftarrow{\widetilde{N}} & A^{\otimes 3} & \xleftarrow{1-\widetilde{t}} & \\
\downarrow{\widetilde{b}} & & \downarrow{-b'} & & \downarrow{\widetilde{b}} & & \\
A^{\otimes 2} & \xleftarrow{1-\widetilde{t}} & A^{\otimes 2} & \xleftarrow{\widetilde{N}} & A^{\otimes 2} & \xleftarrow{1-\widetilde{t}} & \\
\downarrow{\widetilde{b}} & & \downarrow{-b'} & & \downarrow{\widetilde{b}} & & \\
A & \xleftarrow{1-\widetilde{t}} & A & \xleftarrow{\widetilde{N}} & A & \xleftarrow{1-\widetilde{t}} &
\end{array}
$$

where

$$
\widetilde{t}_n(g_0 \otimes \cdots \otimes g_{n-1}) = (-1)^{n-1} g_0 \cdots g_{n-1} \otimes (g_1 \cdots g_{n-1})^{-1} \otimes g_1 \otimes \cdots \otimes g_{n-2}
$$

and $\widetilde{N}_n = \sum_{i=1}^{n} \widetilde{t}^i$; the odd columns are $\widetilde{\mathrm{Bar}}_*(A)[1]$; the even columns are the complex $\widetilde{\mathrm{Bar}}(A) \otimes_A k$ with the differential $\widetilde{b} = b' \otimes_A \mathrm{id}_k$, where $\widetilde{\mathrm{Bar}}(A)$ is the deleted bar resolution. Thus we have

$$
\mathrm{HC}_*(G) = H_*(k \otimes_{kG} \mathrm{Tot}(\widetilde{C}_{*,*}(G)) \simeq H_*(k \otimes_{kG}^{\mathbb{L}} \mathrm{Tot}(\widetilde{C}_{*,*}(G)).
$$

By a spectral sequence argument, we have that $\mathrm{Tot}(\widetilde{C}_{*,*}(G))$ is quasi-isomorphic, as complexes of kG-modules, to the following complex

$$
K_* : k \leftarrow 0 \leftarrow k \leftarrow 0 \leftarrow k \leftarrow \cdots .
$$

So we have a quasi-isomorphism

$$
k \otimes_{kG}^{\mathbb{L}} K_* \simeq k \otimes_{kG}^{\mathbb{L}} \mathrm{Tot}(\widetilde{C}_{*,*}(G)).
$$

Taking the following projective resolution $\widetilde{K}_{*,*}$ of K_*,

$$
\begin{array}{ccccc}
\downarrow{\scriptstyle\widetilde{b}} & & \downarrow & & \downarrow{\scriptstyle\widetilde{b}} \\
A^{\otimes 3} & \longleftarrow & 0 & \longleftarrow & A^{\otimes 3} \longleftarrow \\
\downarrow{\scriptstyle\widetilde{b}} & & \downarrow & & \downarrow{\scriptstyle\widetilde{b}} \\
A^{\otimes 2} & \longleftarrow & 0 & \longleftarrow & A^{\otimes 2} \longleftarrow \\
\downarrow{\scriptstyle\widetilde{b}} & & \downarrow & & \downarrow{\scriptstyle\widetilde{b}} \\
A & \longleftarrow & 0 & \longleftarrow & A \longleftarrow
\end{array}
$$

we get that

$$
\mathrm{HC}_n(G) \simeq H_n(k \otimes_{kG}^{\mathbb{L}} K_*) \simeq H_n(k \otimes_{kG}^{\mathbb{L}} \mathrm{Tot}(\widetilde{K}_{*,*})) \simeq H_n(k \otimes_{kG} \mathrm{Tot}(\widetilde{K}_{*,*})) \simeq \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathrm{H}_{n-2i}(G,k),
$$

where the forth isomorphism comes from the fact that $k \otimes_{kG} \widetilde{\mathrm{Bar}}_*(A) \otimes_{kG} k \simeq C_*(G,k)$. We have the following commutative diagram,

$$
\begin{array}{ccc}
\mathrm{HC}_n(G) & \xrightarrow{\quad S \quad} & \mathrm{HC}_{n-2}(G) \\
\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\
\bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathrm{H}_{n-2i}(G,k) & \xrightarrow{\quad \widetilde{S} \quad} & \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathrm{H}_{n-2-2i}(G,k)
\end{array}
$$

where $\widetilde{S}$ is induced by the identity morphisms from $\mathrm{H}_{n-2i}(G,k)$ to $\mathrm{H}_{n-2i}(G,k)$ for $i > 0$. So the kernel of S is $\mathrm{H}_n(G,k)$. $\qquad\square$

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