ON THE FIRST HOCHSCHILD COHOMOLOGY OF FINITE DIMENSIONAL QUIVER ALGEBRAS UNDER GLUING IDEMPOTENTS

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Abstract. We compare the Lie algebra structures of the first Hochschild cohomology groups of a quiver algebra A and a radical embedding B obtained by gluing two idempotents of A. Under a mild assumption, we show that the first Hochschild cohomology groups of A and B are either isomorphic as Lie algebras or they differ by a one-dimensional Lie ideal. In particular, in the case of stable equivalences obtained by gluing a source and a sink vertex, we prove that either the first Hochschild cohomology groups of A and B are isomorphic or $\mathrm{HH}^1(B)$ is a central extension of $\mathrm{HH}^1(A)$ by a one-dimensional ideal. As a consequence, we obtain a new invariant under stable equivalences induced by gluing a source and a sink. We also compare the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$, as well as the centers of A and B, when gluing two arbitrary idempotents.

1. Introduction

Let k be a field. Let A, B be two finite dimensional k-algebras and let $\mathrm{rad}(A)$, $\mathrm{rad}(B)$ be the Jacobson radicals of A and B, respectively. Let $\phi : B \to A$ be a radical embedding, that is, an algebra monomorphism such that $\phi(\mathrm{rad}(B)) = \mathrm{rad}(A)$. Radical embeddings frequently arise in the study of finite dimensional algebras and their representation theory, for example, in determining the finiteness of the finitistic dimension of algebras [7, 20]. If A is basic and k is algebraically closed, then by Xi's observation in [20, §3] we can assume that B is a subalgebra of A obtained by repeatedly gluing two idempotents of A. Therefore, the gluing of idempotents plays a pivotal role in the study of radical embeddings.

The gluing of idempotents is also essential in the study of stable equivalences. More precisely, Martinez-Villa proves in [16] that the gluing of a source and a sink induces an equivalence $\underline{\mathrm{mod}}A \xrightarrow{\sim} \underline{\mathrm{mod}}B$ between the stable module categories modulo projective modules, see also [11]. Conversely, let $\phi : B \to A$ be a radical embedding obtained by gluing two primitive idempotents. If A and B are stably equivalent and if the Auslander–Reiten conjecture holds, then B is obtained from A by gluing a source and a sink [11, Proposition 4.11]. For this reason, we are particularly interested in this type of gluings.

It is well known that Hochschild cohomology is not functorial, that is, an algebra homomorphism $\phi : B \to A$, does not give rise to a map from $\mathrm{HH}^*(A)$ to $\mathrm{HH}^*(B)$ or from $\mathrm{HH}^*(B)$ to $\mathrm{HH}^*(A)$. This makes Hochschild cohomology difficult to compute since it is not possible in general to reduce the study of Hochschild cohomology to smaller, and potentially easier, algebras. However, there are specific cases for which the functorial properties of Hochschild cohomology have been shown. For example, in the context of fully faithful embeddings of differential graded categories [10]. These arise, for example, for derived equivalences [10] or stable equivalences of Morita type [12] [2]. In particular, these results imply that the (restricted) Lie algebra structure of the first Hochschild cohomology $\mathrm{HH}^1(A)$ of an algebra A is an invariant under derived equivalences, and for self-injective algebras, under stable equivalences of Morita type.

In contrast to the situation for stable equivalences of Morita type, stable equivalences obtained by gluing idempotents are induced by bimodules that are only projective on one side [11]. Therefore,

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no specific invariants are known for these types of stable equivalences beyond those established for general stable equivalences, such as representation dimension [9] and representation type [13]. For this reason, a natural question to ask is if $\mathrm{HH}^1(A)$ is an *invariant under stable equivalences induced by gluing a source and a sink*. If this is not the case, then one could ask if $\mathrm{HH}^1(A)$ *still has some functoriality properties*, that is, if it is possible to define a (restricted) Lie algebra homomorphism between $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$. More generally, similar questions could be asked in the case of gluing of two arbitrary idempotents.

The main aim of this work is to address these questions. Let A and B be two finite dimensional quiver algebras such that B is obtained from A by gluing two arbitrary idempotents. By [17, 15], we can compute $\mathrm{HH}^1(A)$ as the quotient $\mathrm{Ker}(\delta_A^1)/\mathrm{Im}(\delta_A^0)$, where δ_A is the differential of a cochain complex $\mathcal{C}_{para}$ which can be described by the generalized parallel paths method. A similar computation applies to B, therefore we can use the complex $\mathcal{C}_{para}$ to compare the Lie algebra structures of $\mathrm{HH}^1(A)$ with $\mathrm{HH}^1(B)$. To make this comparison, we also define, for a fixed gluing of two idempotents, two subspaces $V_{sp} \subseteq \mathrm{Im}(\delta_B^0)$ and $V_{spp} \subseteq \mathrm{Ker}(\delta_B^1)$, see Definition 3.5 and Definition 3.12 for further details. When gluing a source and a sink, or equivalently, in the case of a stable equivalence, we have that $V_{spp} = V_{sp}$. This condition plays a pivotal role in our main theorems.

Theorem A (Theorem 3.21)**.** *Let A be a quiver algebra and let B be a radical embedding obtained by gluing two idempotents of A. Let* $\mathrm{char}(k)$ *be zero or big enough and assume $V_{spp} = V_{sp}$.*

(1) If we glue from two different blocks of A, then $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)$ as (restricted) Lie algebras.

(2) If we glue from the same block of A, then $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/\mathcal{I}$ as (restricted) Lie algebras, where $\mathcal{I}$ is a one-dimensional (restricted) Lie ideal of $\mathrm{HH}^1(B)$.

As a consequence we obtain:

Theorem B (Corollary 3.22)**.** *Let A be a quiver algebra and let B be a radical embedding obtained by gluing two idempotents of A. Let* $\mathrm{char}(k)$ *be zero or big enough and assume $V_{spp} = V_{sp}$. Then*

$$\mathrm{HH}^1(A)/\mathrm{rad}(\mathrm{HH}^1(A)) \simeq \mathrm{HH}^1(B)/\mathrm{rad}(\mathrm{HH}^1(B)).$$

In particular, for quiver algebras, we obtain a *new invariant* under stable equivalences induced by gluing a source and a sink. Theorem 3.19 addresses also the case $V_{spp} \neq V_{sp}$. In this setting, we show that $\mathcal{I}$ is not a Lie ideal and we give an exact commutative diagram which relates $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$. More general conditions for the validity of the above theorem can be found in Assumption 1. In the particular case of stable equivalences induced by idempotent gluing, we obtain the following result:

Theorem C (Theorem 3.25, Corollary 4.6)**.** *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing a source vertex and a sink vertex from the same block of A. Then the one-dimensional Lie ideal $\mathcal{I}$ lies in the center of $\mathrm{HH}^1(B)$ and $\mathrm{HH}^1(B)$ is a central extension of $\mathrm{HH}^1(A)$ by $\mathcal{I}$. In addition, if* $\mathrm{char}(k) = 0$ *and if A is a monomial algebra, then there is a Lie algebra isomorphism*

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \oplus \mathcal{I}.$$

Let c_A, c_B be the number of blocks of A, B, respectively. We also compare the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ when gluing of two arbitrary idempotents:

Theorem D (Theorem 3.17)**.** *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents of A. If* $\mathrm{char}(k)$ *is zero or big enough, then we have*

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1 - \dim_k V_{spp} + \dim_k V_{sp} + c_A - c_B.$$

In [5, Theorem 1] the authors give a formula to compute the dimension of $\mathrm{HH}^1(A)$ for a monomial algebra A which allows to give another interpretation for the dimension of V_{spp} for monomial algebras, see Remark 3.18 for further details. Furthermore, in Section 4.3 we give an interpretation of Sánchez-Flores' description of the first Hochschild cohomology for radical square zero algebras in terms of gluing operations.

Finally, we study the relation between a radical embedding $\phi : B \to A$ and the centers $Z(A), Z(B)$ of A and B, respectively.

Theorem E (Proposition 5.4, Proposition 5.7). *Let A be an indecomposable quiver k-algebra and let B be a radical embedding of A obtained by gluing two idempotents of A. Then there is an algebra monomorphism:*

- $Z(A) \hookrightarrow Z(B)$, *if we glue from the same block of A.*
- $Z(B) \hookrightarrow Z(A)$, *if we glue from different blocks of A.*

We also provide an explicit combinatorial formula to calculate the difference of the dimensions between $Z(A)$ and $Z(B)$.

Rather interestingly, the authors of this paper have obtained similar results for monomial algebras in the case of gluing arrows [14].

Outline. In Section 2, we introduce some notation that will be used throughout the paper and provide background on various topics. In Section 3 we prove Theorem A, Theorem B, Theorem D and first part of Theorem C. In Section 4.1 we prove the second part of Theorem C. In Section 4.2 we apply our main results to radical square zero algebras. In Section 4.3 we give an interpretation on Sánchez-Flores' description of the first Hochschild cohomology for radical square zero algebras [18] by inverse gluing operations. In Section 5 we prove Theorem E. In Section 6 we provide various examples to illustrate our definitions and results.

2. Preliminaries

2.1. Bound quivers.

All algebras considered are finite dimensional algebras which are isomorphic to kQ/I, where k is a field of arbitrary characteristic, Q is a finite quiver and I is an admissible ideal in the path algebra kQ. Any homomorphism between two algebras sends the identity element to the identity element. For all $n \in \mathbb{N}$, let Q_n be the set of paths of length n of Q and let $Q_{\geq n}$ be the set of paths of length greater than or equal to n. Note that Q_0 is the set of vertices and Q_1 is the set of arrows of Q. The number of vertices and arrows of Q is denoted by $|Q_0|$ and $|Q_1|$, respectively. We denote by $s(\gamma)$ the source vertex of an (oriented) path γ of Q and by $t(\gamma)$ its terminal vertex. The *path algebra* kQ is the k-linear span of the set of paths of Q, where the multiplication of $\beta \in Q_i$ and $\alpha \in Q_j$ is provided by the concatenation $\beta\alpha \in Q_{i+j}$ if $t(\alpha) = s(\beta)$ and 0 otherwise. We denote by $l(p)$ the length of a path p. A path p of length $l \geq 1$ is an *oriented cycle* (or an *oriented l-cycle*) if $s(p) = t(p)$. An oriented 1-cycle is called a *loop*. Two paths ϵ, γ of Q are called *parallel* if $s(\epsilon) = s(\gamma)$ and $t(\epsilon) = t(\gamma)$, denoted by $\epsilon /\!/ \gamma$. If ϵ and γ are not parallel, we denote by $\epsilon \not\!/\!\!/ \gamma$. If X, Y are sets of paths of Q, we denote by $X/\!/Y$ the set of parallel paths consisting of the couples $\epsilon/\!/\gamma$ with $\epsilon \in X$ and $\gamma \in Y$, and denote by $k(X/\!/Y)$ the k-vector space with basis $X/\!/Y$. An element in kQ is called *uniform* if it is a linear combination of parallel paths.

We fix a finite dimensional k-algebra $A = kQ_A/I_A$, where I_A is an admissible ideal in kQ_A. Denote the vertices of Q_A by $e_1, \cdots, e_n$. A vertex e_i is *isolated* if it does not exist any arrow α such that $s(\alpha) = e_i$ or $t(\alpha) = e_i$. A *source vertex* e_i of Q_A is a vertex such that there is no arrow α with $t(\alpha) = e_i$. A *sink vertex* e_j of Q_A is a vertex such that there is no arrow α with $s(\alpha) = e_j$. By abuse of notation, we denote by $e_1, \cdots, e_n$ the corresponding primitive orthogonal idempotents in the algebra A. For a path p in Q_A, we use the same notation to denote its image $\overline{p} = p + I_A$ in A. If $A = A_1 \times \cdots \times A_s$ is a decomposition of A into a product of indecomposable algebras, then A_i's are called *blocks* of A. Note that such a decomposition of A is unique and if $s = 1$, then A

is an indecomposable algebra. We denote by c_A the number of blocks of A which is also equal to number of connected components of the Gabriel quiver Q_A of A.

2.2. **Gröbner basis theory for quiver algebras.**
Let $A = kQ/I$ be a quiver algebra such that the ideal I is contained in $kQ_{\geq 2}$. We briefly recall the Gröbner basis (or Gröbner-Shirshov basis) theory for the ideal I. Recall that $\prec$ is a *well-order* on the k-basis $Q_{\geq 0}$ of the path algebra kQ if $\prec$ is a total order on the k-basis $Q_{\geq 0}$ and every nonempty subset of the k-basis $Q_{\geq 0}$ has a minimal element. First, we fix an admissible well-order $\prec$ on the k-basis $Q_{\geq 0}$ of the path algebra kQ, that is, a well-order on $Q_{\geq 0}$ which is compatible with multiplication. More precisely,

Definition 2.1. ([6, Section 2.2.]) Let kQ be a path algebra with k-basis $Q_{\geq 0}$. We call a well-order $\prec$ on $Q_{\geq 0}$ *admissible* if the following three conditions are satisfied for $p, q, r, s \in Q_{\geq 0}$:

- if $p \prec q$, then $pr \prec qr$ for both $pr \neq 0$ and $qr \neq 0$;
- if $p \prec q$, then $sp \prec sq$ for both $sp \neq 0$ and $sq \neq 0$;
- if $p = qr$, then $p \succeq q$ and $p \succeq r$.

For each path algebra, the *left length-lexicographic order* provides an admissible well-order (cf. [15, Example 2.1]). Unless otherwise specified, we will always use the left length-lexicographic orders in the present paper. Let $r = \sum_{p \in Q_{\geq 0}, \lambda_p \in k} \lambda_p p$ be a k-linear combination of paths and $\mathrm{Supp}(r) = \{\text{path } p \text{ in } r \mid \lambda_p \neq 0\}$. The *tip* of r, denoted by $\mathrm{Tip}(r)$, is the maximal monomial appearing with nonzero coefficient in r. In other words, $\mathrm{Tip}(r) = p$ if $p \in \mathrm{Supp}(r)$ and $\tilde{p} \preceq p$ for all $\tilde{p} \in \mathrm{Supp}(r)$. Moreover, we write $\mathrm{CTip}(r)$ as the coefficient of the tip of r. For a subset X of kQ, we denote by $\mathrm{Tip}(X) = \{\mathrm{Tip}(r) \mid r \in X, r \neq 0\}$ and put $\mathrm{NonTip}(X) := Q_{\geq 0} \setminus \mathrm{Tip}(X)$.

Let $A = kQ/I$ be a quiver algebra. By [6] there is a k-vector space decomposition

$$kQ = I \oplus \mathrm{Span}_k(\mathrm{NonTip}(I)).$$

So $\mathcal{B} := \mathrm{NonTip}(I)$ (modulo I) gives a "monomial" k-basis of the quiver algebra $A = kQ/I$. Let $b_1, b_2 \in kQ$. Then we say that b_1 *divides* b_2, and we denote $b_1 | b_2$, if there are elements $c, d \in kQ$ such that $b_2 = cb_1d$. If b_1 does not divide b_2 we write $b_1 \nmid b_2$. We can give now the definition of a Gröbner basis:

Definition 2.2. ([6, Definition 2.4]) Using the above notation, we say that a subset $\mathcal{G}$ of uniform elements in I is a *Gröbner basis* for the ideal I with respect to the order $\prec$ if

$$\langle \mathrm{Tip}(\mathcal{G}) \rangle = \langle \mathrm{Tip}(I) \rangle,$$

that is, $\mathrm{Tip}(\mathcal{G})$ and $\mathrm{Tip}(I)$ generate the same ideal in kQ.

Note that in this case $I = \langle \mathcal{G} \rangle$. We will see in the next theorem that there is a criterion in [6], called the *Termination Theorem*, to judge whether a set of generators of an ideal I in kQ is a Gröbner basis. Such criterion is based on the overlap relations.

Definition 2.3. ([6, Definition 2.7]) Let kQ be a path algebra, $\prec$ an admissible order on $Q_{\geq 0}$ and $f, g \in kQ$. Suppose $b, c \in Q_{\geq 0}$, such that

- $\mathrm{Tip}(f)c = b\mathrm{Tip}(g)$,
- $\mathrm{Tip}(f) \nmid b$ and $\mathrm{Tip}(g) \nmid c$.

Then the *overlap relation* of f and g by b, c is

$$o(f, g, b, c) = (C\mathrm{Tip}(f))^{-1} \cdot fc - (C\mathrm{Tip}(g))^{-1} \cdot bg.$$

It is clear that $\mathrm{Tip}(o(f, g, b, c)) \prec \mathrm{Tip}(f)c = b\mathrm{Tip}(g)$. We can describe now the Termination Theorem.

Theorem 2.4. ([6, Theorem 2.3]) *Let kQ be a path algebra, $\prec$ an admissible order on $Q_{\geq 0}$ and $\mathcal{G}$ a set of uniform elements of kQ. Suppose for every overlap relation, we have*

$$o(g_1, g_2, p, q) \Rightarrow_{\mathcal{G}} 0,$$

that is, $o(g_1, g_2, p, q)$ can be divided by $\mathrm{Tip}(\mathcal{G})$, with $g_1, g_2 \in \mathcal{G}$ and $p, q \in Q_{\geq 0}$. Then $\mathcal{G}$ is a Gröbner basis of the ideal $\langle \mathcal{G} \rangle$ generated by $\mathcal{G}$.

For the definition of divisibility of $o(g_1, g_2, p, q)$ by $\mathrm{Tip}(\mathcal{G})$, see §2.3.2 and Definition 2.6 in [6]. In general, a Gröbner basis for an ideal I in kQ is not unique. However, we can get a unique one, called the reduced Gröbner basis, if we still require some additional conditions.

Definition 2.5. ([6, Definition 2.5 and Proposition 2.6]) A Gröbner basis $\mathcal{G}$ for the ideal I is *reduced* if the following three conditions are satisfied:

- $\mathcal{G}$ is tip-reduced: $\mathrm{Tip}(g) \nmid \mathrm{Tip}(h)$, for any $g \neq h \in \mathcal{G}$;
- $\mathcal{G}$ is monic: $\mathrm{CTip}(g) = 1$, for any $g \in \mathcal{G}$;
- $g - \mathrm{Tip}(g) \in \mathrm{Span}_k(\mathrm{NonTip}(I))$, for any $g \in \mathcal{G}$.

It is easy to see, under a given admissible order, that I has a unique reduced Gröbner basis $\mathcal{G}$, and in this case $\mathrm{Tip}(\mathcal{G})$ is a minimal generator set of $\langle \mathrm{Tip}(I) \rangle$. We always assume that $\mathcal{G}$ is a reduced Gröbner basis of I in the sequel.

We also recall the following lemma, which will be useful in Section 3.

Lemma 2.6. ([15, Lemma 3.10]) *Let $A = kQ/I$ be a finite dimensional quiver algebra with $\mathcal{G}$ a reduced Gröbner basis for I. If α is a loop in Q, then $\alpha^m \in \mathrm{Tip}(\mathcal{G})$ and $\alpha^{m-1} \in \mathrm{NonTip}(\mathcal{G})$ for some integer $m \geq 2$.*

2.3. **Hochschild cohomology of quiver algebras.**

Let $A = kQ_A/I_A$ be a finite dimensional quiver algebra, where I_A is an admissible ideal in kQ_A. The *Hochschild cohomology*

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^e}(A, A)$$

of the k-algebra A can be computed using different projective resolutions of A over its *enveloping algebra* $A^e := A \otimes_k A^{op}$. The zero-th Hochschild cohomology group $\mathrm{HH}^0(A)$ is identified with the center $Z(A)$ of the algebra A. In particular, $Z(A)$ is a commutative subalgebra of A. The first Hochschild cohomology $\mathrm{HH}^1(A)$ is the quotient of the space of derivations $\mathrm{Der}(A)$ by the space of inner derivations $\mathrm{Inn}(A)$. It is well-known that $\mathrm{Der}(A)$ is a Lie algebra under the Lie bracket $[f, g] = f \circ g - g \circ f$, where $f, g \in \mathrm{Der}(A)$. In addition, $\mathrm{Inn}(A)$ is a Lie ideal of $\mathrm{Der}(A)$, therefore $\mathrm{HH}^1(A)$ has a Lie algebra structure. If the field k has positive characteristic p, then $\mathrm{HH}^1(A)$ is a *restricted Lie algebra*, that is, it is a Lie algebra endowed with a map called *p-power map* that satisfies some compatibility properties with respect to the Lie algebra structure. For further background on restricted Lie algebras see for example [8, Chapter 2]. The p-power map of a derivation f is defined by composing f with itself p-times. The inner derivations form a restricted Lie ideal of space of derivations, therefore $\mathrm{HH}^1(A)$ is a restricted Lie algebra.

In order to compute the first Hochschild cohomology group, one can use the following truncated projective resolution $\mathcal{P}_{min}$ (which is minimal on the degrees 0 and 1) of the A-bimodule A given by Bardzell in [1, Proposition 2.1] (see also Chouhy and Solotar [5]. For a proof using the algebraic Morse theory, see [15, Lemma 3.6].):

$$A \otimes_E k(\mathrm{Tip}(\mathcal{G})) \otimes_E A \xrightarrow{d_1} A \otimes_E kQ_1 \otimes_E A \xrightarrow{d_0} A \otimes_E kQ_0 \otimes_E A \xrightarrow{\mu} A \longrightarrow 0,$$

where $E \simeq kQ_0$ is the separable subalgebra of A and the A-bimodule morphisms are given by

$$\mu(a \otimes_E e_i \otimes_E b) = ae_ib,$$
$$d_0(a \otimes_E \alpha \otimes_E b) = a\alpha \otimes_E s(\alpha) \otimes_E b - a \otimes_E t(\alpha) \otimes_E \alpha b \text{ and}$$
$$d_1(a \otimes_E \operatorname{Tip}(g) \otimes_E b) = \sum_{p=\alpha_n\cdots\alpha_1 \in \operatorname{Supp}(g)} c_g(p) \sum_{i=1}^{n} a\alpha_n \cdots \alpha_{i+1} \otimes_E \alpha_i \otimes_E \alpha_{i-1} \cdots \alpha_1 b$$

for all $a, b \in A, e_i \in Q_0, \alpha, \alpha_n, \cdots, \alpha_1 \in Q_1$ and $g \in \mathcal{G}$ (with the convention $\alpha_{n+1} = t(\alpha_n)$ and $\alpha_0 = s(\alpha_1)$). Applying the contravariant functor $\operatorname{Hom}_{A^e}(-, A)$ to $\mathcal{P}_{min}$ we obtain the following cochain complex $\mathcal{C}_{min}$ (cf. [19, Section 2] in the monomial case):

$$0 \longrightarrow \operatorname{Hom}_{E^e}(kQ_0, A) \xrightarrow{d_0{}^*} \operatorname{Hom}_{E^e}(kQ_1, A) \xrightarrow{d_1{}^*} \operatorname{Hom}_{E^e}(k(\operatorname{Tip}(\mathcal{G})), A),$$

where the differentials are given by

$$(d_0{}^* f)(\alpha) = \alpha f(s(\alpha)) - f(t(\alpha))\alpha,$$
$$(d_1{}^* h)(\operatorname{Tip}(g)) = \sum_{p=\alpha_n\cdots\alpha_1 \in \operatorname{Supp}(g)} c_g(p) \sum_{i=1}^{n} \alpha_n \cdots \alpha_{i+1} h(\alpha_i) \alpha_{i-1} \cdots \alpha_1,$$

where $f \in \operatorname{Hom}_{E^e}(kQ_0, A)$, $\alpha, \alpha_n, \cdots, \alpha_1 \in Q_1$, $h \in \operatorname{Hom}_{E^e}(kQ_1, A)$ and $g \in \mathcal{G}$. In particular, we have $\operatorname{HH}^1(A) \simeq \operatorname{Ker}(d_1{}^*)/\operatorname{Im}(d_0{}^*)$ as k-vector spaces. Similar to [19, Proposition 2.8, Corollary 2.9], we have that $\operatorname{Ker}(d_1{}^*)$ is isomorphic, as a Lie algebra, to the space E^e-derivations of A and $\operatorname{Im}(d_0{}^*)$ is a Lie ideal of $\operatorname{Ker}(d_1{}^*)$ isomorphic to the space of the inner E^e-derivations of A.

By carrying out the identification $k(X/\!/Y) \simeq \operatorname{Hom}_{E^e}(kX, kY)$ in [19, Lemma 2.3], where X and Y are two finite subsets of paths of Q_A, we can rewrite the above cochain complex which gives a more practical way of computing HH^1.

Proposition 2.7. ([15, Proposition 3.7]) *Let $A = kQ_A/I_A$ be a quiver algebra. Let $\mathcal{G}$ be a reduced Gröbner basis of I_A, and denote by $\mathcal{B}$ the k-basis of A given by $\operatorname{NonTip}(I)$ (modulo I). By the above mentioned identifications, the cochain complex $\mathcal{C}_{min}$ is naturally isomorphic to the following complex*

$$\mathcal{C}_{para}: \qquad 0 \longrightarrow k(Q_0/\!/\mathcal{B}) \xrightarrow{\delta^0} k(Q_1/\!/\mathcal{B}) \xrightarrow{\delta^1} k(\operatorname{Tip}(\mathcal{G})/\!/\mathcal{B}) \xrightarrow{\delta^2} \cdots,$$

where the differentials are given by

$$\delta^0 : k(Q_0/\!/\mathcal{B}) \to k(Q_1/\!/\mathcal{B})$$
$$e/\!/\gamma \mapsto \sum_{a \in Q_1 e, a\gamma \in \mathcal{B}} a/\!/a\gamma - \sum_{a \in eQ_1, \gamma a \in \mathcal{B}} a/\!/\gamma a,$$
$$\delta^1 : k(Q_1/\!/\mathcal{B}) \to k(\operatorname{Tip}(\mathcal{G})/\!/\mathcal{B})$$
$$a/\!/\gamma \mapsto \sum_{r \in \mathcal{G}, p \in \operatorname{Supp}(r)} c_r(p) \operatorname{Tip}(r)/\!/p^{a/\!/\gamma},$$

where $r = \sum_{p \in \operatorname{Supp}(r)} c_r(p) p$ with $c_r(p) \in k$ and where $p^{a/\!/\gamma}$ denotes the sum of all paths in $\mathcal{B}$ obtained by replacing each appearance of the arrow a in p by the path γ. In particular, we have $\operatorname{HH}^0(A) \simeq \operatorname{Ker}(\delta^0)$ and $\operatorname{HH}^1(A) \simeq \operatorname{Ker}(\delta^1)/\operatorname{Im}(\delta^0)$ as k-vector spaces.

The isomorphism $\operatorname{HH}^1(A) \simeq \operatorname{Ker}(\delta^1)/\operatorname{Im}(\delta^0)$ in Proposition 2.7 is induced by the following map: send each f in $\operatorname{Hom}_{E^e}(kQ_1, k\mathcal{B})$ to the element $\sum_{a/\!/\gamma \in Q_1/\!/\mathcal{B}} \lambda_{a,\gamma}(a/\!/\gamma)$ in $k(Q_1/\!/\mathcal{B})$, where $f(a) = \sum_{\gamma \in \mathcal{B}} \lambda_{a,\gamma}\gamma$. Moreover, the inverse of the above isomorphism is induced by sending an element $a/\!/\gamma$ in $k(Q_1/\!/\mathcal{B})$ to f in $\operatorname{Hom}_{E^e}(kQ_1, k\mathcal{B})$ with $f(a) = \gamma$ and $f(b) = 0$ for $a \neq b \in Q_1$.

The method of computing HH^1 using parallel paths was first given by Strametz for monomial algebras in [19]. In [17, Section 2.2] and in [15, Section 3.2], this was generalized to arbitrary quiver algebras and called the *generalized parallel paths method* in [15]. Moreover, Theorem 3.8 in [15] shows that the second isomorphism in Proposition 2.7 is an isomorphism as Lie algebras.

Theorem 2.8. *The bracket*
$$[a/\!/\gamma, b/\!/\eta] = b/\!/\eta^{a/\!/\gamma} - a/\!/\gamma^{b/\!/\eta}$$
for all $a/\!/\gamma, b/\!/\eta \in Q_1/\!/\mathcal{B}$ induces a Lie algebra structure on $\mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$ *such that* $\mathrm{HH}^1(A)$ *and* $\mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$ *are isomorphic as Lie algebras.*

For quiver algebras, it is easy to describe the p-power map using the chain map from $\mathcal{C}_{min}$ to $\mathcal{C}_{para}$ and its inverse chain map. For example, for $p = 3$, the p-power map of $a/\!/\gamma$ is $(a/\!/\gamma^{a/\!/\gamma})^{a/\!/\gamma}$. We note that several of results in this paper, such as Proposition 3.10 and Corollary 3.23, can be readily extended from the context of 'Lie' algebras to 'restricted Lie' algebras.

Remark 2.9. The center $Z(A)$ of A is naturally isomorphic to $\mathrm{Ker}(\delta^0)$. For an explicit map between $\mathrm{Ker}(\delta^0)$ and $Z(A)$, see the proof of Proposition 5.4.

2.4. **Gluing of two idempotents and radical embedding subalgebra.**
Let $A = kQ_A/I_A$ be a finite dimensional quiver algebra, where I_A is an admissible ideal in kQ_A. Since each radical embedding reduces to a gluing of two idempotents, from now on we are going to consider B to be a radical embedding which is obtained by gluing only two idempotents of A. More precisely, let $e_1, \dots, e_n$ be a complete set of primitive orthogonal idempotents in A. Let B be a subalgebra of A obtained by gluing two idempotents e_1 and e_n of A. In other words, B is identified as a subalgebra of A generated by $f_1 := e_1 + e_n, f_2 := e_2, \cdots, f_{n-1} := e_{n-1}$ and all arrows in Q_A. Note that $\dim_k B = \dim_k A - 1$. Note also that the choice of idempotents to glue is arbitrary; however, we prefer to fix the notation such that $f_1 := e_1 + e_n$. We denote by Z_{new} the set of all newly formed paths of length 2 of the form $\cdot \to f_1 \to \cdot$.

Lemma 2.10. *Let $A = kQ_A/I_A$ be a finite dimensional quiver algebra and let B be a subalgebra of A obtained by gluing two idempotents e_1 and e_n of A. Then $B \simeq kQ_B/I_B$, where Q_B is the quiver obtained from Q_A by identifying the vertices e_1 and e_n, and I_B is an admissible ideal of kQ_B generated by the elements in $I_A \cup Z_{new}$. In particular, Q_B is the Gabriel quiver of B.*

Proof. Let B' be the algebra of the form kQ_B/I_B. Then there is an algebra monomorphism from B' to A by sending f_1 to $e_1 + e_n$, f_i to e_i for $2 \le i \le n-1$ and each arrow in Q_B to the same arrow in Q_A. It is clear that this map factors through the inclusion $B \hookrightarrow A$, which gives rise to another algebra monomorphism from B' to B. Moreover, since B' has dimension $\dim_k A - 1$, it must be isomorphic to B. □

Note that there is an obvious bijection between the arrows of A and the arrows of B. For each arrow α in Q_A, we denote the corresponding arrow in Q_B by α'. We define the quiver morphism
$$\varphi : Q_A \to Q_B$$
as follows: let $\varphi(e_i) = f_i$ for $2 \le i \le n-1$, let $\varphi(e_1) = \varphi(e_n) = f_1$, and let $\varphi(\alpha) = \alpha'$. By extending the map $\varphi : Q_A \to Q_B$, we define $\varphi_n : (Q_A)_n \to (Q_B)_n$. More precisely, let $p = a_n \dots a_1$ be a path in $(Q_A)_n$. Then $\varphi_n(p) = p' = a'_n \dots a'_1$.

The following proposition shows how a Gröbner basis behaves under gluing of two idempotents.

Proposition 2.11. *Let $A = kQ_A/I_A$ be a finite dimensional quiver algebra and let B be a subalgebra of A obtained by gluing two idempotents e_1 and e_n of A. Let $\mathcal{G}_A$ be a reduced Gröbner basis of I_A under some left length-lexicographic order on $(Q_A)_{\ge 0}$. Consider a left length-lexicographic order on $(Q_B)_{\ge 0}$ defined as follows: order the vertices f_i $(1 \le i \le n-1)$ arbitrarily and let $\alpha' \prec \beta'$ if $\alpha \prec \beta$ for $\alpha, \beta \in (Q_A)_1$. Identify $\mathcal{G}_A$ in Q_A with $\varphi(\mathcal{G}_A)$ in Q_B, and similarly for* $\mathrm{Tip}(\mathcal{G}_A)$*. Then*
$$\mathcal{G}_B := \mathcal{G}_A \cup Z_{new}$$

is a reduced Gröbner basis of I_B under the above left length-lexicographic order on $(Q_B)_{\geq 0}$. In particular,

$$\mathrm{Tip}(\mathcal{G}_B) = \mathrm{Tip}(\mathcal{G}_A) \cup Z_{new}.$$

Proof. First we show that $\mathcal{G}_B := \mathcal{G}_A \cup Z_{new}$ is a Gröbner basis of I_B. Since $\mathcal{G}_A$ is a reduced Gröbner basis, for each $g \in \mathcal{G}_B$, we have that $\mathrm{CTip}(g) = 1$. By Theorem 2.4 and Lemma 2.10, it suffices to show that

$$o(g_1, g_2, p, q) = g_1 q - p g_2 \Rightarrow_{\mathcal{G}_B} 0$$

for $g_1, g_2 \in \mathcal{G}_B$ and $p, q \in (Q_B)_{\geq 0}$. The proof is divided into four cases.

Case 1: Let $g_1, g_2 \in Z_{new}$. Suppose $g_1 = a_2' a_1'$ and $g_2 = b_2' b_1'$ with $a_i', b_i' \in (Q_B)_1$ for $i = 1, 2$. Then $\mathrm{Tip}(g_1)q = p\mathrm{Tip}(g_2)$ is equivalent to $g_1 q = p g_2$. It follows that $o(g_1, g_2, p, q) = g_1 q - p g_2 = 0$.

Case 2: Let $g_1 \in \mathcal{G}_A$ and $g_2 \in Z_{new}$. Then the condition $\mathrm{Tip}(g_1)q = p\mathrm{Tip}(g_2)$ implies that

$$\begin{aligned} o(g_1, g_2, p, q) &= g_1 q - p g_2 \\ &= (g_1 - \mathrm{Tip}(g_1))q - p(g_2 - \mathrm{Tip}(g_2)) \\ &= (g_1 - \mathrm{Tip}(g_1))q \\ &= (\sum_i \lambda_i p_i) q, \end{aligned}$$

where $p_i \in \mathrm{NonTip}(I_A)$ and $\lambda_i \in k$. The last two equalities follow from the facts that $g_2 \in Z_{new}$ whence $g_2 = \mathrm{Tip}(g_2)$ and $g_1 - \mathrm{Tip}(g_1) \in \mathrm{Span}_k(\mathrm{NonTip}(I_A))$. We claim that $q \in (Q_B)_1$, that is, the length $l(q)$ of q equals 1. Indeed, if $l(q) = 0$, then $\mathrm{Tip}(g_1) = p g_2$ should have a preimage in Q_A, which is absurd since $g_2 \in Z_{new}$. Therefore, we have $l(q) \geq 1$. Moreover, $l(q) < 2$, otherwise $p g_2 = \mathrm{Tip}(g_1) q$ and $l(g_2) = 2$ yield that $\mathrm{Tip}(g_1) \mid p$, a contradiction.

Assume that $g_2 = a_2' a_1'$ with $a_1', a_2' \in (Q_B)_1$ and $t(a_1) \neq s(a_2)$. As a consequence, we have $q = a_1'$ and $\mathrm{Tip}(g_1) = p a_2'$ since $p g_2 = \mathrm{Tip}(g_1) q$. It follows that all summands of g_1 are starting from $s(a_2)$, so does for $\mathrm{Tip}(g_1)$. Therefore each $p_i a_1'$ has a subpath in Z_{new} and we have $o(g_1, g_2, p, q) = (\sum_i \lambda_i p_i) q = (\sum_i \lambda_i p_i) a_1' \Rightarrow_{Z_{new}} 0$.

Case 3: Let $g_1 \in Z_{new}$ and $g_2 \in \mathcal{G}_A$. The proof is similar to that of Case 2.

Case 4: Let $g_1, g_2 \in \mathcal{G}_A$. If $l(p) = 0$, then $\mathrm{Tip}(g_1) q = \mathrm{Tip}(g_2)$ which yields $\mathrm{Tip}(g_1) \mid \mathrm{Tip}(g_2)$. Since $\mathcal{G}_A$ is reduced, we have $g_1 = g_2$ and $l(q) = 0$. Consequently, $o(g_1, g_2, p, q) = 0$. If $l(p) > 0$ such that p has a subpath in Z_{new}, then the conditions $\mathrm{Tip}(g_1) q = p\mathrm{Tip}(g_2)$ and $\mathrm{Tip}(g_1) \nmid p$ imply that p is a proper subpath of $\mathrm{Tip}(g_1)$. Hence $\mathrm{Tip}(g_1)$ has a subpath in Z_{new}, a contradiction. Similarly, if $l(q) = 0$ or $l(q) > 0$ such that q has a subpath in Z_{new}, it will lead to a contradiction. So we may assume that $l(p) > 0$, $l(q) > 0$ and both p and q do not contain a subpath in Z_{new}. Then, under our assumption on the admissible order on $(Q_B)_{\geq 0}$, the overlap relation $o(g_1, g_2, p, q)$ in kQ_B becomes an overlap relation in kQ_A. Since $o(g_1, g_2, p, q) \Rightarrow_{\mathcal{G}_A} 0$, then $o(g_1, g_2, p, q) \Rightarrow_{\mathcal{G}_B} 0$.

This proves that $\mathcal{G}_B$ is a Gröbner basis of I_B. Finally, it is obvious that the Gröbner basis $\mathcal{G}_B$ is reduced. □

3. First Hochschild cohomology

In this section we assume that A is a finite dimensional algebra isomorphic to kQ_A/I_A, where k is a field, Q_A is a finite quiver (with vertices $e_1, \cdots, e_n$) and I_A is an admissible ideal in the path algebra kQ_A. We exclude the case in which e_1 or e_n is an isolated vertex. Let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. We denote the vertices of Q_B by $f_1, \cdots, f_{n-1}$, where f_1 is obtained by gluing e_1 and e_n. For the rest of this section, we always assume that A and B are as in Proposition 2.11 so that I_A has a reduced Gröbner basis $\mathcal{G}_A$ and I_B has a reduced Gröbner basis $\mathcal{G}_B = \mathcal{G}_A \cup Z_{new}$ under some appropriate left length-lexicographic orders. Moreover, A has a 'monomial' k-basis $\mathcal{B}_A$ given by $\mathrm{NonTip}(I_A)$ (modulo I_A) and B has a 'monomial' k-basis $\mathcal{B}_B$ given by $\mathrm{NonTip}(I_B)$ (modulo I_B).

We briefly outline the main results of this section. Firstly, we will compare $\mathrm{Im}(\delta_A^0)$ and $\mathrm{Im}(\delta_B^0)$. Then we will study the Lie algebra structures of $\mathrm{Ker}(\delta_A^1)$ and $\mathrm{Ker}(\delta_B^1)$. Lastly, we will compare the dimensions and the Lie structures of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$.

We will use the cochain complex $\mathcal{C}_{\text{para}}$ from the previous section in order to understand the behaviour of the first Hochschild cohomology under idempotent gluings. We start by considering how idempotent gluings behave with respect to parallelism of arrows and paths. Recall from Section 2 that the quiver morphism $\varphi : Q_A \to Q_B$ sends a vertex e_i to f_i for $2 \leq i \leq n-1$ and e_1, e_n to f_1. In addition, φ sends an arrow α in Q_A to an arrow α' in Q_B.

Lemma 3.1. *Let B be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Let $\alpha, \beta \in (Q_A)_1$. If $\alpha /\!/ \beta$, then $\alpha' /\!/ \beta'$.*

Proof. The proof follows from the definition of gluing of two idempotents. □

Lemma 3.2. *Let B be a radical embedding obtained by gluing a source and a sink of A. Let $\alpha, \beta \in (Q_A)_1$. Then $\alpha /\!/ \beta$ if and only if $\alpha' /\!/ \beta'$.*

Proof. The sufficiency is obvious by Lemma 3.1, it suffices to show the necessity. If $\alpha' /\!/ \beta'$, then to show $\alpha /\!/ \beta$ we need to use the assumption that we are gluing a source, say e_1, and a sink, say e_n. We show that if $\alpha \not/\!/ \beta$, then $\alpha' \not/\!/ \beta'$. If $\alpha \not/\!/ \beta$, then either $s(\alpha) \neq s(\beta)$ or $t(\alpha) \neq t(\beta)$. Assume $s(\alpha) = e_i \neq e_j = s(\beta)$, where $i \neq j$. We consider three cases:

a) If $2 \leq i \leq n-1, 1 \leq j \leq n$ and $i \neq j$, then

$$s(\alpha') = f_i \neq s(\beta') = \begin{cases} f_j & \text{for } 2 \leq j \leq n-1 \\ f_1 & \text{for } j = 1 \text{ or } n \end{cases},$$

which means $\alpha' \not/\!/ \beta'$.

b) If $i = 1, 1 \leq j \leq n$ and $i \neq j$, then $s(\alpha') = f_1$ and $s(\beta') = \begin{cases} f_j & \text{for } 2 \leq j \leq n-1 \\ f_1 & \text{for } j = n \end{cases}$. We have $s(\beta') = f_1 = s(\alpha')$ only when $j = n$, that is, if $s(\beta) = e_n$. But this is not possible since e_n is a sink. Hence $s(\alpha') \neq s(\beta')$, which means $\alpha' \not/\!/ \beta'$.

c) We can deduce the same for $i = n$, $1 \leq j \leq n$ and $i \neq j$.

Similar arguments apply if we assume $t(\alpha) \neq t(\beta)$. □

We now partially extend the above results to parallel paths.

Proposition 3.3. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents of A. Then the following hold:*

(1) *The map $\varphi : Q_A \to Q_B$ induces a surjective map, also denoted by $\varphi : \mathcal{B}_A \to \mathcal{B}_B$, such that $\varphi^{-1}(p') = \{p\}$ for $p' \neq f_1$ and $\varphi^{-1}(f_1) = \{e_1, e_n\}$, where we denote $\varphi(p)$ by p' for $p \in \mathcal{B}_A$.*

(2) *Let $p, q \in \mathcal{B}_A$. If $p /\!/ q$ in Q_A, then $p' /\!/ q'$ in Q_B.*

(3) *The map $\varphi : \mathcal{B}_A \to \mathcal{B}_B$ induces k-linear maps*

$$\varphi_0 : k((Q_A)_0 /\!/ \mathcal{B}_A) \to k((Q_B)_0 /\!/ \mathcal{B}_B),$$
$$\varphi_1 : k((Q_A)_1 /\!/ \mathcal{B}_A) \to k((Q_B)_1 /\!/ \mathcal{B}_B),$$
$$\varphi_2 : k(\mathrm{Tip}(\mathcal{G}_A) /\!/ \mathcal{B}_A) \to k(\mathrm{Tip}(\mathcal{G}_B) /\!/ \mathcal{B}_B).$$

Proof. We identify $\mathcal{B}_A$ with $\mathrm{NonTip}(I_A)$ (modulo I_A) and observe that $\mathrm{NonTip}(I_A) := (Q_A)_{\geq 0} \setminus \mathrm{Tip}(I_A)$ consists of monomial elements. The same holds for $\mathcal{B}_B$.

The quiver morphism $\varphi : Q_A \to Q_B$ induces a k-linear map $kQ_A \to kQ_B$ between path algebras by sending a path $p = a_m \cdots a_1$ ($a_i \in (Q_A)_1$ for $1 \leq i \leq m$) in Q_A to a path $p' := a_m' \cdots a_1'$ in Q_B. Clearly, the condition $p \in \mathcal{B}_A$ is equivalent to $p \notin \langle \mathrm{Tip}(I_A) \rangle = \langle \mathrm{Tip}(\mathcal{G}_A) \rangle \subseteq kQ_A$. We deduce that $p' \notin \langle \mathrm{Tip}(I_B)\rangle = \langle \mathrm{Tip}(\mathcal{G}_B)\rangle = \langle \mathrm{Tip}(I_A) \cup Z_{new}\rangle \subseteq kQ_B$ since the elements in the set Z_{new} are the newly formed relations in I_B. Therefore $p' \in \mathcal{B}_B$. The statement (1) follows from the fact that $\dim_k B = \dim_k A - 1$, and the statements (2) and (3) follow from Lemma 3.1. $\square$

We have the following *non-commutative* diagram:

$$\begin{tikzcd}
0 \arrow[r] & k((Q_A)_0/\!/\mathcal{B}_A) \arrow[r, "\delta_A^0"] \arrow[d, "\varphi_0"] & k((Q_A)_1/\!/\mathcal{B}_A) \arrow[r, "\delta_A^1"] \arrow[d, "\varphi_1"] & k(\mathrm{Tip}(\mathcal{G}_A)/\!/\mathcal{B}_A) \arrow[d, "\varphi_2"] \\
0 \arrow[r] & k((Q_B)_0/\!/\mathcal{B}_B) \arrow[r, "\delta_B^0"] & k((Q_B)_1/\!/\mathcal{B}_B) \arrow[r, "\delta_B^1"] & k(\mathrm{Tip}(\mathcal{G}_B)/\!/\mathcal{B}_B)\ .
\end{tikzcd} \qquad (*)$$

Note that the top and the bottom complexes are truncations of the complexes $\mathcal{C}_{para}$ of A and of B, respectively. Although both squares in the diagram $(*)$ are not commutative in general, there are close connections between the coboundary elements (resp. the cocycle elements) of the top complex and the coboundaries (respectively the cocycles) of the bottom complex in the diagram $(*)$.

In order to compare $\mathrm{Im}(\delta_A^0)$ and $\mathrm{Im}(\delta_B^0)$ we need some definitions and a lemma. With Proposition 2.7 in mind, we introduce the following notation:

Notation 1. We denote by $\delta^0_{(A)_0}$ to be the map δ^0_A restricted to the subspace $k((Q_A)_0/\!/(Q_A)_0)$. We denote by $\mathrm{Im}(\delta^0_{(A)_0})$ the k-vector space generated by the image of δ^0_A on $e_i/\!/e_i$, where e_i $(1 \leq i \leq n)$ are idempotents corresponding to vertices of Q_A. We denote by $\mathrm{Ker}(\delta^0_{(A)_0})$ the kernel of the map $\delta^0_{(A)_0}$. Similarly, we denote by $\mathrm{Im}(\delta^0_{(A)_{\geq 1}})$ the k-vector space generated by the image of δ^0_A on $e_i/\!/p$ $(1 \leq i \leq n)$, where $p \in \mathcal{B}_A$ and $p \neq e_i$. We use the same notation for $\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$.

Lemma 3.4. *Let $A = kQ_A/I_A$ be a quiver algebra. Then*

$$\dim_k \mathrm{Im}(\delta^0_{(A)_0}) = n_A - c_A,$$

where $n_A = |(Q_A)_0|$ is the number of vertices of Q_A and c_A is the number of connected components of Q_A.

Proof. It is enough to assume that A is indecomposable. Indeed, if it holds for each block A_i of A, then

$$\dim_k(\mathrm{Im}(\delta^0_{(A)_0})) = \sum_{A_i}(|(Q_{A_i})_0| - 1) = |(Q_A)_0| - c_A.$$

Hence assume A is indecomposable. Note that:

$$\dim_k(k((Q_A)_0/\!/(Q_A)_0) = |(Q_A)_0| = \dim_k(\mathrm{Im}(\delta^0_{(A)_0})) + \dim_k(\mathrm{Ker}(\delta^0_{(A)_0})).$$

Consequently, it is enough to show that $\dim_k(\mathrm{Ker}(\delta^0_{(A)_0})) = 1$. It is straightforward to check that $\sum_{i=1}^{n_A} e_i/\!/e_i$ is in $\mathrm{Ker}(\delta^0_{(A)_0})$. Therefore $\mathrm{Ker}(\delta^0_{(A)_0})$ has dimension at least one. We will prove by contradiction that the dimension of $\mathrm{Ker}(\delta^0_{(A)_0})$ is exactly 1.

Assume the dimension of $\mathrm{Ker}(\delta^0_{(A)_0})$ is greater than 1. Then we can assume without loss of generality that there exists $T \subsetneqq \{1, \dots, n_A\}$ such that $\sum_{i \in T} \lambda_i e_i/\!/e_i$ is an element of $\mathrm{Ker}(\delta^0_{(A)_0})$, where λ_i are non-zero scalars. Indeed, if there exists an element $\sum_{i=1}^{n_A} \lambda_i e_i/\!/e_i$ in $\mathrm{Ker}(\delta^0_{(A)_0})$, then by taking a linear combination with $\sum_{i=1}^{n_A} e_i/\!/e_i$ we can always find such T. Consider the full subquiver $\overline{Q}$ having the vertices indexed by T. Since Q_A is connected and since $T \subsetneqq \{1, \dots, n_A\}$, then $\delta^0_A(\sum_{i \in T} \lambda_i e_i/\!/e_i)$ has one summand of the form $c/\!/c$ where c is an arrow such that $s(c) \in \overline{Q}_0$ and $t(c) \notin \overline{Q}_0$ (or $s(c) \notin \overline{Q}_0$ and $t(c) \in \overline{Q}_0$). Since $c/\!/c$ cannot be written as a linear combination of other elements of $k((Q_A)_1/\!/\mathcal{B}_A)$ and since λ_i are non-zero, then $\sum_{i \in T} \lambda_i e_i/\!/e_i$ is not in $\mathrm{Ker}(\delta^0_{(A)_0})$. The statement follows. $\square$

Let p be a path between e_1 and e_n in $\mathcal{B}_A$. Then p' is an oriented cycle at f_1 in Q_B. If p is a path from e_1 to e_n, then we have

$$\delta_B^0(f_1/\!/p') = \sum_{s(a)=e_n, a\in(Q_A)_1, ap\in\mathcal{B}_A} a'/\!/a'p' - \sum_{t(b)=e_1, b\in(Q_A)_1, pb\in\mathcal{B}_A} b'/\!/p'b'.$$

Note that we have omitted some zero terms in the above sum, for example, if $d \in (Q_A)_1$ is an arrow starting at e_1, then $d'/\!/d'p'$ appears as a term in the above sum, however, it is zero since $d'p'$ lies in I_B. If p is a path from e_n to e_1, then we have

$$\delta_B^0(f_1/\!/p') = \sum_{s(a)=e_1, a\in(Q_A)_1, ap\in\mathcal{B}_A} a'/\!/a'p' - \sum_{t(b)=e_n, b\in(Q_A)_1, pb\in\mathcal{B}_A} b'/\!/p'b'.$$

As in the previous case, we have omitted some zero terms in the above sum. Moreover, in both cases, $\delta_B^0(f_1/\!/p')$ is zero if and only if $ap, pa \in I_A$ for all $a \in (Q_A)_1$. This observation leads to the following definition:

Definition 3.5. Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Let p be a path between e_1 and e_n in $\mathcal{B}_A$. We call p a *special path* between e_1 and e_n in Q_A if $\delta_B^0(f_1/\!/p') \neq 0$, or equivalently, if there exists some $a \in (Q_A)_1$ such that $ap \notin I_A$ or $pa \notin I_A$.

We denote by Sp_1^n the set of special paths between e_1 and e_n in Q_A, and by V_{sp} the k-subspace of $\mathrm{Im}(\delta_B^0)$ generated by the elements $\delta_B^0(f_1/\!/p')$ for $p \in \mathrm{Sp}_1^n$. Furthermore, we denote by sp_1^n the dimension of V_{sp}.

Lemma 3.6. *Let p be a special path between e_1 and e_n in $\mathcal{B}_A$ and let q be a path in $\mathcal{B}_A \backslash \mathrm{Sp}_1^n$. Then the set of the summands of $\delta_B^0(f_1/\!/p')$ and the set of the summands of $\delta_B^0(f_i/\!/q')$ $(1 \leq i \leq n-1)$ are disjoint.*

Proof. Without loss of generality, we assume that p is a special path from e_1 to e_n. Then

$$\delta_B^0(f_1/\!/p') = \sum_{s(\alpha)=e_n, \alpha\in(Q_A)_1, \alpha p\in\mathcal{B}_A} \alpha'/\!/\alpha'p' - \sum_{t(\beta)=e_1, \beta\in(Q_A)_1, p\beta\in\mathcal{B}_A} \beta'/\!/p'\beta',$$

$$\delta_B^0(f_i/\!/q') = \sum_{s(a')=f_i, a'\in(Q_B)_1, a'q'\in\mathcal{B}_B} a'/\!/a'q' - \sum_{t(b')=f_i, b'\in(Q_B)_1, q'b'\in\mathcal{B}_B} b'/\!/q'b'.$$

Note that $\alpha'/\!/\alpha'p' \neq a'/\!/a'q'$, otherwise, $\alpha' = a' \in (Q_B)_1$ and $\alpha'p' = a'q' \in \mathcal{B}_B$ which imply that $\alpha = a \in (Q_A)_1$ and $\alpha p = aq \in \mathcal{B}_A$ by the bijection between $(\mathcal{B}_A)_{\geq 1}$ and $(\mathcal{B}_B)_{\geq 1}$. Moreover, the equality $\alpha p = aq \in \mathcal{B}_A$ implies that $p/\!/q$. Hence q is also a path in $\mathcal{B}_A$ from e_1 to e_n and $aq \notin I_A$ for an arrow a. This means that $q \in \mathrm{Sp}_1^n$, a contradiction. In addition, we have $\alpha'/\!/\alpha'p' \neq b'/\!/q'b'$, otherwise $\alpha = b \in (Q_A)_1$ and $\alpha p = qb \in \mathcal{B}_A$ which implies $e_1 = s(p) = s(\alpha p) = s(q\alpha) = s(\alpha) = e_n$, a contradiction. We can similarly show that $\beta'/\!/p'\beta' \neq a'/\!/a'q'$ and $\beta'/\!/p'\beta' \neq b'/\!/q'b'$. □

Remark 3.7. (1) The dimension of V_{sp} is less than or equal to the number of special paths, that is, $\mathrm{sp}_1^n \leq |\mathrm{Sp}_1^n|$. This follows from the fact that the summands of $\delta_B^0(f_1/\!/p')$ and of $\delta_B^0(f_1/\!/q')$ may cancel each other out for $p, q \in \mathrm{Sp}_1^n$, $p \neq q$ (cf. Example 6.6).

(2) If e_1 and e_n belong to different blocks of A or A is a radical square zero algebra, then $\mathrm{sp}_1^n = 0$.

(3) In general, the number sp_1^n could be arbitrarily large, see Example 6.5.

We can now compare the dimensions of $\mathrm{Im}(\delta_A^0)$ and $\mathrm{Im}(\delta_B^0)$:

Proposition 3.8. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Then*

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 + c_B - c_A - \mathrm{sp}_1^n.$$

In particular, if we glue e_1 and e_n from the same block of A, then

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 - \mathrm{sp}_1^n;$$

if we glue e_1 and e_n from different blocks of A, then

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0).$$

Proof. As usual, the vertices of Q_A are $e_1, \cdots, e_n$ and the vertices of Q_B are $f_1, \cdots, f_{n-1}$, where f_1 is obtained by gluing e_1 and e_n. We begin with describing the basis elements in $\mathrm{Im}(\delta_A^0)$ and in $\mathrm{Im}(\delta_B^0)$.

Let $e_i /\!/ p \in k((Q_A)_0 /\!/ \mathcal{B}_A)$. We consider two cases, depending on whether $p = e_i$ or $p \neq e_i$.

$(a1)$ If $p = e_i$ $(1 \leq i \leq n)$, then we have

$$\delta_A^0(e_i /\!/ e_i) = \sum_{s(a)=e_i, a \in (Q_A)_1} a /\!/ a - \sum_{t(b)=e_i, b \in (Q_A)_1} b /\!/ b.$$

By Lemma 3.4, the subspace $\mathrm{Im}(\delta_{(A)_0}^0)$ of $\mathrm{Im}\delta_A^0$ generated by the elements of the form $\delta_A^0(e_i /\!/ e_i)$ has dimension $n_A - c_A$.

$(a2)$ If $p \neq e_i$, then p is an oriented cycle at e_i and

$$\delta_A^0(e_i /\!/ p) = \sum_{s(a)=e_i, a \in (Q_A)_1, ap \in \mathcal{B}_A} a /\!/ ap - \sum_{t(b)=e_i, b \in (Q_A)_1, pb \in \mathcal{B}_A} b /\!/ pb.$$

It is clear that

$$\mathrm{Im}(\delta_A^0) = \mathrm{Im}(\delta_{(A)_0}^0) \ \oplus \ \mathrm{Im}(\delta_{(A)_{\geq 1}}^0).$$

Similarly, we let $f_i /\!/ q \in k((Q_B)_0 /\!/ \mathcal{B}_B)$ and consider four cases.

(b_1) If $q = f_i$ $(1 \leq i \leq n-1)$, then we have

$$\delta_B^0(f_i /\!/ f_i) = \sum_{s(a')=f_i, a' \in (Q_B)_1} a' /\!/ a' - \sum_{t(b')=f_i, b' \in (Q_B)_1} b' /\!/ b'.$$

By Lemma 3.4, the subspace $\mathrm{Im}(\delta_{(B)_0}^0)$ of $\mathrm{Im}(\delta_B^0)$ generated by the elements of the form $\delta_B^0(f_i /\!/ f_i)$ has dimension $n_B - c_B$.

(b_2) If q is an oriented cycle at f_i and $i \neq 1$, then by Proposition 3.3 we have $q = p'$ for some oriented cycle $p \in \mathcal{B}_A$ at e_i $(2 \leq i \leq n-1)$. Therefore

$$\delta_B^0(f_i /\!/ p') = \sum_{s(a')=f_i, a' \in (Q_B)_1} a' /\!/ a'p' - \sum_{t(b')=f_i, b' \in (Q_B)_1} b' /\!/ p'b' = \varphi_1(\delta_A^0(e_i /\!/ p)).$$

(b_3) If q is an oriented cycle at f_1 such that $q = p'$, for some oriented cycle $p \in \mathcal{B}_A$ at e_1, then

$$\delta_B^0(f_1 /\!/ p') = \sum_{s(a')=f_1, a' \in (Q_B)_1, a'p' \in \mathcal{B}_B} a' /\!/ a'p' - \sum_{t(b')=f_1, b' \in (Q_B)_1, p'b' \in \mathcal{B}_B} b' /\!/ p'b' = \varphi_1(\delta_A^0(e_1 /\!/ p)).$$

If q is an oriented cycle at f_1 such that $q = p'$, for some oriented cycle $p \in \mathcal{B}_A$ at e_n, then

$$\delta_B^0(f_1 /\!/ p') = \sum_{s(a')=f_1, a' \in (Q_B)_1, a'p' \in \mathcal{B}_B} a' /\!/ a'p' - \sum_{t(b')=f_1, b' \in (Q_B)_1, p'b' \in \mathcal{B}_B} b' /\!/ p'b' = \varphi_1(\delta_A^0(e_n /\!/ p)).$$

(b_4) If q is an oriented cycle at f_1 with $q = p'$ for some path p between e_1 and e_n in $\mathcal{B}_A$, then we assume that p is a special path since otherwise $\delta_B^0(f_1 /\!/ p')$ is zero. Note that q is of the form $f_1 \xrightarrow{a'} \cdots \xrightarrow{b'} f_1$ and might be a loop at f_1. If p is a path from e_1 to e_n, then we have

$$\delta_B^0(f_1 /\!/ p') = \sum_{s(a)=e_n, a \in (Q_A)_1, ap \in \mathcal{B}_A} a' /\!/ a'p' - \sum_{t(b)=e_1, b \in (Q_A)_1, pb \in \mathcal{B}_A} b' /\!/ p'b'.$$

If p is a path from e_n to e_1, then we have

$$\delta_B^0(f_1 /\!/ p') = \sum_{s(a)=e_1, a \in (Q_A)_1, ap \in \mathcal{B}_A} a' /\!/ a'p' - \sum_{t(b)=e_n, b \in (Q_A)_1, pb \in \mathcal{B}_A} b' /\!/ p'b'.$$

In both cases, $\delta_B^0(f_1 /\!/ p')$ is nonzero since p is a special path.

In addition, we have
$$\mathrm{Im}(\delta^0_B) = \mathrm{Im}(\delta^0_{(B)_0}) \oplus \mathrm{Im}(\delta^0_{(B)_{\geq 1}}).$$
We claim that
$$\mathrm{Im}(\delta^0_{(B)_{\geq 1}}) = \varphi_1(\mathrm{Im}(\delta^0_{(A)_{\geq 1}})) \oplus V_{sp}.$$
It suffices to show that the set of the summands of $\delta^0_B(f_1/\!/p')$ and the set of the summands of $\varphi_1(\mathrm{Im}(\delta^0_{(A)_{\geq 1}}))$ are disjoint for $p \in \mathrm{Sp}^n_1$. Since an element in $\varphi_1(\mathrm{Im}(\delta^0_{(A)_{\geq 1}}))$ is of the form $\varphi_1(\delta^0_A(e_i \,/\!/\, q)) = \delta^0_B(f_i \,/\!/\, q')$, where q is an oriented cycle at e_i $(1 \leq i \leq n)$ (here we identify f_n with f_1), the statement follows from Lemma 3.6. Note also that the map $\varphi_1 : \mathrm{Im}(\delta^0_{(A)_{\geq 1}}) \to \mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ is clearly injective. Therefore we have

$$\dim_k \mathrm{Im}(\delta^0_{(A)_{\geq 1}}) = \dim_k \mathrm{Im}(\delta^0_{(B)_{\geq 1}}) - \mathrm{sp}^n_1. \tag{1}$$

By (a_1) and (b_1) and since $n_A = n_B + 1$, we get

$$\dim_k \mathrm{Im}(\delta^0_{(A)_0}) = \dim_k \mathrm{Im}(\delta^0_{(B)_0}) + 1 + c_B - c_A. \tag{2}$$

Then we have

$$\begin{aligned}\dim_k \mathrm{Im}(\delta^0_A) &= \dim_k \mathrm{Im}(\delta^0_{(A)_{\geq 1}}) + \dim_k \mathrm{Im}(\delta^0_{(A)_0}) \\ &= \dim_k \mathrm{Im}(\delta^0_{(B)_{\geq 1}}) - \mathrm{sp}^n_1 + \dim_k \mathrm{Im}(\delta^0_{(B)_0}) + 1 + c_B - c_A \\ &= \dim_k \mathrm{Im}(\delta^0_B) + 1 + c_B - c_A - \mathrm{sp}^n_1,\end{aligned} \tag{3}$$

where the second equality follows from Equations (1) and (2). In particular, if we glue e_1 and e_n from the same block of A, then we have $c_B = c_A$. If e_1 and e_n are from two different blocks of A, then $c_B = c_A - 1$ and $\mathrm{sp}^n_1 = 0$. □

We obtain a corollary that will be useful for stable equivalences induced by idempotent gluings.

Corollary 3.9. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Assume one of the following two conditions holds:*

- (i) *e_1 is a source and e_n is a sink;*
- (ii) *A is a radical square zero algebra.*

Then we have
$$\dim_k \mathrm{Im}(\delta^0_A) = \dim_k \mathrm{Im}(\delta^0_B) + 1 + c_B - c_A.$$
In particular, if we glue e_1 and e_n from the same block of A, then
$$\dim_k \mathrm{Im}(\delta^0_A) = \dim_k \mathrm{Im}(\delta^0_B) + 1;$$
if e_1 and e_n are from two different blocks of A, then we have
$$\dim_k \mathrm{Im}(\delta^0_A) = \dim_k \mathrm{Im}(\delta^0_B).$$

Proof. It is clear that under the condition (i) or (ii) there are no special paths between e_1 and e_n. Therefore $\mathrm{sp}^n_1 = 0$. If we glue e_1 and e_n from the same block of A, then $c_B = c_A$; if e_1 and e_n are from two different blocks of A, then $c_B = c_A - 1$. Thus, the result follows from Proposition 3.8. □

We will often use the following assumption on the characteristic of the field k:

Assumption 1. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. For each loop α at e_1 or at e_n with $\alpha^m \in \mathrm{Tip}(\mathcal{G}_A)$, we have that* $\mathrm{char}(k) \nmid m$.

Clearly, Assumption 1 holds if the characteristic of the field k is zero or big enough. We now proceed to compare the Lie structures of $\mathrm{Ker}(\delta^1_A)$ and $\mathrm{Ker}(\delta^1_B)$:

Proposition 3.10. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. If* $\mathrm{char}(k)$ *satisfies Assumption 1, then there exists an injective (restricted) Lie algebra homomorphism* $\mathrm{Ker}(\delta_A^1) \hookrightarrow \mathrm{Ker}(\delta_B^1)$ *induced from $\varphi_1 : k((Q_A)_1/\!/\mathcal{B}_A) \to k((Q_B)_1/\!/\mathcal{B}_B)$, which we still denote by φ_1.*

Proof. First we notice that $I_A = \langle \mathcal{G}_A \rangle$ and $I_B = \langle \mathcal{G}_B \rangle$, and by Proposition 2.11 we can write $\mathcal{G}_B = \mathcal{G}_A \cup Z_{new}$, where $Z_{new} = \{b'c' \mid b', c' \in (Q_B)_1, t(c') = f_1 = s(b'), bc \notin \mathcal{B}_A\}$. Having the diagram $(*)$ in mind, let $\alpha/\!/p \in k((Q_A)_1/\!/\mathcal{B}_A)$ and let $\varphi_1(\alpha/\!/p) = \alpha'/\!/p'$ be the corresponding element in $k((Q_B)_1/\!/\mathcal{B}_B)$. On the one hand, we have

$$\varphi_2(\delta_A^1(\alpha/\!/p)) = \varphi_2(\sum_{r\in\mathcal{G}_A, q\in\mathrm{Supp}(r)} c_r(q)\cdot \mathrm{Tip}(r)/\!/q^{\alpha/\!/p}) = \sum_{r\in\mathcal{G}_A, q\in\mathrm{Supp}(r)} c_r(q)\cdot \mathrm{Tip}(r)'/\!/q'^{\alpha'/\!/p'};$$

On the other hand, we have

$$\begin{aligned}\delta_B^1(\varphi_1(\alpha/\!/p)) &= \delta_B^1(\alpha'/\!/p') \\ &= \sum_{r'\in\mathcal{G}_B, q'\in\mathrm{Supp}(r')} c_{r'}(q')\cdot \mathrm{Tip}(r')/\!/q'^{\alpha'/\!/p'} \\ &= \sum_{r\in\mathcal{G}_A, q\in\mathrm{Supp}(r)} c_r(q)\cdot \mathrm{Tip}(r)'/\!/q'^{\alpha'/\!/p'} + \sum_{r'\in Z_{new}} r'/\!/r'^{\alpha'/\!/p'}.\end{aligned}$$

We consider four cases.

(c1) If α is a loop at e_i, for $2 \le i \le n-1$, and $p = e_i$ or p is an oriented cycle at e_i, then $\sum_{r'\in Z_{new}} r'/\!/r'^{\alpha'/\!/p'} = 0$. Indeed, α' does not appear in any $r' \in Z_{new}$. Therefore $\varphi_2(\delta_A^1(\alpha/\!/p)) = \delta_B^1(\varphi_1(\alpha/\!/p))$.

(c2) If α is a loop at e_1 (respectively e_n) and $p = e_1$ (resp. $p = e_n$). In case $p = e_1$, since A is finite dimensional, by Lemma 2.6 there exists an element r in $\mathcal{G}_A$ such that $\mathrm{Tip}(r) = \alpha^m$ for some integer $m \ge 2$. Hence, $\delta_A^1(\alpha/\!/e_1)$ contains the summand $m\mathrm{Tip}(r)/\!/\alpha^{m-1} = m\alpha^m/\!/\alpha^{m-1}$, which cannot be cancelled in $\mathrm{Im}(\delta_A^1)$ unless $\mathrm{char}(k) \mid m$. That is, if $\mathrm{char}(k) \nmid m$, then $\alpha/\!/e_1$ cannot appear as a summand of an element of $\mathrm{Ker}(\delta_A^1)$. Note that α' appears in some $r' \in Z_{new}$ and therefore $\delta_B^1(\varphi_1(\alpha/\!/e_1)) = \delta_B^1(\alpha'/\!/f_1)$ contains a summand $r'/\!/r'^{\alpha'/\!/f_1}$, which cannot be cancelled in $\mathrm{Im}(\delta_B^1)$. Therefore, $\varphi_1(\alpha/\!/e_1)$ cannot appear as a summand of an element of $\mathrm{Ker}(\delta_B^1)$. A similar result holds if α is a loop at e_n and if $p = e_n$.

(c3) If α is a loop at e_1 (resp. e_n) and p is an oriented cycle at e_1 (resp. e_n), then once we replace α' in any $r' \in Z_{new}$ by p', r' becomes a path in Q_B that still contains some relation in Z_{new}. Hence $\sum_{r'\in Z_{new}} r'/\!/r'^{\alpha'/\!/p'} = 0$. Therefore $\varphi_2(\delta_A^1(\alpha/\!/p)) = \delta_B^1(\varphi_1(\alpha/\!/p))$.

(c4) If α is a an arrow which is not a loop such that α' appears in some $r' \in Z_{new}$ and if $p \in \mathcal{B}_A$ is a path parallel to α, then, by the same argument as in (c3), the element obtained from r' by replacing α' by p' is not in $\mathcal{B}_B$. Hence $\sum_{r'\in Z_{new}} r'/\!/r'^{\alpha'/\!/p'} = 0$ and consequently $\varphi_2(\delta_A^1(\alpha/\!/p)) = \delta_B^1(\varphi_1(\alpha/\!/p))$.

The above discussion shows that, if $\mathrm{char}(k)$ satisfies Assumption 1, then there is a k-linear map $\varphi_1 : \mathrm{Ker}(\delta_A^1) \longrightarrow \mathrm{Ker}(\delta_B^1)$ induced from $\varphi_1 : k((Q_A)_1/\!/\mathcal{B}_A) \to k((Q_B)_1/\!/\mathcal{B}_B)$. It is also clear that $\varphi_1 : \mathrm{Ker}(\delta_A^1) \longrightarrow \mathrm{Ker}(\delta_B^1)$ is injective and preserves the Lie bracket, since $\varphi_1 : k((Q_A)_1/\!/\mathcal{B}_A) \to k((Q_B)_1/\!/\mathcal{B}_B)$ preserves parallel paths. $\square$

Remark 3.11. Since the characteristic condition is only needed in (c2), we do not need Assumption 1 in Proposition 3.10 under one of the following conditions:

(1) There is no loop both at e_1 and at e_n. In particular if e_1 (resp. e_n) is a source vertex and e_n (resp. e_1) is a sink vertex.

(2) A (hence also B) is a radical square zero algebra, excluding the case when we glue e_1 and e_n from different blocks of A such that one of the two blocks is isomorphic to $k[x]/(x^2)$. Indeed, if A

is a radical square zero algebra, then Assumption 1 is equivalent to require $\mathrm{char}(k) \neq 2$. Therefore, if we exclude the case of gluing two blocks such that one of them is isomorphic to $k[x]/(x^2)$, then a loop α will appear in a relation $\alpha\beta$ (where β is an arrow different from α) or in a relation $\gamma\alpha$ (where γ is an arrow different from α). Consequently, $\alpha /\!/ e_1 \notin \mathrm{Ker}(\delta_A^1)$ and $\varphi_1(\alpha /\!/ e_1) \notin \mathrm{Ker}(\delta_B^1)$.

In order to describe the elements in $\mathrm{Ker}(\delta_B^1)$ which are in the complement of the subspace $\varphi_1(\mathrm{Ker}(\delta_A^1))$, we introduce the following definition.

Definition 3.12. Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Let α be an arrow and p be a path in $\mathcal{B}_A$. We call (α, p) is a *special pair* with respect to the gluing of e_1 and e_n if the following two conditions are satisfied:

(1) $\alpha \not\parallel p$ in Q_A;

(2) $\alpha' /\!/ p'$ in Q_B.

We denote by Spp_1^n the set of all special pairs with respect to the gluing of e_1 and e_n and by $\widetilde{\mathrm{Spp}_1^n}$ the k-subspace of $k((Q_B)_1 /\!/ \mathcal{B}_B)$ generated by the elements $\alpha' /\!/ p'$, where $(\alpha, p) \in \mathrm{Spp}_1^n$. Furthermore, we denote by V_{spp} the intersection of $\widetilde{\mathrm{Spp}_1^n}$ and $\mathrm{Ker}(\delta_B^1)$, and by kspp_1^n the dimension of the k-subspace V_{spp} of $\mathrm{Ker}(\delta_B^1)$.

Observe that V_{sp} is a subspace of V_{spp} and therefore we always have $\mathrm{kspp}_1^n \geq \mathrm{sp}_1^n$. Note that every nonzero element of V_{spp} is a linear combination of parallel paths corresponding to special pairs (cf. Example 6.9). Moreover, conditions (1) and (2) imply that α is starting from e_1, or ending at e_1, or starting from e_n, or ending at e_n. Note also that the notion of special pairs leads to various possible configurations of the pairs $(\alpha, p) \in (Q_A)_1 /\!/ \mathcal{B}_A$, see Example 6.8.

Remark 3.13. Although in the radical square zero case there are no special paths in Q_A, there may exist special pairs when we glue e_1 and e_n whether from the same block (see Examples 6.2 and 6.7) or from two different blocks of A. Moreover, if we glue from two different blocks and exclude the case that there are loops both at e_1 and at e_n, then $V_{spp} = 0$. Indeed, when we glue e_1 and e_n from different blocks of A we have

$$\mathrm{Spp}_1^n = \{(\alpha, e_n), (\alpha, \beta), (\beta, e_1), (\beta, \alpha) \mid \alpha \text{ (resp. } \beta) \text{ is a loop at } e_1 \text{ (resp. } e_n)\}.$$

Since e_1 and e_n are not isolated vertices, and B is also a radical square zero algebra, we have neither $\alpha' /\!/ f_1$ nor $\beta' /\!/ f_1$ lies in $\mathrm{Ker}(\delta_B^1)$. Therefore, if we exclude the case that there are loops both at e_1 and at e_n, then $V_{spp} = \langle \alpha' /\!/ \beta', \beta' /\!/ \alpha' \mid \alpha \text{ (resp. } \beta) \text{ is a loop at } e_1 \text{ (resp. } e_n)\rangle$ is zero.

Proposition 3.14. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. If $\mathrm{char}(k)$ satisfies Assumption 1, then we have a decomposition*

$$\mathrm{Ker}(\delta_B^1) = \varphi_1(\mathrm{Ker}(\delta_A^1)) \oplus V_{spp},$$

as k-vector spaces and therefore

$$\dim_k \mathrm{Ker}(\delta_B^1) = \dim_k \mathrm{Ker}(\delta_A^1) + \mathrm{kspp}_1^n.$$

Proof. By Proposition 3.10, we only need to describe the elements θ in $\mathrm{Ker}(\delta_B^1)$ which are in the complement of the subspace $\varphi_1(\mathrm{Ker}(\delta_A^1))$, under Assumption 1. According to the proof of Proposition 3.10, we may assume that θ is a linear combination of elements of the form $\alpha' /\!/ p'$ such that (α, p) is a special pair with respect to the gluing of e_1 and e_n. Clearly in this case $\theta \in V_{spp}$. Therefore, we have the following decomposition: $\mathrm{Ker}(\delta_B^1) = \varphi_1(\mathrm{Ker}(\delta_A^1)) \oplus V_{spp}$. $\square$

Exceptional case 1. Let $\mathrm{char}(k) = 2$, by gluing we obtain a block of B of the form $k[x]/(x^2)$, in other words, A has one block which has a Gabriel quiver of type A_2 and we perform the gluing in this block.

Corollary 3.15. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing a source vertex e_1 and a sink vertex e_n of A. Then we have* $\mathrm{Ker}(\delta^1_A) \simeq \mathrm{Ker}(\delta^1_B)$ *as Lie algebras, except in the Exceptional case 1.*

Proof. By Lemma 3.2, the only possible special pair with respect to the gluing of e_1 and e_n has the form (α, e_1) or (α, e_n) such that α is an arrow from e_1 to e_n. Therefore $\widetilde{\mathrm{Spp}}^n_1$ is generated by the elements of the form $\alpha' /\!/ f_1$. Suppose now that $\alpha' /\!/ f_1 \in \mathrm{Ker}(\delta^1_B)$. Then we consider two cases. If Q_A contains a connected component $e_1 \xrightarrow{\alpha} e_n$ so that B has a block isomorphic to $k[x]/(x^2)$, then $\delta^1_B(\alpha' /\!/ f_1) = 2r' /\!/ \alpha' = 0$ (where $r' = \alpha'\alpha'$) implies that $\mathrm{char}(k) = 2$. If Q_A is not the above case, then either there is an arrow $\beta' \neq \alpha'$ starting from f_1 or there is an arrow $\gamma' \neq \alpha'$ ending at f_1 in Q_B. Therefore $\delta^1_B(\alpha' /\!/ f_1)$ will contain a summand $\beta'\alpha' /\!/ \beta'$ or a summand $\alpha'\gamma' /\!/ \gamma'$, which clearly cannot be cancelled in $\mathrm{Im}(\delta^1_B)$, so $\alpha' /\!/ f_1 \notin \mathrm{Ker}(\delta^1_B)$. It follows that $\alpha' /\!/ f_1 \in \mathrm{Ker}(\delta^1_B)$ if and only if B has a block isomorphic $k[x]/(x^2)$ and $\mathrm{char}(k) = 2$. Summarising the above discussion, we get $\mathrm{kspp}^n_1 = 0$ when gluing a source and a sink and excluding the Exceptional case 1. The statement follows from Proposition 3.14, Proposition 3.10 and Remark 3.11 (1). □

Remark 3.16. For the Exceptional case 1, since the rest of the blocks of A do not change, this reduces to the case when A has only one block which has a Gabriel quiver of type A_2. In this case $\mathrm{char}(k) = 2$ and $B \simeq k[x]/(x^2)$, $A = kQ_A$ where Q_A is given by the quiver $1 \xrightarrow{\alpha} 2$. By a direct computation, we have the following: $\mathrm{Im}(\delta^0_A) = \mathrm{Ker}\delta^1_A$ is 1-dimensional with k-basis $\{\alpha /\!/ \alpha\}$, $\mathrm{Im}(\delta^0_B) = 0$ and $\mathrm{Ker}(\delta^1_B)$ is 2-dimensional with k-basis $\{\alpha' /\!/ f_1, \alpha' /\!/ \alpha'\}$. Note that $\mathrm{Spp}^2_1 = \{(\alpha, e_1), (\alpha, e_2)\}$ and $V_{spp} = \langle \alpha' /\!/ f_1 \rangle$. Therefore, $\dim_k \mathrm{Ker}(\delta^1_B) = \dim_k \mathrm{Ker}(\delta^1_A) +$ kspp^2_1.

We can finally compare the dimensions of $\mathrm{HH}^1(A)$ and of $\mathrm{HH}^1(B)$.

Theorem 3.17. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. If* $\mathrm{char}(k)$ *satisfies Assumption 1, then we have*

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1 - \mathrm{kspp}^n_1 + \mathrm{sp}^n_1 + c_A - c_B.$$

In particular, if we glue e_1 and e_n from the same block of A, then

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1 - \mathrm{kspp}^n_1 + \mathrm{sp}^n_1;$$

if e_1 and e_n are from two different blocks of A, then $\mathrm{HH}^1(A)$ *is a Lie subalgebra of* $\mathrm{HH}^1(B)$ *and*

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - \mathrm{kspp}^n_1.$$

Proof. Since $\mathrm{HH}^1 \simeq \mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$, the statement follows from Propositions 3.8 and 3.14. □

Remark 3.18. In [5, Theorem 1] the authors give a formula to compute the dimension of $\mathrm{HH}^1(A)$ for a monomial algebra A using an exact sequence in [4, Page 98]. They introduce the following notions: an element $a /\!/ p$ in $(Q_A)_1 /\!/ \mathcal{B}$ is *admissible* if $a /\!/ p \in \mathrm{Ker}(\delta^1_A)$. An element $a /\!/ p$ in $(Q_A)_1 /\!/ \mathcal{B}$ is *glued* if p is a vertex or a is the first or the last arrow of p. The image of δ^1 restricted to the subspace spanned by glued elements is denoted $\mathrm{Im}(R_g)$. An element $a /\!/ p$ is called *effective* if it is neither glued nor admissible. We denote by $((Q_A)_1 /\!/ \mathcal{B}_A)_e$ the set of effective elements. Then:

$$\dim \mathrm{HH}^1(A) = |(Q_A)_1 /\!/ \mathcal{B}_A| - |((Q_A)_1 /\!/ \mathcal{B}_A)_e| - \dim(\mathrm{Im}(R_g)) - (|(Q_A)_0 /\!/ \mathcal{B}_A| - \dim(Z(A))).$$

This gives another interpretation for the dimension of V_{spp} for monomial algebras, that is, $\mathrm{kspp}^n_1 = K_A - K_B$, where $K_A := |(Q_A)_1 /\!/ \mathcal{B}_A| - |((Q_A)_1 /\!/ \mathcal{B}_A)_e| - \dim(\mathrm{Im}(R_g))$ and so does for B.

Notation 2. We set

$$Y := \varphi_1(\mathrm{Im}(\delta^0_A)) \oplus V_{sp} \subseteq \mathrm{Ker}(\delta^1_B),$$

where V_{sp} is the subspace of $\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ generated by the elements $\delta^0_B(f_1 /\!/ p')$ for $p \in \mathrm{Sp}^n_1$.

We have the following strengthened form of Theorem 3.17.

Theorem 3.19. *Under the conditions of Theorem 3.17, we have the following exact commutative diagram:*

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & V_{sp} \arrow[u] \arrow[r, "\tilde{\iota}_B"] & V_{spp} \arrow[u] \arrow[r] & \mathrm{Coker}(\varphi) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[u, "\pi^0"] \arrow[r, "\iota_B"] & \mathrm{Ker}(\delta_B^1) \arrow[u, "\pi^1"] \arrow[r] & \frac{\mathrm{Ker}(\delta_B^1)}{Y} \arrow[u, "\pi"] \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Im}(\delta_A^0) \arrow[u, "\varphi_1|_{\mathrm{Im}(\delta_A^0)}"] \arrow[r, "\iota_A"] & \mathrm{Ker}(\delta_A^1) \arrow[u, "\varphi_1"] \arrow[r] & \mathrm{HH}^1(A) \arrow[u, "\varphi"] \arrow[r] & 0, \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \qquad (**)$$

where π^0, π^1 are canonical projections, ι_A and ι_B are canonical injections, φ is an injective map induced from φ_1 and π is a surjective map induced from π^1, $Y := \varphi_1(\mathrm{Im}(\delta_A^0)) \oplus V_{sp}$ is a subspace of $\mathrm{Ker}(\delta_B^1)$. In addition,

- *Y is equal to $\mathrm{Im}(\delta_B^0)$ in the case that e_1 and e_n are from different blocks of A.*
- *Y contains $\mathrm{Im}(\delta_B^0)$ as a codimension 1 subspace in case that e_1 and e_n are from the same block of A.*

Proof. By Proposition 3.10, there exists an injective Lie algebra homomorphism $\varphi_1 : \mathrm{Ker}(\delta_A^1) \hookrightarrow \mathrm{Ker}(\delta_B^1)$, which is induced from the canonical map $\varphi_1 : k((Q_A)_1/\!/\mathcal{B}_A) \to k((Q_B)_1/\!/\mathcal{B}_B)$. Moreover, by Proposition 3.14, we have the decomposition $\mathrm{Ker}(\delta_B^1) = \varphi_1(\mathrm{Ker}(\delta_A^1)) \oplus V_{spp}$.

Therefore by Proposition 3.8 and by the fact that $\delta_B^0(f_1/\!/f_1) = \varphi_1(\delta_A^0(e_1/\!/e_1)) + \varphi_1(\delta_A^0(e_n/\!/e_n))$, we have that $\varphi_1 : \mathrm{Ker}(\delta_A^1) \hookrightarrow \mathrm{Ker}(\delta_B^1)$ restricts to an injective map

$$\varphi_1|_{\mathrm{Im}(\delta_A^0)} : \mathrm{Im}(\delta_A^0) = \mathrm{Im}(\delta^0_{(A)_0}) \oplus \mathrm{Im}(\delta^0_{(A)_{\geq 1}}) \hookrightarrow X \oplus \mathrm{Im}(\delta^0_{(B)_{\geq 1}}) \subseteq \mathrm{Ker}(\delta_B^1),$$

where X is the subspace of $\mathrm{Ker}(\delta_B^1)$ generated by the elements $\varphi_1(\delta_A^0(e_1/\!/e_1))$, $\varphi_1(\delta_A^0(e_n/\!/e_n))$ and $\delta_B^0(f_i/\!/f_i)$ $(2 \leq i \leq n-1)$.

Note that $X \oplus \mathrm{Im}(\delta^0_{(B)_{\geq 1}}) = Y$. In addition, the dimension of X is equal to $\dim_k \mathrm{Im}(\delta^0_{(A)_0})$. It follows from Lemma 3.4 that if e_1 and e_n are from two different blocks of A, then $X = \mathrm{Im}(\delta^0_{(B)_0})$. By the same reasoning, if e_1 and e_n are from the same block of A, then $\mathrm{Im}(\delta^0_{(B)_0}) \subseteq X$ has codimension 1 in X. Therefore $\mathrm{Im}(\delta_B^0)$=$\mathrm{Im}(\delta^0_{(B)_0})$$\oplus$$\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ is equal to Y if e_1 and e_n are from two different blocks of A, and $\mathrm{Im}(\delta_B^0)$ has codimension 1 in Y if e_1 and e_n are from the same block of A.

If p is a special path from e_1 to e_n, then each summand $a'/\!/a'p'$ (or $b'/\!/p'b'$) of $\delta_B^0(f_1/\!/p')$, where a is an arrow starting from e_n such that $ap \in \mathcal{B}_A$ (or where b is an arrow ending at e_1 such that $pb \in \mathcal{B}_A$), is induced from a special pair (a, ap) (or (b, pb)). In the case that p is a special path from e_n to e_1, we have the similar conclusion. Therefore the canonical injective map $Y \hookrightarrow \mathrm{Ker}(\delta_B^1)$ restricts to an injective map $V_{sp} \hookrightarrow V_{spp}$.

Hence we obtain the exact commutative diagram $(**)$. $\square$

Lemma 3.20. *The space Y is a Lie ideal of $\mathrm{Ker}(\delta_B^1)$ if and only if $[\varphi_1(\mathrm{Im}(\delta_A^0)), V_{spp}] \subseteq Y$.*

Proof. By the definition of Y we have that

$$\begin{aligned}
[Y, \mathrm{Ker}(\delta_B^1)] &= [\varphi_1(\mathrm{Im}(\delta_A^0)), \mathrm{Ker}(\delta_B^1)] + [V_{sp}, \mathrm{Ker}(\delta_B^1)] \\
&= [\varphi_1(\mathrm{Im}(\delta_A^0)), \varphi_1(\mathrm{Ker}(\delta_A^1))] + [\varphi_1(\mathrm{Im}(\delta_A^0)), V_{spp}] + [V_{sp}, \mathrm{Ker}(\delta_B^1)] \\
&= \varphi_1([\mathrm{Im}(\delta_A^0), \mathrm{Ker}(\delta_A^1)]) + [\varphi_1(\mathrm{Im}(\delta_A^0)), V_{spp}] + [V_{sp}, \mathrm{Ker}(\delta_B^1)] \\
&\subseteq \varphi_1(\mathrm{Im}(\delta_A^0)) + [\varphi_1(\mathrm{Im}(\delta_A^0)), V_{spp}] + [\mathrm{Im}(\delta_B^0), \mathrm{Ker}(\delta_B^1)] \\
&\subseteq \varphi_1(\mathrm{Im}(\delta_A^0)) + [\varphi_1(\mathrm{Im}(\delta_A^0)), V_{spp}] + \mathrm{Im}(\delta_B^0) \\
&\subseteq Y + [\varphi_1(\mathrm{Im}(\delta_A^0)), V_{spp}] + Y,
\end{aligned}$$

where the second equality follows from Proposition 3.14, the third equality follows from the fact that φ_1 is a Lie algebra homomorphism, and the last three equalities follow from the definition of Y and the fact that $\mathrm{Im}(\delta^0)$ is the Lie ideal of $\mathrm{Ker}(\delta^1)$. □

Theorem 3.21. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Assume* $\mathrm{char}(k)$ *satisfies Assumption 1. If $V_{spp} = V_{sp}$, then*

- *Y is a Lie ideal of* $\mathrm{Ker}(\delta_B^1)$ *and*
- *there is a Lie algebra epimorphism from* $\mathrm{HH}^1(B)$ *to* $\mathrm{Ker}(\delta_B^1)/Y \simeq \mathrm{HH}^1(A)$ *with kernel $\mathcal{I} := Y/\mathrm{Im}(\delta_B^0)$, where $\mathcal{I}$ is zero if e_1 and e_n are from two different blocks of A and $dim_k\mathcal{I} = 1$ if e_1 and e_n are from the same block of A.*

Proof. For the first part, note that if e_1 and e_2 are from two different blocks then by Theorem 3.19 we have $Y = \mathrm{Im}(\delta_B^0)$. Hence Y is a Lie ideal of $\mathrm{Ker}(\delta_B^1)$. If e_1 and e_2 are from the same block, then the statement follows from Lemma 3.20. The second part of the proof follows from Theorem 3.19. □

Corollary 3.22. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Assume* $\mathrm{char}(k)$ *satisfies Assumption 1. If $V_{spp} = V_{sp}$, then*

$$\mathrm{HH}^1(A)/\mathrm{rad}(\mathrm{HH}^1(A)) \simeq \mathrm{HH}^1(B)/\mathrm{rad}(\mathrm{HH}^1(B)).$$

Proof. Since by Theorem 3.21 the ideal $\mathcal{I}$ is at most one-dimensional, then $\mathcal{I}$ is solvable. Since the radical contains every solvable ideal, then $\mathcal{I} \subseteq \mathrm{rad}(\mathrm{HH}^1(B))$. Hence by quotienting by the radical we obtain the desired isomorphism. □

Corollary 3.23. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing a source vertex e_1 and a sink vertex e_n of A. Then we have*

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) + c_A - c_B - 1,$$

except in the Exceptional case 1. In particular, if we glue e_1 and e_n from two different blocks of A, then there is a (restricted) Lie algebra isomorphism

$$\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B);$$

if e_1 and e_n are from the same block of A, then

$$\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/\mathcal{I}$$

as (restricted) Lie algebras, where $\mathcal{I}$ is a one-dimensional (restricted) Lie ideal of $\mathrm{HH}^1(B)$.

Proof. First we notice that by Remark 3.11 (1), we do not need Assumption 1. By Corollary 3.15, we have $V_{spp} = 0$ since we glue a source and a sink. The statement follows from Theorem 3.17 and Theorem 3.21. □

Note that the one-dimensional ideal $\mathcal{I} := Y/\mathrm{Im}(\delta_B^0)$ of $\mathrm{HH}^1(B)$ in Theorem 3.21 is generated by $\varphi_1(\delta_A^0(e_1/\!/e_1)) = \varphi_1(\sum_{\alpha\in(Q_A)_1 e_1} \alpha/\!/\alpha - \sum_{\beta\in e_1(Q_A)_1} \beta/\!/\beta)$. In case we glue a source e_1 and a sink e_n, the ideal $\mathcal{I}$ is generated by $\sum_{\alpha\in(Q_A)_1 e_1} \alpha'/\!/\alpha'$.

Lemma 3.24. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing a source vertex e_1 and a sink vertex e_n of A. Then $\mathcal{I}$ is an ideal in the center of* $\mathrm{HH}^1(B)$.

Proof. An element in $\mathrm{HH}^1(B)$ is a linear combination of elements $\beta'/\!/p'$, where β' is an arrow in Q_B and p' is a path in $\mathcal{B}_B$. We show that $[\sum_{\alpha\in(Q_A)_1 e_1} \alpha'/\!/\alpha', \beta'/\!/p'] = 0$ for every $\beta'/\!/p'$. First observe that p' contains an arrow α', where $s(\alpha) = e_1$, if and only if $p' = p'_n \cdots p'_2\alpha'$ and $p'_i \neq \alpha'$ for $i = 2, \dots, n$.

If $s(\beta') \neq f_1$, then $s(p') \neq f_1$, $\beta \neq \alpha'$, where $s(\alpha) = e_1$, and p' does not contain any arrow α' where $\alpha \in (Q_A)_1 e_1$. Therefore $[\sum_{\alpha\in(Q_A)_1 e_1} \alpha'/\!/\alpha', \beta'/\!/p'] = 0$. If $s(\beta') = f_1$, then $\beta' = \alpha'_j$ for some α'_j, where $s(\alpha_j) = e_1$. In addition, $p' = p'_n \cdots p'_2\alpha'_i$ for some α'_i, where $s(\alpha_i) = e_1$, and $p'_i \neq \alpha'$ for $i = 2, \dots, n$ where $\alpha \in (Q_A)_1 e_1$. Hence $[\sum_{\alpha\in(Q_A)_1 e_1} \alpha'/\!/\alpha', \beta'/\!/p'] = \alpha_j/\!/p' - \alpha_j/\!/p' = 0$. □

Recall that an exact sequence of Lie algebra homomorphisms $0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow 0$ is called a *central extension* of $\mathfrak{g}$ by $\mathfrak{a}$ if $[\mathfrak{a}, \mathfrak{h}] = 0$, where we identify $\mathfrak{a}$ with the corresponding Lie ideal of $\mathfrak{h}$.

Theorem 3.25. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing a source vertex e_1 and a sink vertex e_n of A from the same block of A. Then* $\mathrm{HH}^1(B)$ *is a central extension of* $\mathrm{HH}^1(A)$ *by* $\mathcal{I}$.

Proof. By Theorem 3.21 we have a short exact sequence of Lie algebras:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathrm{HH}^1(B) \longrightarrow \mathrm{HH}^1(A) \longrightarrow 0.$$

By Lemma 3.24 this extension is central. □

4. Monomial algebras

Recall that a finite dimensional k-algebra Λ is called *monomial* if it is a quotient kQ/I of a path algebra, where the two-sided ideal I of kQ is generated by a set Z of paths of length ≥ 2. We assume that Z is *minimal*, that is, no proper subpath of a path in Z is again in Z. Clearly Z is a reduced Gröbner basis of I under any left length-lexicographic order on $Q_{\geq 0}$. Let $\mathcal{B} = \mathcal{B}_\Lambda$ be the set of paths of Q which do not contain any element of Z as a subpath. It is clear that the (classes modulo I of) elements of $\mathcal{B}$ form a basis of Λ. We shall denote by $\mathcal{B}_n$ the subset $Q_n \cap \mathcal{B}$ of $\mathcal{B}$ formed by the paths of length n.

For the quiver Q, the parallelism is an equivalence relation on the set of arrows Q_1; for $\alpha \in Q_1$, $[\alpha]$ denotes the equivalence class of α. We denote $\bar{Q}_1$ the set of equivalence classes of parallel arrows. The quiver which has Q_0 as vertices and $\bar{Q}_1$ as set of arrows, will be denoted by $\bar{Q}$. We denote by $\chi(\bar{Q})$ the *first Betti number* of $\bar{Q}$ which is equal to $|\bar{Q}_1| - |\bar{Q}_0| + c_{\bar{Q}}$, where $c_{\bar{Q}}$ is the number of connected components of $\bar{Q}$.

4.1. A direct sum decomposition of HH^1.

In this subsection, we will show that our Theorem 3.25 can be strengthened to Corollary 4.6 for monomial algebras. More precisely, we will show that when we glue a source and a sink in a monomial algebra A, the central extension is actually a trivial extension, that is, we have a direct sum of Lie algebras.

According to [19, Section 4], the Lie algebra $\mathrm{HH}^1(\Lambda)$ of a monomial algebra $\Lambda = kQ/\langle Z\rangle$ has a natural grading. Indeed, if $a/\!/\gamma \in Q_1/\!/\mathcal{B}_n$ and $b/\!/\epsilon \in Q_1/\!/\mathcal{B}_m$, then the Lie bracket defined in Theorem 2.8 shows that $[a/\!/\gamma, b/\!/\epsilon] \in k(Q_1/\!/\mathcal{B}_{n+m-1})$. Thus, we have a grading on the Lie algebra $k(Q_1/\!/\mathcal{B}) = \oplus_{i\in\mathbb{N}} k(Q_1/\!/\mathcal{B}_i)$ by considering that the elements of $k(Q_1/\!/\mathcal{B}_i)$ have degree $i - 1$ for all $i \in \mathbb{N}$. It is clear that the Lie subalgebra $\mathrm{Ker}(\delta^1)$ of $k(Q_1//\mathcal{B})$ preserves this grading and that $\mathrm{Im}(\delta^0)$ is a graded ideal, which induces a grading on the Lie algebra $\mathrm{HH}^1(\Lambda) \simeq \mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$. More precisely, if we set

$$L_{-1} := k(Q_1//Q_0) \cap \mathrm{Ker}(\delta^1),$$
$$L_0 := (k(Q_1//Q_1) \cap \mathrm{Ker}(\delta^1))/\langle \delta^0(e//e) \mid e \in Q_0\rangle \textit{ and}$$
$$L_i := (k(Q_1//\mathcal{B}_{i+1}) \cap \mathrm{Ker}(\delta^1))/\langle \delta^0(e//p) \mid e//p \in Q_1//\mathcal{B}_i\rangle$$

for all $i \geq 1$, then $\mathrm{HH}^1(\Lambda) = \bigoplus_{i\geq -1} L_i$ and $[L_i, L_j] \subseteq L_{i+j}$ for all $i, j \geq -1$, where $L_{-2} = 0$.

Remark 4.1. Note that if the characteristic of the field k is equal to 0, then $L_{-1} = 0$ by Proposition 4.2 in [19]. It follows that $\bigoplus_{i\geq 1} L_i$ is a solvable Lie ideal of $\mathrm{HH}^1(\Lambda) = \bigoplus_{i\geq 0} L_i$ since $\mathrm{HH}^1(\Lambda)$ is finite dimensional. It is also obvious that L_0 is a Lie subalgebra of $\mathrm{HH}^1(\Lambda)$.

In order to ensure each $L_0^{[\alpha]}$ (in the Lie algebra decomposition (†) of L_0 below) to be a Lie ideal, we need to use the following variation of [19, Proposition 4.7]).

Lemma 4.2. *The basis $\mathcal{B}_{L_0}$ of L_0 is given by the union of the following sets:*

(*i*) *all the elements $a//b \in L_0$ such that $a \neq b$;*

(*ii*) *for every class of parallel arrows $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \in \bar{Q}_1$, all the elements $\alpha_i//\alpha_i - \alpha_n//\alpha_n \in L_0$ such that $i < n$;*

(*iii*) *for each (oriented or undirected) cycle in $\bar{Q}$, choose one class of parallel arrows $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ in this cycle and take $\alpha_n//\alpha_n$. Note that there are $\chi(\bar{Q})$ linearly independent elements in (iii).*

For each class of parallel arrows $[\alpha] \in \bar{Q}_1$ we denote by $L_0^{[\alpha]}$ the Lie ideal of L_0 generated by the elements of the form $\alpha_i//\alpha_j$ and $\alpha_i//\alpha_i - \alpha_n//\alpha_n$ in $\mathcal{B}_{L_0}$, where $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $1 \leq i, j \leq n$. Obviously the Lie algebra L_0 is the direct sum of these Lie algebras:

$$L_0 = \oplus_{[\alpha]\in\bar{Q}} L_0^{[\alpha]}, \qquad (\dagger)$$

where this decomposition depends on the basis $\mathcal{B}_{L_0}$ and $L_0^{[\alpha]}$ may be equal to zero for some $[\alpha]$.

Remark 4.3. Let A be a monomial algebra and let B be a radical embedding obtained by gluing two idempotents in A. Then B is also a monomial algebra, hence both $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ have a canonical grading. Note that the one-dimensional ideal $\mathcal{I} := Y/\mathrm{Im}(\delta_B^0)$ of $\mathrm{HH}^1(B)$ in Theorem 3.21 is generated by $\varphi_1(\delta_A^0(e_1//e_1)) = \varphi_1(\sum_{[\alpha]\in(\bar{Q}_A)_1 e_1} \mathcal{I}_{[\alpha]} - \sum_{[\alpha]\in e_1(\bar{Q}_A)_1} \mathcal{I}_{[\alpha]})$, where $\mathcal{I}_{[\alpha]} := \sum_{i=1}^m \alpha_i//\alpha_i$ for $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$.

We can rewrite the generator $\varphi_1(\delta_A^0(e_1//e_1))$ of $\mathcal{I}$ after introducing the following definition.

Definition 4.4. Let Q_A^c and Q_A^d be two sub-quivers of Q_A such that the arrows of Q_A^c satisfy one of the following two conditions:

(*i*) $t(\alpha) = e_n$;

(*ii*) α lies in a path or an undirected path in the quiver Q_A, which is starting at e_1 and ending at e_n and passes through e_1 just once.

The arrows of Q_A^d are the arrows of Q_A which are not in Q_A^c. We also define the corresponding sub-quivers Q_B^c and Q_B^d via the map φ in Section 2.

Denote by $\Delta := (\bar{Q}_A^c)_1 e_1$ the subset of $(\bar{Q}_A)_1 e_1$ consisting of the equivalence classes of parallel arrows $[\alpha]$ starting from e_1 in $\bar{Q}_A^c$.

For a concrete example for Δ, see Example 6.11.

Lemma 4.5. *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing a source vertex e_1 and a sink vertex e_n of A. If e_1 and e_n are from the same block of A, then the one-dimensional Lie ideal $\mathcal{I}$ in Corollary 3.23 is generated by $\varphi_1(\sum_{[\alpha]\in\Delta}\mathcal{I}_{[\alpha]})$ (modulo an element in $\mathrm{Im}(\delta_B^0)$).*

Proof. Since e_1 is a source vertex, then Remark 4.3 yields that

$$\begin{aligned}\varphi_1(\delta_A^0(e_1/\!/e_1)) &= \varphi_1(\sum_{[\alpha]\in(\bar{Q}_A)_1e_1}\mathcal{I}_{[\alpha]}) = \varphi_1(\sum_{[\alpha]\in(\bar{Q}_A^c)_1e_1}\mathcal{I}_{[\alpha]}) + \varphi_1(\sum_{[\alpha]\in(\bar{Q}_A^d)_1e_1}\mathcal{I}_{[\alpha]})\\ &= \varphi_1(\sum_{[\alpha]\in\Delta}\mathcal{I}_{[\alpha]}) + \varphi_1(\sum_{[\alpha]\in(\bar{Q}_A^d)_1e_1}\mathcal{I}_{[\alpha]}).\end{aligned}$$

Note that Q_A^c and Q_A^d can be obtained from Q_A by splitting in e_1 since Definition 4.4 shows that Q_A^c and Q_A^d are disjoint and they only share the vertex e_1 when Q_A^d is not empty. By combing this with the fact that e_1 is a source vertex, we deduce that $\delta_A^0(\sum_{i=1}^n e_i/\!/e_i) = 0$ if and only if $\delta_A^0(\sum_{e_i\in Q_A^c} e_i/\!/e_i) = 0$ and $\delta_A^0(\sum_{e_i\in Q_A^d} e_i/\!/e_i) = 0$, whence

$$\varphi_1(\sum_{[\alpha]\in(\bar{Q}_A^d)_1e_1}\mathcal{I}_{[\alpha]}) = -\varphi_1(\sum_{e_i\in(Q_A^d)_0, e_i\neq e_1}\delta_A^0(e_i/\!/e_i)) = \sum_{f_i\in(Q_B^d)_0, f_i\neq f_1}\delta_B^0(f_i/\!/f_i)\in\mathrm{Im}(\delta_B^0).$$

□

Now we can give the main result in this subsection.

Corollary 4.6. *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing a source vertex e_1 and a sink vertex e_n of A. If e_1 and e_n are from the same block of A and $\mathrm{char}(k) = 0$, then*

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A)\oplus\mathcal{I} \simeq \mathrm{HH}^1(A)\oplus k$$

as Lie algebras.

Proof. We claim that it is enough to show that we have a decomposition as vector spaces $L_0 = \mathcal{I}\oplus G$, where G is a Lie subalgebra of L_0. Indeed, if this is the case, by the grading on $\mathrm{HH}^1(B)$ we have a decomposition as vector spaces:

$$\mathrm{HH}^1(B) = L_0\oplus\bigoplus_{i\geq1}L_i = (\mathcal{I}\oplus G)\oplus\bigoplus_{i\geq1}L_i = \mathcal{I}\oplus(G\oplus\bigoplus_{i\geq1}L_i) =: \mathcal{I}\oplus L.$$

Note that L is a Lie subalgebra of $\mathrm{HH}^1(B)$ since G is a Lie subalgebra and $\bigoplus_{i\geq1}L_i$ is a Lie ideal of $\mathrm{HH}^1(B)$. In addition, by Theorem 3.25 the ideal $\mathcal{I}$ is in the center of $\mathrm{HH}^1(B)$, hence L is a Lie ideal of $\mathrm{HH}^1(B)$. Therefore we have a direct sum decomposition as Lie algebras: $\mathrm{HH}^1(B) = \mathcal{I}\oplus L$. Since $\mathrm{HH}^1(A)\simeq\mathrm{HH}^1(B)/\mathcal{I}$ as Lie algebras, then $L\simeq\mathrm{HH}^1(A)$ as Lie algebras. Therefore, $\mathrm{HH}^1(B) = \mathcal{I}\oplus\mathrm{HH}^1(A)$ as Lie algebras.

We show that $\mathcal{I}$ is a direct summand of L_0 as vector spaces. From now on, we fix the 'minimal' generator $\varphi_1(\sum_{[\alpha]\in\Delta}\mathcal{I}_{[\alpha]})$ of the Lie ideal $\mathcal{I}$ which is given by Lemma 4.5. We sketch the proof in the case that Δ only contains two equivalence classes of parallel arrows (cf. Definition 4.4), namely $\Delta = \{[\alpha],[\beta]\}$, where $[\alpha] = \{\alpha_1,\cdots,\alpha_m\}$ and $[\beta] = \{\beta_1,\cdots,\beta_t\}$. Then $\mathcal{I} = \langle\varphi_1(\mathcal{I}_{[\alpha]}+\mathcal{I}_{[\beta]})\rangle = \langle\sum_{i=1}^m\alpha_i'/\!/\alpha_i' + \sum_{j=1}^t\beta_j'/\!/\beta_j'\rangle$. Since $L_0^{[\alpha]} = \langle\alpha_i'/\!/\alpha_j', \alpha_l'/\!/\alpha_l' \mid 1\leq l\leq m, 1\leq i\neq j\leq m, \alpha_i/\!/\alpha_j\in\mathrm{Ker}(\delta_A^1)\rangle$ and $L_0^{[\beta]} = \langle\beta_i'/\!/\beta_j', \beta_l'/\!/\beta_l' \mid 1\leq l\leq t, 1\leq i\neq j\leq t, \beta_i/\!/\beta_j\in\mathrm{Ker}(\delta_A^1)\rangle$, it is easy to see that $\mathcal{I}$ is a summand of $\oplus_{[\alpha]\in\Delta}L_0^{[\alpha]} = L_0^{[\alpha]}\oplus L_0^{[\beta]}$, hence it is a summand of L_0. In fact, we have vector space decompositions:

$$\begin{aligned}L_0^{[\alpha]} &= \langle\sum_{i=1}^m\alpha_i'/\!/\alpha_i'\rangle\oplus\langle\alpha_i'/\!/\alpha_j', \alpha_l'/\!/\alpha_l' - \alpha_m'/\!/\alpha_m' \mid 1\leq l\leq m-1, 1\leq i\neq j\leq m, \alpha_i/\!/\alpha_j\in\mathrm{Ker}(\delta_A^1)\rangle\\ &=: \langle\varphi_1(I_{[\alpha]})\rangle\oplus J_1,\end{aligned}$$

$$\begin{aligned}L_0^{[\beta]} &= \langle \sum_{i=1}^{t} \beta_i'/\!/\beta_i' \rangle \oplus \langle \beta_i'/\!/\beta_j', \beta_l'/\!/\beta_l' - \beta_t'/\!/\beta_t' \mid 1 \le l \le t-1, 1 \le i \ne j \le t, \beta_i/\!/\beta_j \in \mathrm{Ker}(\delta_A^1) \rangle \\ &=: \langle \varphi_1(I_{[\beta]}) \rangle \oplus J_2.\end{aligned}$$

As a consequence, there are vector space decompositions

$$\begin{aligned}L_0^{[\alpha]} \oplus L_0^{[\beta]} &= (\langle \varphi_1(\mathcal{I}_{[\alpha]}) \rangle \oplus \langle \varphi_1(\mathcal{I}_{[\beta]}) \rangle) \oplus J_1 \oplus J_2 = (\mathcal{I} \oplus \langle \varphi_1(\mathcal{I}_{[\alpha]}) \rangle) \oplus J_1 \oplus J_2 \\ &= \mathcal{I} \oplus (\langle \varphi_1(\mathcal{I}_{[\alpha]}) \rangle \oplus J_1 \oplus J_2) =: \mathcal{I} \oplus J.\end{aligned}$$

It follows from the definition of the Lie bracket in Theorem 2.8 that J is a subalgebra of L_0. It follows that $L_0 = \oplus_{[\alpha] \in (\bar{Q}_B)_1} L_0^{[\alpha]} = \mathcal{I} \oplus G$ as vector spaces, where $G := J \oplus \bigoplus_{\alpha \in (\bar{Q}_B)_1 \setminus \Delta} L_0^{[\alpha]}$. Note that G is a Lie subalgebra since J is a Lie subalgebra and since we have the direct sum decomposition (†). □

4.2. **Radical square zero algebras.**

We now apply our main results in Section 3 to a subclass of monomial algebras: radical square zero algebras. An application of these results can be found in Subsection 4.3. Throughout this subsection, we let $A = kQ_A/I_A$ be a radical square zero algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A.

Corollary 4.7. *Let A be a radical square zero algebra and let B be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. If $\mathrm{char}(k) \neq 2$, then we have*

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1 - \mathrm{kspp}_1^n - c_B + c_A.$$

In particular, if we glue e_1 and e_n from the same block of A, then

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1 - \mathrm{kspp}_1^n;$$

if we glue e_1 and e_n from two different blocks of A, then

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - \mathrm{kspp}_1^n$$

and $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)$ as Lie algebras if we exclude the case that there are loops both at e_1 and e_n.

Proof. For radical square zero algebras, there are no special paths and Assumption 1 is equivalent to the condition that $\mathrm{char}(k) \neq 2$. The dimension formulas follow immediately from Theorem 3.17 and Theorem 3.19. Moreover, if we glue e_1 and e_n from two different blocks of A and exclude the case that there are loops at e_1 and at e_n simultaneously, then $V_{spp} = 0$ by Remark 3.13. Since V_{sp} is a subspace of V_{spp}, then $V_{sp} = 0$ and by Theorem 3.21 we have $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)$ as Lie algebras. □

Moreover, it is easy to see that if one of the following conditions holds, then the results in Corollary 4.7 still hold in the case $\mathrm{char}(k) = 2$ by Remark 3.11 (2):

- (i) glue e_1 and e_n from the same block of A;
- (ii) glue $e_1 \in A_1$ and $e_n \in A_2$ from the different blocks of A such that both A_1 and A_2 are not isomorphic to $k[x]/(x^2)$.

Remark 4.8. Let A and B as above and let A_1 and A_2 be two different blocks of A. Suppose $e_1 \in A_1$ and $e_n \in A_2$.

(1) If there are loops at e_1 or at e_n, then in general $\mathrm{HH}^1(A)$ is not isomorphic to $\mathrm{HH}^1(B)$ and the difference between the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ can be arbitrarily large, see Example 6.10.

(2) If $\mathrm{char}(k) = 2$ and exactly one of A_1, A_2 is isomorphic to $k[x]/(x^2)$, then

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) + 1.$$

Corollary 4.9. *Let A be a radical square zero algebra and let B be a radical embedding obtained by gluing two idempotents e_1 and e_n from the same block of A. If $V_{spp} = 0$ and $\mathrm{char}(k) = 0$, then we have a Lie algebra isomorphism*

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \oplus k.$$

Proof. We use the notation in Theorem 3.19. Since $V_{spp} = 0$, we have a Lie algebra epimorphism from $\mathrm{HH}^1(B)$ to $\mathrm{Ker}(\delta_B^1)/Y \simeq \mathrm{HH}^1(A)$ with one-dimensional kernel $I := Y/\mathrm{Im}(\delta_B^0)$, where $\mathrm{Ker}(\delta_B^1) = \varphi_1(\mathrm{Ker}(\delta_A^1))$ and Y is a Lie ideal of $\mathrm{Ker}(\delta_B^1)$. Also note that this epimorphism and equality do not depend on the Assumption 1 since we glue e_1 and e_n from the same block, cf. Remark 3.11 (2). Since $V_{spp} = 0$, then $\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1$. Note that by gluing e_1 and e_n from the same block we have $\chi(\bar{Q}_B) = \chi(\bar{Q}_A) + 1$. By Theorem 2.9 in [18] (see also Theorem 4.12) there is an injective Lie algebra homomorphism:

$$\mathrm{HH}^1(A) \simeq \oplus_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \oplus k^{\chi(\bar{Q}_A)} \to \oplus_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \oplus k^{\chi(\bar{Q}_A)} \oplus k \simeq \mathrm{HH}^1(B).$$

Therefore it gives rise to the following Lie algebra isomorphisms:

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \oplus I \simeq \mathrm{HH}^1(A) \oplus k.$$

□

Remark 4.10. Let A and B as above, and suppose that e_1 and e_n are in the same block of A. If we exclude the Exceptional case 1, then it is straightforward to check that $V_{spp} = 0$ under each of the following conditions:

(i) e_1 is a source and e_n is a sink;

(ii) Both e_1 and e_n are sinks such that

$$\{s(\alpha) \mid t(\alpha) = e_1, \alpha \in (Q_A)_1\} \cap \{s(\beta) \mid t(\beta) = e_n, \beta \in (Q_A)_1\} = \emptyset;$$

(iii) Both e_1 and e_n are sources such that

$$\{t(\alpha) \mid s(\alpha) = e_1, \alpha \in (Q_A)_1\} \cap \{t(\beta) \mid s(\beta) = e_n, \beta \in (Q_A)_1\} = \emptyset.$$

Remark 4.11. Let A be a radical square zero algebra having Gabriel quiver Q. By direct computations, we can determine the Lie algebra structure of $\mathrm{HH}^1(A)$ in the following well-known cases, which are recalled here for completeness:

(1) $\mathrm{HH}^1(A) \simeq \mathfrak{gl}_n(k)$ if Q is the quiver with one vertex and n loops , except in the case $n = 1$ and $\mathrm{char}(k) = 2$ (for this exceptional case, see Remark 3.16); The isomorphism sends $\alpha_i /\!/ \alpha_j$ to E_{ji}, where E_{ij} is the matrix that has 1 in position (i, j) and 0 elsewhere. Note that if the characteristic of the field k does not divide n, then $\mathfrak{gl}_n(k) \simeq \mathfrak{sl}_n(k) \oplus k$ as Lie algebras.

(2) $\mathrm{HH}^1(A) \simeq \mathfrak{pgl}_n(k)$ if Q is the n-Kronecker quiver, with the convention that 1-Kronecker quiver is the Dynkin quiver A_2, where $\mathfrak{pgl}_n(k)$ is the quotient of $\mathfrak{gl}_n(k)$ by its center $k \cdot \mathrm{Id}$. Let e be the source vertex of the n-Kronecker quiver. Then the above isomorphism can be obtained by observing that $\mathrm{Ker}(\delta_A^1) \simeq \mathfrak{gl}_n(k)$ via the isomorphism in (1). In addition, this isomorphism sends the unique generator $\sum_{s(\alpha_i)=e} \alpha_i /\!/ \alpha_i$ of $\mathrm{Im}(\delta_A^0)$ to Id. If the characteristic of the field k does not divide n, then $\mathfrak{pgl}_n(k) \simeq \mathfrak{sl}_n(k)$.

4.3. **Sánchez-Flores' decomposition via inverse gluing.**

In this section, we provide an interpretation of Sánchez-Flores' description of the Lie algebra structure of the first Hochschild cohomology for radical square zero algebras [18] using inverse gluing operations.

Given a quiver Q, denote by S a complete set of representatives of the non-trivial classes on the set of arrows Q_1, that is, equivalence classes having at least two arrows, and for $\alpha \in S$, $|\alpha|$ denotes the number of arrows in the equivalence class $[\alpha]$ of α. Sánchez-Flores' description of the first Hochschild cohomology for radical square zero algebras can be stated as follows.

Theorem 4.12. ([18, Theorem 2.9]) *Let k be a field of characteristic zero and let A be an indecomposable radical square zero algebra having Gabriel quiver Q. Then we have an isomrphisms of Lie algebras:*

$$\mathrm{HH}^1(A) \simeq \oplus_{\alpha\in S}\mathfrak{sl}_{|\alpha|}(k) \oplus k^{\chi(\bar{Q})}.$$

Note that intuitively we can say that $\chi(\bar{Q})$ counts the number of holes in $\bar{Q}$. From this point of view we could give an interpretation of the above result by inverse gluing operations. To be more intuitive we will demonstrate our method by an example that includes all possible cases. Note also that the characteristic zero condition in the above result is necessary since the proof uses the Lie algebra decomposition $\mathfrak{gl}_{|\alpha|}(k) \simeq \mathfrak{sl}_{|\alpha|}(k) \oplus k$ when $\mathrm{char}(k) = 0$.

Example 4.13. Let k be a field of characteristic zero and let A be a radical square zero algebra having Gabriel quiver Q_A. Note that in this case $\chi(\bar{Q}_A) = 4$ and $S = \{[\alpha_1], [\beta_1]\}$.

$$Q_A : \quad \begin{tikzcd}
j \arrow[loop left, "\alpha_1"] \arrow[loop above, "\alpha_2"] \arrow[r, "\eta_1"] & i \arrow[r, "\xi_4"] \arrow[d, "\xi_1"] & h \arrow[r, "\eta_2"] & g \\
a \arrow[d, shift left, "\gamma_2"] & d \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] \arrow[dl, "\beta_3"] & e \arrow[l, "\xi_2"] \arrow[u, "\xi_3"'] & f \arrow[l, "\eta_3"] \\
b \arrow[u, shift left, "\gamma_1"]
\end{tikzcd}$$

Step 1 (separate and reduce loops): We separate the loops at the vertex j of Q_A to get Q_B. The algebra B has two blocks, say B_1 and B_2.

$$Q_B : \quad \begin{tikzcd}
j_1 \arrow[r, "\eta_1"] & i \arrow[r, "\xi_4"] \arrow[d, "\xi_1"] & h \arrow[r, "\eta_2"] & g \\
a \arrow[d, shift left, "\gamma_2"] & d \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] \arrow[dl, "\beta_3"] & e \arrow[l, "\xi_2"] \arrow[u, "\xi_3"'] & f \arrow[l, "\eta_3"] \\
b \arrow[u, shift left, "\gamma_1"] & & B_1 &
\end{tikzcd} \qquad \begin{tikzcd}
j_2 \arrow[loop left, "\alpha_1"] \arrow[loop above, "\alpha_2"] \\
B_2
\end{tikzcd}$$

The inverse operation is given by gluing two vertices (one of which has no loops) from two different blocks, that is, we glue $j_1 \in Q_{B_1}$ and $j_2 \in Q_{B_2}$. By Corollary 4.7, this operation does not change the dimension and the Lie structure of $\mathrm{HH}^1(A)$, that is,

$$\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B) \simeq \mathrm{HH}^1(B_1) \oplus \mathrm{HH}^1(B_2).$$

By Remark 4.11 (1) we obtain $\mathrm{HH}^1(B_2) \simeq \mathfrak{gl}_2(k) \simeq sl_2 \oplus k$, where the summand k contributes 1 to the value of $\chi(\bar{Q}_A)$. After this step, we have reduced Q_A to the no loop quiver Q_{B_1}.

Step 2 (reduce oriented l-cycles ($l \geq 2$)): We reduce the oriented cycle $p := \gamma_2\gamma_1$ in Q_{B_1}. Choose the vertex b in p and split it into a source vertex b_1 and a sink vertex b_2:

$$Q_C : \quad \begin{tikzcd}
 & j_1 \arrow[r, "\eta_1"] & i \arrow[r, "\xi_4"] \arrow[d, "\xi_1"] & h \arrow[r, "\eta_2"] & g \\
 & a \arrow[d, "\gamma_2"] & d \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] \arrow[dl, "\beta_3"] & e \arrow[l, "\xi_2"] \arrow[u, "\xi_3"'] & f \arrow[l, "\eta_3"] \\
b_1 \arrow[ur, "\gamma_1"] & b_2
\end{tikzcd}$$

The inverse operation is given by gluing b_1 and b_2 from the same block. By Remark 4.10 (i), by reducing p from Q_{B_1} we get one summand isomorphic to k (cf. Corollary 4.9), which contributes 1 to the value of $\chi(\bar{Q}_B) = \chi(\bar{Q}_A)$. So

$$\mathrm{HH}^1(B_1) \simeq \mathrm{HH}^1(C) \oplus k$$

and we have reduced Q_{B_1} to the no oriented cycle quiver Q_C.

Step 3 (reduce undirected l-cycles ($l \geq 3$)): We first deal with the undirected 3-cycle $q_1 := \beta_3 - \gamma_2 - \beta_1$ in Q_C. We can split b_2 into two sinks, say b_3 and b_4, and denote the corresponding quiver and algebra by Q_D and D, respectively.

$$Q_D: \quad \begin{tikzcd}
& j_1 \arrow[r, "\eta_1"] & i \arrow[r, "\xi_4"] \arrow[d, "\xi_1"] & h \arrow[r, "\eta_2"] & g \\
& a \arrow[d, "\gamma_2"] & d \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] \arrow[d, "\beta_3"] & e \arrow[l, "\xi_2"] \arrow[u, "\xi_3"] & f \arrow[l, "\eta_3"] \\
b_1 \arrow[ur, "\gamma_1"] & b_3 & b_4 & &
\end{tikzcd}$$

The inverse operation is given by gluing b_3 and b_4 from the same block. By Corollary 4.9 and Remark 4.10, by reducing q_1 from Q_C we get a summand isomorphic to k, which again contributes 1 to the value of $\chi(\bar{Q}_A)$. Therefore

$$\mathrm{HH}^1(C) \simeq \mathrm{HH}^1(D) \oplus k.$$

We then reduce another undirected cycle $q_2 := \xi_4 - \xi_3 - \xi_2 - \xi_1$. Choose the vertex i in q_2 and split i into a sink vertex i_1 and a source vertex i_2 to get Q_E, denote the corresponding algebra by E.

$$Q_E: \quad \begin{tikzcd}
j \arrow[r, "\eta_1"] & i_1 & i_2 \arrow[r, "\xi_4"] \arrow[d, "\xi_1"] & h \arrow[r, "\eta_2"] & g \\
& a \arrow[d, "\gamma_2"] & d \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] \arrow[d, "\beta_3"] & e \arrow[l, "\xi_2"] \arrow[u, "\xi_3"] & f \arrow[l, "\eta_3"] \\
b_1 \arrow[ur, "\gamma_1"] & b_3 & b_4 & &
\end{tikzcd}$$

The inverse operation is given by gluing i_1 and i_2 from two different blocks. By Corollary 4.7, this operation does not change the dimension and the Lie structure of $\mathrm{HH}^1(D)$, that is,

$$\mathrm{HH}^1(D) \simeq \mathrm{HH}^1(E).$$

Note that the above reduction produces a new undirected cycle $q_2' := \xi_4 - \xi_3 - \xi_2 - \xi_1$ in Q_E. However, we can reduce q_2' in Q_E by splitting i_2 into two sources, say i_3 and i_4 (the corresponding quiver is Q_F).

$$Q_F: \quad \begin{tikzcd}
i_1 & i_3 \arrow[dr, "\xi_1"] & i_4 \arrow[r, "\xi_4"] & h \arrow[r, "\eta_2"] & g \\
j \arrow[u, "\eta_1"] & a \arrow[d, "\gamma_2"] & d \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] \arrow[d, "\beta_3"] & e \arrow[l, "\xi_2"] \arrow[u, "\xi_3"] & f \arrow[l, "\eta_3"] \\
b_1 \arrow[ur, "\gamma_1"] & b_3 & b_4 & &
\end{tikzcd}$$

The inverse operation is given by gluing two sources from the same block. Again by Corollary 4.9 and Remark 4.10, we get that

$$\mathrm{HH}^1(E) \simeq \mathrm{HH}^1(F) \oplus k,$$

where the summand k also contributes 1 to the value of $\chi(\bar{Q}_A)$. We have reduced to a quiver Q_F that has neither oriented cycles nor undirected cycles.

Step 4 (Split into several m-Kronecker quivers): Since Q_F contains no cycles (whether oriented or undirected), we can split Q_F into several quivers. Note that each of these quivers is a m-Kronecker quiver for some $m \geq 1$.

$$
Q_G: \quad \begin{tikzcd}
 & & i_4 \arrow[r, "\xi_4"'] & h_3 & \\
i_1 & i_3 \arrow[r, "\xi_1"'] & d_1 & h_2 \arrow[r, "\eta_2"] & g \\
j \arrow[u, "\eta_1"'] & a & d_2 \arrow[l, shift right, "\beta_1"'] \arrow[l, shift left, "\beta_2"] & h_1 & e_3 \arrow[l, "\xi_3"] \\
a_1 & a_2 \arrow[d, "\gamma_2"] & d_3 \arrow[d, "\beta_3"] & f & e_2 \arrow[l, "\eta_3"] \\
b_1 \arrow[u, "\gamma_1"'] & b_3 & b_4 & d_4 & e_1 \arrow[l, "\xi_2"']
\end{tikzcd}
$$

The inverse of the above operations are given by repeatedly applying three types of operations: gluing a source and a sink from different blocks, gluing two sources from different blocks, gluing two sinks from different blocks. By Corollary 4.7, these operations do not change the dimension and the Lie structure of $\mathrm{HH}^1(F)$. Therefore,

$$\mathrm{HH}^1(F) \simeq \mathrm{HH}^1(G).$$

By Remark 4.11 (2) the HH^1 of a m-Kronecker algebra is $\mathfrak{sl}_m(k)$, consequently $\mathrm{HH}^1(G) \simeq \mathfrak{sl}_2(k)$.

We conclude that $\mathrm{HH}^1(B_1) \simeq \mathrm{HH}^1(G) \oplus k^3$, therefore

$$\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B_1) \oplus \mathrm{HH}^1(B_2) \simeq \mathfrak{sl}_2(k)^2 \oplus k^4.$$

5. Center

In this section, we study the behaviour of the centers of finite dimensional quiver algebras under gluing idempotents. Throughout this section we will denote by $Z(A)$ the center of an algebra A.

Definition 5.1. Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Let p be a path between e_1 and e_n in $\mathcal{B}_A$. We call p a *non-special path* between e_1 and e_n in Q_A if $\delta_B^0(f_1/\!/p') = 0$, or equivalently, if $ap \in I_A$ and $pb \in I_A$ for arbitrary $a, b \in (Q_A)_1$.

We denote by NSp_1^n the set of non-special paths between e_1 and e_n in Q_A, and by V_{nsp} the k-subspace of $k((Q_B)_0/\!/\mathcal{B}_B)$ generated by the elements $f_1/\!/p'$ for $p \in \mathrm{NSp}_1^n$. Furthermore, we denote by nsp_1^n the dimension of V_{nsp}.

As the name suggests, the notion of non-special path is exactly the opposite notion of special path. It is clear that there are no non-special paths between e_1 and e_n when we glue these two idempotents from different blocks. By Lemma 3.6 the dimension of V_{nsp} equals the number of non-special paths between e_1 and e_n, that is, $\mathrm{nsp}_1^n = |\mathrm{NSp}_1^n|$.

Notation 3. Similarly to Notation 1 and Definition 3.5, we denote by

- $\delta^0_{(A)_{\geq 1}}$ the map δ^0_A restricted to the subspace $k((Q_A)_0/\!/(\mathcal{B}_A)_{\geq 1})$;
- $\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ the kernel of the map $\delta^0_{(A)_{\geq 1}}$;
- $\widetilde{\mathrm{Sp}_1^n}$ the k-subspace of $k((Q_B)_0/\!/\mathcal{B}_B)$ generated by the elements $f_1/\!/p'$ for $p \in \mathrm{Sp}_1^n$;
- $\delta^0_B|_{\widetilde{\mathrm{Sp}_1^n}}$ the map δ^0_B restricted to $\widetilde{\mathrm{Sp}_1^n}$;
- $\mathrm{Ker}(\delta^0_B|_{\widetilde{\mathrm{Sp}_1^n}})$ the kernel of the map $\delta^0_B|_{\widetilde{\mathrm{Sp}_1^n}}$.

Since $V_{sp} = \mathrm{Im}(\delta^0_B|_{\widetilde{\mathrm{Sp}_1^n}})$ and $\dim_k \widetilde{\mathrm{Sp}_1^n} = |\mathrm{Sp}_1^n|$, we have $\dim_k \mathrm{Ker}(\delta^0_B|_{\widetilde{\mathrm{Sp}_1^n}}) = |\mathrm{Sp}_1^n| - \mathrm{sp}_1^n$.

Lemma 5.2. *Let $A = kQ_A/I_A$ be a quiver algebra and let $B = kQ_B/I_B$ be a radical embedding obtained by gluing two idempotents e_1 and e_n of A. Then there is a decomposition as k-vector spaces*

$$\mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \varphi_0(\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})) \oplus V_{nsp} \oplus \mathrm{Ker}(\delta^0_B|_{\widetilde{\mathrm{Sp}^n_1}}).$$

In particular, if we glue e_1 and e_n from the same block, then

$$\dim_k \mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_{\geq 1}}) + \mathrm{nsp}^n_1 + |\mathrm{Sp}^n_1| - \mathrm{sp}^n_1;$$

if we glue e_1 and e_n from different blocks, then $\dim_k \mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$.

Proof. Recall from Proposition 3.3 that there is a k-linear map $\varphi_0 : k((Q_A)_0/\!/\mathcal{B}_A) \to k((Q_B)_0/\!/\mathcal{B}_B)$. A direct computation shows that $\delta^0_B(\varphi_0(e_i/\!/p)) = \varphi_1(\delta^0_A(e_i/\!/p))$ for $1 \leq i \leq n$ and $p \in \mathcal{B}_A \backslash \{e_1, e_n\}$. It follows that φ_0 induces an injective k-linear map from $\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ to $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$.

Let $\theta \in \mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$ be in the complement of the subspace $\varphi_0(\mathrm{Ker}(\delta^0_{(A)_{\geq 1}}))$. Then we assume that θ is a linear combination of the elements of the form $f_1/\!/p'$ such that p is a path between e_1 and e_n. If p is a non-special path, then $f_1/\!/p' \in V_{nsp} \subseteq \mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$. Otherwise, $p \in \mathrm{Sp}^n_1$. Note that there may exist another special path $q \neq p$ such that the summands of $\delta^0_B(f_1/\!/p')$ and the summands of $\delta^0_B(f_1/\!/q')$ can be cancelled by each other. Consequently, θ can be a linear combination of the elements in V_{nsp} and in $\mathrm{Ker}(\delta^0_B|_{\widetilde{\mathrm{Sp}^n_1}})$. Therefore, the formula $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \varphi_0(\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})) \oplus V_{nsp} \oplus \mathrm{Ker}(\delta^0_B|_{\widetilde{\mathrm{Sp}^n_1}})$ follows. □

Remark 5.3. Note that in the monomial case, Lemma 5.2 can be simplified since the space $\mathrm{Ker}(\delta^0_B|_{\widetilde{\mathrm{Sp}^n_1}})$ vanishes. This is because in this case Lemma 3.6 holds for any path $q \in \mathcal{B}_A$ with $q \neq p$. Moreover, by the same reason, we have $\mathrm{sp}^n_1 = |\mathrm{Sp}^n_1|$, that is, the dimension of V_{sp} is equal to the number of special paths. In addition, the number of special pairs is greater than or equal to the number of special paths. Note that in general these statements are not true (cf. Example 6.6).

First, we deal with the case that the algebra A is indecomposable.

Proposition 5.4. *Let A be an indecomposable finite dimensional quiver k-algebra and let B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then there is an algebra monomorphism $Z(A) \hookrightarrow Z(B)$. Moreover,*

$$\dim_k Z(B) = \dim_k Z(A) + \mathrm{nsp}^n_1 + |\mathrm{Sp}^n_1| - \mathrm{sp}^n_1.$$

Proof. We adopt the notation in Proposition 3.3 and identify the centers $Z(A)$, $Z(B)$ as $\mathrm{Ker}(\delta^0_A)$, $\mathrm{Ker}(\delta^0_B)$ respectively. Also notice that $\mathrm{Ker}(\delta^0_A) = \mathrm{Ker}(\delta^0_{(A)_0}) \oplus \mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ as k-vector spaces and a similar decomposition applies for $\mathrm{Ker}(\delta^0_B)$.

By Lemma 5.2 we have that φ_0 induces an injective k-linear map from $\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ to $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$, and $\dim_k \mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_{\geq 1}}) + \mathrm{nsp}^n_1 + |\mathrm{Sp}^n_1| - \mathrm{sp}^n_1$. By using the fact that $\mathrm{Ker}(\delta^0_{(A)_0}) = \langle \sum_{1 \leq i \leq n} e_i/\!/e_i \rangle$ and $\mathrm{Ker}(\delta^0_{(B)_0}) = \langle \sum_{1 \leq i \leq n-1} f_i/\!/f_i \rangle$, cf. proof of Lemma 3.4, we deduce that $\dim_k \mathrm{Ker}(\delta^0_{(B)_0}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_0})$. Hence the second statement follows. Moreover, there is an injective k-linear map $\varphi_0 : \mathrm{Ker}(\delta^0_A) \to \mathrm{Ker}(\delta^0_B)$. Note that we can identify $\mathrm{Ker}(\delta^0_A)$ with $Z(A)$ by $\sum e_i/\!/p \mapsto \sum p$ and $\sum_{i=1}^n e_i/\!/e_i \mapsto 1_A$, so does for $\mathrm{Ker}(\delta^0_B)$ and $Z(B)$. Then, by the fact that $p'q' = (pq)'$ for $p, q \in (\mathcal{B}_A \backslash \{e_1, \cdots, e_n\})$, φ_0 gives an algebra monomorphism, and the first statement follows. □

Corollary 5.5. *Let A be an indecomposable finite dimensional quiver k-algebra and let B be a radical embedding of A obtained by gluing a source vertex e_1 and a sink vertex e_n. Then φ_0 : $\mathrm{Ker}(\delta^0_A) \hookrightarrow \mathrm{Ker}(\delta^0_B)$ is an isomorphism if and only if there is no path from e_1 to e_n.*

Proof. Note that in this case, $\mathrm{Sp}_1^n = \emptyset$ and p is a non-special path between e_1 and e_n if and only if p is a path from e_1 to e_n. Thus the result follows from Proposition 5.4. $\square$

Corollary 5.6. *Let A be a radical square zero indecomposable finite dimensional algebra and let B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then $\varphi_0 : \mathrm{Ker}(\delta_A^0) \hookrightarrow \mathrm{Ker}(\delta_B^0)$ is isomorphism if and only if there are no arrows between e_1 and e_n in Q_A.*

Proof. For radical square zero algebras, the set NSp_1^n consists of all arrows between e_1 and e_n in Q_A and there is no special path between e_1 and e_n in Q_A. $\square$

Note that Cibils has shown in [3] that the dimension of the center of an indecomposable radical square zero algebra is given by $|Q_1/\!/Q_0| + 1$. Indeed, by the proof of Proposition 5.4, we know that the basis of the center of an indecomposable radical square zero algebra is provided by the set of loops together with the unit element of the algebra.

Next we deal with the case that the algebra A is not indecomposable. Without loss of generality we assume that A has two blocks, say A_1 and A_2, and assume that B is an algebra obtained from A by gluing $e_1 \in A_1$ and $e_n \in A_2$.

Proposition 5.7. *Let A be a finite dimensional quiver algebra with two blocks A_1 and A_2. Let B be a radical embedding of A obtained by gluing idempotents $e_1 \in A_1$ and $e_n \in A_2$. Then the radical embedding $B \to A$ restricts to a radical embedding $Z(B) \to Z(A)$. In particular, $\dim_k Z(A) = \dim_k Z(B) + 1$.*

Proof. Let $\mathcal{B}_A = \{e_1, \cdots, e_n, p_1, \cdots, p_u \mid \text{the length of each } p_i \text{ is } \geq 1\}$ denotes a k-basis of the quiver algebra A (cf. Section 2). Then the subalgebra B of A has a k-basis $\mathcal{B}_B = \{e_1 + e_n, e_2, \cdots, e_{n-1}, p_1, \cdots, p_u\}$. We identify the centers $Z(A), Z(B)$ with $\mathrm{Ker}(\delta_A^0), \mathrm{Ker}(\delta_B^0)$ respectively. Let $Z(A) = Z(A)_0 \oplus Z(A)_{\geq 1}$ be the decomposition corresponding to $\mathrm{Ker}(\delta_A^0) = \mathrm{Ker}(\delta^0_{(A)_0}) \oplus \mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ as k-vector spaces, so does for $Z(B)$.

By Lemma 5.2, we obtain that $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) \simeq \mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$, hence

$$Z(A)_{\geq 1} = \langle \sum p \mid p \text{ is a cycle in } \mathcal{B}_A \rangle = Z(B)_{\geq 1}.$$

Note that $Z(A)_0 = \langle 1_{A_1}, 1_{A_2} \rangle$, where 1_{A_j} denotes the unit element in A_j for $j = 1, 2$, and $Z(B)_0 = \langle 1_B = 1_{A_1} + 1_{A_2} \rangle$. Therefore, there is an embedding from $Z(B)$ to $Z(A)$ which sends 1_B to $1_{A_1} + 1_{A_2}$ and each element in $Z(B)_{\geq 1}$ to the corresponding element in $Z(A)_{\geq 1}$.

It is clear that this embedding from $Z(B)$ to $Z(A)$ is an injection of algebras and preserves the radical, hence, by gluing $e_1 \in A_1$ and $e_n \in A_2$, we get a radical embedding from $Z(B)$ to $Z(A)$. In particular, we have $\dim_k Z(A) = \dim_k Z(B) + 1$. $\square$

6. Examples

We give some examples concerning the main results in this paper. The first one shows that our gluing technique is useful for computing the HH^1 of non-monomial algebras.

Example 6.1. Let the algebra B be obtained from A by gluing source e_1 and sink e_4:

$$Q_A: \quad \begin{tikzcd} e_1 \bullet \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \bullet e_3 \arrow[d, "\eta"] \\ e_2 \bullet \arrow[r, "\beta"'] & \bullet e_4 \end{tikzcd} \qquad\qquad Q_B: \quad \begin{tikzcd} f_1 \bullet \arrow[r, shift left, "\gamma'"] \arrow[d, shift right, "\alpha'"'] & \bullet f_3 \arrow[l, shift left, "\eta'"] \\ f_2 \bullet \arrow[u, shift right, "\beta'"'] & \end{tikzcd}$$

Where $Z_A = \{\beta\alpha - \eta\gamma\}, Z_{new} = \{\alpha'\beta', \gamma'\beta', \alpha'\eta', \gamma'\eta'\}$ and $Z_B = Z_A \cup Z_{new}$. We fix the order on $(Q_A)_1$ by $\eta \succ \gamma \succ \beta \succ \alpha$, then $\mathrm{Tip}(Z_A) = \{\eta\gamma\}$. It follows that $\mathcal{G}_A = \{\eta\gamma - \beta\alpha\}$ and $\mathcal{G}_B = \mathcal{G}_A \cup Z_{new}$. A direct computation based on Theorem 2.7 shows that $\delta_A^1(\alpha/\!/\alpha) = \eta\gamma/\!/(\eta\gamma)^{\alpha/\!/\alpha} -$

$\eta\gamma /\!/ (\beta\alpha)^{\alpha/\!/\alpha} = -\eta\gamma/\!/\beta\alpha = \delta_A^1(\beta/\!/\beta), \delta_A^1(\gamma/\!/\gamma) = \eta\gamma/\!/\eta\gamma = \delta_A^1(\eta/\!/\eta)$. Similarly we can compute δ_A^0, δ_B^0 and δ_B^1. Note that in this case $V_{sp} = V_{spp} = 0$. Observe that $\beta\alpha = \eta\gamma$ in $\mathcal{B}_A$, we obtain that

$$\mathrm{Im}(\delta_A^0) \simeq \langle \beta/\!/\beta - \alpha/\!/\alpha, \eta/\!/\eta - \gamma/\!/\gamma, \alpha/\!/\alpha + \gamma/\!/\gamma \rangle \simeq \mathrm{Ker}(\delta_A^1) \simeq \mathrm{Ker}(\delta_B^1),$$
$$\mathrm{Im}(\delta_B^0) \simeq \langle \beta'/\!/\beta' - \alpha'/\!/\alpha', \eta'/\!/\eta' - \gamma'/\!/\gamma' \rangle.$$

Therefore,

$$\mathrm{HH}^1(A) \simeq \mathrm{Ker}(\delta_A^1)/\mathrm{Im}(\delta_A^0) = 0,$$
$$\mathrm{HH}^1(B) \simeq \mathrm{Ker}(\delta_B^1)/\mathrm{Im}(\delta_B^0) \simeq \langle \alpha'/\!/\alpha' + \gamma'/\!/\gamma' \rangle \simeq \mathrm{HH}^1(A) \oplus k.$$

It is clear that $\beta\alpha = \eta\gamma$ is a non-special path between e_1 and e_4 in Q_A, hence

$$Z(A) \simeq \mathrm{Ker}(\delta_A^0) \simeq \langle \sum_{i=1}^{4} e_i/\!/e_i \rangle \hookrightarrow Z(B) \simeq \mathrm{Ker}(\delta_B^0) \simeq \langle \sum_{i=1}^{3} f_i/\!/f_i, f_1/\!/\beta'\alpha' \rangle.$$

The second example shows a particular instance of Corollary 4.7 in which B is not obtained from A by gluing a source and a sink:

Example 6.2. Assume $\mathrm{char}(k) \neq 2$. The algebra B is obtained from A by gluing e_1 and e_3:

$$Q_A: \quad \begin{tikzcd} e_1\bullet & \bullet e_2 \arrow[l, "\alpha_1"'] \arrow[r, "\alpha_2"] & \bullet e_3 \end{tikzcd} \qquad Q_B: \quad \begin{tikzcd} f_2\bullet \arrow[r, shift left, "\alpha_1'"] \arrow[r, shift right, "\alpha_2'"'] & \bullet f_1 \end{tikzcd}$$

Where $Z_A = Z_B = \emptyset$. Note that A is hereditary, and the underlying graph of Q_A is a tree, therefore $\mathrm{HH}^1(A) = 0$. We have that $V_{sp} = 0$ and V_{spp} has a k-basis given by $\{\alpha_1'/\!/\alpha_2', \alpha_2'/\!/\alpha_1'\}$. By Corollary 4.7 the dimension of $\mathrm{HH}^1(B)$ is 3. Indeed, a direct computation shows $\mathrm{HH}^1(B) \cong \mathfrak{sl}_2(k)$ having a k-basis given by $\{\alpha_1'/\!/\alpha_1', \alpha_1'/\!/\alpha_2', \alpha_2'/\!/\alpha_1'\}$.

The next two examples show that the characteristic condition in Proposition 3.10 is necessary.

Example 6.3. Assume that $\mathrm{char}(k) = 2$, and that B is obtained from A by gluing e_1 and e_2:

$$Q_A: \quad \begin{tikzcd} e_1\bullet \arrow[loop above, "\alpha"] \arrow[r, shift left, "\beta"] & \bullet e_2 \arrow[l, shift left, "\gamma"] \end{tikzcd} \qquad Q_B: \quad \begin{tikzcd} f_1\bullet \arrow[loop above, "\beta'"] \arrow[loop left, "\alpha'"] \arrow[loop right, "\gamma'"] \end{tikzcd}$$

Where $Z_A = \{r_1 = \alpha^2 - \gamma\beta, r_2 = \beta\alpha\gamma, \beta\gamma\}$, $Z_{new} = \{r_3 = \alpha'\beta', r_4 = \gamma'\alpha', (\gamma')^2, (\beta')^2\}$ and $Z_B = Z_A \cup Z_{new}$. We fix the order on $(Q_A)_1$ by $\gamma \prec \beta \prec \alpha$. Then $\mathcal{G}_A = Z_A$ and $\mathcal{G}_B = Z_B$. A direct computation shows that $\delta_A^1(\alpha/\!/e_1) = 2\alpha^2/\!/\alpha + r_2/\!/\beta\gamma = 2\alpha^2/\!/\alpha = 0$ since $\mathrm{char}(k) = 2$. However, $\delta_B^1(\alpha'/\!/f_1) = 2(\alpha')^2/\!/\alpha' + r_3'/\!/\beta' + r_4'/\!/\gamma' \neq 0$, which means that although $\alpha/\!/e_1 \in \mathrm{Ker}(\delta_A^1)$, $\varphi_1(\alpha/\!/e_1) = \alpha'/\!/f_1 \notin \mathrm{Ker}(\delta_B^1)$. Hence φ_1 does not induce an injective k-linear map from $\mathrm{Ker}(\delta_A^1)$ to $\mathrm{Ker}(\delta_B^1)$.

Example 6.4. Let A be given by two blocks A_1 and A_2 such that A_1 is isomorphic to $k[x]/(x^2)$ and A_2 is isomorphic to $k[y]/(y^2)$. Let B be obtained by gluing the units of A_1 and A_2. Then $\mathrm{Ker}(\delta_B^1) = \mathrm{HH}^1(B) \simeq \mathfrak{gl}_2(k)$ and has k-basis given by $\{x/\!/x,\ x/\!/y,\ y/\!/x,\ y/\!/y\}$. However, there are two cases for A.

(1) If $\mathrm{char}(k) \neq 2$, then $\mathrm{Ker}(\delta_A^1) = \mathrm{HH}^1(A) \simeq k \oplus k$ has k-basis given by $x/\!/x$ and $y/\!/y$, and there is an injective Lie algebra homomorphism $\mathrm{Ker}(\delta_A^1) \hookrightarrow \mathrm{Ker}(\delta_B^1)$.

(2) If $\mathrm{char}(k) = 2$, then $\mathrm{Ker}(\delta_A^1) = \mathrm{HH}^1(A)$ has k-basis given by $\{x/\!/x,\ x/\!/e_1,\ y/\!/y,\ y/\!/e_2\}$. Clearly in this case we cannot get an injective Lie algebra homomorphism from $\mathrm{Ker}(\delta_A^1)$ to $\mathrm{Ker}(\delta_B^1)$.

In the following example, we compute explicitly the special paths and the k-space V_{sp} (resp. the special pairs and the k-space V_{spp}) appeared in Definition 3.5 and Proposition 3.8 (resp. in Definition 3.12 and Proposition 3.14).

Example 6.5. Let B be obtained from A by gluing e_1 and e_4:

$$
\begin{tikzcd}
Q_A: & e_2\bullet \arrow[r, "a"] & e_1\bullet \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_n"'] \arrow[r, phantom, "\vdots"] & e_4\bullet \arrow[r, "b"] & e_3\bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & f_3\bullet & \\
Q_B: & f_2\bullet \arrow[r, "a'"] & \bullet f_1 \arrow[ul, "b'"'] \arrow[loop above, "\alpha'_1"] \arrow[loop right, "\alpha'_n"]
\end{tikzcd}
$$

Where $Z_A = \emptyset$, $Z_B = Z_{new} = \{\alpha'_i\alpha'_j \mid 1 \leq i, j \leq n\}$. Since $\alpha_i a \notin I_A$ for $1 \leq i \leq n$, α_i is a special path from e_1 to e_4 for $1 \leq i \leq n$, we have $\mathrm{Sp}_1^4 = \{\alpha_i \mid 1 \leq i \leq n\}$ and

$$
\begin{aligned}
V_{sp} &= \langle \delta_B^0(f_1/\!/\alpha'_i) \mid 1 \leq i \leq n\rangle \\
&= \langle b'/\!/b'\alpha'_i - a'/\!/\alpha'_i a' \mid 1 \leq i \leq n\rangle.
\end{aligned}
$$

Hence $\mathrm{sp}_1^4 = n = \dim_k V_{sp}$. Since $a'/\!/\alpha'_i a', b'/\!/b'\alpha'_i, \alpha'_i/\!/f_1$, $a \not/\!/ \alpha_i a, b \not/\!/ b\alpha_i, \alpha_i \not/\!/ e_1, \alpha_i \not/\!/ e_4$, we know that $(a, \alpha_i a), (b, b\alpha_i), (\alpha_i, e_1), (\alpha_i, e_n)$ are special pairs with respect to the gluing of e_1 and e_4 for $1 \leq i \leq n$, and $\mathrm{Spp}_1^4 = \{(a, \alpha_i a), (b, b\alpha_i), (\alpha_i, e_1), (\alpha_i, e_n) \mid 1 \leq i \leq n\}$. As a result we get

$$
\langle \mathrm{Spp}_1^4\rangle = \langle a'/\!/\alpha'_i a', b'/\!/b'\alpha'_i, \alpha'_i/\!/f_1 \mid 1 \leq i \leq n\rangle,
$$

$$
\begin{aligned}
V_{spp} &= \langle \mathrm{Spp}_1^4\rangle \cap \mathrm{Ker}(\delta_B^1) \\
&= \langle a'/\!/\alpha'_i a', b'/\!/b'\alpha'_i \mid 1 \leq i \leq n\rangle.
\end{aligned}
$$

Hence $\mathrm{kspp}_1^4 = \dim_k V_{spp} = 2n$. A direct computation shows that $\mathrm{Im}(\delta_A^0), \mathrm{Im}(\delta_B^0)$ are 3-dimensional and $(n+2)$-dimensional, respectively, since

$$
\mathrm{Im}(\delta_A^0) = \langle a/\!/a, b/\!/b, \sum_{i=1}^{n} \alpha_i/\!/\alpha_i\rangle,
$$

$$
\mathrm{Im}(\delta_B^0) = \langle a'/\!/a', b'/\!/b', b'/\!/b'\alpha'_i - a'/\!/\alpha'_i a' \mid 1 \leq i \leq n\rangle.
$$

Therefore,

$$
\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 - \mathrm{sp}_1^4.
$$

In addition,

$$
\mathrm{Ker}(\delta_A^1) = \langle a/\!/a, b/\!/b, \alpha_i/\!/\alpha_j \mid 1 \leq i, j \leq n\rangle
$$

is (n^2+2)-dimensional and

$$
\mathrm{Ker}(\delta_B^1) = \langle a'/\!/a', b'/\!/b', \alpha'_i/\!/\alpha'_j, b'/\!/b'\alpha'_i, a'/\!/\alpha'_i a' \mid 1 \leq i, j \leq n\rangle
$$

is (n^2+2n+2)-dimensional. Hence

$$
\dim_k \mathrm{Ker}(\delta_B^1) = \dim_k \mathrm{Ker}(\delta_A^1) + \mathrm{kspp}_1^4.
$$

One can verify that $\mathrm{HH}^1(A)$ is isomorphic to $\mathfrak{pgl}_n(k)$ and $\mathrm{HH}^1(B)$ contains a subalgebra isomorphic to $\mathfrak{gl}_n(k)$. Note also that by the notations in the proof of Theorem 3.19, in this example the subspace Y of $\mathrm{Ker}(\delta_B^1)$ is equal to $\mathrm{Im}(\delta_B^0) \oplus \langle \sum_{i=1}^{n} \alpha'_i/\!/\alpha'_i\rangle$ and Y is not a Lie ideal of $\mathrm{Ker}(\delta_B^1)$.

The following example shows that in non-monomial case, the dimension of V_{sp} is not equal to the number of special paths and the number of special pairs may be smaller than the number of special paths in general.

Example 6.6. The algebra B is obtained from A by gluing e_1 and e_4:

$$
\begin{tikzcd}
Q_A: & e_2\bullet \arrow[r, "a"] & \bullet e_1 \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & \bullet e_3 \arrow[r, "c"] & \bullet e_4
\end{tikzcd}
\qquad
\begin{tikzcd}
Q_B: & f_2\bullet \arrow[r, "a'"] & \bullet f_1 \arrow[r, shift left=2, "b'_1"] \arrow[r, dashed, "b'_2" description] & \bullet f_3 \arrow[l, shift left=2, "c'"]
\end{tikzcd}
$$

Where $Z_A = \{cb_1a - cb_2a\}$, $Z_{new} = \{b_1'c', b_2'c'\}$ and $Z_B = Z_A \cup Z_{new}$. We fix the order on $(Q_A)_1$ by $c \prec b_2 \prec b_1 \prec a$. Then it is clear that $\mathcal{G}_A = Z_A$ and $\mathcal{G}_B = Z_B$. Moreover, we have $\mathrm{Sp}_1^4 = \{cb_1, cb_2\}$ and $\mathrm{Spp}_1^4 = \{(a, cb_1a) = (a, cb_2a)\}$, $\delta_B^0(f_1/\!/c'b_1') = -a'/\!/c'b_1'a'$ and $\delta_B^0(f_1/\!/c'b_2') = -a'/\!/c'b_2'a'$. Note that $c'b_1'a' = c'b_2'a'$ in $\mathcal{B}_B$, we get $V_{sp} = \langle a'/\!/c'b_1'a'\rangle = V_{spp}$ and $f_1/\!/c'b_1' - f_1/\!/c'b_2' \in \mathrm{Ker}(\delta_B^0)$. Therefore, $\mathrm{Ker}(\delta_B^0|_{\langle \mathrm{Sp}_1^4\rangle}) = \langle f_1/\!/c'b_1' - f_1/\!/c'b_2'\rangle$ is non-empty, $\mathrm{sp}_1^4 = \dim_k V_{sp} = 1 < |\mathrm{Sp}_1^4| = 2$ and the number of special pairs $|\mathrm{Spp}_1^4| = 1$ is less than the number of special paths $|\mathrm{Sp}_1^4| = 2$.

By Corollary 3.15, if B is a radical embedding obtained by gluing a source vertex e_1 and a sink vertex e_n of A (in case $\mathrm{char}(k) = 2$, we assume that B has no block isomorphic to $k[x]/(x^2)$), then $\mathrm{Ker}(\delta_B^1) \simeq \mathrm{Ker}(\delta_A^1)$. However, the converse of Corollary 3.15 is not true in general as the following example shows.

Example 6.7. Let B be obtained from A by gluing e_1 and e_4:

$$Q_A: \quad e_2\bullet \xrightarrow{a} e_1\bullet \underset{\alpha_n}{\overset{\alpha_1}{\vdots}} e_4\bullet \xleftarrow{b} e_3\bullet \qquad\qquad Q_B: \quad \begin{array}{l} f_3\bullet \xrightarrow{b'} \bullet f_1 \\ f_2\bullet \xrightarrow{a'} \bullet f_1 \circlearrowleft \alpha_1', \ldots, \alpha_n' \end{array}$$

Where $Z_A = \{\alpha_i a \mid 1 \le i \le n\}$, $Z_{new} = \{\alpha_i' b', \alpha_i'\alpha_j' \mid 1 \le i, j \le n\}$ and $Z_B = Z_A \cup Z_{new}$. Note that although $\mathrm{Spp}_1^4 = \{(\alpha_i, e_1), (\alpha_i, e_4) \mid 1 \le i \le n\}$, we have $V_{spp} = \langle \mathrm{Spp}_1^4\rangle \cap \mathrm{Ker}(\delta_B^1) = \langle \alpha_i'/\!/f_1\rangle \cap \mathrm{Ker}(\delta_B^1) = 0$. By Proposition 3.14 we have $\dim_k \mathrm{Ker}(\delta_B^1) = \dim_k \mathrm{Ker}(\delta_A^1)$. In fact, a direct computation shows that both

$$\mathrm{Ker}(\delta_A^1) = \langle a/\!/a, b/\!/b, \alpha_i/\!/\alpha_j \mid 1 \le i, j \le n\rangle$$

and

$$\mathrm{Ker}(\delta_B^1) = \langle a'/\!/a', b'/\!/b', \alpha_i'/\!/\alpha_j' \mid 1 \le i, j \le n\rangle$$

are (n^2+2)-dimensional. Hence although we do not glue a source and a sink, we have $\mathrm{Ker}(\delta_B^1) \simeq \mathrm{Ker}(\delta_A^1)$.

The following example shows various types of special pairs.

Example 6.8. In this example we always assume that B is obtained from A by gluing e_1 and e_n, and that α is an arrow in Q_A and p is a path in $\mathcal{B}_A$. It can be proved that the special pairs (α,p) rise exclusively from the following seven cases and their dual cases:

(i) : α is a loop at e_1 or e_n, assume that $\overset{\alpha}{\circlearrowright} e_1\bullet$. (The case that $\overset{\alpha}{\circlearrowright} e_n\bullet$ is dual.)

Case 1: $p = a_n \cdots a_1$ is an oriented cycle at e_n or $p = e_n$, such as:

$$\overset{\alpha}{\circlearrowright} e_1\bullet \longrightarrow \bullet \quad \cdots \quad \bullet \longleftarrow \bullet e_n, \qquad \bullet e_n \xrightarrow{a_1} \cdots \xrightarrow{a_n} \bullet e_n \;;$$

Case 2: $p = a_n \cdots a_1$ is a path between e_1 and e_n, such as:

$$\overset{\alpha}{\circlearrowright} e_1\bullet \xrightarrow{a_1} \bullet \quad \cdots \quad \bullet \xrightarrow{a_n} \bullet e_n \;;$$

(ii) : α is an arrow between e_1 and e_n, assume that $e_1\bullet \xrightarrow{\alpha} \bullet e_n$. (The case that $e_n\bullet \xrightarrow{\alpha} \bullet e_1$ is dual.)

Case 3: $p = a_n \cdots a_1$ is an oriented cycle at e_1 or e_n or $p = e_1$ or e_n, such as:

$$\begin{tikzcd} & \cdots \arrow[dl, "a_n"] \\ \bullet \arrow[from=d, "a_1"] & \\ e_1 \bullet \arrow[r, "\alpha"] & \bullet e_n \end{tikzcd} ;$$

Case 4: $p = a_n \cdots a_1$ is a path from e_n to e_1, such as:

$$\begin{tikzcd} e_1 \bullet \arrow[rr, "\alpha"] & & \bullet e_n \arrow[dl, "a_1"] \\ & \cdots \arrow[ul, "a_n"] & \end{tikzcd} ;$$

(iii) : Exactly one of the vertex of α is e_1 or e_n, assume that $e_1 \bullet \xrightarrow{\alpha} \bullet$. (The other cases are dual.)

Case 5: $p = a_n \cdots a_1$ is a path from e_n to $t(\alpha)$, such as:

$$\begin{tikzcd} e_1 \bullet \arrow[r, "\alpha"] & \bullet \arrow[r] & \bullet \cdots \bullet & \bullet e_n \arrow[l] \arrow[dl, "a_1"] \\ & & \cdots \arrow[ull, "a_n"] & \end{tikzcd} ;$$

Case 6: $p = \alpha p_1$, where $p_1 = a_n \cdots a_1$ is a path from e_n to e_1, such as:

$$\begin{tikzcd} e_1 \bullet \arrow[r, "\alpha"] & \bullet \cdots \bullet & \bullet e_n \arrow[l] \arrow[dl, "a_1"] \\ & \cdots \arrow[ul, "a_n"] & \end{tikzcd} ;$$

Case 7: $p = p_2 \alpha p_1$, where $p_1 = a_n \cdots a_1$ is a path from e_n to e_1 and $p_2 = b_m \cdots b_1$ is a cycle at $t(\alpha)$, such as:

$$\begin{tikzcd} & \cdots \arrow[d, "b_m"] & & \\ e_1 \bullet \arrow[r, "\alpha"] & \bullet \arrow[u, bend left, "b_1"] \quad \cdots \quad \bullet & & \bullet e_n \arrow[ll] \arrow[dll, "a_1"] \\ & \cdots \arrow[ul, "a_n"] & & \end{tikzcd} .$$

After giving relations in specific examples, we can show that the special pair (α, p) in each of the above cases can appear. Indeed, the following example covers all the above 7 cases:

$$Q_A: \begin{tikzcd} e_2 \bullet \arrow[loop left, "d"] & e_1 \bullet \arrow[l, "\gamma"'] \arrow[loop above, "\alpha"] \arrow[r, shift left, "\beta"] & \bullet e_3 \arrow[l, shift left, "a"] \arrow[ll, bend left, "c"] \end{tikzcd} \qquad Q_B: \begin{tikzcd} f_2 \bullet \arrow[loop left, "d'"] & \bullet f_1 \arrow[l, shift right, "\gamma'"'] \arrow[l, shift left, "c'"] \arrow[loop above, "\alpha'"] \arrow[loop right, "\beta'"] \arrow[loop below, "a'"] \end{tikzcd}$$

Where Z_A consists of all paths in Q_A of length 3 except $d\gamma a$, $Z_B = Z_A \cup Z_{new}$ where $Z_{new} = \{a'\alpha', c'\alpha', \alpha'\beta', (\beta')^2, \gamma'\beta', (a')^2, c'a'\}$. We list all special pairs (α, p) for each case as follows:

Case 1: $(\alpha, \beta a)$, (α, e_3);

Case 2: (α, a), (α, β), $(\alpha, \beta\alpha)$, $(\alpha, \alpha a)$;

Case 3: (β, α), $(\beta, a\beta)$, $(\beta, \beta a)$, (β, e_1), (β, e_3) (a, α), $(a, a\beta)$, $(a, \beta a)$, (a, e_1), (a, e_3);

Case 4: (β, a), (a, β);

Case 5: (γ, c), (γ, dc), $(c, \gamma\alpha)$, $(c, d\gamma)$;

Case 6: $(\gamma, \gamma a)$, $(c, c\beta)$;

Case 7: $(\gamma, d\gamma a)$.

By check one by one, we have Spp_1^3 is the set consisting of these 25 special pairs and $\langle \mathrm{Spp}_1^3 \rangle = \langle \alpha' /\!/ p' \mid (\alpha, p) \in \mathrm{Spp}_1^3 \rangle$ and therefore

$$\begin{aligned} V_{spp} &= \langle \mathrm{Spp}_1^3 \rangle \cap \mathrm{Ker}(\delta_B^1) \\ &= \langle a'/\!/\beta' a', \alpha'/\!/\beta'\alpha', \alpha'/\!/\alpha' a', \beta'/\!/a'\beta', \beta'/\!/\beta' a', a'/\!/a'\beta', \\ &\quad a'/\!/\beta' a', \gamma'/\!/d'c', c'/\!/\gamma'\alpha', c'/\!/d'\gamma', \gamma'/\!/\gamma' a', c'/\!/c'\beta', \gamma'/\!/d'\gamma' a' \rangle. \end{aligned}$$

Hence $\mathrm{kspp}_1^3 = 13$. Note also that the special paths in this example are β and a, so $\mathrm{sp}_1^3 = 2$.

It worth to mention that, although the k-space $\langle \mathrm{Spp}_1^n \rangle$ is generated by the elements of the form $\alpha'/\!/p'$ (where α is an arrow and p is a path), an element in V_{spp} is usually a k-linear combination of such elements.

Example 6.9. Let B be obtained from A by gluing e_1 and e_5:

$$Q_A : e_2\bullet \xrightarrow{\ b\ } e_1\bullet \xrightarrow{\ c\ } e_3\bullet \xrightarrow{\ d\ } e_5\bullet \xrightarrow{\ a\ } \bullet e_4 \qquad \begin{tikzcd} & \bullet f_4 & \\ Q_B : f_2\bullet \arrow[r, "b'"] & \bullet f_1 \arrow[u, "a'"] \arrow[r, shift left, "c'"] & \bullet f_3 \arrow[l, shift left, "d'"] \end{tikzcd}$$

Where $Z_A = \emptyset$ and $Z_B = Z_{new} = \{a'b', c'd'\}$. It follows from a direct calculation that

$$\mathrm{Im}(\delta_A^0) = \langle a/\!/a, b/\!/b, c/\!/c, d/\!/d \rangle = \mathrm{Ker}(\delta_A^1).$$

Hence $\mathrm{HH}^1(A) = 0$. Similarly we have

$$\mathrm{Im}(\delta_B^0) = \langle a'/\!/a', b'/\!/b', d'/\!/d' - c'/\!/c', a'/\!/a'd'c' - b'/\!/d'c'b' \rangle,$$

$$\mathrm{Ker}(\delta_B^1) = \langle a'/\!/a', b'/\!/b', c'/\!/c', d'/\!/d', a'/\!/a'd'c' - b'/\!/d'c'b' \rangle,$$

hence $\mathrm{HH}^1(B) \simeq \langle c'/\!/c' \rangle$. Using the notation in Theorem 3.21, we get the ideal $\mathcal{I} \simeq \langle \varphi_1(\delta_A^0(e_1/\!/e_1)) \rangle = \langle c'/\!/c' - b'/\!/b' \rangle$ and $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/\mathcal{I}$. It is clear that $\mathrm{Spp}_1^5 = \{(a, adc), (b, dcb)\}$, therefore $\langle \mathrm{Spp}_1^5 \rangle = \langle a'/\!/a'd'c', b'/\!/d'c'b' \rangle$ and

$$\begin{aligned} V_{spp} &= \langle \mathrm{Spp}_1^5 \rangle \cap \mathrm{Ker}(\delta_B^1) \\ &= \langle a'/\!/a'd'c' - b'/\!/d'c'b' \rangle. \end{aligned}$$

The following example shows that the difference between the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ can be arbitrarily large.

Example 6.10. Let A be given by two blocks A_1 and A_2 such that A_1 and A_2 are radical square zero local algebras having m-loops and n-loops respectively. If we exclude the case that $m = 1$ and $n = 1$ in $\mathrm{char}(k) = 2$ (for this case, see Example 6.4), then the dimension of $\mathrm{HH}^1(A)$ is the sum of the dimensions of $\mathrm{HH}^1(A_1) \simeq \mathfrak{gl}_m(k)$ and $\mathrm{HH}^1(A_2) \simeq \mathfrak{gl}_n(k)$, that is, $m^2 + n^2$. Let B be obtained by gluing the units of A_1 and A_2. Then $\mathrm{HH}^1(B) \simeq \mathfrak{gl}_{m+n}(k)$ and consequently has dimension $(m+n)^2$.

We use the following example to show a particular case of Corollary 4.6.

Example 6.11. Suppose $\mathrm{char}(k) = 0$. Let B be obtained from A by gluing e_1 and e_4:

$$\begin{tikzcd} & e_2\bullet \arrow[dr, "\beta"] & \\ Q_A : \quad e_3\bullet & e_1\bullet \arrow[l, "\eta"] \arrow[u, shift left, "\alpha_1"'] \arrow[u, shift right, "\alpha_2"] \arrow[r, "\gamma"] & \bullet e_4 \end{tikzcd} \qquad \begin{tikzcd} Q_B : \quad f_3\bullet & f_1\bullet \arrow[l, "\eta'"] \arrow[loop above, "\gamma'"] \arrow[r, shift left=2, "\alpha_1'"] \arrow[r, dashed, "\alpha_2'"] & \bullet f_2 \arrow[l, shift left=2, "\beta'"] \end{tikzcd}$$

Where $Z_A = \{\beta\alpha_1\}$ and $Z_{new} = \{(\gamma')^2, \alpha_i'\gamma', \alpha_i'\beta', \gamma'\beta', \eta'\gamma', \eta'\beta' \mid i = 1, 2\}$. From a straightforward computation we have

$$\mathrm{Im}(\delta_A^0) = \langle \alpha_1/\!/\alpha_1 + \alpha_2/\!/\alpha_2 + \gamma/\!/\gamma, \beta/\!/\beta + \gamma/\!/\gamma, \eta/\!/\eta\rangle,$$
$$\mathrm{Im}(\delta_B^0) = \langle \alpha_1'/\!/\alpha_1' + \alpha_2'/\!/\alpha_2' - \beta'/\!/\beta', \eta'/\!/\eta'\rangle,$$
$$\mathrm{Ker}(\delta_A^1) = \langle \alpha_2/\!/\alpha_1, \alpha_1/\!/\alpha_1, \alpha_2/\!/\alpha_2, \beta/\!/\beta, \gamma/\!/\gamma, \gamma/\!/\beta\alpha_2, \eta/\!/\eta\rangle.$$

Since we glue a source and a sink, Corollary 3.15 shows that $\mathrm{Ker}(\delta_B^1) \simeq \mathrm{Ker}(\delta_A^1)$. As a consequence,

$$\mathrm{HH}^1(A) \simeq \langle \alpha_2/\!/\alpha_1, \alpha_1/\!/\alpha_1, \alpha_2/\!/\alpha_2, \gamma/\!/\beta\alpha_2\rangle,$$
$$\mathrm{HH}^1(B) \simeq \langle \alpha_2'/\!/\alpha_1', \alpha_1'/\!/\alpha_1', \alpha_2'/\!/\alpha_2', \gamma'/\!/\gamma', \gamma'/\!/\beta'\alpha_2'\rangle.$$

Using the notation in Theorem 3.21, we get the ideal $\mathcal{I} = \langle \varphi_1(\delta_A^0(e_1/\!/e_1))\rangle = \langle \alpha_1'/\!/\alpha_1' + \alpha_2'/\!/\alpha_2' + \gamma'/\!/\gamma' + \eta'/\!/\eta'\rangle$ and $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/\mathcal{I}$. In this case $L'' = 0$. Then G is generated by $\gamma'/\!/\beta'\alpha_2'$.

Note that in this case Δ in Definition 4.4 is equal to $\{[\alpha], [\gamma]\}$, where $[\alpha] = \{\alpha_1, \alpha_2\}$ and $[\gamma] = \{\gamma\}$. We can rewrite the generator of $\mathcal{I}$ as $\varphi_1(\mathcal{I}_{[\alpha]} + \mathcal{I}_{[\gamma]}) = \alpha_1'/\!/\alpha_1' + \alpha_2'/\!/\alpha_2' + \gamma'/\!/\gamma'$ since $\eta'/\!/\eta' \in \mathrm{Im}(\delta_B^0)$. Also $L_0^{[\alpha']} = \langle \alpha_2'/\!/\alpha_1', \alpha_1'/\!/\alpha_1', \alpha_2'/\!/\alpha_2'\rangle$, $L_0^{[\gamma']} = \langle \gamma'/\!/\gamma'\rangle$, hence

$$\begin{aligned} L_0 = L_0^{[\alpha']} \oplus L_0^{[\gamma']} &= \langle \alpha_2'/\!/\alpha_1', \alpha_1'/\!/\alpha_1', \alpha_2'/\!/\alpha_2'\rangle \oplus \langle \gamma'/\!/\gamma'\rangle \\ &= \langle \alpha_2'/\!/\alpha_1', \alpha_1'/\!/\alpha_1', \alpha_2'/\!/\alpha_2'\rangle \oplus \langle \alpha_1'/\!/\alpha_1' + \alpha_2'/\!/\alpha_2' + \gamma'/\!/\gamma'\rangle = L_0^{[\alpha']} \oplus \mathcal{I} \end{aligned}$$

as Lie algebras. Since $L_1 = \langle \gamma'/\!/\beta'\alpha_2'\rangle$,

$$\mathrm{HH}^1(B) = L_0 \oplus L_1 = (L_0^{[\alpha']} \oplus \mathcal{I}) \oplus L_1 = (L_0^{[\alpha']} \oplus L_1) \oplus \mathcal{I} \simeq \mathrm{HH}^1(A) \oplus \mathcal{I} \simeq \mathrm{HH}^1(A) \oplus k$$

as Lie algebras.

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