The Hochschild cohomology groups under gluing idempotents

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Abstract: We compare the first Hochschild cohomology groups of finite dimensional monomial algebras under gluing two idempotents. We also compare the fundamental groups and the Hochschild cohomology groups in other degrees. In particular, we will study the case of gluing a source and a sink, that is, when we obtain a stable equivalence.

1 Introduction

Let A be a finite dimensional algebra of the form kQ_A/I_A, where k is a field, Q_A is a finite quiver and I_A is an admissible ideal in the path algebra kQ_A. Let B be a finite dimensional algebra such that there is a radical embedding $\phi : B \to A$, that is, ϕ is an algebra monomorphism with $\phi(\mathrm{rad}B) = \mathrm{rad}A$, where $\mathrm{rad}A$ (resp. $\mathrm{rad}B$) denotes the (Jacobson) radical of A (resp. B). The radical embedding is a common construction in the studies of finite dimensional algebras and their representation theory (see, for example, [6, 9, 16]). By Xi's observation in [16] we can without loss of generality assume that B is a subalgebra of A and obtained from A by repeatedly gluing two idempotents. Recall that if B is obtained from A by gluing two idempotents e_1 and e_2, then B has the form kQ_B quotient by the admissible ideal I_B, where the quiver Q_B is obtained from Q_A by identifying e_1 and e_2 with a new vertex f_1, I_B is generated by the elements in I_A plus all newly formed paths, if any, (of length 2) through the vertex f_1. By a result of the first author jointly with Koenig (see [9, Theorem 4.10]), when we glue a source and a sink we obtain a stable equivalence $\underline{\mathrm{mod}}A \xrightarrow{\sim} \underline{\mathrm{mod}}B$ between the stable module categories modulo projective modules.

The main aim of this note is to study the behaviour of Hochschild cohomology and of the fundamental groups when we glue two idempotents from a monomial algebra A.

It is well known that Hochschild cohomology is not functorial, that is, if we have an algebra homomorphism $\phi : B \to A$ then we do not know how to construct a map from $\mathrm{HH}^*(A)$ to $\mathrm{HH}^*(B)$ or from $\mathrm{HH}^*(B)$ to $\mathrm{HH}^*(A)$. This makes Hochschild cohomology difficult to compute since it is not possible in general to reduce the study of Hochschild cohomology to some smaller, and potentially easier, algebras. However, when we have a radical embedding $\phi : B \to A$ which is obtained by gluing two idempotents of a monomial algebra A there is some sort of functoriality, at least for HH^1. In fact very often we can construct a map (even a Lie algebra homomorphism) from $\mathrm{HH}^1(B)$ to $\mathrm{HH}^1(A)$ or from $\mathrm{HH}^1(A)$ to $\mathrm{HH}^1(B)$. We will distinguish two types of gluings: gluing two idempotents from the same block of A

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and from different blocks of A. It turns out that the behaviour of the latter is much simpler than of the former. In both types we will study the case of gluing a source and a sink, that is, when we obtain a stable equivalence.

Our main result in this direction is Theorem 3.21, which states that under some mild condition on the characteristic of the field k (Assumption 3.11), there is a close connection between the Lie algebras $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$, once there is a radical embedding $\phi : B \to A$ which is obtained by gluing two idempotents of a monomial algebra A. In particular, we can explicitly compare the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ in terms of some combinatorial datum, see Theorem 3.20 for the details.

Of particular interest is the case of gluing a source and a sink, in which case we get a stable equivalence $F : \underline{\mathrm{mod}}A \xrightarrow{\sim} \underline{\mathrm{mod}}B$, and Theorem 3.21 and Theorem 3.20 take their nicest forms, see Corollary 3.24. According to [9], we know that in contrast to the situation for a stable equivalences of Morita type, the above stable equivalence F is induced by bimodules that are only projective on one side, but not on the other. For this reason, we do not know how to set up a (restricted) Lie algebra homomorphism between $\mathrm{HH}^*(A)$ and $\mathrm{HH}^*(B)$ under such a stable equivalence using the above mentioned bimodules, as it was set in [10] [3]. Nevertheless, we can compare $\mathrm{HH}^*(A)$ and $\mathrm{HH}^*(B)$ directly using the algebra monomorphism (that is, the radical embedding) $\phi : B \to A$ and Strametz's description of Hochschild cochain complex of monomial algebras. This allows us to 'control' the behaviours of Hochschild cohomologies far beyond the scope of stable equivalences, although in stable equivalence case we get the best results.

The radical embedding $\phi : B \to A$ also gives a close connection between the centers $Z(A)$ and $Z(B)$. More precisely, if A is an indecomposable monomial algebra and $\phi : B \to A$ is a radical embedding which is obtained by gluing two idempotents of A, then there is an algebra monomorphism $Z(A) \hookrightarrow Z(B)$ and the difference between $\dim_k Z(A)$ and $\dim_k Z(B)$ can be described by some combinatorial data (Proposition 6.3); if A is a monomial algebra with two blocks and $\phi : B \to A$ is a radical embedding which is obtained by gluing two idempotents from different blocks, then $\phi : B \to A$ restricts to a radical embedding $Z(B) \hookrightarrow Z(A)$, in particular $\dim_k Z(A) = \dim_k Z(B) + 1$ (Proposition 6.7).

We also study the relation between the fundamental groups and the idempotent gluings. More precisely, we consider the π_1-rank(A) which has been introduced in [2]. The π_1-rank(A) is the maximal dimension of a dual fundamental group for some (minimal) presentation of A. For a monomial algebra A the π_1-rank(A) coincides with the first Betti number of the Gabriel quiver Q_A of A which intuitively count the number of holes of Q_A. We compare π_1-rank(A) and π_1-rank(B) for the two types of gluings, see Lemma 5.1.

We also compare higher Hochschild cohomology groups for radical square zero algebras. More precisely, we show that when we glue a source and a sink, there is always an injective map from cocycles (respectively coboundaries) of A to cocycles (respectively coboundaries) of B, see Proposition 6.11.

Rather interestingly, our results can be easily generalized to the situation of gluing arrows. We will discuss this and other generalizations in a subsequent paper.

Outline. In Section 2 we give some notations and terminologies which we keep throughout the paper. It also provides some background on various topics. In Section 3 we focus on the study of the behaviour of the first Hochschild cohomology under gluing idempotents. The main results in this section are Theorem 3.20 and its strengthened form Theorem 3.21. Some applications of our main results are presented in Corollaries 3.24, 3.26 and 3.28. Moreover, in Subsection 3.1 we give an interpretation on Sánchez-Flores' description of the first Hochschild cohomology for radical square zero algebras by inverse gluing operations. In Section 4 we give various examples to illustrate our definitions and results in Section 3. In Section 5 we study the relation between gluing idempotents and the π_1-ranks. Finally in Section 6 we study how gluing changes the center and higher Hochschild cohomology groups.

2 Preliminaries

Bound quivers. All algebras considered are finite dimensional algebras which are isomorphic to kQ/I, where k is a field of arbitrary characteristic, Q is a finite quiver and I is an admissible ideal in the path algebra kQ. Any homomorphism between two algebras is requested to send the identity to the identity. For all $n \in \mathbb{N}$, let Q_n be the set of paths of length n of Q and let $Q_{\geq n}$ be the set of paths of length greater than or equal to n. Note that Q_0 is the set of vertices and Q_1 is the set of arrows of Q. We denote by $s(\gamma)$ the source vertex of an (oriented) path γ of Q and by $t(\gamma)$ its terminal vertex. The path algebra kQ is the k-linear span of the set of paths of Q where the multiplication of $\beta \in Q_i$ and $\alpha \in Q_j$ is provided by the concatenation $\beta\alpha \in Q_{i+j}$ if $t(\alpha) = s(\beta)$ and 0 otherwise. A path p of length $l \geq 1$ is said to be an oriented cycle (or an oriented l-cycle) if $s(p) = t(p)$. An oriented 1-cycle is called a loop. Two paths ϵ, γ of Q are called parallel if $s(\epsilon) = s(\gamma)$ and $t(\epsilon) = t(\gamma)$, denoted by $\epsilon \| \gamma$. If ϵ and γ are not parallel, we denote by $\epsilon \nparallel \gamma$. If X, Y are sets of paths of Q, we denote by $X \| Y$ the set of parallel paths consisting of the couples $\epsilon \| \gamma$ with $\epsilon \in X$ and $\gamma \in Y$, and denote by $k(X \| Y)$ the k-vector space with basis $X \| Y$.

We now fix a finite dimensional k-algebra $A = kQ_A/I_A$ (where I_A is an admissible ideal in kQ_A) and denote the vertices of Q_A by $e_1, \cdots, e_n$. A vertex e_i is isolated if it does not exist any arrow α such that $s(\alpha) = e_i$ or $t(\alpha) = e_i$. By a source vertex e_i of Q_A, we mean that there is no arrow α with $t(\alpha) = e_i$; by a sink vertex e_j of Q_A, we mean that there is no arrow α with $s(\alpha) = e_j$. By abuse of notation, we denote by $e_1, \cdots, e_n$ the corresponding primitive idempotents in the algebra A, and for a path p in Q_A, we use the same notation to denote its image $\overline{p} = p + I_A$ in A. If $A = A_1 \times \cdots \times A_s$ is a decomposition of A into a product of indecomposable algebras, then A_i's are called blocks of A. Note that such a decomposition of A is unique and if $s = 1$ then A is an indecomposable algebra.

Radical embeddings. Let B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. That is, B is identified as a subalgebra of A generated by $f_1 := e_1 + e_n, f_2 := e_2, \cdots, f_{n-1} := e_{n-1}$ and all arrows in Q_A. By abuse of notation, we denote the vertices of B by $f_1, \cdots, f_{n-1}$. Then the algebra B has the form kQ_B/I_B, where the quiver Q_B is obtained from Q_A by identifying the vertices e_1 and e_n, and the admissible ideal I_B is generated by the elements in I_A plus all newly formed paths of the form $\cdot \to f_1 \to \cdot$ (each of such a path has length 2). In particular, we have $\dim_k B = \dim_k A - 1$.

Note that there is an obvious bijection between the arrows of A and the arrows of B. For each arrow α in Q_A, we denote the corresponding arrow in Q_B by α^*. We define a quiver morphism

$$\varphi : Q_A \to Q_B$$

as follows: let $\varphi(e_i) = f_i$ for $2 \leq i \leq n-1$ and $\varphi(e_1) = \varphi(e_n) = f_1$, and let $\varphi(\alpha) = \alpha^*$. It is possible also to define $\varphi_n : (Q_A)_n \to (Q_B)_n$ by extending the map $\varphi : Q_A \to Q_B$. More precisely, let $p = a_n \ldots a_1$ be a path in $(Q_A)_n$. Then $\varphi_n(p) = p^* = a_n^* \ldots a_1^*$. It is known that the above radical embedding induces a stable equivalence (modulo projective modules) $\underline{\text{mod}}A \to \underline{\text{mod}}B$ if and only if B is obtained from A by gluing a sink vertex and a source vertex, at least when we modulo the Auslander-Reiten conjecture for stable equivalences (see [9]).

Monomial algebras. We denote by Λ a finite dimensional monomial k-algebra, that is, a finite dimensional k-algebra which is isomorphic to a quotient kQ/I of a path algebra where the two-sided ideal I of kQ is generated by a set Z of paths of length ≥ 2. We shall assume that Z is minimal, that is, no proper subpath of a path in Z is again in Z. Let $\mathcal{B} = \mathcal{B}_\Lambda$ be the set of paths of Q which do not contain any element of Z as a subpath. It is clear that the (classes modulo I of) elements of $\mathcal{B}$ form a basis of Λ. We shall denote by $\mathcal{B}_n$ the subset $Q_n \cap \mathcal{B}$ of $\mathcal{B}$ formed by the paths of length n. Moreover, we shall use $E \simeq kQ_0$ to denote the separable subalgebra of Λ generated by the (classes modulo I of the) vertices of Q.

Hochschild cohomology of monomial algebras. The Hochschild cohomology

$$\mathrm{H}^*(\Lambda, \Lambda) := \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$$

of the k-algebra Λ can be computed using different projective resolutions of Λ over its enveloping algebra $\Lambda^e = \Lambda \otimes_k \Lambda^{op}$. We usually denote $\mathrm{H}^*(\Lambda, \Lambda)$ by $\mathrm{HH}^*(\Lambda)$. The zero-th Hochschild cohomology group $\mathrm{HH}^0(\Lambda)$ is identified as the center $Z(\Lambda)$ of the algebra Λ. In particular, $Z(\Lambda)$ is a commutative subalgebra of Λ. In order to compute the first Hochschild cohomology groups $\mathrm{HH}^1(\Lambda)$ of a finite dimensional monomial algebra Λ, one can use the minimal projective resolution of the Λ-bimodule Λ given by Bardzell in [1]. The part of this resolution $\mathcal{P}_{min}$ in which we are interested is the following:

$$\cdots \longrightarrow \Lambda \otimes_E kZ \otimes_E \Lambda \xrightarrow{\delta_1} \Lambda \otimes_E kQ_1 \otimes_E \Lambda \xrightarrow{\delta_0} \Lambda \otimes_E \Lambda \xrightarrow{\pi} \Lambda \longrightarrow 0,$$

where the Λ-bimodule morphisms are given by

$$\begin{aligned}
\pi(\lambda \otimes_E \mu) =& \lambda\mu \\
\delta_0(\lambda \otimes_E a \otimes_E \mu) =& \lambda a \otimes_E \mu - \lambda \otimes_E a\mu \text{ and} \\
\delta_1(\lambda \otimes_E p \otimes_E \mu) =& \sum_{d=1}^{n} \lambda a_n \cdots a_{d+1} \otimes_E a_d \otimes_E a_{d-1} \cdots a_1 \mu
\end{aligned}$$

for all $\lambda, \mu \in \Lambda, a, a_n, \cdots, a_1 \in Q_1$ and $p = a_n \cdots a_1 \in Z$ (with the conventions $a_{n+1} = t(a_n)$ and $a_0 = s(a_1)$).

Applying the contravariant functor $\mathrm{Hom}_{\Lambda^e}(-, \Lambda)$ to $\mathcal{P}_{min}$ we obtain the following cochain complex $\mathcal{C}_{min}$ (see [14, Section 2]):

$$0 \longrightarrow Hom_{E^e}(kQ_0, \Lambda) \xrightarrow{\delta_0{}^*} Hom_{E^e}(kQ_1, \Lambda) \xrightarrow{\delta_1{}^*} Hom_{E^e}(kZ, \Lambda) \xrightarrow{\delta_2{}^*} \cdots,$$

where the differentials are given by

$$\begin{aligned}
(\delta_0{}^* f)(a) &= a f(s(a)) - f(t(a))a \quad \textit{and} \\
(\delta_1{}^* g)(p) &= \sum_{d=1}^{n} a_n \cdots a_{d+1} g(a_d) a_{d-1} \cdots a_1.
\end{aligned}$$

where $f \in \mathrm{Hom}_{E^e}(kQ_0, \Lambda)$, $a, a_n, \cdots, a_1 \in Q_1$, $g \in \mathrm{Hom}_{E^e}(kQ_1, \Lambda)$ and $p = a_n \cdots a_1 \in Z$. In particular, we have $\mathrm{HH}^1(\Lambda) \simeq \mathrm{Ker}(\delta_1{}^*)/\mathrm{Im}(\delta_0{}^*)$ as k-spaces, where the elements in $\mathrm{Ker}(\delta_1{}^*)$ can be interpreted as the E^e-derivations of kQ_1 into Λ and the elements in $\mathrm{Im}(\delta_0{}^*)$ can be interpreted as the inner E^e-derivations of kQ_1 into Λ. It is well-known that $\mathrm{Ker}(\delta_1{}^*)$ is a Lie algebra under the Lie bracket $[f, g] = f \circ g - g \circ f$ and $\mathrm{Im}(\delta_0{}^*)$ is a Lie ideal of $\mathrm{Ker}(\delta_1{}^*)$, so that $\mathrm{HH}^1(\Lambda)$ has a Lie algebra structure.

A practical way of computing $\mathrm{HH}^1(\Lambda)$ of a monomial algebra Λ is given by Strametz in [14].

Proposition 2.1 ([14, Proposition 2.6]) *Let Λ be a finite dimensional monomial algebra. Then the above cochain complex $\mathcal{C}_{min}$ is isomorphic to the following cochain complex $\mathcal{C}_{mon}$:*

$$0 \longrightarrow k(Q_0 \| \mathcal{B}) \xrightarrow{\delta^0} k(Q_1 \| \mathcal{B}) \xrightarrow{\delta^1} k(Z \| \mathcal{B}) \xrightarrow{\delta^2} \cdots,$$

where the differentials are given by

$$\begin{aligned}
\delta^0 &: k(Q_0 \| \mathcal{B}) \to k(Q_1 \| \mathcal{B}) \\
&\qquad e \| \gamma \mapsto \sum_{s(a)=e, a\gamma \in \mathcal{B}} a \| a\gamma - \sum_{t(a)=e, \gamma a \in \mathcal{B}} a \| \gamma a, \\
\delta^1 &: k(Q_1 \| \mathcal{B}) \to k(Z \| \mathcal{B}) \\
&\qquad a \| \gamma \mapsto \sum_{r \in Z} r \| r^{a \| \gamma},
\end{aligned}$$

where $r^{a\|\gamma}$ denotes the sum of all paths in $\mathcal{B}$ obtained by replacing each appearance of the arrow a in r by the path γ. In particular, we have $\mathrm{HH}^1(\Lambda) \simeq \mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$ as k-spaces, where this isomorphism is induced by the following map: send each f in $Hom_{E^e}(kQ_1, k\mathcal{B})$ to be an element $\sum_{a\|\gamma \in Q_1\|\mathcal{B}} \lambda_{a,\gamma}(a\|\gamma)$

in $k(Q_1\|\mathcal{B})$*, where* $f(a) = \sum_{\gamma\in\mathcal{B}} \lambda_{a,\gamma}\gamma$*. Moreover, the inverse of the above isomorphism is induced by the following map: send an element* $a\|\gamma$ *in* $k(Q_1\|\mathcal{B})$ *to be a map* f *in* $Hom_{E^e}(kQ_1, k\mathcal{B})$ *with* $f(a) = \gamma$ *and* $f(b) = 0$ *for* $a \neq b \in Q_1$*.*

Theorem 2.2 ([14]) *Let* Λ *be a finite dimensional monomial algebra. Then the bracket*

$$[a\|\gamma, b\|\epsilon] = b\|\epsilon^{a\|\gamma} - a\|\gamma^{b\|\epsilon} \qquad (a\|\gamma, b\|\epsilon \in Q_1\|\mathcal{B})$$

induces a Lie algebra structure on $\mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$ *such that the natural isomorphism*

$$\mathrm{HH}^1(\Lambda) \simeq \mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$$

is a Lie algebra isomorphism.

If the field k has positive characteristic p, then $\mathrm{HH}^1(A)$ is a restricted Lie algebra, that is, it is a Lie algebra endowed with a map called p-power map that satisfies some compatibility properties with respect to the Lie algebra structure. For further background on restricted Lie algebras see for example [7, Chapter 2]. The p-power map of a derivation f is defined by composing f with itself p-times. The inner derivations form a restricted Lie ideal of space of derivations, therefore $\mathrm{HH}^1(A)$ is a restricted Lie algebra. For monomial algebras, it is easy to describe the p-power map using the chain map from $\mathcal{C}_{min}$ to $\mathcal{C}_{mon}$ and its inverse chain map. For example, for $p = 3$, the p-power map of $a\|\gamma$ is $(a\|\gamma^{a\|\gamma})^{a\|\gamma}$. We remark that some of results in the present paper, such as Proposition 3.12 and Corollary 3.24, can be easily generalized from 'Lie' level to 'restricted Lie' level.

Remark 2.3 *The center* $Z(\Lambda)$ *of* Λ *is naturally isomorphic to* $\mathrm{Ker}(\delta^0)$*. For a concrete map between* $\mathrm{Ker}(\delta^0)$ *and* $Z(\Lambda)$*, see the proof of Proposition 6.3.*

According to [14, Section 4], the Lie algebra $\mathrm{HH}^1(\Lambda)$ of a monomial algebra $\Lambda = kQ/\langle Z\rangle$ has a natural graduation. Actually, if $a\|\gamma \in Q_1\|\mathcal{B}_n$ and $b\|\epsilon \in Q_1\|\mathcal{B}_m$, then the Lie bracket in Theorem 2.2 shows that $[a\|\gamma, b\|\epsilon] \in k(Q_1\|\mathcal{B}_{n+m-1})$. Thus, we have a graduation on the Lie algebra $k(Q_1\|\mathcal{B}) = \oplus_{i\in\mathbb{N}} k(Q_1\|\mathcal{B}_i)$ by considering that the elements of $k(Q_1\|\mathcal{B}_i)$ have degree $i - 1$ for all $i \in \mathbb{N}$. It is clear that the Lie subalgebra $\mathrm{Ker}(\delta^1)$ of $k(Q_1\|\mathcal{B})$ preserves this graduation and that $\mathrm{Im}(\delta^0)$ is a graded ideal, which induces a graduation on the Lie algebra $\mathrm{HH}^1(\Lambda) \simeq \mathrm{Ker}(\delta^1)/\mathrm{Im}(\delta^0)$. More precisely, if we set

$$L_{-1} := k(Q_1\|Q_0) \cap \mathrm{Ker}(\delta^1),$$

$$L_0 := (k(Q_1\|Q_1) \cap \mathrm{Ker}(\delta^1))/\langle \delta^0(e\|e) \mid e \in Q_0\rangle \text{ and}$$

$$L_i := (k(Q_1\|\mathcal{B}_{i+1}) \cap \mathrm{Ker}(\delta^1))/\langle \delta^0(e\|p) \mid e\|p \in Q_1\|\mathcal{B}_i\rangle$$

for all $i \geq 1, i \in \mathbb{N}$, then $\mathrm{HH}^1(\Lambda) = \bigoplus_{i\geq -1} L_i$ and $[L_i, L_j] \subset L_{i+j}$ for all $i, j \geq -1$, where $L_{-2} = 0$.

Remark 2.4 *Note that if the characteristic of the field* k *is equal to 0, then* $L_{-1} = 0$ *since there exists for every loop* $a\|e \in Q_1\|Q_0$ *a relation* $r = a^m \in Z$ *for some* $m \geq 2$ *such that* $\delta^1(a\|e)$ *has a summand* $mr\|a^{m-1}$ *which can not be cancelled, whence* $\delta^1(a\|e) \neq 0$*. It follows that* $\bigoplus_{i\geq 1} L_i$ *is a solvable Lie ideal of* $\mathrm{HH}^1(\Lambda) = \bigoplus_{i\geq 0} L_i$ *since* $\mathrm{HH}^1(\Lambda)$ *is finite dimensional. It is also obvious that* L_0 *is a Lie subalgebra of* $\mathrm{HH}^1(\Lambda)$*.*

For the quiver Q, the parallelism is an equivalence relation on the set of arrows Q_1; for $\alpha \in Q_1$, $[\alpha]$ denotes the equivalence class of α. We denote $\bar{Q}_1$ the set of equivalence classes of parallel arrows. The quiver which has Q_0 as vertices and $\bar{Q}_1$ as set of arrows, will be denoted by $\bar{Q}$. We denote by $\chi(\bar{Q})$ the first Betti number of $\bar{Q}$ (see Section 5). In order to ensure each $L_0^{[\alpha]}$ (in the Lie algebra decomposition (†) of L_0 below) to be a Lie ideal, we need to use the following variation of [14, Proposition 4.7]).

Proposition 2.5 *The basis* $\mathcal{B}_{L_0}$ *of* L_0 *is given by the union of the following sets:*

(*i*) *all the elements* $a\|b \in L_0$ *such that* $a \neq b$;

(*ii*) *for every class of parallel arrows* $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \in \bar{Q}_1$, *all the elements* $\alpha_i\|\alpha_i - \alpha_n\|\alpha_n \in L_0$ *such that* $i < n$;

(*iii*) *for each (oriented or undirected) cycle in* $\bar{Q}$, *choose one class of parallel arrows* $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ *in this cycle and take* $\alpha_n\|\alpha_n$. *Note that there are* $\chi(\bar{Q})$ *linearly independent elements in (iii).*

For each class of parallel arrows $[\alpha] \in \bar{Q}_1$ we denote by $L_0^{[\alpha]}$ the Lie ideal of L_0 generated by the elements $\alpha_i\|\alpha_j \in \mathcal{B}_{L_0}$ and $\alpha_i\|\alpha_i - \alpha_n\|\alpha_n \in \mathcal{B}_{L_0}$ where $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Obviously the Lie algebra L_0 is the product of these Lie algebras:

$$L_0 = \prod_{[\alpha]\in\bar{Q}} L_0^{[\alpha]}, \qquad (\dagger)$$

where this decomposition depends on the basis $\mathcal{B}_{L_0}$ and $L_0^{[\alpha]}$ may be equal to zero for some $[\alpha]$.

To describe the generators of the center $Z(L_0)$ of L_0, we adopt the definitions introduced by Strametz in [14]: for every class of parallel arrows $[\alpha]$ of $\bar{Q}_1$ we call a set $C \subset [\alpha]$ connected, if for every two arrows α_1 and α_m of C there exist arrows $\alpha_2, \cdots, \alpha_{m-1} \in C$ such that we have $r^{\alpha_{i+1}\|\alpha_i} = 0$ or $r^{\alpha_i\|\alpha_{i+1}} = 0$ for all $1 \leq i \leq m-1$ and all $r \in Z$. A connected set $C \subset [\alpha]$ is called a connected component of $[\alpha]$ if it is maximal for the connection, that is, for every arrow $\beta \in [\alpha]\backslash C$ there is no arrow $\alpha \in C$ such that $r^{\beta\|\alpha} = 0$ or $r^{\alpha\|\beta} = 0$ for all $r \in Z$.

Proposition 2.6 ([14, Lemma 4.20]) *Let* Q *be a connected quiver and* $\Lambda = kQ/\langle Z\rangle$ *a finite dimensional monomial algebra.*

(*i*) *The center* $Z(L_0)$ *of the Lie algebra* L_0 *is generated by the elements* $\sum_{a\in C} a\|a$, *where* C *denotes a connected component of a class of parallel arrows of* Q.

(*ii*) *If the field* k *has characteristic 0, then the center* $Z(\mathrm{HH}^1(\Lambda))$ *of the Lie algebra* $\mathrm{HH}^1(\Lambda)$ *is contained in the center* $Z(L_0)$ *of* L_0.

Since $L_0 = \prod\limits_{[\alpha]\in\bar{Q}_1} L_0^{[\alpha]}$ as Lie algebras, we have $Z(L_0) = \prod\limits_{[\alpha]\in\bar{Q}_1} Z(L_0^{[\alpha]})$. And we have the parallel conclusion for $Z(L_0^{[\alpha]})$, in fact, $Z(L_0^{[\alpha]}) = \bigoplus\limits_C \langle \sum\limits_{a\in C} a\|a \mid \sum\limits_{a\in C} a\|a \in L_0^{[\alpha]}\rangle$ as abelian Lie algebras for each $[\alpha] \in \bar{Q}_1$, where C takes over all the connected components of $[\alpha]$.

Let A be a finite dimensional algebra. By the Wedderburn–Malcev theorem we have $A = E \oplus rad(A)$. Another complex which computes Hochschild cohomology and which very helpful for calculations is the reduced bar complex:

$$0 \to A^E \to \mathrm{Hom}_{E^e}(r, A) \to \cdots \to \mathrm{Hom}_{E^e}(r^{\otimes_E^n}, A) \to \mathrm{Hom}_{E^e}(r^{\otimes_E^{n+1}}, A) \to \dots$$

where $A^E = \{a \in A | ae = ea \text{ for all } e \in E\}$, $r := rad(A)$ and $r^{\otimes_E^n}$ denotes the n-th fold tensor product over E. The differential is described in [4, Proposition 2.2].

For radical square zero algebras, Cibils [4] provides an isomorphic complex to the reduced bar complex which can be described entirely in terms of the combinatorics of the quiver.

We recall very briefly some constructions from [4]: The author denotes by $(kQ)_2$ the quotient of the path algebra of Q by the two-sided ideal generated by paths of length 2. From Gabriel's theorem it follows that every radical square zero algebra Λ over an algebraically closed field k is Morita equivalent to an algebra $(kQ)_2$. The algebra Λ is assumed to be indecomposable, therefore the Gabriel quiver Q of Λ is connected and finite. By Lemma 2.1 in [4] we have that $r^{\otimes_E^n}$ has basis given by Q_n. By Lemma 2.2 in [4] the vector space $\mathrm{Hom}_{E^e}(r^{\otimes_E^n}, \Lambda)$ is isomorphic to $k(Q_n\|Q_0) \oplus k(Q_n\|Q_1)$. The differential is defined as follows (cf. [5, Proposition 2.4]):

$$k(Q_n\|Q_0) \oplus k(Q_n\|Q_1) \xrightarrow{\delta^n} k(Q_{n+1}\|Q_0) \oplus k(Q_{n+1}\|Q_1),$$

where δ^n is the 2×2-matrix that has D_n in position $(2, 1)$ and 0 elsewhere, where D_n is defined as:

$$D_n(\gamma \| e) = \sum_{s(a)=e, a \in (Q_A)_1} a\gamma \| a + (-1)^{n+1} \sum_{t(b)=e, b \in (Q_A)_1} \gamma b \| b.$$

Fundamental groups of a bound quiver. The elements of I of an ideal of the path algebra kQ are usually called relations. A relation r is minimal in I if it is a nonzero relation $r = \sum_{i=1}^{s} a_i p_i$, where the p_i are distinct parallel paths in Q and $a_i \in k \setminus \{0\}$, such that there is no proper nonempty subset $T \subset \{1, \ldots, s\}$ for which $\sum_{i \in T} a_i p_i \in I$. In order to construct the fundamental group of a bound quiver, the first step is to consider for each arrow $\alpha \in Q_1$ the formal inverse α^{-1} which is an arrow such that $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$. Then we consider the double quiver $\bar{\bar{Q}}$ where $\bar{\bar{Q}}_0 = Q_0$ and $\bar{\bar{Q}}_1 = Q_1 \cup \{\alpha^{-1} \mid \alpha \in Q_1\}$. A walk is an oriented path in $\bar{\bar{Q}}$. As for the classical fundamental group, one needs to define an homotopy relation $\sim_I$ which is an equivalence relation on the set of walks in $\bar{\bar{Q}}$ generated by:

- $\alpha\alpha^{-1} \sim_I t(\alpha)$, $\alpha^{-1}\alpha \sim_I s(\alpha)$,
- if $v \sim_I v^{'}$, then $wvu \sim_I wv^{'}u$,
- if u and v are paths which occur with a nonzero coefficient in the same minimal relation, then $u \sim_I v$.

It is worth noting that the definition of the fundamental group does depend on the relation, that is, two different presentations I and J of a finite dimensional algebra A have different fundamental groups, see for example [11, Example 1.2]. Let $e_i \in Q_0$, then we denote by $\pi(Q, I, e_i)$ the set of equivalence classes of walks having source and target e_i. This set is endowed with a group structure given by concatenation of walks. The unit is the trival walk e_i. The group $\pi(Q, I, e_i)$ is called the fundamental group of the bound quiver (Q, I) based at e_i.

3 First Hochschild cohomology

We assume that A is a finite dimensional algebra which is isomorphic to kQ_A/I, where k is a field with characteristic ≥ 0, Q_A is a finite quiver (with vertices $e_1, \cdots, e_n$) and I is an admissible ideal in the path algebra kQ_A. Throughout we assume that $e_1 \neq e_n$ and we exclude the case that e_1 or e_n is an isolated vertex.

Having in mind the cochain complex $\mathcal{C}_{mon}$ in the previous section, in order to understand the behaviour of the first Hochschild cohomology under idempotent gluings, we should start by considering how idempotent gluings behave with respect to parallelism of arrows and paths.

Lemma 3.1 *Let B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Let $\alpha, \beta \in (Q_A)_1$. If $\alpha \| \beta$, then $\alpha^* \| \beta^*$, where $\varphi : Q_A \to Q_B$ sends α to α^* (cf. Notations in Section 2).*

Proof Recall that we denote the vertices of Q_B by $f_1, \cdots, f_{n-1}$, where f_1 is obtained by gluing e_1 and e_n. Let $\alpha, \beta \in (Q_A)_1$ and $\alpha \| \beta$. We may assume that $s(\alpha) = s(\beta) = e_i$ and $t(\alpha) = t(\beta) = e_j$. It is clear that

$$s(\alpha^*) = \begin{cases} f_i & for\ 2 \leq i \leq n-1 \\ f_1 & for\ i = 1\ or\ n \end{cases},$$

and

$$t(\alpha^*) = \begin{cases} f_j & for\ 2 \leq j \leq n-1 \\ f_1 & for\ j = 1\ or\ n \end{cases}.$$

Similarly, for β^*, we have

$$s(\beta^*) = \begin{cases} f_i & for\ 2 \leq i \leq n-1 \\ f_1 & for\ i = 1\ or\ n \end{cases},$$

and

$$t(\beta^*) = \begin{cases} f_j & for\ 2 \leq j \leq n-1 \\ f_1 & for\ j = 1\ or\ n \end{cases}.$$

Hence, if $\alpha \| \beta$, then $s(\alpha^*) = s(\beta^*)$ and $t(\alpha^*) = t(\beta^*)$, that is to say $\alpha^* \| \beta^*$. □

Lemma 3.2 *Let B be a radical embedding of A obtained by gluing a source and a sink of A. Let $\alpha, \beta \in (Q_A)_1$. Then $\alpha \| \beta$ if and only if $\alpha^* \| \beta^*$.*

Proof The sufficiency is obvious by Lemma 3.1, it suffices to show the necessity. If $\alpha^* \| \beta^*$, then to show $\alpha \| \beta$ we need to use the assumption that we are gluing a source, say e_1, and a sink, say e_n. We show that if $\alpha \nparallel \beta$, then $\alpha^* \nparallel \beta^*$. If $\alpha \nparallel \beta$, then either $s(\alpha) \neq s(\beta)$ or $t(\alpha) \neq t(\beta)$. Assume $s(\alpha) = e_i \neq e_j = s(\beta)$ where $i \neq j$. We consider three cases:

a) If $2 \leq i \leq n-1, 1 \leq j \leq n$ and $i \neq j$, then

$$s(\alpha^*) = f_i \neq s(\beta^*) = \begin{cases} f_j & for\ 2 \leq j \leq n-1 \\ f_1 & for\ j = 1\ or\ n \end{cases},$$

which means $\alpha^* \nparallel \beta^*$.

b) If $i = 1, 1 \leq j \leq n$ and $i \neq j$, then $s(\alpha^*) = f_1$, $s(\beta^*) = \begin{cases} f_j & for\ 2 \leq j \leq n-1 \\ f_1 & for\ j = 1\ or\ n \end{cases}$. Since $i \neq j, s(\beta^*) = \begin{cases} f_j & for\ 2 \leq j \leq n-1 \\ f_1 & for\ j = n \end{cases}$. Only when $j = n$, we have $s(\beta^*) = f_1 = s(\alpha^*)$. That is to say $s(\beta) = e_n$. But this is not possible since e_n is a sink, so there no exists β such that $s(\beta) = e_n$. Hence $s(\alpha^*) \neq s(\beta^*)$, which means $\alpha^* \nparallel \beta^*$.

c) We can deduce the same for $i = n$, $1 \leq j \leq n$ and $i \neq j$.

Similar arguments apply if we assume $t(\alpha) \neq t(\beta)$. □

We now extend partly the above results to parallel paths for monomial algebras. Recall that for a monomial algebra $A = kQ_A/I_A$, there is a k-basis $\mathcal{B}_A$ of A consisting of paths of Q_A which do not contain any element of Z_A as a subpath, where Z_A is a minimal generating set of the ideal I_A.

Proposition 3.3 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents of A. Let $\varphi : Q_A \to Q_B$ be the quiver morphism defined as in Section 2. Then we have the following.*

(1) $\varphi : Q_A \to Q_B$ induces a surjective map $\widetilde{\varphi} : \mathcal{B}_A \to \mathcal{B}_B$ such that $\widetilde{\varphi}^{-1}(p^*) = \{p\}$ for $p^* \neq f_1$ and $\widetilde{\varphi}^{-1}(f_1) = \{e_1, e_n\}$, where we denote $\widetilde{\varphi}(p)$ by p^* for $p \in \mathcal{B}_A$.

(2) Let $p, q \in \mathcal{B}_A$. Then $p \| q$ in Q_A implies $p^* \| q^*$ in Q_B.

(3) $\widetilde{\varphi} : \mathcal{B}_A \to \mathcal{B}_B$ induces k-linear maps $\psi_0 : k((Q_A)_0 \| \mathcal{B}_A) \to k((Q_B)_0 \| \mathcal{B}_B)$, $\psi_1 : k((Q_A)_1 \| \mathcal{B}_A) \to k((Q_B)_1 \| \mathcal{B}_B)$, $\psi_2 : k(Z_A \| \mathcal{B}_A) \to k(Z_B \| \mathcal{B}_B)$.

Proof As before, suppose that the vertices of Q_A are given by $e_1, \cdots, e_n$ and that the vertices of Q_B are given by $f_1, \cdots, f_{n-1}$, where f_1 is obtained by gluing e_1 and e_n; for each arrow α in Q_A, we denote the corresponding arrow in Q_B by α^*. Then the quiver morphism $\varphi : Q_A \to Q_B$ is given by the following formula: $\varphi(e_i) = f_i$ for $2 \leq i \leq n-1$, $\varphi(e_1) = \varphi(e_n) = f_1$, and $\varphi(\alpha) = \alpha^*$ for each arrow α. Notice that since A is monomial, B is also monomial.

The quiver morphism $\varphi : Q_A \to Q_B$ extends to a k-linear map $kQ_A \to kQ_B$ between path algebras by sending a path $p = a_1 \cdots a_m$ ($a_i \in (Q_A)_1$ for $1 \leq i \leq m$) in Q_A to a path $p^* := a_1^* \cdots a_m^*$ in Q_B. Since the newly formed relations in I_B are of the forms $\cdot \to f_1 \to \cdot$, $p \in \mathcal{B}_A$ implies $p^* \in \mathcal{B}_B$. Now the statement (1) follows from the fact that $\dim_k B = \dim_k A - 1$, and the statements (2) and (3) follow from Lemma 3.1. □

From now on, we fix $A = kQ_A/I_A$ and $B = kQ_B/I_B$ to be the monomial algebras as in Proposition 3.3. Then we have the following diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & k((Q_A)_0\|\mathcal{B}_A) & \xrightarrow{\delta_A^0} & k((Q_A)_1\|\mathcal{B}_A) & \xrightarrow{\delta_A^1} & k(Z_A\|\mathcal{B}_A) & \\ & \downarrow \psi_0 & & \downarrow \psi_1 & & \downarrow \psi_2 & (*) \\ 0 \longrightarrow & k((Q_B)_0\|\mathcal{B}_B) & \xrightarrow{\delta_B^0} & k((Q_B)_1\|\mathcal{B}_B) & \xrightarrow{\delta_B^1} & k(Z_B\|\mathcal{B}_B)\ , & \end{array}$$

where $\psi_0 : k((Q_A)_0\|\mathcal{B}_A) \to k((Q_B)_0\|\mathcal{B}_B)$, $\psi_1 : k((Q_A)_1\|\mathcal{B}_A) \to k((Q_B)_1\|\mathcal{B}_B)$, $\psi_2 : k(Z_A\|\mathcal{B}_A) \to k(Z_B\|\mathcal{B}_B)$ are the induced k-linear maps from the quiver morphism $\varphi : Q_A \to Q_B$ as mentioned in Proposition 3.3. Note that the top and the bottom complexes are truncations of the complexes $\mathcal{C}_{mon}$ of A and of B, respectively. Both squares in the diagram $(*)$ are not commutative in general, however, there are close connections between the coboundary elements (resp. the cocycle elements) of the top complex and the coboundaries (respectively the cocycles) of the bottom complex in the diagram $(*)$.

We briefly outline the structure of the rest of this section. Firstly, we will compare $\text{Im}(\delta_A^0)$ and $\text{Im}(\delta_B^0)$. Then we will study the Lie algebra structure of $\text{Ker}(\delta_A^1)$ and $\text{Ker}(\delta_B^1)$. Lastly, we will compare the dimensions and the Lie structures of $\text{HH}^1(A)$ and $\text{HH}^1(B)$. In order to express these connections more precisely, we need some definitions and a lemma.

Definition 3.4 *With Proposition 2.1 in mind, we define $\delta^0_{(A)_0}$ to be the map δ_A^0 restricted to the subspace $k((Q_A)_0||(Q_A)_0)$. Then $\text{Im}(\delta^0_{(A)_0})$ is the k-vector space generated by the image of δ_A^0 on $e_i\|e_i$, where e_i $(1 \le i \le n)$ are idempotents corresponding to vertices of Q_A. Similarly, we define $\text{Ker}(\delta^0_{(A)_0})$ to be the kernel of the map $\delta^0_{(A)_0}$.*

Lemma 3.5 *Let A be a monomial algebra. Then $\dim_k\text{Im}(\delta^0_{(A)_0}) = n_A - c_A$, where n_A is the number of vertices of Q_A and c_A is the number of connected components of Q_A.*

Proof Note that it is enough to assume that A is indecomposable. In fact, if it holds for each block A_i of A, then

$$\dim_k(\text{Im}(\delta^0_{(A)_0})) = \sum_{A_i}(n_{A_i} - c_{A_i}) = n_A - c_A.$$

Hence assume A is indecomposable. Note that:

$$\dim_k(k((Q_A)_0||(Q_A)_0) = n_A = \dim_k(\text{Im}(\delta^0_{(A)_0})) + \dim_k(\text{Ker}(\delta^0_{(A)_0})).$$

Consequently it is enough to show that $\dim_k(\text{Ker}(\delta^0_{(A)_0})) = 1$. It is straightforward to check that $\sum_{i=1}^{n_A} e_i||e_i$ is in $\text{Ker}(\delta^0_{(A)_0})$. Consequently $\text{Ker}(\delta^0_{(A)_0})$ has dimension at least one. We will prove by contradiction that the dimension of $\text{Ker}(\delta^0_{(A)_0})$ is exactly 1.

Assume the dimension of $\text{Ker}(\delta^0_{(A)_0})$ is greater than 1. Then we can assume without loss of generality that there exists $T \subsetneqq \{1, \dots, n_A\}$ such that $\sum_{i\in T} \lambda_i e_i||e_i$ is an element of $\text{Ker}(\delta^0_{(A)_0})$ where λ_i are non-zero scalars. Indeed, if there exists an element $\sum_{i=1}^{n_A} \lambda_i e_i||e_i$ in $\text{Ker}(\delta^0_{(A)_0})$, then by taking a linear combination with $\sum_{i=1}^{n_A} e_i||e_i$ we can always find such T. Consider the full subquiver $\overline{Q}$ having the vertices indexed by T. Since Q_A is connected and since $T \subsetneqq \{1, \dots, n_A\}$, then $\delta_A^0(\sum_{i\in T} \lambda_i e_i||e_i)$ has one summand of the form $c||c$ where c is an arrow such that $s(c) \in \overline{Q}_0$ and $t(c) \notin \overline{Q}_0$ (or $s(c) \notin \overline{Q}_0$ and $t(c) \in \overline{Q}_0$). Since $c||c$ cannot be written as a linear combination of other elements of $k((Q_A)_1||\mathcal{B}_A)$ and since λ_i are non-zero, then $\sum_{i\in T} \lambda_i e_i||e_i$ is not in $\text{Ker}(\delta^0)$. The statement follows. □

Let p be a path either from e_1 to e_n or from e_n to e_1 in Q_A. Then p^* is an oriented cycle at f_1 in Q_B. If p is a path from e_1 to e_n, then we have

$$\delta_B^0(f_1\|p^*) = \sum_{s(a)=e_n, a\in(Q_A)_1, ap\in\mathcal{B}_A} a^*\|a^*p^* - \sum_{t(b)=e_1, b\in(Q_A)_1, pb\in\mathcal{B}_A} b^*\|p^*b^*.$$

Note that we have omitted some zero terms in the above sum, for example, if $d \in (Q_A)_1$ is an arrow starting at e_1, then $d^*\|d^*p^*$ appears as a term in the above sum, however, it is zero since d^*p^* lies in I_B. If p is a path from e_n to e_1, then we have

$$\delta_B^0(f_1\|p^*) = \sum_{s(a)=e_1, a\in(Q_A)_1, ap\in\mathcal{B}_A} a^*\|a^*p^* - \sum_{t(b)=e_n, b\in(Q_A)_1, pb\in\mathcal{B}_A} b^*\|p^*b^*.$$

As in the previous case, we have omitted some zero terms in the above sum. Moreover, in both cases, $\delta_B^0(f_1\|p^*)$ is zero if and only if $ap, pa \in I_A$ for all $a \in (Q_A)_1$. This observation leads to the following definition:

Definition 3.6 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A and let p be a path either from e_1 to e_n or from e_n to e_1 in Q_A. We call p is a special path between e_1 and e_n in Q_A if $\delta_B^0(f_1\|p^*) \neq 0$, or equivalently, if there exists some $a \in (Q_A)_1$ such that $ap \notin I_A$ or $pa \notin I_A$.*

We denote by $\mathrm{Sp}(1,n)$ the set of special paths between e_1 and e_n in Q_A, and by $\mathrm{sp}(1,n)$ the number of these special paths. Furthermore, we denote by Z_{sp} the k-subspace of $\mathrm{Im}(\delta_B^0)$ generated by the elements $\delta_B^0(f_1\|p^*)$, where $p \in \mathrm{Sp}(1,n)$.

Lemma 3.7 *The summands of $\delta_B^0(f_1\|p^*)$ and $\delta_B^0(f_i\|q^*)$ $(1 \leq i \leq n-1)$ are disjoint, where p is a path either from e_1 to e_n or from e_n to e_1 in $\mathcal{B}_A$, q is a path in $\mathcal{B}_A$ but different from p.*

Proof Without loss of generality, we assume that p is a path from e_1 to e_n. Then

$$\delta_B^0(f_1\|p^*) = \sum_{s(\alpha)=e_n, \alpha\in(Q_A)_1, \alpha p\in\mathcal{B}_A} \alpha^*\|\alpha^*p^* - \sum_{t(\beta)=e_1, \beta\in(Q_A)_1, p\beta\in\mathcal{B}_A} \beta^*\|p^*\beta^*,$$

$$\delta_B^0(f_i\|q^*) = \sum_{s(a^*)=f_i, a^*\in(Q_B)_1, a^*q^*\in\mathcal{B}_B} a^*\|a^*q^* - \sum_{t(b^*)=f_i, b^*\in(Q_B)_1, q^*b^*\in\mathcal{B}_B} b^*\|q^*b^*.$$

Note that $\alpha^*\|\alpha^*p^* \neq a^*\|a^*q^*$ since $p \neq q$. Also $\alpha^*\|\alpha^*p^* \neq b^*\|q^*b^*$, otherwise $\alpha = b$ and $\alpha p = qb$ which cause p to be of the form $x_m \cdots x_2\alpha$ where $x_2, \cdots, x_m \in (Q_A)_1$. It follows that $s(p) = s(\alpha) = e_n$, which is a contradiction. We can similarly show that $\beta^*\|p^*\beta^* \neq a^*\|a^*q^*$ and $\beta^*\|p^*\beta^* \neq b^*\|q^*b^*$. □

Remark 3.8 (1) *The dimension of Z_{sp} is equal to $\mathrm{sp}(1,n)$. This follows from the fact that the summands of $\delta_B^0(f_1\|p^*)$ and $\delta_B^0(f_1\|q^*)$ are disjoint for $p, q \in \mathrm{Sp}(1,n)$ $(p \neq q)$ by Lemma 3.7.*

(2) *If e_1 and e_n belong to different blocks of A or A is a radical square zero algebra, then $\mathrm{sp}(1,n) = 0$.*

(3) *In general, the number $\mathrm{sp}(1,n)$ could be arbitrarily large, see Example 4.3.*

We can now compare the dimensions of $\mathrm{Im}(\delta_A^0)$ and $\mathrm{Im}(\delta_B^0)$:

Proposition 3.9 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Let $\mathrm{sp}(1,n)$ be the number of special paths between e_1 and e_n in Q_A. Then*

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 + c_B - c_A - \mathrm{sp}(1,n),$$

where c_A and c_B are the number of connected components of Q_A and Q_B respectively. In particular, if we glue e_1 and e_n from the same block of A, then

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 - \mathrm{sp}(1,n);$$

if we glue e_1 and e_n from different blocks of A, then

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0).$$

Proof We keep the notations as in the proof of Proposition 3.3, that is, the vertices of Q_A are given by $e_1, \cdots, e_n$ and the vertices of Q_B are given by $f_1, \cdots, f_{n-1}$, where f_1 is obtained by gluing e_1 and e_n. The quiver morphism $\varphi : Q_A \to Q_B$ is given by the following formula: $\varphi(e_i) = f_i$ for $2 \le i \le n-1$, $\varphi(e_1) = \varphi(e_n) = f_1$, and $\varphi(\alpha) = \alpha^*$ for each arrow α. We begin with describing the basis elements in $\mathrm{Im}(\delta_A^0)$ and in $\mathrm{Im}(\delta_B^0)$.

Let $e_i \| p \in k((Q_A)_0 \| \mathcal{B}_A)$. We consider two cases depending if $p = e_i$ or $p \neq e_i$.

$(a1)$ $p = e_i$ $(1 \le i \le n)$. We have

$$\delta_A^0(e_i \| e_i) = \sum_{s(a)=e_i, a\in (Q_A)_1} a\|a - \sum_{t(b)=e_i, b\in(Q_A)_1} b\|b.$$

By Lemma 3.5, the subspace $\mathrm{Im}(\delta^0_{(A)_0})$ of $\mathrm{Im}\delta_A^0$ generated by the elements of the form $\delta_A^0(e_i\|e_i)$ has dimension $n_A - c_A$.

$(a2)$ $p \neq e_i$. Then p is an oriented cycle at e_i and

$$\delta_A^0(e_i\|p) = \sum_{s(a)=e_i, a\in(Q_A)_1, ap\in\mathcal{B}_A} a\|ap - \sum_{t(b)=e_i, b\in(Q_A)_1, pb\in\mathcal{B}_A} b\|pb.$$

Let denote $\mathrm{Im}(\delta^0_{(A)_{\ge 1}})$ the k-vector space generated by the image of δ_A^0 on $e_i\|p$ $(1 \le i \le n)$, where $p \in \mathcal{B}_A$ and $p \neq e_i$. It is clear that

$$\mathrm{Im}(\delta_A^0) = \mathrm{Im}(\delta^0_{(A)_0}) \ \oplus \ \mathrm{Im}(\delta^0_{(A)_{\ge 1}}).$$

Similarly, we let $f_i\|q \in k((Q_B)_0\|\mathcal{B}_B)$ and consider four cases.

(b_1) $q = f_i$ $(1 \le i \le n-1)$. We have

$$\delta_B^0(f_i\|f_i) = \sum_{s(a^*)=f_i, a^*\in(Q_B)_1} a^*\|a^* - \sum_{t(b^*)=f_i, b^*\in(Q_B)_1} b^*\|b^*.$$

By Lemma 3.5, the subspace $\mathrm{Im}(\delta^0_{(B)_0})$ of $\mathrm{Im}\delta_B^0$ generated by the elements of the form $\delta_B^0(f_i\|f_i)$ has dimension $n_B - c_B$.

(b_2) q is an oriented cycle at f_i and $i \neq 1$. By Proposition 3.3, $q = p^*$ for some oriented cycle $p \in \mathcal{B}_A$ at e_i $(2 \le i \le n-1)$. We have

$$\delta_B^0(f_i\|p^*) = \sum_{s(a^*)=f_i, a^*\in(Q_B)_1} a^*\|a^*p^* - \sum_{t(b^*)=f_i, b^*\in(Q_B)_1} b^*\|p^*b^* = \psi_1(\delta_A^0(e_i\|p)).$$

(b_3) q is an oriented cycle at f_1 such that $q = p^*$ for some oriented cycle $p \in \mathcal{B}_A$ at e_1 or e_n. If p is an oriented cycle at e_1, then we have

$$\delta_B^0(f_1\|p^*) = \sum_{s(a^*)=f_1, a^*\in(Q_B)_1, a^*p^*\in\mathcal{B}_B} a^*\|a^*p^* - \sum_{t(b^*)=f_1, b^*\in(Q_B)_1, p^*b^*\in\mathcal{B}_B} b^*\|p^*b^* = \psi_1(\delta_A^0(e_1\|p)).$$

If p is an oriented cycle at e_n, then we have

$$\delta_B^0(f_1\|p^*) = \sum_{s(a^*)=f_1, a^*\in(Q_B)_1, a^*p^*\in\mathcal{B}_B} a^*\|a^*p^* - \sum_{t(b^*)=f_1, b^*\in(Q_B)_1, p^*b^*\in\mathcal{B}_B} b^*\|p^*b^* = \psi_1(\delta_A^0(e_n\|p)).$$

(b_4) q is an oriented cycle at f_1 with $q = p^*$ for some $p \in \mathcal{B}_A$, where p is a path either from e_1 to e_n or from e_n to e_1. We may assume p is a special path since otherwise $\delta_B^0(f_1\|p^*)$ is zero. Note that q is of the form $f_1 \overset{a^*}{\to} \cdots \overset{b^*}{\to} f_1$ and might be a loop at f_1. If p is a path from e_1 to e_n, then we have

$$\delta_B^0(f_1\|p^*) = \sum_{s(a)=e_n, a\in(Q_A)_1, ap\in\mathcal{B}_A} a^*\|a^*p^* - \sum_{t(b)=e_1, b\in(Q_A)_1, pb\in\mathcal{B}_A} b^*\|p^*b^*.$$

If p is a path from e_n to e_1, then we have

$$\delta_B^0(f_1\|p^*) = \sum_{s(a)=e_1, a\in(Q_A)_1, ap\in\mathcal{B}_A} a^*\|a^*p^* - \sum_{t(b)=e_n, b\in(Q_A)_1, pb\in\mathcal{B}_A} b^*\|p^*b^*.$$

In both cases, $\delta_B^0(f_1\|p^*)$ is nonzero since p is a special path.

Let denote $\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ the k-vector space generated by the image of δ_B^0 on $f_i\|q$ $(1 \leq i \leq n-1)$, where $q \in \mathcal{B}_B$ and $q \neq f_i$. Then we have

$$\mathrm{Im}(\delta_B^0) = \mathrm{Im}(\delta^0_{(B)_0}) \oplus \mathrm{Im}(\delta^0_{(B)_{\geq 1}}).$$

We now claim that

$$\mathrm{Im}(\delta^0_{(B)_{\geq 1}}) = \psi_1(\mathrm{Im}(\delta^0_{(A)_{\geq 1}})) \oplus Z_{sp}.$$

It suffices to show that the summands of $\delta_B^0(f_1\|p^*)$ and the summands of elements in $\psi_1(\mathrm{Im}(\delta^0_{(A)_{\geq 1}}))$ are disjoint for $p \in \mathrm{Sp}(1,n)$. Since the element in $\psi_1(\mathrm{Im}(\delta^0_{(A)_{\geq 1}}))$ is of the form $\psi_1(\delta_A^0(e_i \parallel q)) = \delta_B^0(f_i \parallel q^*)$, where q is an oriented cycle at e_i $(1 \leq i \leq n)$ (here we identify f_n with f_1), the statement follows from Lemma 3.7.

From the above claim we see that there is a bijection between the basis elements of $\mathrm{Im}(\delta^0_{(A)_{\geq 1}})$ and the basis elements of $\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ that are not appeared in the above case (b_4). According to Remark 3.8 (1), the dimension of $\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ is equal to the dimension of $\mathrm{Im}(\delta^0_{(A)_{\geq 1}})$ plus $\mathrm{sp}(1,n)$. Combining this and (a_1), (b_1) and the fact that $n_A = n_B + 1$, we get that the dimension of $\mathrm{Im}(\delta^0_{(A)_0})$ is equal to the dimension of $\mathrm{Im}(\delta^0_{(B)_0})$ plus $1 + c_B - c_A$. Therefore we have

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 + c_B - c_A - \mathrm{sp}(1,n).$$

In particular, if we glue e_1 and e_n from the same block of A, then we have $c_B = c_A$. However, if e_1 and e_n are from two different blocks of A, then we have $c_B = c_A - 1$ and $sp(1,n) = 0$. We are done. □

We obtain an important corollary that it will be useful for stable equivalences induced by idempotent gluings.

Corollary 3.10 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then we have*

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 + c_B - c_A$$

under each of the following two conditions:

(i) e_1 is a source and e_n is a sink;

(ii) A is a radical square zero algebra.

In particular, if we glue e_1 and e_n from the same block of A, then

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1;$$

if e_1 and e_n are from two different blocks of A, then we have

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0).$$

Proof It is clear that under the condition (i) or (ii) there is no special path between e_1 and e_n, so we have $\mathrm{sp}(1,n) = 0$. If we glue e_1 and e_n from the same block of A, then $c_B = c_A$; if e_1 and e_n are from two different blocks of A, then $c_B = c_A - 1$. Hence the result follows from Proposition 3.9. □

From now on, we often use the following assumption on the characteristic of the filed k.

Assumption 3.11 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. For each loop α at e_1 or at e_n with $\alpha^m \in Z_A$ for some $m \geq 2$, $char(k) \nmid m$.*

Clearly Assumption 3.11 holds when the characteristic of the field k is zero or big enough. We now proceed to compare the Lie structures of $\mathrm{Ker}(\delta_A^1)$ and $\mathrm{Ker}(\delta_B^1)$:

Proposition 3.12 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. If $\mathrm{char}(k)$ satisfies Assumption 3.11, then there exists an injective (restricted) Lie algebra homomorphism $\mathrm{Ker}(\delta_A^1) \hookrightarrow \mathrm{Ker}(\delta_B^1)$ induced from $\psi_1 : k((Q_A)_1 \| \mathcal{B}_A) \to k((Q_B)_1 \| \mathcal{B}_B)$, which we still denote by ψ_1.*

Proof First we notice that $I_A = \mathrm{Span}(Z_A)$ and $I_B = \mathrm{Span}(Z_B)$, and by obvious identification we can write $Z_B = Z_A \cup Z_{new}$, where $Z_{new} = \{b^*c^* \mid b^*, c^* \in (Q_B)_1, t(c^*) = f_1 = s(b^*), bc \notin \mathcal{B}_A\}$. Having the diagram $(*)$ in mind, let $\alpha\|p \in k((Q_A)_1\|\mathcal{B}_A)$ and let $\psi_1(\alpha\|p) = \alpha^*\|p^*$ be the corresponding element in $k((Q_B)_1\|\mathcal{B}_B)$. On the one hand, we have

$$\psi_2(\delta_A^1(\alpha\|p)) = \psi_2(\sum_{r\in Z_A} r\|r^{\alpha\|p}) = \sum_{r\in Z_A} r\|r^{\alpha^*\|p^*};$$

On the other hand, we have

$$\delta_B^1(\psi_1(\alpha\|p)) = \delta_B^1(\alpha^*\|p^*) = \sum_{r\in Z_A} r\|r^{\alpha^*\|p^*} + \sum_{r'\in Z_{new}} r'\|r'^{\alpha^*\|p^*}.$$

We consider four cases.

(c1) α is a loop at e_i $(2 \le i \le n-1)$, $p = e_i$ or p is an oriented cycle at e_i. Then $\sum_{r'\in Z_{new}} r'\|r'^{\alpha^*\|p^*} = 0$ since α^* does not appear in any $r' \in Z_{new}$. Therefore $\psi_2(\delta_A^1(\alpha\|e_i)) = \delta_B^1(\psi_1(\alpha\|e_i))$.

(c2) α is a loop at e_1 (respectively e_n) and $p = e_1$ (resp. $p = e_n$). In case $p = e_1$, since A is finite dimensional, Z_A contains an element $r = \alpha^m$ for some $m \ge 2$, $\delta_A^1(\alpha\|e_1)$ contains a summand $mr\|\alpha^{m-1}$, which can not be cancelled in $\mathrm{Im}(\delta_A^1)$ unless $\mathrm{char}(k) \mid m$. That is, if $\mathrm{char}(k) \nmid m$, then $\alpha\|e_1 \notin \mathrm{Ker}(\delta_A^1)$. On the other hand, α^* must appears in some $r' \in Z_{new}$ and therefore $\delta_B^1(\psi_1(\alpha\|e_1)) = \delta_B^1(\alpha^*\|f_1) \neq 0$. Therefore, if $\mathrm{char}(k) \nmid m$, then both $\alpha\|e_1 \notin \mathrm{Ker}(\delta_A^1)$ and $\psi_1(\alpha\|e_1) \notin \mathrm{Ker}(\delta_B^1)$ hold. The similar result holds in case $p = e_n$.

(c3) α is a loop at e_1 (resp. e_n) and p is an oriented cycle at e_1 (resp. e_n). Since once we replace α^* in any $r' \in Z_{new}$ by p^*, r' becomes a path in Q_B that still contains some relation in Z_{new}, we have $\sum_{r'\in Z_{new}} r'\|r'^{\alpha^*\|p^*} = 0$. Therefore $\psi_2(\delta_A^1(\alpha\|p)) = \delta_B^1(\psi_1(\alpha\|p))$.

(c4) α is a non-loop arrow and $p \in \mathcal{B}_A$ is a parallel path to α. In this case it is easy to see that once α^* appears in some $r' \in Z_{new}$ (note that $r' \notin \mathcal{B}_B$), the element obtained from r' by replacing α^* by p^* is again not in $\mathcal{B}_B$. We still have $\sum_{r'\in Z_{new}} r'\|r'^{\alpha^*\|p^*} = 0$. Therefore $\psi_2(\delta_A^1(\alpha\|p)) = \delta_B^1(\psi_1(\alpha\|p))$.

The above discussion shows that, if $\mathrm{char}(k)$ satisfies Assumption 3.11, then there is a k-linear map $\psi_1 : \mathrm{Ker}(\delta_A^1) \longrightarrow \mathrm{Ker}(\delta_B^1)$ induced from the following mapping: $\alpha\|e_i \mapsto \alpha^*\|f_i$ $(i \neq 1, n)$, $\alpha\|p \mapsto \alpha^*\|p^*$ ($p \in \mathcal{B}_A$ has length ≥ 1). It is also clear that $\psi_1 : \mathrm{Ker}(\delta_A^1) \longrightarrow \mathrm{Ker}(\delta_B^1)$ is injective and preserves the Lie bracket, since ψ_1 preserves the parallel paths. □

Remark 3.13 *Since the characteristic condition only makes sense in the case (c2), if there is no loop both at e_1 and at e_n, then we do not need Assumption 3.11 in Proposition 3.12. In particular, we do not need Assumption 3.11 in Proposition 3.12 when we glue a source vertex and a sink vertex.*

Remark 3.14 (1) *If A (hence also B) is a radical square zero algebra, then Assumption 3.11 is equivalent to $\mathrm{char}(k) \neq 2$ and we do not need this assumption in Proposition 3.12 when we glue e_1 and e_n from the same block of A or when we glue e_1 and e_n from different blocks (say A_1 and A_2) of A such that both A_1 and A_2 are not isomorphic to $k[x]/(x^2)$. In fact, the characteristic condition only makes sense in the case (c2), however, in each of the above two cases, the loop α must appear in a relation $\alpha\beta$ (where β is an arrow different from α) or in a relation $\gamma\alpha$ (where γ is an arrow different from α), and so both $\alpha\|e_1 \notin \mathrm{Ker}(\delta_A^1)$ and $\psi_1(\alpha\|e_1) \notin \mathrm{Ker}(\delta_B^1)$ hold.*

(2) *It is easy to see that $\mathrm{Ker}(\delta_A^1) = k((Q_A)_1\|(Q_A)_1)$ when A is radical square zero, except in the case that $\mathrm{char}(k) = 2$ and one of the blocks of A is isomorphic to $k[x]/(x^2)$ (for this case, see Remark 3.19).*

In order to describe the elements in $\mathrm{Ker}(\delta_B^1)$ which are in the complement of the subspace $\psi_1(\mathrm{Ker}(\delta_A^1))$, we introduce some further notation.

Definition 3.15 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Let α be an arrow and p be a path in $\mathcal{B}_A$. We call (α, p) is a special pair with respect to the gluing of e_1 and e_n if the following three conditions are satisfied:*

(1) *α is starting from e_1, or ending at e_1, or starting from e_n, or ending at e_n;*

(2) *$\alpha^* \| p^*$ in Q_B;*

(3) *$\alpha \nparallel p$ in Q_A.*

We denote by $\mathrm{Spp}(1,n)$ the set of all special pairs with respect to the gluing of e_1 and e_n and by $\langle \mathrm{Spp}(1,n) \rangle$ the k-subspace of $k((Q_B)_1 \| \mathcal{B}_B)$ generated by the elements $\alpha^* \| p^*$, where $(\alpha, p) \in \mathrm{Spp}(1,n)$. Furthermore, we denote by Z_{spp} the intersection of $\langle \mathrm{Spp}(1,n) \rangle$ and $\mathrm{Ker}(\delta_B^1)$, and by $\mathrm{kspp}(1,n)$ the dimension of the k-subspace Z_{spp} of $\mathrm{Ker}(\delta_B^1)$.

Note that every nonzero element of Z_{spp} is a linear combination of the parallel paths corresponding to special pairs and it gives a E^e-derivation in $\mathrm{Ker}(\delta_B^1)$ which lies in the complement of the subspace $\psi_1(\mathrm{Ker}(\delta_A^1))$. Moreover, the condition (1) can be derived from conditions (2) and (3) in Definition 3.15, but for convenience we include condition (1). Note also that there are usually many types of special pairs, see Example 4.5.

Remark 3.16 *Note that although in radical square zero case there is no special path in Q_A, there may exists special pair when we glue e_1 and e_n whether from the same block or from two different blocks. In fact, if we glue two idempotents e_1 and e_n from the same block and there is an arrow α connects e_1 and e_n, then both (α, e_1) and (α, e_n) are special pairs. More precisely, in this case the special pairs (α, p) have the following 5 types (cf. Example 4.5):*

(*i*) *α is a loop at e_1 or e_n, p is an arrow between e_1 and e_n;*

(*ii*) *α is an arrow between e_1 and e_n, $p = e_1$ (or e_n) or p is a loop at e_1 (or e_n);*

(*iii*) *$s(\alpha) = s(p) = e_i$ for $i \neq 1, n$, and $t(\alpha), t(p)$ are both in $\{e_1, e_n\}$ but $t(\alpha) \neq t(p)$, also the dual case;*

(*iv*) *α is an arrow from e_1 to e_n, and p is an arrow from e_n to e_1, also the dual case;*

(*v*) *α is a loop at e_1, and $p = e_n$ or p is a loop at e_n, also the dual case.*

For the gluing different blocks case, $\mathrm{Spp}(1,n)$ consists of all special pairs of the form $(\alpha, e_n), (\alpha, \beta), (\beta, e_1)$ and (β, α), where α is a loop at e_1 and β is a loop at e_n. Then under the condition $\mathrm{char}(k) \neq 2$, neither $\alpha^* \| f_1$ nor $\beta^* \| f_1$ belongs to $\mathrm{Ker}(\delta_B^1)$, hence Z_{spp} must be generated by all special pairs $\alpha^* \| p^*$ and $\beta^* \| q^*$. However, once we exclude the case that there are loops at e_1 and e_n simultaneously, these generators vanish and Z_{spp} is zero.

Proposition 3.17 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. If $\mathrm{char}(k)$ satisfies Assumption 3.11, then we have $\dim_k \mathrm{Ker}(\delta_B^1) = \dim_k \mathrm{Ker}(\delta_A^1) + \mathrm{kspp}(1,n)$.*

Proof By Proposition 3.12, we only need to describe the elements θ in $\mathrm{Ker}(\delta_B^1)$ which are in the complement of the subspace $\psi_1(\mathrm{Ker}(\delta_A^1))$, under Assumption 3.11. According to the proof of Proposition 3.12, we may assume that θ is a linear combination of the elements of the form $\alpha^* \| p^*$ such that (α, p) is a special pair with respect to the gluing of e_1 and e_n. Clearly in this case $\theta \in Z_{spp}$, where Z_{spp} is the subspace of $\mathrm{Ker}(\delta_B^1)$ defined in Definition 3.15. Therefore, we have the following decomposition: $\mathrm{Ker}(\delta_B^1) = \psi_1(\mathrm{Ker}(\delta_A^1)) \oplus Z_{spp}$. Hence the dimension formula follows. □

Corollary 3.18 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing a source vertex e_1 and a sink vertex e_n of A. Then we have $\mathrm{Ker}(\delta_A^1) \simeq \mathrm{Ker}(\delta_B^1)$ as Lie algebras, except in the case that by gluing we obtain a block of B of the form $k[x]/(x^2)$ in $\mathrm{char}(k) = 2$.*

Proof By Lemma 3.2, in this case the only possible special pair with respect to the gluing of e_1 and e_n has the form (α, e_1) or (α, e_n) such that α^* is a loop at f_1 but α is neither a loop at e_1 nor a loop at e_n. Therefore $\langle \mathrm{Spp}(1,n)\rangle$ is generated by the elements of the form $\alpha^*||f_1$. Suppose now that $\alpha^*||f_1 \in \mathrm{Ker}(\delta_B^1)$. Then we consider two cases. If Q_A contains a connected component $e_1 \xrightarrow{\alpha} e_n$ so that B has a block isomorphic to $k[x]/(x^2)$, then $\delta_B^1(\alpha^*\|f_1) = 2r'\|\alpha^* = 0$ (where $r' = \alpha^*\alpha^*$) implies that $\mathrm{char}(k) = 2$. If Q_A is not the above case, then either there is an arrow $\beta^* \neq \alpha^*$ starting from f_1 or there is an arrow $\gamma^* \neq \alpha^*$ ending at f_1 in Q_B. Therefore $\delta_B^1(\alpha^*\|f_1)$ will contain a summand $\alpha^*\beta^*\|\beta^*$ or a summand $\gamma^*\alpha^*\|\gamma^*$, which clearly can not be cancelled in $\mathrm{Im}\delta_B^1$, so $\alpha^*\|f_1 \notin \mathrm{Ker}(\delta_B^1)$. It follows that $\alpha^*\|f_1 \in \mathrm{Ker}(\delta_B^1)$ if and only if B has a block isomorphic $k[x]/(x^2)$ and $\mathrm{char}(k) = 2$. Summarizing the above discussion we get $\mathrm{kspp}(1,n) = 0$ when gluing a source and a sink and excluding the case that by gluing we obtain a block of B of the form $k[x]/(x^2)$ in $\mathrm{char}(k) = 2$. Now the result follows from Proposition 3.17, Proposition 3.12 and Remark 3.13. □

Remark 3.19 *The case that we exclude in Corollary 3.18 occurs in $\mathrm{char}(k) = 2$ when A has one block of the form A_2 and we perform the gluing in this block. Since the rest of the blocks do not change, this reduces to the case when A has only the block A_2. In this case $\mathrm{char}(k) = 2$ and $B \simeq k[x]/(x^2)$, $A = kQ_A$ where Q_A is given by the quiver $1 \xrightarrow{\alpha} 2$. By a direct computation, we have the following: $\mathrm{Im}(\delta_A^0) = \mathrm{Ker}\delta_A^1$ is 1-dimensional with k-basis $\{\alpha\|\alpha\}$, $\mathrm{Im}(\delta_B^0) = 0$ and $\mathrm{Ker}(\delta_B^1)$ is 2-dimensional with k-basis $\{\alpha^*\|f_1, \alpha^*\|\alpha^*\}$. Here $\mathrm{Spp}(1,2) = \{(\alpha, e_1), (\alpha, e_2)\}$ and $Z_{spp} = \langle \alpha^*\|f_1\rangle$, hence the formula $\dim_k\mathrm{Ker}(\delta_B^1) = \dim_k\mathrm{Ker}(\delta_A^1) + \mathrm{kspp}(1,2)$ still holds in this case.*

We can finally compare the dimensions of $\mathrm{HH}^1(A)$ and of $\mathrm{HH}^1(B)$.

Theorem 3.20 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. If char(k) satisfies Assumption 3.11, then we have*

$$\dim_k\mathrm{HH}^1(A) = \dim_k\mathrm{HH}^1(B) - 1 - \mathrm{kspp}(1,n) + \mathrm{sp}(1,n) + c_A - c_B.$$

In particular, if we glue e_1 and e_n from the same block of A, then

$$\dim_k\mathrm{HH}^1(A) = \dim_k\mathrm{HH}^1(B) - 1 - \mathrm{kspp}(1,n) + \mathrm{sp}(1,n);$$

if e_1 and e_n are from two different blocks of A, then $\mathrm{HH}^1(A)$ is a Lie subalgebra of $\mathrm{HH}^1(B)$ and

$$\dim_k\mathrm{HH}^1(A) = \dim_k\mathrm{HH}^1(B) - \mathrm{kspp}(1,n).$$

Proof Since $\mathrm{HH}^1(A) \simeq \mathrm{Ker}(\delta_A^1)/\mathrm{Im}(\delta_A^0)$, this is a direct consequence of Proposition 3.9, Proposition 3.12 and Proposition 3.17. □

Concerning the relation between Lie structures of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$, we have the following strengthened form of Theorem 3.20.

Theorem 3.21 *Under the conditions of Theorem 3.20, we have the following exact commutative diagram:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Z_{sp} & \xrightarrow{\tilde{\iota}_B} & Z_{spp} & \longrightarrow & \mathrm{Coker}(\varphi) & \longrightarrow & 0 \\
 & & {\scriptstyle \pi^0}\uparrow & & {\scriptstyle \pi^1}\uparrow & & {\scriptstyle \pi}\uparrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\iota_B} & \mathrm{Ker}(\delta_B^1) & \longrightarrow & \frac{\mathrm{Ker}(\delta_B^1)}{Y} & \longrightarrow & 0 \\
 & & {\scriptstyle \psi_1|_{\mathrm{Im}(\delta_A^0)}}\uparrow & & {\scriptstyle \psi_1}\uparrow & & {\scriptstyle \varphi}\uparrow & & \\
0 & \longrightarrow & \mathrm{Im}(\delta_A^0) & \xrightarrow{\iota_A} & \mathrm{Ker}(\delta_A^1) & \longrightarrow & \mathrm{HH}^1(A) & \longrightarrow & 0, \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad (**)
$$

where π^0, π^1 are canonical projections, ι_A and ι_B are canonical injections, φ is an injective map induced from ψ_1 and π is a surjective map induced from π^1, and where $Y = \psi_1(\mathrm{Im}(\delta_A^0)) \oplus Z_{sp}$ is a subspace of $\mathrm{Ker}(\delta_B^1)$ which is equal to $\mathrm{Im}(\delta_B^0)$ in case that e_1 and e_n are from two different blocks of A and which contains $\mathrm{Im}(\delta_B^0)$ as a codimension 1 subspace in case that e_1 and e_n are from the same block of A. In particular, if $Z_{spp} = Z_{sp}$, then Y is a Lie ideal of $\mathrm{Ker}(\delta_B^1)$ and there is a Lie algebra epimorphism from $\mathrm{HH}^1(B)$ to $\mathrm{Ker}(\delta_B^1)/Y \simeq \mathrm{HH}^1(A)$ with kernel $I := Y/\mathrm{Im}(\delta_B^0)$, where I is zero if e_1 and e_n are from two different blocks of A and $\dim_k I = 1$ if e_1 and e_n are from the same block of A.

Proof By Proposition 3.12, there exists an injective Lie algebra homomorphism $\psi_1 : \mathrm{Ker}(\delta_A^1) \hookrightarrow \mathrm{Ker}(\delta_B^1)$, which is induced from the canonical map $\psi_1 : k((Q_A)_1 \| \mathcal{B}_A) \to k((Q_B)_1 \| \mathcal{B}_B)$ $(\alpha \| p \mapsto \alpha^* \| p^*)$. Moreover, by Proposition 3.17, we have the decomposition $\mathrm{Ker}(\delta_B^1) = \psi_1(\mathrm{Ker}(\delta_A^1)) \oplus Z_{spp}$, where Z_{spp} denotes the intersection of $\langle \mathrm{Spp}(1,n) \rangle$ and $\mathrm{Ker}(\delta_B^1)$ (cf. Definition 3.15).

Combining with Proposition 3.9 and noting the fact $\delta_B^0(f_1 \| f_1) = \psi_1(\delta_A^0(e_1 \| e_1)) + \psi_1(\delta_A^0(e_n \| e_n))$, we have that $\psi_1 : \mathrm{Ker}(\delta_A^1) \hookrightarrow \mathrm{Ker}(\delta_B^1)$ restricts to an injective map

$$
\psi_1|_{\mathrm{Im}(\delta_A^0)} : \mathrm{Im}(\delta_A^0) = \mathrm{Im}(\delta^0_{(A)_0}) \oplus \mathrm{Im}(\delta^0_{(A)_{\geq 1}}) \hookrightarrow X \oplus \mathrm{Im}(\delta^0_{(B)_{\geq 1}}) \subseteq \mathrm{Ker}(\delta_B^1),
$$

where X is the subspace of $\mathrm{Ker}(\delta_B^1)$ generated by the elements $\psi_1(\delta_A^0(e_1 \| e_1))$, $\psi_1(\delta_A^0(e_n \| e_n))$ and $\delta_B^0(f_i \| f_i)$ $(2 \leq i \leq n-1)$. If we denote $X \oplus \mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ by Y, then we have the decomposition $Y = \psi_1(\mathrm{Im}(\delta_A^0)) \oplus Z_{sp}$, where Z_{sp} (cf. Definition 3.6) is the subspace of $\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ generated by the elements $\delta_B^0(f_1 \| p^*)$ for $p \in \mathrm{Sp}(1,n)$.

Note that the dimension of X is equal to $\dim_k \mathrm{Im}(\delta^0_{(A)_0})$, $X = \mathrm{Im}(\delta^0_{(B)_0})$ if e_1 and e_n are from two different blocks of A, and $\mathrm{Im}(\delta^0_{(B)_0}) \subseteq X$ has codimension 1 in X if e_1 and e_n are from the same block of A (cf. Lemma 3.5). It follows that $\mathrm{Im}(\delta_B^0)$=$\mathrm{Im}(\delta^0_{(B)_0})$$\oplus$$\mathrm{Im}(\delta^0_{(B)_{\geq 1}})$ is equal to Y if e_1 and e_n are from two different blocks of A and has codimension 1 in Y if e_1 and e_n are from the same block of A. However, when $Y \supsetneqq \mathrm{Im}(\delta_B^0)$, Y is usually not a Lie ideal of $\mathrm{Ker}(\delta_B^1)$ since in general $[Y, Z_{spp}]$ is not contained in Y (In fact, Y is a Lie ideal of $\mathrm{Ker}(\delta_B^1)$ if and only if $[\psi_1(\mathrm{Im}(\delta_A^0)), Z_{spp}] \subset Y$). It is clear that if p is a special path from e_1 to e_n, then each summand $a^* \| a^* p^*$ (or $b^* \| p^* b^*$) of $\delta_B^0(f_1 \| p^*)$, where a is an arrow starting from e_n such that $ap \in \mathcal{B}_A$ (or where b is an arrow ending at e_1 such that $pb \in \mathcal{B}_A$), is induced from a special pair (a, ap) (or (b, pb)). In the case that p is a special path from e_n to e_1, we have the similar conclusion. Therefore the canonical injective map $Y \hookrightarrow \mathrm{Ker}(\delta_B^1)$ restricts to an injective map $Z_{sp} \hookrightarrow Z_{spp}$.

Summarizing the above discussion we obtain the exact commutative diagram $(**)$. From this we know that the conclusions under the condition $Z_{spp} = Z_{sp}$ are also clear. $\square$

Remark 3.22 *Note that the one-dimensional ideal $I := Y/\mathrm{Im}(\delta_B^0)$ of $\mathrm{HH}^1(B)$ in Theorem 3.21 is generated by $\psi_1(\delta_A^0(e_1 \| e_1)) = \psi_1(\sum_{[\alpha] \in (\bar{Q}_A)_1 e_1} I_{[\alpha]} - \sum_{[\alpha] \in e_1 (\bar{Q}_A)_1} I_{[\alpha]})$, where $I_{[\alpha]} := \sum_{i=1}^{m} \alpha_i \| \alpha_i$ for $[\alpha] = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. If we glue a source e_1 and a sink e_n from the same block of A to get B, then we can rewrite the generator $\psi_1(\delta_A^0(e_1 \| e_1))$ of I as $\psi_1(\sum_{[\alpha] \in (\bar{Q}_A)_1 e_1} I_{[\alpha]})$. Specifically, $\psi_1(\delta_A^0(e_1 \| e_1)) =$*

$\psi_1(\sum_{[\alpha]\in\Delta} I_{[\alpha]})$ (modulo an element in $\mathrm{Im}(\delta_B^0)$), where Δ is a subset of $(\bar{Q}_A)_1e_1$ consisting of some equivalence classes of parallel arrows $[\alpha]$ starting from e_1 such that α satisfies one of the following two conditions:

(i) $t(\alpha)=e_n$;

(ii) α lies in a path or an undirected path in the quiver Q_A, which is starting at e_1 and ending at e_n and just through e_1 once.

For a concrete example for Δ, see Example 4.8.

In order to define Δ we could proceed as follows. We define two sub-quivers of Q_A: Q_A^c and Q_A^d. The arrows of Q_A^c satisfy one of the following two conditions:

(i) $t(\alpha)=e_n$;

(ii) α lies in a path or an undirected path in the quiver Q_A, which is starting at e_1 and ending at e_n and just through e_1 once.

The arrows of Q_A^d are the arrows of Q_A which are not in Q_A^c. Note that by construction the sets of arrows of Q_A^c and of Q_A^d are disjoint and that both quivers only share the vertex e_1 when Q_A^d is not empty. Then we define Δ to be a set $(\bar{Q}_A^c)_1e_1$. We also define the corresponding sub-quivers Q_B^c and Q_B^d via the map φ in Section 2. In order to show that $\psi_1(\delta_A^0(e_1\|e_1))$ is equal to $\psi_1(\sum_{[\alpha]\in\Delta} I_{[\alpha]})$ (modulo an element in $\mathrm{Im}(\delta_B^0)$) we proceed as follows. We have

$$\begin{aligned}\psi_1(\delta_A^0(e_1\|e_1)) &= \psi_1(\sum_{[\alpha]\in(\bar{Q}_A)_1e_1} I_{[\alpha]}) = \psi_1(\sum_{[\alpha]\in(\bar{Q}_A^c)_1e_1} I_{[\alpha]}) + \psi_1(\sum_{[\alpha]\in(\bar{Q}_A^d)_1e_1} I_{[\alpha]})\\ &= \psi_1(\sum_{[\alpha]\in\Delta} I_{[\alpha]}) + \psi_1(\sum_{[\alpha]\in(\bar{Q}_A^d)_1e_1} I_{[\alpha]}).\end{aligned}$$

In order to show that $\psi_1(\delta_A^0(e_1\|e_1))$ is equal to $\psi_1(\sum_{[\alpha]\in\Delta} I_{[\alpha]})$ (modulo an element in $\mathrm{Im}(\delta_B^0)$) we proceed as follows: Note that Q_A^c and Q_A^d can be obtained from Q_A by splitting in e_1 since the fact that Q_A^c and Q_A^d are disjoint and they only share the vertex e_1 when Q_A^d is not empty. By combing this with the fact that e_1 is a source vertex, we deduce that $\delta_A^0(\sum_{i=1}^{i=n} e_i\|e_i)=0$ if and only if $\delta_A^0(\sum_{e_i\in Q_A^c} e_i\|e_i)=0$ and $\delta_A^0(\sum_{e_i\in Q_A^d} e_i\|e_i)=0$, whence

$$\psi_1(\sum_{[\alpha]\in(\bar{Q}_A^d)_1e_1} I_{[\alpha]}) = -\psi_1(\sum_{e_i\in(Q_A^d)_0, e_i\neq e_1} \delta_A^0(e_i\|e_i)) = \sum_{f_i\in(Q_B^d)_0, f_i\neq f_1} \delta_B^0(f_i\|f_i)\in\mathrm{Im}(\delta_B^0).$$

Corollary 3.23 *Under the conditions of Theorem 3.20, we have that* $\mathrm{kspp}(1,n)\geq\mathrm{sp}(1,n)$. *In particular, the number of special pairs is bigger than or equal to the number of special paths. Actually, the above fact does not depend on Assumption 3.11.*

Corollary 3.24 *Let $A=kQ_A/I_A$ be a monomial algebra and let $B=kQ_B/I_B$ be a radical embedding of A obtained by gluing a source vertex e_1 and a sink vertex e_n of A. Then we have*

$$\dim_k\mathrm{HH}^1(A)=\dim_k\mathrm{HH}^1(B)+c_A-c_B-1,$$

except in the case that by gluing we obtain a block of B of the form $k[x]/(x^2)$ in $\mathrm{char}(k)=2$. In particular, if we glue e_1 and e_n from two different blocks of A, then there is a (restricted) Lie algebra isomorphism $\mathrm{HH}^1(A)\simeq\mathrm{HH}^1(B)$; if e_1 and e_n are from the same block of A, then $\mathrm{HH}^1(A)\simeq\mathrm{HH}^1(B)/I$ as (restricted) Lie algebras, where I is a one-dimensional (restricted) Lie ideal of $\mathrm{HH}^1(B)$. Moreover, if $\mathrm{char}(k)=0$, then the one-dimensional Lie ideal I lies in the center of $\mathrm{HH}^1(B)$ and

$$\mathrm{HH}^1(B)\simeq\mathrm{HH}^1(A)\times I\simeq\mathrm{HH}^1(A)\times k$$

as Lie algebras.

Proof First we notice that by Remark 3.13, we do not need Assumption 3.11 here. By Corollary 3.18, we have $Z_{spp} = 0$ since we glue a source and a sink. Now our conclusions except for the last one follow directly from Theorem 3.20 and Theorem 3.21.

It suffices to show the last statement of the Corollary. From now on, we fix the 'minimal' generator $\psi_1(\sum_{[\alpha]\in\Delta} I_{[\alpha]})$ of the Lie ideal I by Remark 3.22. We claim that I is contained in the center $Z(L_0)$ of L_0 and I is a direct summand of L_0 as Lie algebras. Then the statement follows. Indeed, if we assume that $L_0 = I \oplus G$ as Lie algebras, then by the graduation of $\mathrm{HH}^1(B)$ we have

$$\mathrm{HH}^1(B) = L_0 \oplus \bigoplus_{i\geq 1} L_i = (I \oplus G) \oplus \bigoplus_{i\geq 1} L_i = I \oplus (G \oplus \bigoplus_{i\geq 1} L_i) =: I \oplus L,$$

where L is a Lie ideal of $\mathrm{HH}^1(B)$. Indeed, $[I, L] = [I, G] + [I, \bigoplus_{i\geq 1} L_i] = [I, \bigoplus_{i\geq 1} L_i] \subset \bigoplus_{i\geq 1} L_i$ by Remark 2.4 and the above claim. Consequently, to show $\mathrm{HH}^1(B) = I \oplus L$ as Lie algebras is equivalent to show $0 = [I, L] = [I, \bigoplus_{i\geq 1} L_i]$. By Remark 2.4 we know that both $\bigoplus_{i\geq 1} L_i$ and I are Lie ideals of $\mathrm{HH}^1(B)$, hence $[I, \bigoplus_{i\geq 1} L_i] = 0$. Therefore, $\mathrm{HH}^1(B) = I \oplus L$ as Lie algebras. Since $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/I$ as Lie algebras, then $L \simeq \mathrm{HH}^1(A)$ as Lie algebras.

Now we prove the above claim. Firstly, we show that I is contained in the center $Z(L_0)$ of L_0. We also adopt symbol $[\alpha]$ rather than $[\alpha^*]$ in $(\bar{Q}_B)_1$ for every $[\alpha]$ in $(\bar{Q}_A)_1$. According to Proposition 2.5, without loss of generality, we can always take $L_0^{[\alpha]} = \langle \alpha_i^* \| \alpha_j^*, \alpha_l^* \| \alpha_l^* \mid 1 \leq l \leq m, 1 \leq i \neq j \leq m, \alpha_i^* \| \alpha_j^* \in L_0 \rangle$ for each $[\alpha] = \{\alpha_1, \cdots, \alpha_m\}$ in Δ. Then by Proposition 2.6 we have $Z(L_0^{[\alpha]}) = \oplus_C \langle \sum_{\alpha_i \in C} \alpha_i^* \| \alpha_i^* \rangle$, where C traverses connected components of $[\alpha]$ and $[\alpha] \in \Delta$. And it is easy to see that $I = \langle \psi_1(\sum_{[\alpha]\in\Delta} I_{[\alpha]}) \rangle = \langle \sum_{[\alpha]\in\Delta} \sum_{i=1}^m \alpha_i^* \| \alpha_i^* \rangle \subset \oplus_{[\alpha]\in\Delta} (\oplus_C \langle \sum_{\alpha_i \in C} \alpha_i^* \| \alpha_i^* \rangle) = \oplus_{[\alpha]\in\Delta} Z(L_0^{[\alpha]}) \subset Z(L_0)$.

Secondly, we show that I is a direct summand of L_0 as Lie algebras. We sketch the proof in case that Δ only contains two equivalence classes of parallel arrows (cf. Remark 3.22), namely $\Delta = \{[\alpha], [\beta]\}$, where $[\alpha] = \{\alpha_1, \cdots, \alpha_m\}$ and $[\beta] = \{\beta_1, \cdots, \beta_t\}$. Then $I = \langle \psi_1(I_{[\alpha]} + I_{[\beta]}) \rangle = \langle \sum_{i=1}^m \alpha_i^* \| \alpha_i^* + \sum_{j=1}^t \beta_j^* \| \beta_j^* \rangle$. Since $L_0^{[\alpha]} = \langle \alpha_i^* \| \alpha_j^*, \alpha_l^* \| \alpha_l^* \mid 1 \leq l \leq m, 1 \leq i \neq j \leq m, \alpha_i \| \alpha_j \in \mathrm{Ker}(\delta_A^1) \rangle$ and $L_0^{[\beta]} = \langle \beta_i^* \| \beta_j^*, \beta_l^* \| \beta_l^* \mid 1 \leq l \leq t, 1 \leq i \neq j \leq t, \beta_i \| \beta_j \in \mathrm{Ker}(\delta_A^1) \rangle$, it is easy to see that I is a summand of $\oplus_{[\alpha]\in\Delta} L_0^{[\alpha]} = L_0^{[\alpha]} \oplus L_0^{[\beta]}$, hence it is a summand of L_0. In fact, we have Lie algebra decompositions:

$$\begin{aligned} L_0^{[\alpha]} &= \langle \sum_{i=1}^m \alpha_i^* \| \alpha_i^* \rangle \oplus \langle \alpha_i^* \| \alpha_j^*, \alpha_l^* \| \alpha_l^* - \alpha_m^* \| \alpha_m^* \mid 1 \leq l \leq m-1, 1 \leq i \neq j \leq m, \alpha_i \| \alpha_j \in \mathrm{Ker}(\delta_A^1) \rangle \\ &=: \langle \psi_1(I_{[\alpha]}) \rangle \oplus J_1, \end{aligned}$$

$$L_0^{[\beta]} = \langle \sum_{i=1}^t \beta_i^* \| \beta_i^* \rangle \oplus \langle \beta_i^* \| \beta_j^*, \beta_l^* \| \beta_l^* - \beta_t^* \| \beta_t^* \mid 1 \leq l \leq t-1, 1 \leq i \neq j \leq t, \beta_i \| \beta_j \in \mathrm{Ker}(\delta_A^1) \rangle =: \langle \psi_1(I_{[\beta]}) \rangle \oplus J_2.$$

As a consequence, we can easily check that there are Lie algebra decompositions

$$\begin{aligned} L_0^{[\alpha]} \oplus L_0^{[\beta]} &= (\langle \psi_1(I_{[\alpha]}) \rangle \oplus \langle \psi_1(I_{[\beta]}) \rangle) \oplus J_1 \oplus J_2 = (I \oplus \langle \psi_1(I_{[\alpha]}) \rangle) \oplus J_1 \oplus J_2 \\ &= I \oplus (\langle \psi_1(I_{[\alpha]}) \rangle \oplus J_1 \oplus J_2) =: I \oplus J. \end{aligned}$$

It follows that $L_0 = \oplus_{[\alpha]\in(\bar{Q}_B)_1} L_0^{[\alpha]} = I \oplus G$ as Lie algebras, where $G := J \oplus \bigoplus_{\alpha\in(\bar{Q}_B)_1\setminus\Delta} L_0^{[\alpha]}$. □

Remark 3.25 (1) *From Corollary 3.24 we deduce that although the (restricted) Lie algebra structure of* HH^1 *is an invariant under stable equivalences of Morita type,* HH^1 *is not invariant under stable equivalences obtained by gluing a source and a sink from the same block. However, there is still a close relation between these two Lie algebras, namely,* $\mathrm{HH}^1(A)$ *is a quotient of* $\mathrm{HH}^1(B)$ *by an (often splitting) one-dimensional Lie ideal.*

(2) *It would be interesting to know when there is a Lie algebra decomposition*

$$\mathrm{HH}^1(B) = \mathrm{HH}^1(A) \times I \simeq \mathrm{HH}^1(A) \times k$$

if we glue two idempotents from the same block of A such that $Z_{spp} = Z_{sp}$. When $\mathrm{char}(k) \neq 0$, we can easily find examples (cf. Remark 3.30 (1)) in which I is not a Lie algebra direct summand of $\mathrm{HH}^1(B)$ and therefore $\mathrm{HH}^1(B) \not\cong \mathrm{HH}^1(A) \times k$. When $\mathrm{char}(k) = 0$, Corollary 3.24 shows that $\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \times k$ when we glue a source and a sink. Moreover, at least for radical square zero algebras over a field of characteristic zero, when we glue two idempotents from the same block such that $Z_{spp} = 0$, we have $\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \times k$ as Lie algebras (see Corollary 3.28 and Remark 3.29).

Corollary 3.26 *Let $A = kQ_A/I_A$ be a radical square zero algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. If $\mathrm{char}(k) \neq 2$, then we have*

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1 - \mathrm{kspp}(1, n) - c_B + c_A.$$

In particular, if we glue e_1 and e_n from the same block of A, then

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1 - \mathrm{kspp}(1, n);$$

if we glue e_1 and e_n from two different blocks of A, then

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - \mathrm{kspp}(1, n)$$

and then $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)$ as Lie algebras if we exclude the case that there are loops both at e_1 and e_n. Moreover, if one of the following conditions holds, then the above results still hold in the case $\mathrm{char}(k) = 2$:

- *(i) glue e_1 and e_n from the same block of A;*
- *(ii) glue $e_1 \in A_1$ and $e_n \in A_2$ from the different blocks of A such that both A_1 and A_2 are not isomorphic to $k[x]/(x^2)$.*

Proof In the radical square zero case, the number of special paths is zero, and by Remark 3.14, Assumption 3.11 is equivalent to $\mathrm{char}(k) \neq 2$. Thus the dimension formulas follow immediately from Remark 3.16, Theorem 3.20 and Theorem 3.21. Moreover, if we glue e_1 and e_n from two different blocks of A and exclude the case that there are loops at e_1 and e_n simultaneously, then $Z_{spp} = 0$ and $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)$ as Lie algebras again by Theorem 3.21. Finally, it is also easy to see that we do not need the assumption $\mathrm{char}(k) \neq 2$ under the condition (i) or (ii). □

Remark 3.27 *Let B be a radical embedding of a radical square zero algebra A obtained by gluing two idempotents $e_1 \in A_1$ and $e_n \in A_2$ from the different blocks A_1, A_2.*

1. *If there are loops at e_1 or at e_n, then in general $\mathrm{HH}^1(A)$ is not isomorphic to $\mathrm{HH}^1(B)$ and the difference between the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ can be arbitrarily large, see Example 4.7.*
2. *If $\mathrm{char}(k) = 2$ and exactly one of A_1, A_2 is isomorphic to $k[x]/(x^2)$, then by a direct computation the dimension formula should be changed as*

$$\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) + 1.$$

Corollary 3.28 *Let $A = kQ_A/I_A$ be a radical square zero algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n from the same block of A. If $Z_{spp} = 0$ and $\mathrm{char}(k) = 0$, then we have a Lie algebra isomorphism*

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \times k.$$

Proof We use the notations in Theorem 3.21. Since $Z_{spp} = 0$, we have a Lie algebra epimorphism from $\mathrm{HH}^1(B)$ to $\mathrm{Ker}(\delta_B^1)/Y \simeq \mathrm{HH}^1(A)$ with one-dimensional kernel $I := Y/\mathrm{Im}(\delta_B^0)$, where $\mathrm{Ker}(\delta_B^1) = \psi_1(\mathrm{Ker}(\delta_A^1))$ and Y is a Lie ideal of $\mathrm{Ker}(\delta_B^1)$. Also note that this epimorphism and equality do not depend on the Assumption 3.11 since we glue e_1 and e_n from the same block (cf. Remark 3.14). Furthermore, since A is radical square zero, we can apply Theorem 2.9 in [13]. More precisely, since $Z_{spp} = 0$, $\dim_k \mathrm{HH}^1(A) = \dim_k \mathrm{HH}^1(B) - 1$. Note that by gluing e_1 and e_n from the same block we have $\chi(\bar{Q}_B) = \chi(\bar{Q}_A) + 1$. Then by Theorem 2.9 in [13] (see also Theorem 3.31) there is an injective Lie algebra homomorphism:

$$\mathrm{HH}^1(A) \simeq \prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \times k^{\chi(\bar{Q}_A)} \to \prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \times k^{\chi(\bar{Q}_A)} \times k \simeq \mathrm{HH}^1(B).$$

Therefore it gives rise to the following Lie algebra isomorphisms:

$$\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(A) \times I \simeq \mathrm{HH}^1(A) \times k.$$

Remark 3.29 *Let $A = kQ_A/I_A$ be a radical square zero algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n from the same block of A. If we exclude the case that $\mathrm{char}(k) = 2$ and by gluing we obtain a block of B of the form $k[x]/(x^2)$ then it is straightforward to check that $Z_{spp} = 0$ under each of the following conditions (cf. Remark 3.16):*

(i) e_1 is a source and e_n is a sink;

(ii) Both e_1 and e_n are sinks such that

$$\{s(\alpha) \mid t(\alpha) = e_1, \alpha \in (Q_A)_1\} \cap \{s(\beta) \mid t(\beta) = e_n, \beta \in (Q_A)_1\} = \emptyset;$$

(iii) Both e_1 and e_n are sources such that

$$\{t(\alpha) \mid s(\alpha) = e_1, \alpha \in (Q_A)_1\} \cap \{t(\beta) \mid s(\beta) = e_n, \beta \in (Q_A)_1\} = \emptyset.$$

Remark 3.30 *Let A be a radical square zero algebra defined by some quiver Q. By direct computations, we can determine the Lie algebra structure of $\mathrm{HH}^1(A)$ in the following special situations.*

(1) $\mathrm{HH}^1(A) \simeq \mathfrak{gl}_n(k)$ if Q is the n-multiple loop quiver (the quiver with one vertex and n loops) , except in the case $n = 1$ and $\mathrm{char}(k) = 2$ (for this exceptional case, see Remark 3.19); The isomorphism sends $\alpha_i||\alpha_j$ to E_{ji} where E_{ij} is the matrix that has 1 in position (i, j) and 0 elsewhere. Note that if the characteristic of the field k does not divide n, then $\mathfrak{gl}_n(k) \simeq \mathfrak{sl}_n(k) \times k$ as Lie algebras.

(2) $\mathrm{HH}^1(A) \simeq \mathfrak{pgl}_m(k)$ if Q is the m-Kronecker quiver (the quiver with m parallel arrows), with the convention that 1-Kronecker quiver is the Dnykin quiver A_2 where $\mathfrak{pgl}_m(k)$ is defined as $\mathfrak{gl}_m(k)/k \cdot Id$. In other words, $\mathfrak{pgl}_m(k)$ is the quotient $\mathfrak{gl}_m(k)$ by its center. Let e be the source vertex of the m-Kronecker quiver. Then the above isomorphism can be obtained by observing that $\mathfrak{gl}_m(k) \simeq \mathrm{Ker}(\delta_A^1)$. The isomorphism sends $\alpha_i||\alpha_j$ to E_{ji} where E_{ij} is the matrix that has 1 in position (i, j) and 0 elsewhere. In addition, $\mathrm{Im}(\delta_A^0)$ is one dimensional and it is generated by the element $\sum_{s(\alpha_i)=e} \alpha_i||\alpha_i$ which is sent to Id via the isomorphism above. If the characteristic of the field k does not divide m, then $\mathfrak{pgl}_m(k) \simeq \mathfrak{sl}_m(k)$.

3.1 An interpretation of the Lie algebra structure of HH^1 for radical square zero algebras by inverse gluing operations

Given a quiver Q we have that the parallelism is an equivalence relation on the set of arrows Q_1. The set S denotes a complete set of representatives of the non-trivial classes, that is, equivalence classes having at least two arrows, and for $\alpha \in S$, $|\alpha|$ denotes the number of arrows in the equivalence class $[\alpha]$ of α. We denote $\bar{Q}_1$ the set of equivalence classes of parallel arrows. The quiver which has Q_0 as vertices and $\bar{Q}_1$ as set of arrows, will be denoted by $\bar{Q}$. We denote by $\chi(\bar{Q})$ the first Betti number of $\bar{Q}$ (see also Section 5), which is equal to $|\bar{Q}_1| - |\bar{Q}_0| + c_{\bar{Q}}$, where $c_{\bar{Q}}$ is the number of connected components of $\bar{Q}$. Using these notations Sánchez-Flores' description of the first Hochschild cohomology for radical square zero algebras can be stated as follows.

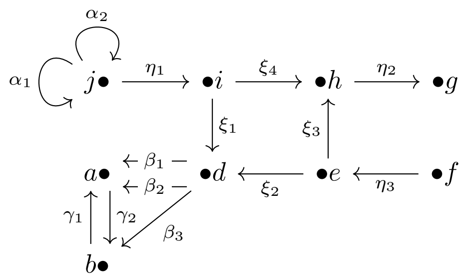
Theorem 3.31 ([13, Theorem 2.9]) *Let A be an indecomposable radical square zero algebra defined by some quiver Q over a field k of characteristic zero. Then (as Lie algebras)*

$$\mathrm{HH}^1(A) \simeq \prod_{\alpha \in S} \mathfrak{sl}_{|\alpha|}(k) \times k^{\chi(\bar{Q})}.$$

Note that intuitively we can say that $\chi(\bar{Q})$ counts the number of holes in $\bar{Q}$. From this point of view we could give an interpretation of the above result by inverse gluing operations. To be more intuitive we will demonstrate our method by a typical example. Note also that the characteristic zero condition in the above result is necessary since the proof uses the Lie algebra decomposition $\mathfrak{gl}_{|\alpha|}(k) \simeq \mathfrak{sl}_{|\alpha|}(k) \times k$ when $\mathrm{char}(k) = 0$.

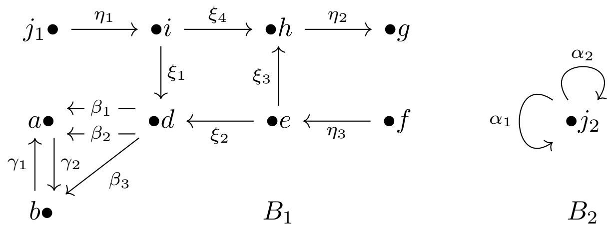
Example 3.32 *Let A be a radical square zero algebra defined by the following quiver Q_A over a field k of characteristic zero. Note that here $\chi(\bar{Q}_A) = 4$ and $S = \{[\alpha_1], [\beta_1]\}$.*

Q_A :



Step 1 (reduce loops): *We separate loops at vertex j of Q_A to get Q_B. The new algebra B has two blocks, say B_1 and B_2.*

Q_B :



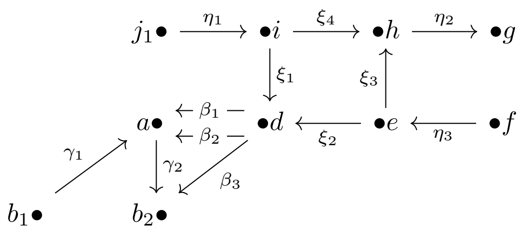
The inverse operation is given by gluing two vertices (one of which has no loops) from two different blocks, that is, we glue $j_1 \in Q_{B_1}$ and $j_2 \in Q_{B_2}$. By Corollary 3.26, this operation does not change the dimension and the Lie structure of $\mathrm{HH}^1(A)$, that is,

$$\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B) \simeq \mathrm{HH}^1(B_1) \times \mathrm{HH}^1(B_2).$$

Combining Remark 3.30 (1) we obtain $\mathrm{HH}^1(B_2) \simeq \mathfrak{gl}_2(k) \simeq sl_2 \times k$, where the summand k contributes 1 for the number $\chi(\bar{Q}_A)$. We have reduced Q_A to the no loop quiver Q_{B_1}.

Step 2 (reduce oriented l-cycles ($l \geq 2$)): *We reduce the oriented cycle $p := \gamma_2\gamma_1$ in Q_{B_1}. Choose the vertex b in p and split it into a source vertex b_1 and a sink vertex b_2:*

Q_C :



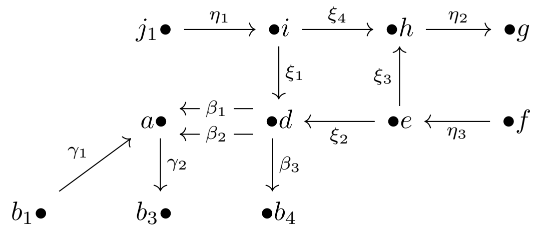
The inverse operation is given by gluing b_1 and b_2 from the same block. By Remark 3.29 (i), reducing p from Q_{B_1} we get one summand isomorphic to k (cf. Corollary 3.28), which contributes 1 for the number $\chi(\bar{Q}_B) = \chi(\bar{Q}_A)$. So

$$\mathrm{HH}^1(B_1) \simeq \mathrm{HH}^1(C) \times k$$

and we have reduced Q_{B_1} to the no oriented cycle quiver Q_C.

Step 3 (reduce undirected l-cycles ($l \geq 3$)): *We first deal with the undirected 3-cycle $q_1 := \beta_3 - \gamma_1 - \beta_1$ in Q_C. Actually, we can split b_2 into two sinks, say b_3 and b_4, denote the corresponding quiver and algebra by Q_D and D, respectively.*

Q_D :

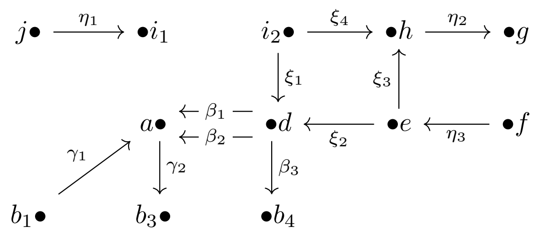


The inverse operation is given by gluing b_3 and b_4 from the same block. By Corollary 3.28 and Remark 3.29, reducing q_1 from Q_C we get a summand isomorphic to k, which again contributes 1 for the number $\chi(\bar{Q}_A)$. Therefore

$$\mathrm{HH}^1(C) \simeq \mathrm{HH}^1(D) \times k.$$

After this, we reduce another undirected cycle $q_2 := \xi_4 - \xi_3 - \xi_2 - \xi_1$. Choose the vertex i in q_2 and split i into a sink vertex i_1 and a source vertex i_2 to get Q_E, denote the corresponding algebra by E.

Q_E :

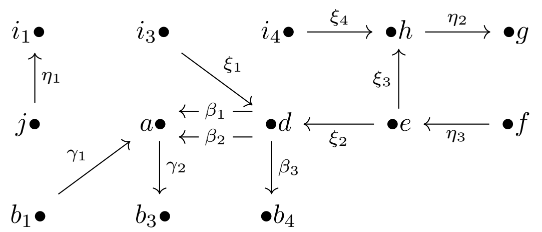


The inverse operation is given by gluing i_1 and i_2 from two different blocks. By Corollary 3.26, this operation does not change the dimension and the Lie structure of $\mathrm{HH}^1(D)$*, that is,*

$$\mathrm{HH}^1(D) \simeq \mathrm{HH}^1(E).$$

Note that the above reduction produces a new undirected cycle $q_2' := \xi_4 - \xi_3 - \xi_2 - \xi_1$ in Q_E. However, we can reduce q_2' in Q_E by splitting i_2 into two sources, say i_3 and i_4 (the corresponding quiver is Q_F).

Q_F :

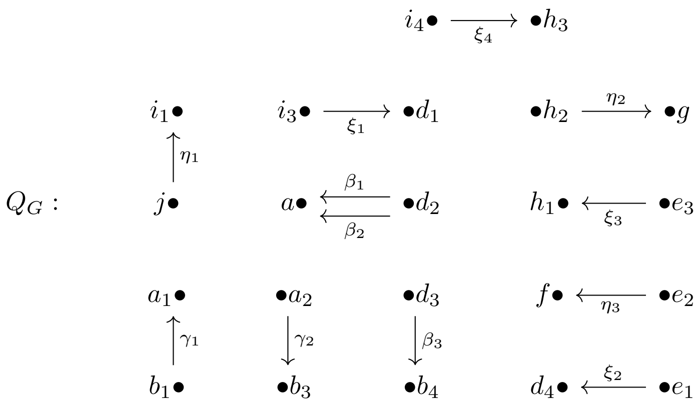


The inverse operation is given by gluing two sources from the same block. Again by Corollary 3.28 and Remark 3.29, we get that

$$\mathrm{HH}^1(E) \simeq \mathrm{HH}^1(F) \times k,$$

where the summand k also contributes 1 for the number $\chi(\bar{Q}_A)$. Now we have reduced to a quiver Q_F that has neither oriented cycles nor undirected cycles.

Step 4 (Split into several m-Kronecker quivers): *Since Q_F contains no cycles (whether oriented or undirected), we can do the last step to split Q_F into several quivers, each of these quivers is a m-Kronecker quiver for some $m \geq 1$.*



The inverse of the above operations are given by repeatedly applying three kinds of operations: gluing a source and a sink from different blocks, gluing two sources from different blocks, gluing two sinks from different blocks. By Corollary 3.26, these operations do not change the dimension and the Lie structure of $\mathrm{HH}^1(F)$. *Therefore,*

$$\mathrm{HH}^1(F) \simeq \mathrm{HH}^1(G).$$

Since the HH^1 *of a m-Kronecker algebra is* $\mathfrak{sl}_m(k)$ *by Remark 3.30 (2),* $\mathrm{HH}^1(G) \simeq \mathfrak{sl}_2(k)$.

We conclude that $\mathrm{HH}^1(B_1) \simeq \mathrm{HH}^1(G) \times k^3$, *therefore*

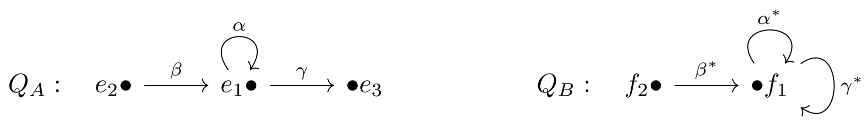
$$\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B_1) \times \mathrm{HH}^1(B_2) \simeq \mathfrak{sl}_2(k)^2 \times k^4.$$

We are done.

4 Examples

The first two examples show that the characteristic condition in Proposition 3.12 is necessary.

Example 4.1 *Here we assume that* $char(k) = 2$, *and that* B *is obtained from* A *by gluing* e_1 *and* e_3*:*



Where $Z_A = \{r = \alpha^2\}$, $Z_{new} = \{r_1 = (\gamma^*)^2, r_2 = \alpha^*\gamma^*\}$ *and* $Z_B = Z_A \cup Z_{new}$. *Then* $\delta^1_A(\alpha\|e_1) = r\|r^{\alpha\|e_1} = 2r\|\alpha = 0$ *since* $char(k) = 2$, *but* $\delta^1_B(\alpha^*\|f_1) = 2r^*\|\alpha^* + r_2\|\gamma^* = r_2\|\gamma^* \neq 0$, *which means that although* $\alpha\|e_1 \in \mathrm{Ker}(\delta^1_A)$, $\psi_1(\alpha\|e_1) = \alpha^*\|f_1 \notin \mathrm{Ker}(\delta^1_B)$, *hence* ψ_1 *does not induce an injective* k*-linear map* $\mathrm{Ker}(\delta^1_A) \hookrightarrow \mathrm{Ker}(\delta^1_B)$.

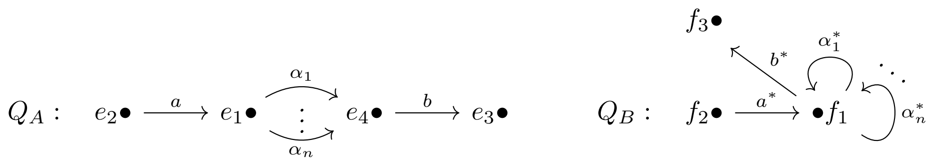
Example 4.2 *Let* A *be given by two blocks* A_1 *and* A_2 *such that* A_1 *is isomorphic to* $k[x]/(x^2)$ *and* A_2 *is isomorphic to* $k[y]/(y^2)$. *Let* B *be obtained by gluing the units of* A_1 *and* A_2. *Then* $\mathrm{Ker}(\delta^1_B) = \mathrm{HH}^1(B) \simeq \mathfrak{gl}_2(k)$ *and has* k*-basis given by* $\{x\|x,\ x\|y,\ y\|x,\ y\|y\}$. *However, there are two cases for* A.

(1) *If* $\mathrm{char}(k) \neq 2$, *then* $\mathrm{Ker}(\delta^1_A) = \mathrm{HH}^1(A) \simeq k \times k$ *has* k*-basis given by* $x\|x$ *and* $y\|y$, *and there is an injective Lie algebra homomorphism* $\mathrm{Ker}(\delta^1_A) \hookrightarrow \mathrm{Ker}(\delta^1_B)$.

(2) *If* $\mathrm{char}(k) = 2$, *then* $\mathrm{Ker}(\delta^1_A) = \mathrm{HH}^1(A)$ *has* k*-basis given by* $\{x\|x,\ x\|e_1,\ y\|y,\ y\|e_2\}$. *Clearly in this case we can not get an injective Lie algebra homomorphism from* $\mathrm{Ker}(\delta^1_A)$ *to* $\mathrm{Ker}(\delta^1_B)$.

We verify the special paths and the k-space Z_{sp} (resp. the special pairs and the k-space Z_{spp}) appeared in Definition 3.6 and Proposition 3.9 (resp. in Definition 3.15 and Proposition 3.17) by the following example.

Example 4.3 B *is obtained from* A *by gluing* e_1 *and* e_4*:*



Where $Z_A = \emptyset$, $Z_B = Z_{new} = \{\alpha_i^* \alpha_j^* \mid 1 \leq i, j \leq n\}$. *Since* $\alpha_i a \notin I_A$ *for* $1 \leq i \leq n$, α_i *is a special path from* e_1 *to* e_4 *for* $1 \leq i \leq n$, *we have* $\mathrm{Sp}(1,4) = \{\alpha_i \mid 1 \leq i \leq n\}$ *and*

$$\begin{aligned} Z_{sp} &= \langle \delta_B^0(f_1 \| \alpha_i^*) \mid 1 \leq i \leq n \rangle \\ &= \langle b^* \| b^* \alpha_i^* - a^* \| \alpha_i^* a^* \mid 1 \leq i \leq n \rangle. \end{aligned}$$

Hence $\mathrm{sp}(1,4) = n = \dim_k Z_{sp}$. *Since* $a^* \| \alpha_i^* a^*, b^* \| b^* \alpha_i^*, \alpha_i^* \| f_1$, $a \nparallel \alpha_i a, b \nparallel b\alpha_i, \alpha_i \nparallel e_1, \alpha_i \nparallel e_4$, *we know that* $(a, \alpha_i a), (b, b\alpha_i), (\alpha_i, e_1), (\alpha_i, e_n)$ *are special pairs with respect to the gluing of* e_1 *and* e_4 *for* $1 \leq i \leq n$, *and* $\mathrm{Spp}(1,4) = \{(a, \alpha_i a), (b, b\alpha_i), (\alpha_i, e_1), (\alpha_i, e_n) \mid 1 \leq i \leq n\}$. *As a result we get*

$$\langle \mathrm{Spp}(1,4) \rangle = \langle a^* \| \alpha_i^* a^*, b^* \| b^* \alpha_i^*, \alpha_i^* \| f_1 \mid 1 \leq i \leq n \rangle,$$

$$\begin{aligned} Z_{spp} &= \langle \mathrm{Spp}(1,4) \rangle \cap \mathrm{Ker}(\delta_B^1) \\ &= \langle a^* \| \alpha_i^* a^*, b^* \| b^* \alpha_i^* \mid 1 \leq i \leq n \rangle. \end{aligned}$$

Hence $\mathrm{kspp}(1,4) = \dim_k Z_{spp} = 2n$. *A direct computation shows that* $\mathrm{Im}(\delta_A^0), \mathrm{Im}(\delta_B^0)$ *are 3-dimensional and* $(n+2)$*-dimensional, respectively, since*

$$\mathrm{Im}(\delta_A^0) = \langle a \| a, b \| b, \sum_{i=1}^{n} \alpha_i \| \alpha_i \rangle,$$

$$\mathrm{Im}(\delta_B^0) = \langle a^* \| a^*, b^* \| b^*, b^* \| b^* \alpha_i^* - a^* \| \alpha_i^* a^* \mid 1 \leq i \leq n \rangle.$$

Therefore

$$\dim_k \mathrm{Im}(\delta_A^0) = \dim_k \mathrm{Im}(\delta_B^0) + 1 - \mathrm{sp}(1,4).$$

$$\mathrm{Ker}(\delta_A^1) = \langle a \| a, b \| b, \alpha_i \| \alpha_j \mid 1 \leq i, j \leq n \rangle$$

is (n^2+2)*-dimensional and*

$$\mathrm{Ker}(\delta_B^1) = \langle a^* \| a^*, b^* \| b^*, \alpha_i^* \| \alpha_j^*, b^* \| b^* \alpha_i^*, a^* \| \alpha_i^* a^* \mid 1 \leq i, j \leq n \rangle$$

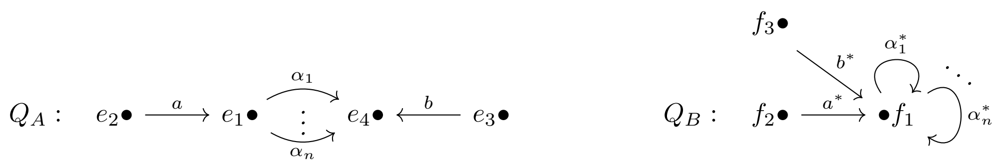
is (n^2+2n+2)*-dimensional, hence*

$$\dim_k \mathrm{Ker}(\delta_B^1) = \dim_k \mathrm{Ker}(\delta_A^1) + \mathrm{kspp}(1,4).$$

One can verify that $\mathrm{HH}^1(A)$ *is isomorphic to* $\mathfrak{pgl}_n(k)$ *and* $\mathrm{HH}^1(B)$ *contains a subalgebra isomorphic to* $\mathfrak{gl}_n(k)$. *Note also that by the notations in the proof of Theorem 3.21, in this example the subspace* Y *of* $\mathrm{Ker}(\delta_B^1)$ *is equal to* $\mathrm{Im}(\delta_B^0) \oplus \langle \sum_{i=1}^{n} \alpha_i^* \| \alpha_i^* \rangle$ *and* Y *is not a Lie ideal of* $\mathrm{Ker}(\delta_B^1)$.

By Corollary 3.18, if B is a radical embedding of A obtained by gluing a source vertex e_1 and a sink vertex e_n of A (in case $\mathrm{char}(k) = 2$, we assume that B has no block isomorphic to $k[x]/(x^2)$), then $\mathrm{Ker}(\delta_B^1) \simeq \mathrm{Ker}(\delta_A^1)$. However, the converse of Corollary 3.18 is not true in general which reveals by the following example.

Example 4.4 *B is obtained from A by gluing* e_1 *and* e_4*:*



Where $Z_A = \{\alpha_i a \mid 1 \le i \le n\}$, $Z_{new} = \{\alpha_i^* b^*, \alpha_i^* \alpha_j^* \mid 1 \le i,j \le n\}$ *and* $Z_B = Z_A \cup Z_{new}$. *Note that although* $\mathrm{Spp}(1,4) = \{(\alpha_i, e_1), (\alpha_i, e_4) \mid 1 \le i \le n\}$, *we have* $Z_{spp} = \langle \mathrm{Spp}(1,4)\rangle \cap \mathrm{Ker}(\delta_B^1) = \langle \alpha_i^* \| f_1\rangle \cap \mathrm{Ker}(\delta_B^1) = 0$. *We infer from Proposition 3.17 that we have* $\dim_k \mathrm{Ker}(\delta_B^1) = \dim_k \mathrm{Ker}(\delta_A^1)$. *In fact, a direct computation shows that both*

$$\mathrm{Ker}(\delta_A^1) = \langle a\|a, b\|b, \alpha_i\|\alpha_j \mid 1 \le i,j \le n\rangle$$

and

$$\mathrm{Ker}(\delta_B^1) = \langle a^*\|a^*, b^*\|b^*, \alpha_i^*\|\alpha_j^* \mid 1 \le i,j \le n\rangle$$

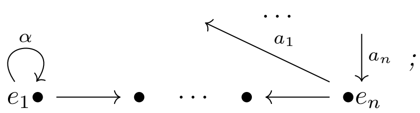
are (n^2+2)*-dimensional. Hence although we do not glue a source and a sink, we have* $\mathrm{Ker}(\delta_B^1) \simeq \mathrm{Ker}(\delta_A^1)$.

In order to have an intuitive feeling, we give various types of special pairs in the following example.

Example 4.5 *In this example we always assume that* B *is obtained from* A *by gluing* e_1 *and* e_n, *and that* α *is an arrow in* Q_A *and* p *is a path in* $\mathcal{B}_A$. *It can be proved that the special pairs* (α,p) *are exclusively from the following 7 typical cases and their dual cases:*

(i) : α *is a loop at* e_1 *or* e_n, *assume that* $e_1 \bullet \overset{\alpha}{\circlearrowleft}$. *(The case that* $e_n \bullet \overset{\alpha}{\circlearrowleft}$ *is dual.)*

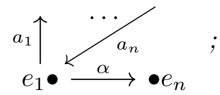
Case 1: $p = a_n \cdots a_1$ *is an oriented cycle at* e_n *or* $p = e_n$, *such as:*



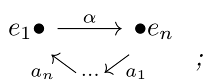
Case 2: $p = a_n \cdots a_1$ *is a path between* e_1 *and* e_n, *such as:*

(ii) : α *is an arrow between* e_1 *and* e_n, *assume that* $e_1\bullet \xrightarrow{\alpha} \bullet e_n$. *(The case that* $e_n\bullet \xrightarrow{\alpha} \bullet e_1$ *is dual.)*

Case 3: $p = a_n \cdots a_1$ *is an oriented cycle at* e_1 *or* e_n *or* $p = e_1$ *or* e_n, *such as:*

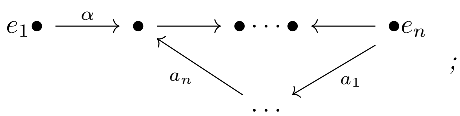


Case 4: $p = a_n \cdots a_1$ *is a path from* e_n *to* e_1, *such as:*

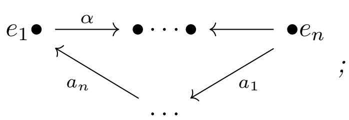


(iii) : Exactly one of the vertex of α *is* e_1 *or* e_n, *assume that* $e_1\bullet \xrightarrow{\alpha} \bullet$. *(The other cases are dual.)*

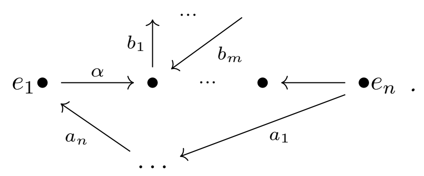
Case 5: $p = a_n \cdots a_1$ *is a path from* e_n *to* $t(\alpha)$, *such as:*



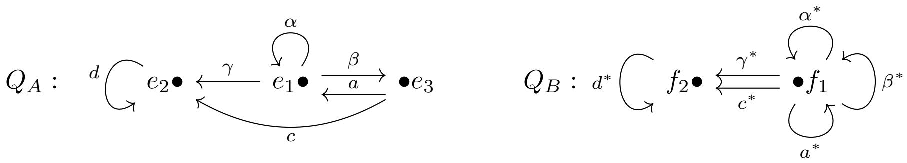
Case 6: $p = \alpha p_1$, *where* $p_1 = a_n \cdots a_1$ *is a path from* e_n *to* e_1, *such as:*



Case 7: $p = p_2\alpha p_1$, where $p_1 = a_n \cdots a_1$ is a path from e_n to e_1 and $p_2 = b_m \cdots b_1$ is a cycle at $t(\alpha)$, such as:



After giving relations in specific examples, we can show that the special pair (α, p) in each of the above cases can appear. Indeed, the following example covers all the above 7 cases:



Where Z_A consists of all paths in Q_A of length 3 except $d\gamma a$, $Z_{new} = \{a^*\alpha^*, c^*\alpha^*, \alpha^*\beta^*, (\beta^*)^2, \gamma^*\beta^*, (a^*)^2, c^*a^*\}$ and $Z_B = Z_A \cup Z_{new}$. We list all special pairs (α, p) for each case as follows:

Case 1: $(\alpha, \beta a)$, (α, e_3);

Case 2: (α, a), (α, β), $(\alpha, \beta\alpha)$, $(\alpha, \alpha a)$;

Case 3: (β, α), $(\beta, a\beta)$, $(\beta, \beta a)$, (β, e_1), (β, e_3) (a, α), $(a, a\beta)$, $(a, \beta a)$, (a, e_1), (a, e_3);

Case 4: (β, a), (a, β);

Case 5: (γ, c), (γ, dc), $(c, \gamma\alpha)$, $(c, d\gamma)$;

Case 6: $(\gamma, \gamma a)$, $(c, c\beta)$;

Case 7: $(\gamma, d\gamma a)$.

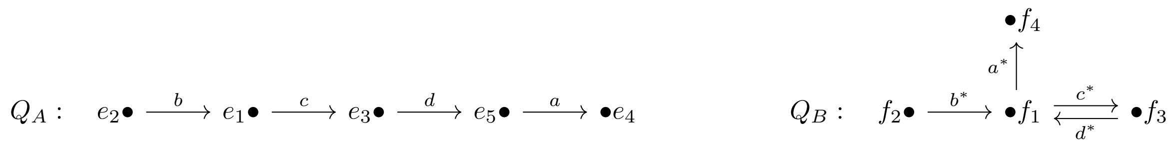
By check one by one, we have $\mathrm{Spp}(1,3)$ is the set consisting of these 25 special pairs and $\langle \mathrm{Spp}(1,3) \rangle = \langle \alpha^* \| p^* \mid (\alpha, p) \in \mathrm{Spp}(1,3) \rangle$ and therefore

$$\begin{aligned} Z_{spp} &= \langle \mathrm{Spp}(1,3) \rangle \cap \mathrm{Ker}(\delta_B^1) \\ &= \langle a^* \| \beta^* a^*, \alpha^* \| \beta^* \alpha^*, \alpha^* \| \alpha^* a^*, \beta^* \| a^* \beta^*, \beta^* \| \beta^* a^*, a^* \| a^* \beta^*, \\ &\quad a^* \| \beta^* a^*, \gamma^* \| d^* c^*, c^* \| \gamma^* \alpha^*, c^* \| d^* \gamma^*, \gamma^* \| \gamma^* a^*, c^* \| c^* \beta^*, \gamma^* \| d^* \gamma^* a^* \rangle. \end{aligned}$$

Hence $\mathrm{kspp}(1,3) = 13$. Note also that the special paths in this example are β and a, so $\mathrm{sp}(1,3) = 2$.

It deserves to mention that although the k-space $\langle \mathrm{Spp}(1,n) \rangle$ is generated by the elements of the form $\alpha^* \| p^*$ (where α is an arrow and p is a path), an element in Z_{spp} is usually a k-linear combination of such elements.

Example 4.6 *B is obtained from A by gluing e_1 and e_5:*



Where $Z_A = \emptyset$ and $Z_B = Z_{new} = \{a^*b^*, c^*d^*\}$. It follows from a direct calculation that

$$\mathrm{Im}(\delta_A^0) = \langle a\|a, b\|b, c\|c, d\|d \rangle = \mathrm{Ker}(\delta_A^1),$$

hence $\mathrm{HH}^1(A) = 0$. Similarly we have

$$\mathrm{Im}(\delta_B^0) = \langle a^*\|a^*, b^*\|b^*, d^*\|d^* - c^*\|c^*, a^*\|a^*d^*c^* - b^*\|d^*c^*b^* \rangle,$$

$$\mathrm{Ker}(\delta_B^1) = \langle a^*\|a^*, b^*\|b^*, c^*\|c^*, d^*\|d^*, a^*\|a^*d^*c^* - b^*\|d^*c^*b^* \rangle,$$

hence $\mathrm{HH}^1(B) \simeq \langle c^* \| c^* \rangle$. *Using the notation in Theorem 3.21, we get the ideal* $I \simeq \langle \psi_1(\delta^0_A(e_1 \| e_1)) \rangle = \langle c^* \| c^* - b^* \| b^* \rangle$ *and* $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/I$. *It is clear that* $\mathrm{Spp}(1,5) = \{(a, adc), (b, dcb)\}$, *therefore* $\langle \mathrm{Spp}(1,5) \rangle = \langle a^* \| a^* d^* c^*, b^* \| d^* c^* b^* \rangle$ *and*

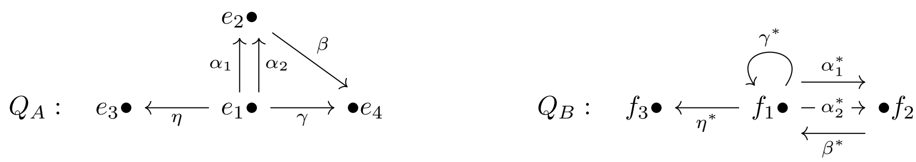
$$\begin{aligned} Z_{spp} &= \langle \mathrm{Spp}(1,5) \rangle \cap \mathrm{Ker}(\delta^1_B) \\ &= \langle a^* \| a^* d^* c^* - b^* \| d^* c^* b^* \rangle. \end{aligned}$$

The following example shows that the difference between the dimensions of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ can be arbitrarily large.

Example 4.7 *Let A be given by two blocks A_1 and A_2 such that A_1 and A_2 are radical square zero local algebras having m-loops and n-loops respectively. If we exclude the case that $m = 1$ and $n = 1$ in* $\mathrm{char}(k) = 2$ *(for this case, see Example 4.2), then the dimension of* $\mathrm{HH}^1(A)$ *is the sum of the dimensions of* $\mathrm{HH}^1(A_1) \simeq \mathfrak{gl}_m(k)$ *and* $\mathrm{HH}^1(A_2) \simeq \mathfrak{gl}_n(k)$*, that is, $m^2 + n^2$. Let B be obtained by gluing the units of A_1 and A_2. Then* $\mathrm{HH}^1(B) \simeq \mathfrak{gl}_{m+n}(k)$ *and consequently has dimension $(m+n)^2$.*

We use the following example to demonstrate the last statement of Corollary 3.24. This example also shows that in general the center of $\mathrm{HH}^1(B)$ is properly contained in the center of L_0 (cf. Proposition 2.6).

Example 4.8 *Suppose* $\mathrm{char}(k) = 0$, *B is obtained from A by gluing e_1 and e_4:*



Where $Z_A = \{\beta\alpha_1\}$ *and* $Z_{new} = \{(\gamma^*)^2, \alpha_i^* \gamma^*, \alpha_i^* \beta^*, \gamma^* \beta^*, \eta^* \gamma^*, \eta^* \beta^* \mid i = 1, 2\}$. *From a straightforward computation we have*

$$\mathrm{Im}(\delta^0_A) = \langle \alpha_1 \| \alpha_1 + \alpha_2 \| \alpha_2 + \gamma \| \gamma, \beta \| \beta + \gamma \| \gamma, \eta \| \eta \rangle,$$

$$\mathrm{Im}(\delta^0_B) = \langle \alpha_1^* \| \alpha_1^* + \alpha_2^* \| \alpha_2^* - \beta^* \| \beta^*, \eta^* \| \eta^* \rangle,$$

$$\mathrm{Ker}(\delta^1_A) = \langle \alpha_2 \| \alpha_1, \alpha_1 \| \alpha_1, \alpha_2 \| \alpha_2, \beta \| \beta, \gamma \| \gamma, \gamma \| \beta\alpha_2, \eta \| \eta \rangle.$$

Since we glue a source and a sink, Corollary 3.18 shows that $\mathrm{Ker}(\delta^1_B) \simeq \mathrm{Ker}(\delta^1_A)$. *As a consequence,*

$$\mathrm{HH}^1(A) \simeq \langle \alpha_2 \| \alpha_1, \alpha_1 \| \alpha_1, \alpha_2 \| \alpha_2, \gamma \| \beta\alpha_2 \rangle,$$

$$\mathrm{HH}^1(B) \simeq \langle \alpha_2^* \| \alpha_1^*, \alpha_1^* \| \alpha_1^*, \alpha_2^* \| \alpha_2^*, \gamma^* \| \gamma^*, \gamma^* \| \beta^* \alpha_2^* \rangle.$$

Using the notation in Theorem 3.21, we get the ideal $I = \langle \psi_1(\delta^0_A(e_1 \| e_1)) \rangle = \langle \alpha_1^* \| \alpha_1^* + \alpha_2^* \| \alpha_2^* + \gamma^* \| \gamma^* + \eta^* \| \eta^* \rangle$ *and* $\mathrm{HH}^1(A) \simeq \mathrm{HH}^1(B)/I$. *Note that the symbol Δ in Remark 3.22 is exactly equal to $\{[\alpha], [\gamma]\}$, where $[\alpha] = \{\alpha_1, \alpha_2\}$ and $[\gamma] = \{\gamma\}$, we can rewrite the generator of I as* $\psi_1(I_{[\alpha]} + I_{[\gamma]}) = \alpha_1^* \| \alpha_1^* + \alpha_2^* \| \alpha_2^* + \gamma^* \| \gamma^*$ *since* $\eta^* \| \eta^* \in \mathrm{Im}(\delta^0_B)$. *Also* $L_0^{[\alpha^*]} = \langle \alpha_2^* \| \alpha_1^*, \alpha_1^* \| \alpha_1^*, \alpha_2^* \| \alpha_2^* \rangle$, $L_0^{[\gamma^*]} = \langle \gamma^* \| \gamma^* \rangle$, *hence*

$$L_0 = L_0^{[\alpha^*]} \oplus L_0^{[\gamma^*]} = \langle \alpha_2^* \| \alpha_1^*, \alpha_1^* \| \alpha_1^*, \alpha_2^* \| \alpha_2^* \rangle \oplus \langle \gamma^* \| \gamma^* \rangle$$

$$= \langle \alpha_2^* \| \alpha_1^*, \alpha_1^* \| \alpha_1^*, \alpha_2^* \| \alpha_2^* \rangle \oplus \langle \alpha_1^* \| \alpha_1^* + \alpha_2^* \| \alpha_2^* + \gamma^* \| \gamma^* \rangle = L_0^{[\alpha^*]} \oplus I$$

as Lie algebras, and therefore $Z(L_0) = Z(L_0^{[\alpha^*]}) \oplus Z(L_0^{[\gamma^*]}) = \langle \alpha_1^* \| \alpha_1^* + \alpha_2^* \| \alpha_2^* \rangle \oplus \langle \gamma^* \| \gamma^* \rangle$. *Clearly* $Z(\mathrm{HH}^1(B)) = I \subsetneq Z(L_0)$. *Since* $L_1 = \langle \gamma^* \| \beta^* \alpha_2^* \rangle$,

$$\mathrm{HH}^1(B) = L_0 \oplus L_1 = (L_0^{[\alpha^*]} \oplus I) \oplus L_1 = (L_0^{[\alpha^*]} \oplus L_1) \oplus I \simeq \mathrm{HH}^1(A) \times I \simeq \mathrm{HH}^1(A) \times k$$

as Lie algebras.

5 Fundamental group

Let $\pi_1(Q, I)$ be a fundamental group of a bound quiver (Q, I). Suppose that a quiver Q has n vertices and m edges and c connected components. We adopt the notation that the first Betti number of Q, denoted by $\chi(Q)$, equals $m-n+c$. Note that the first Betti number is equal to the dimension of the first cohomology group of the underlying graph of Q, see for example [15, Lemma 8.2]. Intuitively we can say that $\chi(Q)$ counts the number of holes in Q.

Recall from [2, Lemma 1.7] that for a bound quiver (Q, I) we have $\dim_k \mathrm{Hom}(\pi_1(Q,I), k^+) \leq \chi(Q)$. Equality holds if I is a monomial ideal, and more generally if I is semimonomial [8, Section 1] and in positive characteristic if I is p-semimonomial; see after Remark 1.8 in [2]. Therefore by Theorem C in [2] we have that

$$\pi_1\text{-rank}(A) := \max\{\dim_k \pi_1(Q,I)^\vee \ : \ A \simeq kQ/I,\ I \text{ is an admissible ideal}\}$$

is equal to $\chi(Q_A)$.

Note that the maximum should be over the minimal presentations, however, since for monomial algebras the maximum is obtained for an admissible ideal it is enough to restrict to this subset. The π_1-rank(A) is a derived invariant and an invariant under stable equivalences of Morita type for selfinjective algebras, see [2, Theorem B]. However, the next lemma shows that it is not an invariant under stable equivalences induced by gluing idempotents.

Lemma 5.1 *Let $A = kQ_A/I_A$ be a finite dimensional monomial (or semimonomial) algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents of A. Then*

$$\pi_1\text{-rank}(A) = \pi_1\text{-rank}(B) + c_A - c_B - 1.$$

In particular, if we glue two idempotents from different blocks, then

$$\pi_1\text{-rank}(A) = \pi_1\text{-rank}(B);$$

and if we glue two idempotents from the same block, then

$$\pi_1\text{-rank}(A) = \pi_1\text{-rank}(B) - 1.$$

Proof Since A and B are monomial algebras we have by Theorem C in [2] that π_1-rank$(A) = \chi(Q_A)$ and π_1-rank$(B) = \chi(Q_B)$. The statement follows from the fact the number of arrows of Q_A and Q_B is the same, that is, $m_A = m_B$, and from the observation that $n_A = n_B + 1$. Also note that the number of connected components have the relation $c_A = c_B + 1$ when we glue two idempotents from different blocks but $c_A = c_B$ when we glue two idempotents from the same block. The same argument applies if A and B are semimonomial algebras. □

Remark 5.2 *When the characteristic of the field is positive, Lemma 5.1 holds also for p-semimonomial algebras since the π_1-rank coincides with the first Betti number.*

Remark 5.3 *Note that Lemma 5.1 and Lemma 3.5 are intimately related. In fact π_1-rank$(A) = m_A - \dim_k(\mathrm{Im}(\delta^0_{(A)_0}))$.*

Remark 5.4 *It is worth noting that if we glue a source and a sink from the same block we have that π_1-rank$(B) - \pi_1$-rank$(A) = 1$ and this quantity coincides with the difference $\dim_k \mathrm{HH}^1(B) - \dim_k \mathrm{HH}^1(A) = 1$. The reason why we obtain such equality is because if we glue a source and a sink, then $\mathrm{HH}^1(A)$ is isomorphic to the quotient of $\mathrm{HH}^1(B)$. More precisely, the ideal of this quotient is a 1-dimensional torus $\psi_1(\mathrm{Im}(\delta^0_A))/\mathrm{Im}(\delta^0_B)$ where $\psi_1 : \mathrm{Ker}(\delta^1_A) \to \mathrm{Ker}(\delta^1_B)$. Therefore, we can say that the π_1-rank controls the change of the dimensions between $\mathrm{HH}^1(B)$ and $\mathrm{HH}^1(A)$. This process also changes the Lie algebra structure. For example, in characteristic different from 2, from a reductive Lie algebra $\mathfrak{gl}_2(k)$ we pass to a simple Lie algebra $\mathfrak{sl}_2(k)$, or for example from a one dimensional abelian Lie algebra to the trivial Lie algebra. It follows from Theorem 3.20 that for a general gluing we do not obtain the equality above.*

6 Centers and Hochschild cohomology groups in higher degrees

6.1 Center

In this subsection, we study the behaviour of the centers of finite dimensional monomial algebras under gluing idempotents. Throughout we will denote by $Z(A)$ the center of an algebra A.

Definition 6.1 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A and let p be a path either from e_1 to e_n or from e_n to e_1 in Q_A. We call that p is a non-special path between e_1 and e_n in Q_A if $\delta^0_B(f_1\|p^*) = 0$, or equivalently, if $ap \in I_A$ and $pb \in I_A$ for arbitrary $a, b \in (Q_A)_1$.*

We denote by $\mathrm{NSp}(1,n)$ the set of non-special paths between e_1 and e_n in Q_A, and by $\mathrm{nsp}(1,n)$ the number of these non-special paths. Furthermore, we denote by Z_{nsp} the k-subspace of $k((Q_B)_0\|\mathcal{B}_B)$ generated by the elements $f_1\|p^*$, where $p \in \mathrm{NSp}(1,n)$.

Actually, as the name shows, non-special path is exactly the opposite notion of special path, and it is clear that there is no non-special path between e_1 and e_n when we glue these two idempotents from different blocks.

Similarly to Definition 3.4, let $\delta^0_{(A)_{\geq 1}}$ denote the map δ^0_A restricted to the subspace $k((Q_A)_0\|(\mathcal{B}_A)_{\geq 1})$ and define $\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ to be the kernel of the map $\delta^0_{(A)_{\geq 1}}$.

Lemma 6.2 *Let $A = kQ_A/I_A$ be a monomial algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then there is a decomposition as k-vector spaces*

$$\mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \psi_0(\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})) \oplus Z_{nsp}.$$

In particular, if we glue e_1 and e_n from the same block, then $\dim_k \mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_{\geq 1}}) + nsp(1,n)$; if we glue e_1 and e_n from different blocks, then $\dim_k \mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$.

Proof Recall from Proposition 3.3 that there is a k-linear map $\psi_0 : k((Q_A)_0\|\mathcal{B}_A) \to k((Q_B)_0\|\mathcal{B}_B)$. A direct computation shows that $\delta^0_B(\psi_0(e_i\|p)) = \psi_1(\delta^0_A(e_i\|p))$ for $1 \leq i \leq n$ and $p \in \mathcal{B}_A\backslash\{e_1, e_n\}$. It follows that ψ_0 induces an injective k-linear map from $\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ to $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$.

Let $\theta \in \mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$ lies in the complement of the subspace $\psi_0(\mathrm{Ker}(\delta^0_{(A)_{\geq 1}}))$. Then we may assume that θ is a linear combination of the elements of the form $f_1\|p^*$ such that p is a path between e_1 and e_n. If p is a non-special path, then $f_1\|p^* \in Z_{nsp} \subseteq \mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$. Moreover, by Lemma 3.7, every element $\sum_{p\in\mathrm{Sp}(1,n)} \lambda_p f_1\|p^*$ does not belong to $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$. It follows easily that $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \psi_0(\mathrm{Ker}(\delta^0_{(A)_{\geq 1}}))\oplus Z_{nsp}$. □

First, we deal with the case that the algebra A is indecomposable.

Proposition 6.3 *Let A be an indecomposable finite dimensional monomial k-algebra and let B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then there is an algebra monomorphism $Z(A) \hookrightarrow Z(B)$. Moreover, we have the formula*

$$\dim_k Z(B) = \dim_k Z(A) + nsp(1,n).$$

Proof We keep the notations as in Proposition 3.3 and identify the centers $Z(A), Z(B)$ as $\mathrm{Ker}(\delta^0_A), \mathrm{Ker}(\delta^0_B)$ respectively. Also notice that $\mathrm{Ker}(\delta^0_A) = \mathrm{Ker}(\delta^0_{(A)_0}) \oplus \mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ as k-vector spaces and the similar decomposition applies for $\mathrm{Ker}(\delta^0_B)$.

By Lemma 6.2 we know that ψ_0 induces an injective k-linear map from $\mathrm{Ker}(\delta^0_{(A)_{\geq 1}})$ to $\mathrm{Ker}(\delta^0_{(B)_{\geq 1}})$, and $\dim_k \mathrm{Ker}(\delta^0_{(B)_{\geq 1}}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_{\geq 1}}) + nsp(1,n)$. Combine the fact (cf. Proof of Lemma 3.5) that $\mathrm{Ker}(\delta^0_{(A)_0}) = \langle \sum_{1\leq i\leq n} e_i\|e_i\rangle$ and $\mathrm{Ker}(\delta^0_{(B)_0}) = \langle \sum_{1\leq i\leq n-1} f_i\|f_i\rangle$, we deduce that $\dim_k \mathrm{Ker}(\delta^0_{(B)_0}) = \dim_k \mathrm{Ker}(\delta^0_{(A)_0})$, hence the second statement follows. Moreover, there is an injective k-linear map ψ_0 :

$\mathrm{Ker}(\delta_A^0) \to \mathrm{Ker}(\delta_B^0)$. Note that we can identify $\mathrm{Ker}(\delta_A^0)$ with $Z(A)$ by $\sum e_i \| p \mapsto \sum p$ and $\sum_{i=1}^n e_i \| e_i \mapsto 1_A$, so does for $\mathrm{Ker}(\delta_B^0)$ and $Z(B)$. Then, by the fact that $p^* q^* = (pq)^*$ for $p, q \in (\mathcal{B}_A \backslash \{e_1, \cdots, e_n\})$, ψ_0 gives an algebra monomorphism, and the first statement follows. □

Corollary 6.4 *Let A be an indecomposable finite dimensional monomial k-algebra and let B be a radical embedding of A obtained by gluing a source vertex e_1 and a sink vertex e_n. Then $\psi_0 : \mathrm{Ker}(\delta_A^0) \hookrightarrow \mathrm{Ker}(\delta_B^0)$ is an isomorphism if and only if there is no path from e_1 to e_n.*

Proof Note that in this case, p is a non-special path between e_1 and e_n if and only if p is a path from e_1 to e_n. Thus the result follows from Proposition 6.3. □

Corollary 6.5 *Let A be a radical square zero indecomposable finite dimensional algebra and let B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then $\psi_0 : \mathrm{Ker}(\delta_A^0) \hookrightarrow \mathrm{Ker}(\delta_B^0)$ is isomorphism if and only if there are no arrows between e_1 and e_n in Q_A.*

Proof This is because in radical square zero case, the set $\mathrm{NSp}(1, n)$ consists of all arrows between e_1 and e_n in Q_A. □

Note that Cibils has shown in [4] that the dimension of the center of an indecomposable radical square zero algebra is given by $|Q_1 \| Q_0| + 1$. Indeed, by the proof of Proposition 6.3, we know that the basis of the center of an indecomposable radical square zero algebra is provided by the set of loops together with the unit element of the algebra.

Corollary 6.6 *Let A be a radical square zero indecomposable finite dimensional algebra and let B be a radical embedding of A obtained by gluing two idempotents e_1 and e_n of A. Then we have*

$$|(Q_B)_1 \| (Q_B)_0| = |(Q_A)_1 \| (Q_A)_0| + nsp(1, n),$$

where $nsp(1, n)$ is equal to the number of arrows between e_1 and e_n.

Proof This is obvious by Corollary 6.5 and by Cibils' dimension formula as mentioned above. □

Next we deal with the case that the algebra A is not indecomposable. Without loss of generality we assume that A has two blocks, say A_1 and A_2, and assume that B is an algebra obtained from A by gluing $e_1 \in A_1$ and $e_n \in A_2$.

Proposition 6.7 *Let A be a finite dimensional monomial algebra with two blocks A_1 and A_2. Let B be a radical embedding of A obtained by gluing idempotents $e_1 \in A_1$ and $e_n \in A_2$. Then the radical embedding $B \to A$ restricts to a radical embedding $Z(B) \to Z(A)$. In particular, $\dim_k Z(A) = \dim_k Z(B) + 1$.*

Proof Let $\mathcal{B}_A = \{e_1, \cdots, e_n, p_1, \cdots, p_u \mid \text{the length of each } p_i \text{ is } \geq 1\}$ denotes the properly chosen k-basis of the monomial algebra A (cf. Section 2). Then the subalgebra B of A has a k-basis $\mathcal{B}_B = \{e_1 + e_n, e_2, \cdots, e_{n-1}, p_1, \cdots, p_u\}$. We identify the centers $Z(A), Z(B)$ as $\mathrm{Ker}(\delta_A^0), \mathrm{Ker}(\delta_B^0)$ respectively. Let $Z(A) = Z(A)_0 \oplus Z(A)_{\geq 1}$ be the decomposition corresponding to $\mathrm{Ker}(\delta_A^0) = \mathrm{Ker}(\delta_{(A)_0}^0) \oplus \mathrm{Ker}(\delta_{(A)_{\geq 1}}^0)$ as k-vector spaces, so does for $Z(B)$.

By Lemma 6.2, we obtain that $\mathrm{Ker}(\delta_{(B)_{\geq 1}}^0) \simeq \mathrm{Ker}(\delta_{(A)_{\geq 1}}^0)$, hence $Z(A)_{\geq 1} = \langle \sum p \mid p \, \textit{is a cycle in} \, \mathcal{B}_A \rangle = Z(B)_{\geq 1}$. Note that $Z(A)_0 = \langle 1_{A_1}, 1_{A_2} \rangle$, where 1_{A_j} denotes the unit element in A_j for $j = 1, 2$, and $Z(B)_0 = \langle 1_B = 1_{A_1} + 1_{A_2} \rangle$. Therefore there is an embedding from $Z(B)$ to $Z(A)$ which sends 1_B to $1_{A_1} + 1_{A_2}$ and let the elements in $Z(B)_{\geq 1}$ one-corresponding-one to $Z(A)_{\geq 1}$.

It is clear that this embedding from $Z(B)$ to $Z(A)$ is an injection of algebras and preserves the radical, hence we get a radical embedding from $Z(B)$ to $Z(A)$ by gluing $e_1 \in A_1$ and $e_n \in A_2$. In particular, we have $\dim_k Z(A) = \dim_k Z(B) + 1$. □

6.2 Higher degrees

In this subsection, we assume that all algebras considered are indecomposable and radical square zero.

Definition 6.8 ([4]) *A n-crown is a quiver with n vertices cyclically labeled by the cyclic group of order n, and n arrows $a_0, \dots, a_{n-1}$ such that $s(a_i) = i$ and $t(a_i) = i+1$. A 1-crown is a loop, and a 2-crown is an oriented 2-cycle.*

Theorem 2.1 in [4] provides the dimension of Hochschild cohomology: Let Q be a connected quiver which is not a crown. The dimension of Hochschild cohomology is:

$$\dim_k \mathrm{HH}^n(A) = |Q_n||Q_1| - |Q_{n-1}||Q_0|$$

For the n-crown case, see Proposition 2.3 in [4]. Note that there is a typo in [4] since the formula above holds for $n > 1$ and not for $n > 0$. On page 24 of Sánchez Flores' PhD thesis [12] this is corrected.

Lemma 6.9 *Let B be a radical embedding of A obtained by gluing two idempotents and let $n \geq 2$. Let $\alpha \in (Q_A)_1$ and $p \in (Q_A)_n$. If $p \| \alpha$, then $p^* \| \alpha^*$, where $\varphi_n : (Q_A)_n \to (Q_B)_n$ sends p to p^* (cf. Notations in Section 2). In addition, φ_n is injective.*

Proof The first statement is easy. The map φ_n is injective because the map $\varphi : (Q_A)_1 \to (Q_B)_1$ sending α to α^* is injective. □

Lemma 6.10 *Let $A = kQ_A/I_A$ be an indecomposable radical square zero algebra and let $B = kQ_B/I_B$ be a radical embedding of A obtained by gluing two idempotents of A. Then $\varphi_n : (Q_A)_n \to (Q_B)_n$ induces k-linear maps $\psi_{n,0} : k((Q_A)_n \| (Q_A)_0) \to k((Q_B)_n \| (Q_B)_0)$ and $\psi_{n,1} : k((Q_A)_n \| (Q_A)_1) \to k((Q_B)_n \| (Q_B)_1)$ for $n \geq 2$. In addition, $\psi_{n,1}$ in injective.*

Proof Follow the same arguments of Proposition 3.3 (3). The injectivity of $\psi_{n,1}$ follows from the injectivity of φ_n. □

Proposition 6.11 *Let A be an indecomposable radical square zero algebra and let B be a radical embedding of A obtained by gluing a source and a sink of A. Then there is an injective map $\psi_n : \mathrm{Ker}(\delta_A^n) \hookrightarrow \mathrm{Ker}(\delta_B^n)$ which restricts to $\mathrm{Im}(\delta_A^{n-1}) \hookrightarrow \mathrm{Im}(\delta_B^{n-1})$ for $n \geq 2$ (cf. Notation in Section 2). In addition, $\dim_k \mathrm{HH}^n(B) - \dim_k \mathrm{HH}^n(A) \geq 0$.*

Proof Note that $\mathrm{Ker}(\delta_A^n) = k((Q_A)_n \| (Q_A)_1) \oplus \mathrm{Ker}(D_n)$, by the proof of Theorem 2.1 in [4] we know that D_n is injective for $n \geq 2$ since the Gabriel quiver of A is not a n-crown, hence $\mathrm{Ker}(\delta_A^n) = k((Q_A)_n \| (Q_A)_1)$, and the map $\psi_n : \mathrm{Ker}(\delta_A^n) \to \mathrm{Ker}(\delta_B^n)$ is well defined by Lemma 6.9 and Lemma 6.10. The injectivity of ψ_n follows from Lemma 6.10. In order to check that ψ_n restricts to $\mathrm{Im}(\delta_A^{n-1}) \to \mathrm{Im}(\delta_B^{n-1})$, it is enough to check that $\psi_{n,1} \circ D_{n-1} = D_{n-1} \circ \psi_{n-1,0}$ (cf. Notations in Section 2). Let e be a vertex of Q_A and let $\gamma \in (Q_A)_{n-1}$ such that γ is parallel to e. On the one hand

$$\psi_{n,1} \circ D_{n-1}(\gamma \| e) = \sum_{s(a)=e, a \in (Q_A)_1} a^* \gamma^* \| a^* + (-1)^n \sum_{t(b)=e, b \in (Q_A)_1} \gamma^* b^* \| b^*.$$

On the other hand

$$D_{n-1} \circ \psi_{n-1,0}(\gamma \| e) = \sum_{s(a^*)=e, a^* \in (Q_B)_1} a^* \gamma^* \| a^* + (-1)^n \sum_{t(b^*)=e, b^* \in (Q_B)_1} \gamma^* b^* \| b^*.$$

Note that the vertex e cannot be a source or a sink. Since for the rest of the vertices there is a bijection between the number of incoming (respectively outcoming) arrows of Q_A and Q_B, then $\psi_{n,1} \circ D_{n-1} = D_{n-1} \circ \psi_{n-1,0}$ for every $n \geq 2$.

Assume that the Gabriel quiver of B is not a n-crown. Then by [4, Theorem 2.1] the expected inequality can be written as:

$$|(Q_B)_n||(Q_B)_1| - |(Q_A)_n||(Q_A)_1| \geq |(Q_B)_{n-1}||(Q_B)_0| - |(Q_A)_{n-1}||(Q_A)_0|.$$

Let $q||f$ be an element of $k((Q_B)_{n-1}||(Q_B)_0)$ which is not in $\mathrm{Im}(\psi_{n-1,0})$ restricted to $k((Q_A)_{n-1}||(Q_A)_0)$. This means that either $p = 0$ or p is not an oriented cycle, that is, $s(p) \neq t(p)$ where $p^* = q \in (Q_B)_{n-1}$. Consider now $a_1^* q||a_1^*$ where $q = a_n^* \dots a_1^*$. Then $a_1^* q||a_1^* \in k((Q_B)_n||(Q_B)_1)$ but it is not an element of $\mathrm{Im}(\psi_{n,1})$. In fact, if $p = 0$, then $a_1 p = 0$. If $s(a_1) = s(p) \neq t(p)$, then $a_1 p = 0$. This proves the above inequality.

Assume that the Gabriel quiver of B is a n-crown for $n \geq 1$. Then A is an A_{n+1}-quiver. For A_{n+1}, the dimensions of Hochschild cohomology groups is zero since A_{n+1} is hereditary. By Proposition 2.3 and Proposition 2.4 in [4] the statement follows. □

Assume that we glue a source and a sink from the same block. The following two examples show that the difference of dimensions of higher Hochschild cohomology groups is not always one. This is very different from the case $n = 1$ where $\dim_k \mathrm{HH}^1(B) - \dim_k \mathrm{HH}^1(A) = 1$, see Remark 3.24.

Example 6.12 *Let A be radical square zero algebra with Gabriel quiver given by a zig-zag type A_n quiver such that e_1 is a source vertex and e_{2n} is a sink vertex.*

$$e_1 \longrightarrow e_2 \longleftarrow e_3 \longrightarrow \dots \longleftarrow e_{2n-1} \longrightarrow e_{2n}$$

Let B be the radical embedding obtained by gluing e_1 and e_{2n}. Then $\mathrm{HH}^n(A)$ and $\mathrm{HH}^n(B)$ are zero for $n > 1$ since there are no elements of the form $|Q_n||Q_1|$.

Example 6.13 *Let m be a positive integer greater than 1. Let Q_A be the m-Kronecker quiver and Q_B be the m-multiple loop quiver. For $n > 1$ the dimension of each HH^n of the m-multiple loop quiver is $|Q_n||Q_1| - |Q_{n-1}||Q_0| = m^{n+1} - m^{n-1}$ whilst for m-Kronecker quiver the Hochschild cohomology groups are zero.*

Example 6.13 shows that any $n > 1$ although we glue a source and a sink vertex from the same block, the difference between the dimensions of $\mathrm{HH}^n(B)$ and $\mathrm{HH}^n(A)$ can be arbitrarily large. In other words, for any $n > 1$ and $M > 0$ we can always find a radical embedding obtained by gluing a source and a sink vertex from the same block such that $\dim_k \mathrm{HH}^n(B) - \dim_k \mathrm{HH}^n(A) > M$.

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